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THE TOPOLOGICAL STRUCTURE OF MAXIMAL LATTICE FREE CONVEX BODIES: THE GENERAL CASE

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The Topological Structure of Maximal Lattice Free Convex Bodies: The General Case

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Abstract. Given a generic $m \times n$ matrix A, the simplicial complex $\mathcal{K}(A)$ is defined to be the collection of simplices representing maximal lattice point free convex bodies of the form $\{x : Ax \leq b\}$. The main result of this paper is that the topological space associated with $\mathcal{K}(A)$ is homeomorphic with R^{m-1}.

1 Introduction

The major question in integer programming is to decide whether or not a given convex body contains integral points. The convex body is usually given as the set of solutions to a system of linear inequalities

$$Ax \leq b \tag{1.1}$$

where A is an m by n matrix $(m > n)$ and $b \in R^m$. In this paper we prove a theorem describing the topological structure of the collection of maximal lattice point free convex bodies of the above form when the matrix A is fixed and b varies.

Let a_i denote the ith row of A, so $a_i \in R^n$. We need the following conditions on A.

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A1. There is a strictly positive row vector $\lambda \in R^m$ with $\lambda A = 0$.

A2. If, for some $i \in \{1, ..., m\}$ and $z \in \mathbb{Z}^n$ $a_i z = 0$, then $z = 0$.

A3. The $n \times n$ minors of A are all nonsingular.

The first and the third condition here imply that for any $b \in R^m$ the convex set

$$K_b = \{x \in R^m \ : \ Ax \leq b\} \tag{1.2}$$

is bounded. Condition A2 asserts that the hyperplane $a_i x = \beta_i$ contains at most one lattice point. This condition is more stringent than necessary for our analysis and and can be relaxed to allow an open set of matrices containing A in its interior.

Definition. K_b is a maximal lattice free convex body (or MLFC body, for short), if

(1) K_b has no lattice points in its interior,

(2) any closed convex body which properly contains K_b does have a lattice point in its interior.

By A1 and A3, K_b is a convex polytope. Notice that if K_b is a MLFC body, then so is $z + K_b$ for every $z \in \mathbb{Z}^n$.

Condition (2) implies that every facet of a MLFC body K_b contains a unique lattice point in its relative interior. Let z^i be this lattice point when the facet is defined by the ith inequality $a_i x \leq \beta_i$. Some inequalities $a_i x \leq \beta_i$ may not define a facet of K_b in which case the inequality $a_i x \leq \beta_i$ can be replaced by $a_i x \leq \overline{\beta}_i$ with any $\overline{\beta}_i > \beta_i$ without changing K_b. Thus different right-hand sides (i.e., different b's) may give rise to the same MLFC body.

To avoid this ambiguity we set $\overline{\beta}_i = +\infty$ for an inequality that does not define a facet. A convenient way to do this is to introduce "ideal points" $w^1, w^2, ..., w^m$ by defining

$$a_i w^j = \begin{cases} +\infty & \text{if } i = j, \\ -\infty & \text{otherwise.} \end{cases}$$

Let $W = \{w^1, ..., w^m\}$.

Assume now that K_b is a MLFC body. We shall represent it by an m-element set $\sigma \subset \mathbb{Z}^n \cup W$ in the following way. For $i = 1,\ 2,\ \ldots,\ m$ define

$$s^i = \begin{cases} z^i & \text{if } a_i x \leq \beta_i \text{ defines a facet, and } z^i \in \mathbb{Z}^n \text{ is on this facet,} \\ w^i & \text{otherwise.} \end{cases}$$

Let $\sigma = \{s^1,\ s^2,\ \ldots,\ s^m\}$.

On the other hand, an m-element set $\sigma \subset \mathbb{Z}^n \cup W$ determines a convex set K_b via

$$\beta_i = \max\{a_i s\ :\ s \in \sigma\},\ \text{and}\ b = (\beta_1,\ \ldots,\ \beta_m)^T.$$

The set K_b is a MLFC body if the elements of σ can be indexed as $\sigma = \{s^1,\ s^2,\ \ldots,\ s^m\}$ so that the following holds: $\beta_i = a_i s^i$ $(i = 1,\ \ldots,\ m)$, $a_i s^j < \beta_i$, if $j \neq i$, and there is no $z \in \mathbb{Z}^n$ with $a_i z < \beta_i$ for all $i = 1,\ \ldots,\ m$.

Define now the *complex* $\mathcal{K}(A)$ associated with this collection of MLFC bodies as the simplicial complex whose simplices are the finite sets σ representing MLFC bodies together with their subsimplices. The vertex set of $\mathcal{K}(A)$ is $\mathbb{Z}^n \cup W$ so it is infinite. Given a simplex $\sigma = \{z^1,\ \ldots,\ z^p, w^{j_1},\ \ldots,\ w^{j_q}\} \in \mathcal{K}(A)$ with $p \geq 1$, its *cell*, $|\sigma|$, is the set of all *abstract mixed* combinations from σ that are defined as

$$x = \sum_{k=1}^{p} \gamma(k) z^k + \sum_{\ell=1}^{q} \beta(j_\ell) w^{j_\ell} \tag{1.3}$$

where $\gamma(k), \beta(j_\ell) \geq 0$ and $\sum_1^p \gamma(k) = 1$. Notice that $|\sigma|$ is not a subset of R^n since the points z^i and w^j are thought of as abstract points.

