COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
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COWLES FOUNDATION DISCUSSION PAPER NO. 1089

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December 1994

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Abstract

We show that if y is an odd integer between 1 and $2^n - 1$, there is an $n \times n$ bimatrix game with exactly y Nash equilibria (NE). We conjecture that this $2^n - 1$ is a tight upper bound on the number of NEs in a "nondegenerate" $n \times n$ game. We prove the conjecture for $n \leq 3$, and provide bounds on the number of NEs in $m \times n$ nondegenerate games when $min(m, n) \leq 4$.

1. Introduction

In two-person game theory, perhaps the most important model is the so-called "strategic form" or "bimatrix game form" of a game. The game is represented as a two-dimensional matrix, in which the rows represent the (pure) strategies for one player and the columns those for the other. In each cell is placed a pair of numbers representing the payoffs to the two players if the corresponding pair of strategies is chosen.

In the analysis of bimatrix games, perhaps the most basic solution concept is that of Nash Equilibrium (NE). A pair of (mixed) strategies (p^*, q^*) is a NE provided the first player cannot do any better than to play p^* against the second player's q^*, while, likewise, the second player's "best response" against p^* is q^*. In Nash's seminal papers (1950, 1953), he proved that every bimatrix game has a NE in mixed strategies.

Since then, of course, the study of bimatrix game NEs has gone off in many directions, from applications and refinements of the NE concept to mathematical analyses into the structure of the set of NEs for a given game. In this paper we are concerned with the number of NEs in a bimatrix game. Obviously, Nash's result gives a first insight: this number must be at least one. Subsequently, Lemke and Howson (1964), showed that, under a certain "nondegeneracy assumption," the number of NEs must be finite and odd. This "oddness" result, in various guises, has also appeared in Eaves (1971), Shapley (1974), and Jansen (1981).[1] Finally, in a recent paper of Gul, Pearce, and Stacchetti (1993), it was shown that if a nondegenerate game has $2y-1$ NEs, at most y of them are pure-strategy NEs.

In Shapley's paper, in a footnote he observes (without providing a proof) that the maximum number of NEs in a "nondegenerate" 3×3 game is seven. In addition, it is a relatively simple exercise to display examples of 3×3 games having one, three, five and seven NEs. However, for $n \times n$ bimatrix games with $n > 3$, the situation is not so clear, and it is an open problem to characterize fully the numbers of NE that can occur. In this paper, we provide a partial answer to the question, by proving:

Theorem: Let n be given, and let y be any odd integer between 1 and $2^n - 1$. Then there is an $n \times n$ bimatrix game with exactly y NEs.

Hence, if one could show that it was impossible to attain more than $2^n - 1$ NEs, one would have a complete answer to the problem. In fact, we conjecture this to be so.

The paper is organized as follows. In the next section, we provide definitions and background on bimatrix games and NEs. In the following section we prove the above Theorem. In Section 4, we prove that the above conjecture is true for $n \leq 3$, and provide a discussion concerning it for higher dimensional cases. Finally, the last section consists of some concluding comments.

2. Bimatrix Games and Nash Equilibria

Let there be two players in a game, denoted by I and II. Player I has m pure strategies at his disposal, denoted by $I = \{1, ..., m\}$, while II has pure strategy set $J = \{1, ..., n\}$. A mixed strategy for player I is an element p of the $m-1$-dimensional simplex P, in which p_i is interpreted to be the probability that he plays pure strategy i. Similarly, the set of mixed strategies for II is the $n-1$-dimensional simplex Q. Given $p \in P$, define the support

[1] The "nondegeneracy" assumptions made in the three papers, however, are all stated slightly differently from one another.

of p, or $supp(p)$, to be the set $\{i \in I : p_i > 0\}$, and define $supp(q)$ for $q \in Q$ similarly. Finally, denote by e^i the mixed strategy in which I plays i with probability 1, and by e^j that in which II plays j with probability 1.

We are also given two $m \times n$ payoff matrices A and B, where a_{ij} and b_{ij} represent the payoffs for players I and II respectively, if I plays mixed strategy e^i and II plays e^j. Hence, if I chooses mixed strategy $p \in P$ and II chooses $q \in Q$, the expected payoff for I is $pAq = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i a_{ij} q_j$, while that for II is $pBq = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i b_{ij} q_j$. Since the two payoff matrices are sufficient to define a bimatrix game, we shall use the terminology "bimatrix game (A, B)."

Given $q \in Q$, p^* is a best response for I against q if $p^* Aq \geq pAq \; \forall p \in P$. Similarly, q^* is a best response for II against p if $pBq^* \geq pBq \; \forall q \in Q$. Denote by $BR_I(q)$ the set of all best responses for I against q, and by $BR_{II}(p)$ the set of all best responses for II against p. A Nash Equilibrium (NE) is a pair $(p^*, q^*) \in P \times Q$ where $p^* \in BR_I(q^*)$ and $q^* \in BR_{II}(p^*)$.