The *body* of $\mathcal{K}(A)$, $|\mathcal{K}(A)|$, is the union of cells of simplices σ containing at least one non–ideal point. This is not the usual definition of the body of a simplicial complex but it suits our purposes well.

We will show later (Lemma 2 in Section 5) that every point of $|\mathcal{K}(A)|$ is contained in finitely many cells of $\mathcal{K}(A)$, i.e., $\mathcal{K}(A)$ is locally finite except possibly at the ideal points. This implies that the topology of $|\mathcal{K}(A)|$ is well defined.

Now we can state our main result.

Theorem 1. *$|\mathcal{K}(A)|$ is homeomorphic to R^{m-1}.*

This theorem is a generalization of a result from [1] where the case $m = n + 1$ is considered. The constructions and the proofs of this paper take their origin from [1], but a different and novel approach is needed here at several places: the well conditioning assumption A3 is necessary here to ensure local finiteness of $\mathcal{K}(A)$; there are no ideal points when $m = n + 1$: and the geometric realization of $\mathcal{K}(A)$ (see Section 7) is simpler in [1].

2 Examples

Before presenting further theorems and the proofs it is instructive to consider a few examples.

When $m = n + 1$, ideal points are not needed since every MLFC body is a simplex. When $n = 2$ and $m = 3$, $\mathcal{K}(A)$ has a particularly simple structure (cf. [7]). Namely, there is a basis, e^1, e^2, of the lattice $\mathbb{Z}^2$ such that the simplices of $\mathcal{K}(A)$ are lattice translates of $\{0, e^1, e^1 + e^2\}$ and $\{0, e^2, e^1 + e^2\}$. The corresponding triangles and their lattice translates form a tiling of the whole plane and constitute a simple geometric realization of $\mathcal{K}(A)$ as R^2 (see Figure 1).

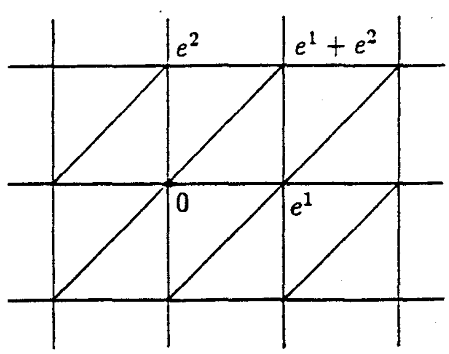

Figure 1. The 3×2 case

When $n = 1$ and $m = 3$ the inequalities in the system (1.1) can be put in the form $-x \leq \beta_1$, $x \leq \beta_2$, $x \leq \beta_3$. The MLFC bodies are the intervals $[k, k + 1]$ ($k \in \mathbb{Z}$). They are represented by simplices of $\mathcal{K}(A)$ of the form

$$\{k,\ w^2,\ k+1\} \text{ and } \{k,\ k+1,\ w^3\}.$$

The ideal point w^1 does not appear in any simplex of $\mathcal{K}(A)$. $|\mathcal{K}(A)|$ is given in two ways in Figure 2: first the ideal points are in the plane, and, second, they are placed at infinity.

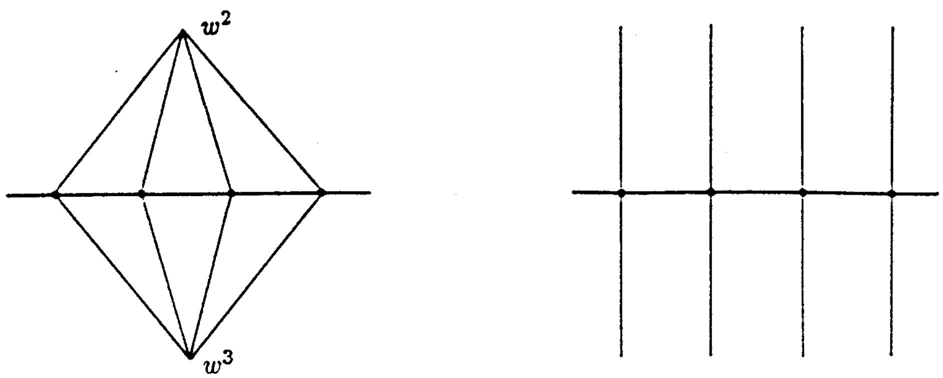

Figure 2. The 3×1 case

The case $n = 2$, $m = 4$ can be treated using results of [7]. In this case some three of the inequalities in (1.1), $a_1 x \leq \beta_1$, $a_2 x \leq \beta_2$, $a_3 x \leq \beta_3$, say, determine a bounded region and the 3 by 2 case applies. Each of the two types of simplices obtained from these three inequalities alone is augmented by w^4 in order to get a maximal simplex in $\mathcal{K}(A)$. Some other three inequalities, $a_2 x \leq \beta_2$, $a_3 x \leq \beta_3$, $a_4 x \leq \beta_3$ say, also determine a bounded region, and the 3 by 2 case applies again. Of the ideal points only w^1 and w^4 are needed and they only appear in this way. The remaining maximal lattice free bodies do not involve the ideal points; the four lines corresponding to the four inequalities are placed at four lattice points z^1, z^2, z^3, z^4 whose convex hull is a