To aid us in finding NEs, let us define the sets $M_I(q)$ and $M_{II}(p)$ as follows: $M_I(q) = \{i \in I : e^i Aq \geq e^k Aq \; \forall k \in I\}$ and $M_{II}(p) = \{j \in J : pBe^j \geq pBe^k \; \forall k \in J\}$. In words, $M_I(q)$ is the set of best pure strategy responses for I against q, while a similar interpretation holds for $M_{II}(p)$. The following Lemma is then readily apparent (see, e.g., Shapley (1974) or Jansen (1981)):

Lemma 2.1: A mixed strategy pair (p, q) is a NE of bimatrix game (A, B) iff $supp(p) \subseteq M_I(q)$ and $supp(q) \subseteq M_{II}(p)$.

3. A Theorem on Possible Numbers of NEs in an $n \times n$ Bimatrix Game

In this section we state and prove the Theorem stated in the Introduction, i.e.,

Theorem 3.1: Let n be given, and let y be any odd integer between 1 and $2^n - 1$. Then there is an $n \times n$ bimatrix game with exactly y NEs.

Before proving this, we make two remarks:

Remark 1: Because one may judiciously append dominated strategies, we may generalize the Theorem as follows:

Corollary 3.2: Let m and n be given, set $M = min(m,n)$ and let y be any odd integer between 1 and $2^M - 1$. Then there is an $m \times n$ bimatrix game with exactly y NEs.

Remark 2: We should note that the proof of Theorem 3.1 is constructive, i.e., it gives an actual procedure for finding a bimatrix game with the desired number of NEs.

To prove Theorem 3.1, we first make a digression into cooperative game theory. Let $N = \{1, ..., n\}$ be the players of an n-person cooperative game, and consider the class of weighted majority games, i.e., games in which each player i is assigned a nonnegative weight w_i, there is a nonnegative quota q, and the characteristic function V is given by:

$$V(S) = \begin{cases} 1 & \text{if } \sum_{i \in S} w_i > q; \\ 0 & \text{otherwise.} \end{cases}$$

An S for which $V(S) = 1$ is called winning; otherwise coalition S is losing.

Lemma 3.3: Given n and an integer $z \in 1, ..., 2^n$, there exists an n-player weighted majority game

a) with exactly z losing coalitions,

b) for which $\sum_{i \in S} w_i \neq q \ \forall S \subseteq N$, and

c) for which $q = 1$.

Proof: To show this, we will show how to define the vector w. First, define the weight vector $\hat{w} \in \Re^n$ by $\hat{w}_i = 2^{i-1}$ for $i = 1, ..., n$. Given these weights, it is easy to see that, for $x = 0, 1, 2, ..., 2^n - 1$, there is exactly one coalition S with $\sum_{i \in S} \hat{w}_i = x$. So, if the quota were $z - \epsilon$, there would be exactly z losing coalitions. Now, [in order for c) of the Lemma to hold] we normalize, setting $w_i = \frac{\hat{w}_i}{z - \epsilon}$ for $i = 1, ..., n$. QED.

Now, given n and y as in the hypothesis of the Theorem, let $z = \frac{y+1}{2}$. It is clear that since y is an odd integer between 1 and $2^n - 1$, z is an integer between 1 and 2^{n-1}. So, let $w_1, ..., w_{n-1}$ be the weights in an $n-1$-player weighted majority game having z losing coalitions and satisfying the conclusions of Lemma 3.3. Now consider the $n \times n$ bimatrix game[2] given by

$$(A,B) = \begin{pmatrix} (1,1) & (0,0) & (0,0) & \cdots & (0,0) & (0,0) \\ (0,0) & (1,1) & (0,0) & \cdots & (0,0) & (0,0) \\ (0,0) & (0,0) & (1,1) & \cdots & (0,0) & (0,0) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (0,0) & (0,0) & (0,0) & \cdots & (1,1) & (0,0) \\ (w_1,0) & (w_2,0) & (w_3,0) & \cdots & (w_{n-1},0) & (1,1) \end{pmatrix}.$$

Let us label the set of pure strategies for player I as $I = \{1, ..., n\}$ and that for II as $J = \{1, ..., n\}$. For any $p \in P$, we define $suppn(p)$ as the set of *numbers* (not strategies!) $\{k :$ pure strategy k is an element of $supp(p)\}$; define $suppn(q)$ similarly.[3]

Lemma 3.4: Let (p, q) be any NE of (A, B). Then $suppn(p) = suppn(q)$.

Proof: First we argue that, if (p, q) is a NE, $suppn(q) \subseteq suppn(p)$. For, if not, there is a number k for which $k \in suppn(q)$ but $k \notin suppn(p)$. But Lemma 2.1 implies that pure strategy k is an element of $M_{II}(p)$, and this is impossible because if $k \notin suppn(p)$, we have $pBe^k = 0$.