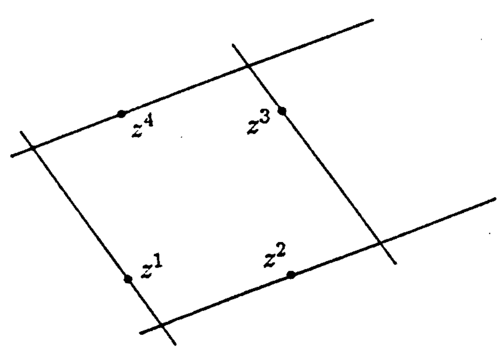

Figure 3. The 4×2 case

parallelogram of unit area. One can visualize the abstract simplicial complex $\mathcal{K}(A)$ as the collection of "3–dimensional" parallelograms, with vertices coming from $\mathbb{Z}^2$. The boundary of their union consists of two pieces: each piece is homeomorphic to R^2 and corresponds to the tiling (of R^2) by triangles from the 3 by 2 subcases. (Above each tiling there is a suspension to infinity by w^1 and w^4.) This is what we like to call the quilted paplan.

As these simple examples show, not all ideal points belong to simplices of $\mathcal{K}(A)$. On the other hand, a result of Doignon [3], Scarf [6], and Bell [2] states that a MLFC body can have at most 2^n facets. Thus for a maximal dimensional simplex $\sigma = \{z^1, z^2, ..., z^k, w^{j_1}, ... w^{j_{m-k}}\} \in \mathcal{K}(A)$ one has $n+1 \leq k \leq 2^n$.

As we mentioned, the well-conditioning assumption A3 ensures the local finiteness of $\mathcal{K}(A)$. An example due to Lovász [5] shows that if A3 does not hold, then $\mathcal{K}(A)$ may not be locally finite.

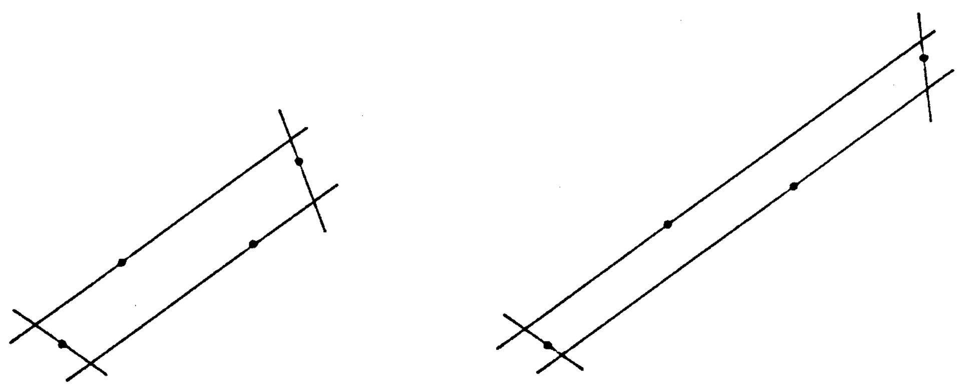

Figure 4. $\mathcal{K}(A)$ is not locally finite

The example is for the 4 by 2 case: two of the vectors, say a_1 and a_2 are opposite ($a_1 + a_2 = 0$) and have irrational slope. Figure 4 depicts two parallelograms $\{z^1, z^2, z^3, z^4\} \in \mathcal{K}(A)$, from an infinite sequence of parallelograms that contain the point $z^4 = 0$ and correspond to MLFC bodies. A3 is violated here by the 2 by 3 minor $[a_1, a_2]^T$ of A.

We mention further that the same well-conditioning assumption A3 was needed in [5] in order to show that the "shapes" of the MLFC bodies of the type K_b (with A fixed, again) can be approximated by the shapes of a finite subset of this type. Details can be found in [5].

3 The Exponential Map

The proof of Theorem 1 will be based on a geometric realization of $\mathcal{K}(A)$. The key construction is the *exponential map* $E : R^n \times (0, \infty) \to R^m$ defined by

$$E(x, t) = (\exp\{ta_1x\}, \exp\{ta_2x\}, ..., \exp\{ta_mx\})^T.$$

Quite often the parameter $t \in (0, \infty)$ is not important and we simply write $E_t(x)$ or $E(x)$.