Hence, if the Lemma is not true, then it must be that $suppn(q)$ is a proper subset of $suppn(p)$. If there is a $k \neq n$ with $k \in suppn(p)$ but $k \notin suppn(q)$, we get a contradiction because $k \in M_I(q)$ but $e^k Aq = 0$. So, we are down to the case where there is some $S \subseteq N/n$ for which $suppn(q) = S$ and $suppn(p) = S \cup \{n\}$. But in this case, in order for $M_I(q)$ to contain all pure strategies $k : k \in S$, one must have $q_j = \frac{1}{|S|}$ $\forall j \in S$. This gives an expected payoff of $\frac{1}{|S|}$ for all $p = e^k, k \in S$. But mixed strategy e^n then gives expected payoff $\frac{1}{|S|} \sum_{i \in S} w_i$ for player I, which is not equal to $\frac{1}{|S|}$ because of conditions b) and c) of Lemma 3.3. Hence, not all elements of $supp(p)$ can be elements of $M_I(q)$, we have a final contradiction, and the Lemma is proven.

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[2] Note that this game is nondegenerate in the sense described in Section 4.

[3] The distinction between $supp(p)$ and $suppn(p)$ is that the former is a set of pure strategies, while the latter is a subset of the numbers $\{1, ..., n\}$. Hence, while it is impossible to compare $supp(p)$ and $supp(q)$ for a mixed strategy pair (p, q), it is possible to compare $suppn(p)$ and $suppn(q)$–in fact, we do this in Lemmas 3.4 and 3.5.

Lemma 3.5: Given (A, B) as defined above, and suppose $S \subseteq N = \{1, ..., n\}$ satisfies $S \neq \emptyset$ and $S \not\ni n$. Then,

1) If $\sum_{i \in S} w_i < 1$, there is exactly 1 NE (p, q) in which $suppn(p) = S$.

2) If $\sum_{i \in S} w_i > 1$, there are no NEs (p, q) in which $suppn(p) = S$.

3) If $\sum_{i \in S} w_i < 1$, there is exactly 1 NE (p, q) in which $suppn(p) = S \cup \{n\}$.

4) If $\sum_{i \in S} w_i > 1$, there are no NEs (p, q) in which $suppn(p) = S \cup \{n\}$.

Proof: Suppose (p, q) is to be a NE in which $suppn(p) = S$. By Lemma 3.3, it is necessarily true that $suppn(q) = S$ also. In fact, in order for $supp(p)$ to be a subset of $M_I(q)$, and for $supp(q)$ to be a subset of $M_{II}(p)$, it must be that $p_i = \frac{1}{|S|}\ \forall i \in supp(p)$ and $q_j = \frac{1}{|S|}\ \forall j \in supp(q)$. This gives expected payoff $\frac{1}{|S|}$ for both players, and will clearly be a NE iff Player I has no incentive to deviate to pure strategy n. But the expected payoff for I of playing e^n is $\frac{1}{|S|} \sum_{i \in S} w_i$, so we have proven 1) and 2).

Now suppose (p, q) is to be a NE in which $suppn(p) = S \cup \{n\} = suppn(q)$. Since $supp(q) \subseteq M_{II}(p)$, we must have $p_i = \frac{1}{|S|+1}\ \forall i \in supp(p)$. Also, since $i \in M_I(q)$ for $i \in S$, we have $q_j = c$ for $j \in S$, and so $q_n = 1 - c|S|$. This gives expected payoff $\frac{1}{c}$ for I if he plays e^i, for any $i \in S$. Hence, in order for (p, q) to be a NE, I's payoff from playing e^n must also be $\frac{1}{c}$. Hence, $1 * (1 - c|S|) + c * \sum_{i \in S} w_i$ must be equal to $\frac{1}{c}$, which gives $c = \frac{1}{|S|+1-\sum_{i \in S} w_i}$. If $\sum_{i \in S} w_i < 1$, we have $c < \frac{1}{|S|}$, the vector q is in the simplex Q, and (p, q) is indeed a NE. If $\sum_{i \in S} w_i > 1$, we have $c > \frac{1}{|S|}$, and $\sum_{j \in S} q_j > 1$. Hence $q \notin Q$, and there is no NE.

Finally, let us count up the NE for game (A, B). We do this by considering each possible $T \subseteq N$, and seeing if there is a NE (p, q) in which $supp(p) = T$. First, if $T \not\ni n$, Lemma 3.5 says a NE exists iff $T \neq \emptyset$ and $\sum_{i \in T} w_i < 1$; by w's definition there are $z - 1$ such T's. On the other hand, if $T \ni n$, let $S = T/n$. If $T \neq \{n\}$, we know a NE exists iff $\sum_{i \in S} w_i < 1$; again this gives $z - 1$ NEs. But there is one more NE, for the case where $T = \{n\}$. Hence, in total we have $2z - 1$ NEs, which is exactly y. QED.

It is interesting to examine the two "endpoint cases" for our construction, i.e., where $y = 1$ and where $y = 2^n - 1$. In the former case, the construction makes pure strategy n a dominant strategy for Player I, i.e., *Player I's pure strategy best response does not depend on what Player II does.* On the other hand, the construction for the latter case calls for very low w_i's, and so Player I's best strategy is to try and play the same pure strategy as II, i.e, *his pure strategy best response is completely dependent on what Player II does.* The fact that the cases $y = 1$ and $y = 2^n - 1$ represent opposite extremes in this way leads us to believe that 1 and $2^n - 1$ represent bounds on the number of NEs one could obtain in a "nondegenerate" game. Obviously, the lower bound of 1 is known to be true; hence we are left with:

Conjecture: If an $n \times n$ bimatrix game is "nondegenerate", one cannot obtain more than $2^n - 1$ NEs.

In the next section we explore this conjecture.[4]

4. On the Maximum number of NEs in a Nondegenerate Bimatrix Game

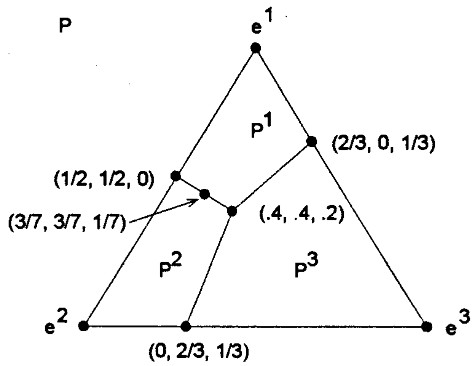
For any bimatrix game, define the sets $Q^i = \{q \in Q : i \in M_I(q)\}$ and $P^j = \{p \in P : j \in M_{II}(p)\}$. Q^i is the set of q's against which pure strategy i is a best response for I, and P^j is interpreted similarly. It is clear that the sets $\{Q^i\}_{i \in I}$ and $\{P^j\}_{j \in J}$ are convex–in fact they are polytopes and they "partition" Q and P respectively.

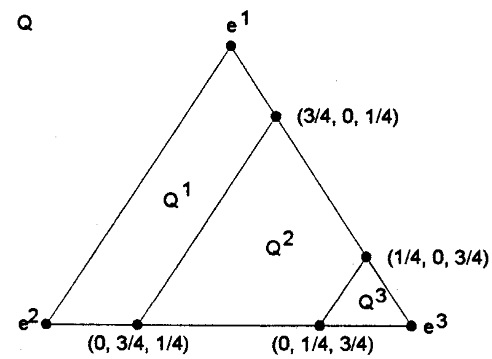
Example 4.1: $m = n = 3$

$$(A, B) = \begin{pmatrix} (4,1) & (4,0) & (0,0) \\ (3,0) & (3,1) & (3,0) \\ (0,0) & (0,0) & (4,2) \end{pmatrix}$$

Then the regions Q^1, Q^2, Q^3, P^1, P^2, and P^3 are as below:

[4] Recently the authors have proved the conjecture in the special case where $A = B$; see Quint-Shubik (1995).





The next result follows from Lemma 2.1 and the definitions of Q^i and P^j:

<u>Lemma 4.2</u> (Shapley): A mixed strategy pair (p, q) is a NE of game (A, B) iff

a) For each $i \in I$, either $p_i = 0$ or $q \in Q^i$, and

b) For each $j \in J$, either $q_j = 0$ or $p \in P^j$.

Lemma 4.2 gives us a way for graphically representing NEs. Consider simplices P and Q. For each $p \in P$, give p a label of "$j = j^*$" if $p \in P^{j^*}$, and a label of "$i = i^*$" if $p_{i^*} = 0$. Every $p \in P$ must have at least one label (because $p \in P^j$ for some j), but it is possible for a point p to have more than one label. For instance, in Example 4.1, the point $p = (\frac{3}{7}, \frac{3}{7}, \frac{1}{7})$ has labels $j = 1$ and $j = 2$, while the point $p = (0, \frac{2}{3}, \frac{1}{3})$ has labels $j = 2$, $j = 3$, and $i = 1$.

Let $L_P(p)$ be the set of labels for point p. Similarly, we can define $L_Q(q)$ as the union of the sets $\{i : q \in Q^i\} \cup \{j : q_j = 0\}$. We can see that, as long as the boundaries of the regions $\{P^j\}$, $\{Q^i\}$ and those of the simplices are in "general position", no point in P will have more than m labels, and no point in Q will have more than n. This, then, is our (=Shapley's [1974]) nondegeneracy condition:

<u>Nondegeneracy Condition</u> (Shapley 1974): No point in simplex P has more than m labels; no point in Q has more than n labels.[5]

We remark that the game in Example 4.1 satisfies this condition. In addition, we note the following consequences of the Nondegeneracy Condition:

[5] Equivalently, we may say that if $p \in P$ satisfies $|supp(p)| = k$, then there are no more than k pure strategy best responses for II against p. Similarly, if $|supp(q)| = k$, we have $|M_I(q)| \leq k$.

a) It is impossible to have a continuum of points in P, all with the same m labels (this would imply the existence of a point with $m+1$ labels).

b) Hence, for every possible set of m labels, there is at most one point in P that has it (due to the convexity of the regions $\{P^j\}$).

c) Similar conclusions can be made concerning Q.