Consider now $\lambda \in R^m_+$ from condition A1 and set

$$M = \{y \in R^m_+ : \prod_{i=1}^m y_i^{\lambda_i} = 1\}. \tag{3.1}$$

Notice that M is the boundary of the strictly convex set $\{y \in R^m_+ : \Pi y_i^{\lambda_i} \geq 1\}$. Further, $E_t(x) \in M$ for every $x \in R^m$.

We remark that, more generally, for a row vector $\mu \in R^m_+$ with $\mu A = 0$ one could define

$$M(\mu) = \{y \in R^m_+ : \prod_1^m y_i^{\mu_i} = 1\}.$$

It follows then that $E_t(x) \in M(\mu)$ for every such μ so that E_t maps R^n to $\bigcap M(\mu)$. In what follows, however, we will only make use of this fact with $\mu = \lambda$.

Define now $V_t = E_t(\mathbb{Z}^n)$, obviously $V_t \subset M$. Moreover, no point of V_t is contained in the convex hull of other points of V_t. Define

$$C_t = R^m_+ + \text{conv } V_t,$$

a convex set that has extreme points $y \in V_t$. Denote the standard basis of R^m by $\{e(1), ..., e(m)\}$.

Let $v^1, ..., v^p \in V_t$ $(p \geq 1)$ and $j_1, ..., j_q \in \{1, ..., m\}$ $(q \geq 0)$ and define

$$F = \text{conv}\{v^1, ..., v^p\} + \text{pos}\{e(j_1), ..., e(j_q)\} \tag{3.2}$$

where conv$X$ and pos$X$ denote the set of convex combinations and non–negative combinations, respectively, of the elements of X. Clearly, F lies in C_t and is a convex polyhedron. F will be called a *face* of C_t if it is the intersection of C_t with a supporting hyperplane. In this way we can define vertices, edges, ..., facets of C_t as well. It is easy to see that the vertices of C_t are the points in V_t.

The connection between $\mathcal{K}(A)$ and the facets of C_t is established in the following theorem.

Theorem 2. *There is a $t_0 > 0$ such that for $t > t_0$ the following statements are equivalent.*

(1) $\sigma = \{z^1, ..., z^p, w^{j_1}, ..., w^{j_q}\}$ *is a maximal simplex of* $\mathcal{K}(A)$ *(i.e., $p + q = m$).*

(2) $F = \text{conv}\{E_t(z^1), ..., E_t(z^p)\} + \text{pos}\{e(j_1), ..., e(j_q)\}$ *is a facet of* C_t.

It follows from Theorem 2 that for $t \geq t_0$, $p + q = m$ holds for the facet F in (3.2).

The boundary of C_t is going to be a geometric realization of the complex $\mathcal{K}(A)$. In order to show this we have to prove that the boundary of C_t consists of faces of the type (3.2).

Theorem 3. *C_t is a closed set. Its boundary consists of faces of the form (3.2) with $v^i = E_t(z^i)$ for some $z^i \in \mathbb{Z}^n$ $(i = 1, ..., p)$.*

Notice that every point of C_t is of the form $\sum \alpha_i E_t(z^i) + \sum \beta_j e(j)$ where the first sum is a convex combination and the second is a nonnegative combination. Thus the first part of Theorem 3 implies the second. We mention further that Theorems 2 and 3 show that the combinatorial structure of the face lattice of C_t stabilizes after $t > t_0$.

4 $\mathbb{Z}^n$ Acts on $\mathcal{K}(A)$ and C

We mentioned earlier that $\mathcal{K}(A)$ is invariant under translations by integers. Precisely, given $z \in \mathbb{Z}^n$ define

$$T_z(x) = \begin{cases} z + x & \text{when } x \in R^n, \\ x & \text{when } x \in W. \end{cases}$$

The group of translations $T^n = \{T_z \ : \ z \in \mathbb{Z}^n\}$ is isomorphic to $\mathbb{Z}^n$ and leaves $\mathcal{K}(A)$ invariant, i.e., if $\sigma \in \mathcal{K}(A)$, then $T_z(\sigma) = \{s + z \ : \ s \in \sigma\} \in \mathcal{K}(A)$ as well. The *orbit* of $\sigma \in \mathcal{K}(A)$ under T^n is the set of all simplices of the form $T\sigma$ with $T \in T^n$. Moreover, T^n acts transitively on the vertices of $\mathcal{K}(A)$ (belonging to $\mathbb{Z}^n$), i.e., for every pair $z, v \in \mathbb{Z}^n$ there is a $T \in T^n$ with $z = Tv$. So we have the following simple

Lemma 1. *The orbit of every $\sigma \in \mathcal{K}(A)$ with $\sigma \cap \mathbb{Z} \neq \emptyset$ contains a simplex with a vertex at the origin.*