We say the pair (p,q) is <u>completely labelled</u> if $I \cup J \subseteq L_Q(q) \cup L_P(p)$. From Lemma 4.2, we immediately have:

<u>Theorem 4.3</u> (Shapley, 1974): For bimatrix game (A,B), mixed strategy pair (p,q) is a NE iff it is completely labelled.

In addition, if a bimatrix game satisfies the Nondegeneracy Condition, we see that the set $L_Q(q) \cup L_P(p)$ has at most $m+n$ elements. Hence

<u>Corollary 4.4</u>: For bimatrix game (A,B) satisfying the Nondegeneracy Condition, pair (p,q) is a NE iff $L_Q(q)$ and $L_P(p)$ partition $I \cup J$.

In Example 4.1, the pair $(p,q) = ((0,\frac{2}{3},\frac{1}{3}),(0,\frac{1}{4},\frac{3}{4}))$ is completely labelled, because $L_Q(q) = \{j=1, i=2, i=3\}$ and $L_P(p) = \{i=1, j=2, j=3\}$. Hence (p,q) is a NE. There are two other NEs in this game, namely $(p^1,q^1) = ((1,0,0),(1,0,0))$ and $(p^2,q^2) = ((0,0,1),(0,0,1))$.

Now suppose (A,B) satisfies the Nondegeneracy Condition. For either simplex (P or Q), we define a <u>vertex</u> as a point in the simplex which has the maximal allowable number of labels (i.e., m if the point is in P, n if in Q). [Note that vertices by this definition correspond to vertices in the graph- theoretic sense in the diagram accompanying Example 4.1.] Corollary 4.4 implies that any NE can be represented as a pair of vertices, one from each simplex. The labels for the two vertices must be "complementary". Since every vertex in Q has a different label, it is clear that each vertex in P corresponds to at most one NE.

Hence, the number of NEs will be bounded by the number of vertices in P.[6] In particular, since there are $\binom{m+n}{m} - 1$[7] possible labellings for a vertex in P, we have

Proposition 4.5: If bimatrix game (A, B) satisfies the Nondegeneracy Condition, the number of NEs is bounded by $\binom{m+n}{m} - 1$.

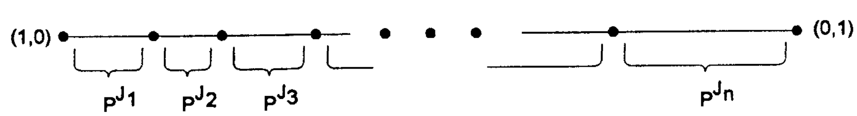
However, in the case where $m = n$, the bound expressed in Proposition 4.5 is much higher than the desired $2^n - 1$. However, it is the best bound (that we know of) that exists for arbitrarily large bimatrices. For now on, we center the analysis on the cases where one player (say, Player I) has relatively few strategies, i.e., m is a low number.

THE CASE m=1.

In this case a bimatrix game (A, B) satisfies the Nondegeneracy Condition if and only if $argmax_j b_{1j}$ is a one-element set. If so, there is exactly one NE.

THE CASE m=2.[8]

In this case P is the line segment from $p = e^1 = (1,0)$ to $p = e^2 = (0,1)$. The regions $\{P^j\}$ are subsegments of P, which don't intersect one another except possibly at endpoints. In addition, the endpoints e^1 and e^2 have labels $i = 2$ and $i = 1$ respectively. Hence there are at most $n + 1$ vertices (see below).



This implies

Lemma 4.6: If a $2 \times n$ bimatrix game satisfies the Nondegeneracy Condition, it has no more than $n + 1$ NEs.

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[6] and, using a similar argument, by the number of vertices in Q.
[7] the "-1" comes about because it is impossible for $p_i = 0\ \forall i$, so the label $i = 1, i = 2, ..., i = m$ cannot occur.
[8] According to Peter Sudholter (personal communication, July 1994), the results in the $m = 2$ case are already known but have never been published. We suspect the same is true of Proposition 4.5.

In particular, we get a bound of 3 in the important 2×2 case. For general n, we can actually improve upon Lemma 4.6 by combining it with the "oddness" result mentioned in the Introduction:

Corollary 4.7: If a $2 \times n$ bimatrix game satisfies the Nondegeneracy Condition, it has no more than $2 * int(\frac{n}{2}) + 1$ NEs.

As the following construction shows, the bound expressed in Corollary 4.7 is tight:

Example 4.8:[9] (A, B) is equal to

$$\begin{pmatrix} (n,n) & (0,n-1) & (n-2,n-2) & (0,n-3) & \dots & (1,1) \\ (0,0) & (2,1) & (0,1+\frac{1}{2}) & (4,1+\frac{1}{2}+\frac{1}{3}) & \dots & (0,\sum_{x=1}^{n-1}\frac{1}{x}) \end{pmatrix} \text{ if } n \text{ is odd,}$$

$$\begin{pmatrix} (n,n) & (0,n-1) & (n-2,n-2) & (0,n-3) & \dots & (0,1) \\ (0,0) & (2,1) & (0,1+\frac{1}{2}) & (4,1+\frac{1}{2}+\frac{1}{3}) & \dots & (n,\sum_{x=1}^{n-1}\frac{1}{x}) \end{pmatrix} \text{ if } n \text{ is even.}$$

Claim: Let $k \in 1, ..., n-1$. [n may be odd or even.] Then there is a NE (p, q) in the game above in which $suppn(p) = \{1, 2\}$ and $suppn(q) = \{k, k+1\}$.

Proof: Suppose k is odd. Consider the mixed strategies (p, q), where $p = (\frac{1}{k+1}, \frac{k}{k+1})$ and where $q_k = \frac{k+1}{n+2}$, $q_{k+1} = \frac{n-k+1}{n+2}$, and $q_j = 0$ for $j \neq k, k+1$. We claim this is a NE. Certainly player I obtains an expected payoff of $\frac{(n-k+1)(k+1)}{n+2}$ by playing either of his pure strategies against q. In addition, player II obtains an expected payoff of

$$\frac{n-k+1}{k+1} + \frac{k}{k+1}\sum_{x=1}^{k-1}\frac{1}{x} \tag{4.1}$$

by playing either pure strategy k or $k+1$ against p. So, if we can show that every other pure strategy for II pays off less than this amount, Lemma 2.1 will imply the Claim.

Consider pure strategy j, where $j \neq k, k+1$. Then, against p, j pays off $\frac{n-j+1}{k+1} + \frac{k}{k+1}\sum_{x=1}^{j-1}\frac{1}{x}$. If $j < k$, the first term of this expression is $\frac{k-j}{k+1}$ more than the corresponding term in (4.1). The second term is $\frac{k}{k+1}\sum_{x=j}^{k-1}\frac{1}{x}$ less than than its corresponding term. But $\frac{k}{k+1}\sum_{x=j}^{k-1}\frac{1}{x} > \frac{k}{k+1}\sum_{x=j}^{k-1}\frac{1}{k} = \frac{k-j}{k+1}$, so indeed j pays off less than k (or $k+1$). Similarly, if

[9] The reader may verify that this example satisfies the Nondegeneracy Condition by verifying the conditions expressed in footnote 5.

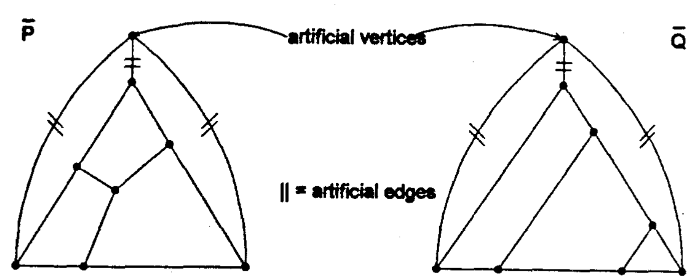
$j > k+1$, the first term in the expression is $\frac{i-k}{k+1}$ *smaller* than the first term in (4.1). The second term is greater than the second term in (4.1), but by an amount less than $\frac{i-k}{k+1}$.

A similar argument can be used for the case where k is even. QED.

Hence, in total we have found $n-1$ NEs having the form laid out in the Claim. In addition, in the n odd case there is one more pure NE, while in the n even case, there are two more pure NEs. Hence the total is exactly that stated in the bound of Corollary 4.7.

THE CASE m=3.

To analyze cases where $m \geq 3$, we employ Lemke-Howson diagrams (see Shapley 1974). We start with the diagrams in which P and Q are divided into the regions $\{P^j\}_{j\in J}$ and $\{Q^i\}_{i\in I}$ respectively (see Example 4.1). We then add an "artificial vertex" to the P-simplex, together with m "artificial edges", one each connecting the artificial vertex with an extreme point e^i of P. We denote the resulting structure as $\overline{P}$. Similarly, we add an artificial vertex and n artificial edges to Q, creating $\overline{Q}$. As an example, we present the Lemke-Howson diagrams for Example 4.1 below:



With the example as a guide, we note the following:

Observation (Shapley): Each vertex ("real" or "artificial") in $\overline{P}$ is incident to m edges, while each vertex in $\overline{Q}$ is incident to n edges.

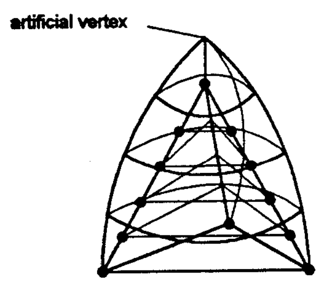
Now we can analyze the $3 \times n$ case. Viewing $\overline{P}$ as a graph, we let V be the number of vertices, E the number of edges, and F the number of faces. By Euler's Theorem, we know that $V - E + F = 2$. In addition, by the Observation, every vertex in $\overline{P}$ is incident to three edges; hence, $E = \frac{3V}{2}$. This implies $F - \frac{V}{2} = 2$, or $V = 2F - 4$. But from the construction, we see that $\overline{P}$ divides $\Re^2$ up into at most $n+3$ regions;[10] hence $F \leq n+3$. So we have $V \leq 2n+2$. Now one of these vertices is artificial, but the rest may correspond to NEs; hence we have

Lemma 4.9: If a $3 \times n$ bimatrix game satisfies the Nondegeneracy Condition, there are no more than $2n+1$ NEs.