$\mathbb{Z}^n$ acts on the convex set C_t as well in the following way. Given $z \in \mathbb{Z}^n$ define the $m \times m$ diagonal matrix D_z as

$$D_z = \mathrm{diag}(\exp\{ta_1 z\}, \ \ldots, \ \exp\{ta_m z\}).$$

$D_z : R^m \to R^m$ is a nonsingular linear map and $D^n = \{D_z : z \in \mathbb{Z}^n\}$ is a group isomorphic to $\mathbb{Z}^n$. Notice that D_z leaves V_t and R^m_+ invariant since

$$D_z E_t(z_0) = E_t(z + z_0) \text{ and } D_z R^m_+ = R^m_+.$$

It follows that $D_z C_t = C_t$ so that C_t is invariant under the group $\mathbb{D}^n$ of linear transformation. This implies that if F is a face of C_t then so is $D_z F$. It is clear, moreover, that $\mathbb{D}^n$ acts transitively on the vertices of C_t and therefore C_t looks the same at every one of its vertices. Thus C_t is a highly symmetric convex set which is, as we shall see later, locally a polytope.

As the group T^n acts on $|\mathcal{K}(A)|$ one can factor it out to obtain the topological space $|\mathcal{K}(A)|/T^n$. We shall prove

Theorem 4. *$|\mathcal{K}(A)|/T^n$ is homeomorphic to the direct product of the n–torus and R^{m-n-1}.*

This result is the natural extension of Theorem 2 from [1]. Its proof uses equivariance as well but this time the exponential map is not simplicial and we have to use an unusual extension of E, cf. (8.1).

5 Auxiliary Results and Proof of Theorem 3

We will need a few properties of the complex $\mathcal{K}(A)$. The first is local finiteness which we state in the form of

Lemma 2. *Each lattice point $z \in \mathbb{Z}^n$ is contained in a finite number of simplices of $\mathcal{K}(A)$.*

Proof. It is enough to prove this for $z = 0$. Assume, to the contrary, that an infinite number of maximal dimensional simplices, $\sigma(1)$, $\sigma(2)$, ... $\in \mathcal{K}(A)$ contain 0. We can further assume (after possibly reordering the rows of A and deleting some of the $\sigma(k)$) that each $\sigma(k)$ is of the form

$$\sigma(k) = \{z^1(k),\ ...,\ z^p(k),\ w^{p+1},\ ...,\ w^m\}$$

where $z^1(k) = 0$ $(\forall k)$ and

$$\max_{j=1,\dots,p} a_i z^j(k) = a_i z^i(k) =: \beta_i(k)\ (i = 1,\ ...,\ p).$$

As the sequence $\sigma(k)$ is infinite, some of the $\beta_i(k)$ cannot be bounded. Assume (again by deleting some of the $\sigma(k)$) that

$$\beta_i(k) \to \beta_i \text{ for } i = 1,\ 2,\ ...,\ p', \text{ and}$$

$$\beta_i(k) \nearrow \infty \text{ for } i = p' + 1,\ ...,\ p$$

where $0 < p' < p$ and $\beta_i < \infty$ for $i = 1,\ ...,\ p'$. Notice that $\beta_i(k) = a_i z^i(k) > a_i z^0(k)$ so that

$$\beta_i \geq 0 \text{ for } i = 1,\ ...,\ p'.$$

Moreover, the sets

$$Q(k) = \{x \in R^n\ :\ a_i x \leq \beta_i(k),\ i = 1,\ ...,\ p'\}$$

cannot be bounded (they contain the infinite sequence $z^p(k)$). Consequently the cone

$$Q(0) = \{x \in R^n\ :\ a_i x \leq 0,\ i = 1,\ ...,\ p'\} \subset Q(k)$$

is not bounded. Now condition A3 readily implies that int $Q(0) \neq \phi$. Then $Q(0)$ contains infinitely many lattice points. But the sets

$$Q(0) \cap \{x \in R^n\ :\ a_i x \leq \beta_i(k),\ i = p' + 1,\ ...,\ p\}$$

form an increasing sequence as $k \to \infty$ (since $\beta_i(k) \nearrow \infty$) and cannot be lattice point free. This contradiction demonstrates Lemma 2. □

Remark. The Lemma is equivalent to the fact that the number of one-dimensional simplices of the form $\{0, z\} \in \mathcal{K}(A)$ is finite. Such a $z \in \mathbb{Z}^n$ is a neighbor of the origin (cf. [7]). Therefore Lemma 2 says that there are finitely many neighbors of the origin if A is well conditioned, i.e., it satisfies A3; similar statements were proved in [9], [7].

We mention further that Theorems 2, 3, and Lemma 2 show that C_t is locally a polytope (when $t > t_0$). Indeed, every point of ∂C_t belongs to some facet by Theorem 3: and every facet comes from a maximal simplex of $\mathcal{K}(A)$ by Theorem 2. Then, by Lemma 2, any vertex v of C_t is contained in finitely many facets; C_t has the structure of a polytope at any one of its vertices.

We need two more properties of the sets K_b. Both of them are stated in [1] for the $n+1$ by n case. The proof given there extends without difficulty and is, therefore, omitted.