Again we note that in the 3×3 case, this gives an upper bound of 7 NEs, which is indeed $2^n - 1$.

THE CASE $m = 4$.

We can use the same type of analysis to derive a bound on the number of NEs in a $4 \times n$ game. We need to view $\overline{P}$ as a three-dimensional object in the following way. The simplex (=tetrahedron) P is again divided up into the n regions $\{P^j\}_{j=1}^n$ (in the example below $n = 5$). The artificial vertex is at the top. The 4 artificial edges determine two-dimensional "buttresses" lying directly on top of each upward- sloping edge of the tetrahedron. Finally, we lay (three) surfaces from the buttresses, creating three "interior" artificial regions. The fourth artificial region is the rest of $\Re^3$.



The corresponding "Euler's Theorem" for this object is $V - E + F - R = 0$.[11] Since each vertex in $\overline{P}$ is incident to four edges, we have $E = 2V$. Hence $V = F - R$. Furthermore, we know that $5 \leq R \leq n+4$. Finally, from the geometry each two- dimensional facet

[10] There are at most n regions inside of P corresponding to the $\{P^j\}$, plus the three "artificial" regions outside of P; compare with the construction in the $m = 4$ case.

[11] This holds because the Euler Characteristic is invariant under subdivision. See e.g., Massey (1991), Chapter 9. Here "F" stands for the number of two-dimensional facets, and "R" for the number of three-dimensional regions. In the above example $V = 17$, $E = 34$, $F = 26$, and $R = 9$.

divides a *different* pair of three-dimensional regions, and so $F \leq \binom{R}{2}$. Hence $F - R \leq \max_{r:5 \leq r \leq n+4} \binom{r}{2} - r$. Now $\binom{r}{2} - r$ is an increasing function of r, so it is maximized at $r = n+4$. Hence we have $F - R \leq \binom{n+4}{2} - (n+4)$. Subtracting 1 for the artificial vertex, we have that $V \leq \binom{n+4}{2} - (n+5) = \frac{n^2+5n+2}{2}$.

Lemma 4.10: If a $4 \times n$ bimatrix game satisfies the Nondegeneracy Condition, there are no more than $\frac{n^2+5n+2}{2}$ NEs.

Unfortunately, when we evaluate the bound in Lemma 4.10 in the 4×4 case, we get an upper bound of 19 for the number of NEs, not 15.

In general, we will not be able to bound the number of NEs at $2^n - 1$ (in the $n \times n$ case) merely by bounding the number of vertices in Lemke-Howson diagrams. [In fact, we have already found an example of a 6×6 bimatrix game where both the P and Q simplices have 66 vertices.] Indeed, in order to prove our conjectured bound, we will have to somehow use the fact that not every vertex in P need correspond to a NE. One possibility in this vein is to use index theory (Shapley, 1974), from which the following Lemma may be deduced:

Lemma 4.11: Let (A, B) be a bimatrix game satisfying the Nondegeneracy Condition. Consider the graph G_P whose vertices and edges are those of P in the Lemke-Howson diagram. Let G_P^* be the subgraph of G_P induced by the set of vertices corresponding to NEs. Then G_P^* is bipartite. Furthermore, one part of G_P^* will have one more node than the other part.

Proof: Follows directly from Shapley (1974), especially the discussion at the bottom of page 188.

In a sense, Lemma 4.11 places limits on the set of vertices which can correspond to NEs. Perhaps results like this could be used to bound the maximum number of NEs.

5. Notes and Open Questions

SOME COUNTING

Although there has been extensive work done on NEs, there are still many open questions concerning their frequency of occurrence in bimatrix games.

The class of 2×2 games is essentially the only case of interest for which one can do an exhaustive enumeration. There are 78 strategically different games with payoffs without ties (see Barany-Lee-Shubik, 1993). [The number of strategically different 3×3 games is on the order of $\frac{9!9!}{3!3!2} = 2 \times 10^9$.]

Concerning the 2×2 games, the distribution of NEs is as follows:

	Pure	Mixed	Number of Games
Games with 1 NE	1	0	58
Games with 1 NE	0	1	9
Games with 3 NE	2	1	11

For an $n \times n$ matrix, we know that the lower bound on the number of pure strategy NEs is 0, the upper bound is n, the expected number of pure NEs for a "randomly selected game" is 1, and this number is Poisson distributed for large n (see Powers 1990, Stanford 1993). Hence, for large n, the fraction of games with 1 or more pure strategy NE remains large, tending toward $1 - \frac{1}{e}$.