Lemma 3. *There is a $\delta_1 > 0$ (depending only on A) with the following property. Let S be a finite set of lattice points and define*

$$K = \{x \in R^n \ : \ Ax \leq b\} \text{ where } \beta_i = \max\{a_i z \ : \ z \in S\}.$$

If K contains a lattice point in its interior, then it contains a lattice point z such that $a_i z < \beta_i - \delta_1$ for all $i = 1, ..., m$.

Lemma 4. *There is $\delta_2 > 0$ (depending only on A) such that if $\sigma = \{z^1, ..., z^p, w^{j_1}, ..., w^{j_q}\} \in \mathcal{K}(A)$ with $p + q = m$ and z is a lattice point different from $z^1, ..., z^p$, then for some $i \in \{1, ..., m\} \setminus \{j_1, ..., j_q\}$*

$$a_i z \geq \max_{j=1,\dots,p} a_i z^j + \delta_2.$$

□

Proof of Theorem 3. We prove that C_t is closed. We may assume $t = 1$.

Notice that V is discrete, i.e., every compact set contains only finitely many elements of V. By the definition of C, every element $c \in C$ can be written as a mixed combination $\sum \alpha_i v^i + \sum \beta_j e(j)$, i.e., the first sum is a convex combination of some $v^i \in V$ and the second is a nonnegative combination. As $V \subset R^m_+$, $\sum \beta_j e(j)$ and every $\alpha_i v^i$ is less (componentwise) than c.

Assume now that c is from the boundary of C. Then $c = \lim_{k\to\infty} c(k)$ with $c(k) = v(k) + f(k)$ where $v(k) \in \text{conv } V$ and $f(k) \in R^m_+$ for all $k = 1, 2, \ldots$. The sequence $f(k)$ must be bounded so we may assume (by considering a subsequence if necessary) that $\lim f(k)$ exists and equals $f \in R^m_+$, say. Then $\lim v(k)$ exists and equals $v = c - f$. As $v(k) \in \text{conv } V \subset R^m$, every $v(k)$ can be written as a convex combination of $m+1$ elements of V:

$$v(k) = \sum_{i=0}^{m} \alpha_i(k) v^i(k).$$

Considering a subsequence if necessary we assume that $\lim \alpha_i(k) = \alpha_i$ for $i = 0, 1, \ldots, m$. Clearly $\alpha_i \geq 0$ and $\sum_0^m \alpha_i = 1$. To have convenient notation assume $\alpha_i > 0$ for $i = 0, 1, \ldots, j$ and $\alpha_i = 0$ for $i = j+1, \ldots, m$. Then, for $i = 0, 1, \ldots, j$, the sequence $v^i(k)$ must be bounded and we may assume that $\lim v^i(k) = v^i$. Since V is discrete, $v^i \in V$. Thus $\lim \sum_0^j \alpha_i(k) v^i(k) = \sum_0^j \alpha_i v^i = u$, say. Consequently $v - u = \lim \sum_{j+1}^m \alpha_i(k) v^i(k)$ and the limit is in R^m_+ since every summand is there. Thus $c = u + (v - u) + f$ and here u is of the form $\sum_0^j \alpha_i v^i$, a convex combination, and $(v - u) + f \in R^m_+$. □

6 Proof of Theorem 2

We essentially repeat the argument for the $(n+1) \times n$ case from [1] with the necessary modifications.

We show first that (2) implies (1). Let h be the normal to C at F, i.e.,

$$hy \geq 1 \text{ for all } y \in C, \text{ with equality for } y \in F. \tag{6.1}$$

Clearly $h = (h_1, \ldots, h_m)^T$ is nonnegative and $h_i = 0$ if and only if F is parallel with $e(i)$. To simplify notation assume $j_1 = m$, $j_2 = m-1$, ..., $j_q = m - q + 1$. Thus $h_i = 0$ if $i \geq m - q + 1$ and we rewrite (6.1) as

$$\sum_{i=1}^{m-q} h_i \ \exp\{ta_i z\} \geq 1 \text{ for all } z \in \mathbb{Z}^n, \text{ with equality for } z = z^1, \ldots, z^p. \tag{6.2}$$

It follows from the equality case that $h_i \exp\{ta_i z^j\} \leq 1$ $(i = 1, \ldots, m-q, \ j = 1, \ldots, p)$, implying $a_i z^j \leq -\frac{1}{t}\log h_i$. So

$$\max_{j=1,\dots,p} a_i z^j \leq -\frac{1}{t}\log h_i \ (i = 1, \dots, m-q). \tag{6.3}$$

We wish to show that $\sigma = \{z^1, \dots, z^p, w^{m-q+1}, \dots, w^m\} \in \mathcal{K}(A)$ (in particular $p + q = m$), i.e., there are no lattice points other than $z^1, \dots, z^p$ in

$$K = \{x \in R^n : a_i x \leq \beta_i, \ i = 1, \dots, m-q\}$$

where $\beta_i = \max\{a_i z^j : j = 1, \dots, p\}$ and, further, that $z^1, \dots, z^p$ are on distinct facets of K. Let z be a lattice point satisfying $a_i z < \beta_i$ for $i = 1, \dots, m-q$ ($z = z^j$ is possible). Then, by Lemma 3, for $i = 1, \dots, m-q$

$$a_i z \leq \max\{a_i z^j : j = 1, \dots, p\} - \delta_1. \tag{6.4}$$

On the other hand, (6.2) shows that there is an $i \in \{1, \dots, m-q\}$ with

$$h_i \ \exp\{t a_i z\} \geq \frac{1}{m-q}, \text{ or, } a_i z \geq -\frac{1}{t}(\log \ h_i + \log\,(m-q)).$$

Thus by (6.3)

$$\begin{aligned} a_i z &\geq -\frac{1}{t}\log h_i - \frac{1}{t}\log(m-q) \\ &\geq \max\{a_i z^j : j = 1, \dots, p\} - \frac{1}{t}\log(m-q), \end{aligned}$$

contradicting (6.4) if $t > t_1 = \frac{1}{\delta_1}\log(m-q)$.

It follows that K is a MLFC body and there is at most one z_i on every one of its facets implying $p \leq m-q$. Finally, $p + q \geq m$ follows from the fact that F is a facet.

We now turn to the second part of the argument and show that (1) implies (2). Assume

$$\sigma = \{z^1, \dots, z^p, \ w^{p+1}, \dots, w^m\} \in \mathcal{K}(A) \tag{6.5}$$

(using convenient notation, again). Let $h \in R^m_+$ satisfy $h_i = 0$ for $i = p+1, \dots, m$ and

$$hE_t(z^j) = 1 \text{ for } j = 1, \dots, p. \tag{6.6}$$

We will show the existence of a t_2 such that $hE_t(z) \geq 2$ for every $t > t_2$ and $z \in \mathbb{Z}^n$, different from $z^1, \ldots, z^p$. Assume the vertices have been permuted so that $a_i z^i = \max\{a_i z^j : j = 1, \ldots, p\}$.

We compute $h_1, \ldots, h_p$ from the system of linear equations (6.6). By Cramer's rule we have

$$h_1 = \frac{\det N}{\det(\exp\{ta_i z^j\})}$$

where N is the matrix obtained by replacing the first row by $(1, \ldots, 1)$ in the matrix appearing in the denominator. The determinant in the denominator can be written as the sum of $p!$ terms, each one based on a permutation of $\{1, \ldots, p\}$. But for each permutation π, other than the identity, the corresponding term is $(\Pi \ \exp\{a_i z^{\pi(i)}\})^t$ which is strictly less than $(\Pi \ \exp\{a_i z^i\})^t$ so that for large t this single term will be the asymptotic value of the denominator. Similarly, the numerator is asymptotically equal to the same product with index ranging from 2 to p. Thus we get that

$$h_1 = (1 + \varepsilon_1(t)) \exp\{-ta_1 z^1\}$$

with $\varepsilon_1(t) \to 0$ as $t \to \infty$. An identical argument gives that for $i = 1, \ldots, p$

$$h_i = (\ 1 + \varepsilon_i(t)) \exp\{-ta_i z^i\}$$

with $\varepsilon_i(t) \to \infty$ as $t \to \infty$. In particular, there is a t_2 so that for all $t \geq t_2$ we have

$$h_i \geq 2 \ \exp\{-ta_i z^i - t\delta_2\} \text{ for } i = 1, \ldots, p \tag{6.7}$$

with δ_2 the constant in Lemma 4 since $1 + \varepsilon_i(t) \geq 2\exp\{-t\delta_2\}$ for large enough t.

Assume now that $v = E_t(z)$ and $z \in \mathbb{Z}^n$ is distinct from $z^1, \ldots, z^p$. We have to show that $hv \geq 2$ for $t \geq t_2$. But using Lemma 4 we get that

$$hv = \Sigma h_i v_i \geq \Sigma\, 2\exp\{-t(a_i z^i + \delta_2)\}\exp\{ta_i z\} \geq 2.$$

In this argument the value of t_2 depends on the particular simplex $\sigma \in \mathcal{K}(A)$. In order to complete the proof of Theorem 3 we must show that a single value suffices for all simplices. To see this recall that if σ is the simplex in (6.5), then $\sigma_0 = T_z\sigma$ is a simplex of $\mathcal{K}(A)$ again. It is an easy matter to check now that if t_2 is the value given by the above argument for σ, then the same value will do for σ_0 as well. This means that a single value of t_2 suffices for the orbit (under the group T^n) of a simplex. By Lemma 1 every such orbit contains a simplex with one vertex at the origin. Lemma 2 implies that there are finitely many simplices in $\mathcal{K}(A)$ containing 0 and consequently, finitely many such orbits. □