The lower bound on the number of mixed strategy NEs is 0, and we conjecture that the upper bound is $2^n - 1 - n$. [From our analysis in Section 4, it is easy to show that this upper bound holds tightly for $n = 1, 2, 3$.] We are not aware of any results concerning the expected number of mixed strategy NEs or about the distribution of games with different numbers of mixed strategy NEs.

Considering the class of strategically different 2×2 games, $\frac{58}{78} = 74.36\%$ have a unique NE which is pure, and $\frac{9}{78} = 14.10\%$ have a unique NE which is mixed. These are fairly large percentages. For larger bimatrices, it appears that the percentage of games with unique NEs quickly becomes much smaller.

A reason for being concerned with uniqueness is that the chances to find a tractable dynamic are far better if there is a unique attractor in the system than if there are many.

METRICS, SMOOTHNESS, ROUGHNESS, AND APPLICATIONS

By selecting the set of all strategically different games, we are considering (almost) all possible smooth or rough payoff surfaces for the two players. But in applications, there is usually some structure to the payoffs which imposes constraints on the form of the payoff functions. In economics, for example, a natural metric on the strategy sets is available when we consider a strategy to be an offer of an amount of a good or money. Conditions on preferences transform the outcomes from markets into payoffs in a structured way. Models for biology may show extrememly different forms of regularity.

From the viewpoint of mathematical curiosity, the investigation of the set of all strategically different bimatrix games may be of interest, but in applications the selection of the appropriate set of games is a matter of understanding context and the substantive questions at stake.

WHY NASH EQUILIBRIUM AS A SOLUTION?

Although we believe there are many useful applications of noncooperative equilibrium theory, we suggest they may depend on the relevance of the context of contract, communication, and coordination available to individuals. The $n \times n$ example below is provided to raise questions concerning the plausibility or desirability of the NE solution concept.

$$\begin{pmatrix} (1+\frac{\epsilon}{n},1-\frac{\epsilon}{n}) & (1-\frac{\epsilon}{n},1+\frac{\epsilon}{n}) & (0,0) & \dots & (0,0) \\ (0,0) & (1+\frac{\epsilon}{n},1-\frac{\epsilon}{n}) & (1-\frac{\epsilon}{n},1+\frac{\epsilon}{n}) & \dots & (0,0) \\ (0,0) & (0,0) & (1+\frac{\epsilon}{n},1-\frac{\epsilon}{n}) & \dots & (0,0) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (1-\frac{\epsilon}{n},1+\frac{\epsilon}{n}) & (0,0) & (0,0) & \dots & (1+\frac{\epsilon}{n},1-\frac{\epsilon}{n}) \end{pmatrix}$$

Consider the set of all games for $n \geq 3$ and odd. As n becomes larger, the expected payoff for the unique mixed strategy NE[12] approaches 0 for both players. However, there are $2n$ pure strategy ϵ-equilibrium points, which could be achieved with some form of communication or coordination. Each of these gives both players payoffs approaching 1.

[12] The unique mixed strategy NE is for both players to play each of their n strategies with probability $\frac{1}{n}$, giving expected payoffs $(\frac{2}{n}, \frac{2}{n})$.

REFERENCES

Barany, I., J. Lee, and M. Shubik (1992), "Classification of Two-Person Ordinal Bimatrix Games", International Journal of Game Theory, vol. 21, pp. 267-290.

Eaves, B.C. (1971) "The Linear Complementarity Problem", Management Science, vol. 17, pp. 612- 634.

Gul, F., D. Pearce, and E. Stacchetti (1993), "A Bound on the Proportion of Pure Strategy Equilibria in Generic Games", Mathematics of Operations Research, vol. 18, pp. 548-552.

Jansen, M. (1981) "Regularity and Stability of Equilibrium Points of Bimatrix Games", Mathematics of Operations Research, vol. 6, pp. 530-550.

Lemke, C.E., and J.T. Howson (1964), "Equilibrium Points in Bimatrix Games", SIAM Journal of Applied Math, vol. 12, pp. 413-423.

Massey, W. (1991), A Basic Course in Algebraic Topology, Springer-Verlag, New York.

Nash, J.F. (1950), "Equilibrium Points in n-person Games", Proceedings of the National Academy of Sciences USA, vol. 36, pp. 48-9.

Nash, J.F. (1953), "Noncooperative Games", Annals of Mathematics, vol. 54, pp. 286-95.

Powers, I. (1990), "Limiting Distributions on the Number of Pure Strategy Nash Equilibria in n-person Games", International Journal of Game Theory, vol. 19, pp. 277-286.

Quint, T. and M. Shubik (1995), "A Bound on the Number of Nash Equilibria in a Coordination Game", Technical Report, Yale University.

Shapley, L.S. (1974), "A Note on the Lemke-Howson Algorithm", Mathematical Programming Study, vol. 1, pp. 175-89.

Stanford, W. (1993), "The Poisson Distribution and Probability of k Pure Equilibria in Matrix Games", Technical Report, Department of Economics, University of Illinois-Chicago.