7 Proof of Theorem 1

Assuming $t > t_0$ we suppress t from the notation. Theorem 1 gives a geometric realization of $|\mathcal{K}(A)|$ as the boundary of the convex set C in the following way. We define a map $f : |\mathcal{K}(A)| \to C$. Let

$$\sigma = \{z^1, \ldots, z^p, w^{j_1}, \ldots, w^{j_q}\} \in \mathcal{K}(A)$$

be a simplex with $p \geq 1$. The abstract mixed combination from (1.3)

$$x = \sum_{k=1}^{p} \gamma(k) z^k + \sum_{\ell=1}^{q} \beta(j_\ell) w^{j_\ell} \tag{7.1}$$

(which is a point of the cell $|\sigma|$ in $|\mathcal{K}(A)|$) is mapped to

$$f(x) = \sum_{k=1}^{p} \gamma(k) E(z^k) + \sum_{\ell=1}^{q} \beta(j_\ell) e(j_\ell). \tag{7.2}$$

One can see easily that f is well defined, i.e., if x belongs to two simplices of $\mathcal{K}(A)$ then the corresponding definitions coincide. Now $f : |\mathcal{K}(A)| \to \partial C$ is one-to-one by Theorem 3. Moreover f is continuous in both directions as one can readily check. Thus f is a geometric realization of $|\mathcal{K}(A)|$, and so $|\mathcal{K}(A)|$ and ∂C are homeomorphic. But ∂C is homeomorphic to R^{m-1} so Theorem 1 follows. □

8 Proof of Theorem 4

Assume again $t > t_0$. We need to define an equivariant extension

$$E^* : |\mathcal{K}(A)| \to \partial C$$

of the exponential map $E : \mathcal{K}(A) \to \partial C$. Equivariance here simply means that $E^*(T_z x) = D_z E^*(x)$ for all $x \in \mathcal{K}(A)$ and all $z \in \mathbb{Z}^n$.

It is easy to see that f in (7.2) is not equivariant since $D_z e(j) = \exp\{a_j z\} e(j)$. As E is simplicial on the simplices σ without ideal points, for these simplices the extension of E is the usual simplicial one: for x in (7.1) with $q = 0$ we have $E^*(x) = \sum_{k=1}^{p} \gamma(k) E(z^k)$. For a generic point $x \in |\mathcal{K}(A)|$ which is of the form (7.1) define

$$E^*(x) = \sum_{k=1}^{p} \gamma(k) E(z^k) + \sum_{\ell=1}^{q} \beta(j_\ell) \sum_{k=1}^{p} \gamma(k) \exp\{a_{j_\ell} z^k\} e(j_\ell). \tag{8.1}$$

It is not difficult to check that E^* is equivariant, one–to–one, and continuous in both directions.

Next, we define a map $g : \partial C \to M$ which is equivariant with respect to D_z, i.e., $D_z g(y) = g(D_z y)$ for every $y \in \partial C$ and every $z \in \mathbb{Z}^n$. Let $R(y)$ be the ray starting at the origin and passing through y and define simply

$$g(y) = M \cap R(y)$$

which is clearly a point in M. g is equivariant since $R(D_z y) = D_z R(y)$ and M is invariant under D_z. We see now that the following diagram

$$\begin{array}{ccccc} |\mathcal{K}(A)| & \xrightarrow{E^*} & \partial C & \xrightarrow{g} & M \\ T_z \downarrow & & D_z \downarrow & & D_z \downarrow \\ |\mathcal{K}(A)| & \xrightarrow{E^*} & \partial C & \xrightarrow{g} & M \end{array}$$

commutes for every $z \in \mathbb{Z}^n$ implying that the quotient space $|\mathcal{K}(A)|/T^n$ is homeomorphic to M/D^n.

M is homeomorphic to R^{m-1} and a natural homeomorphism $M \to R^{m-1}$ is the componentwise logarithm of $y \in M$. Write D^* for the set of all m by m diagonal matrices whose diagonal entries, d_1, ..., d_m, are positive and satisfy $\Pi_1^m d_k^{\lambda_k} = 1$ (cf. (3.1)). D^* acts on M as the group T^* of all translations acts on R^{m-1}. D_n is a discrete subgroup of D^* and the natural isomorphism $D^* \to T^*$ (taking componentwise logarithm of the diagonal entries) maps D_n onto an n–dimensional lattice of T^*, isomorphic to $\mathbb{Z}^n$. Thus the quotient space M/D_n is homeomorphic to $R^{m-1}/\mathbb{Z}^n$ proving the theorem. $\square$

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References

[1] I. Bárány, R. Howe, H. E. Scarf, "The complex of maximal lattice-free simplices," *Mathematical Programming*, 66 (1994) 273–282.

[2] D. E. Bell, "A theorem concerning the integer lattice", *Studies in Applied Mathematics*, 56 (1977) 187–188.

[3] J–P. Doignon, "Convexity in Cristallographic lattices", *Journal of Geometry*, 3 (1973) 77–85.

[4] R. Kannan, L. Lovász, H. E. Scarf, "The shapes of polyhedra", *Mathematics of Operations Research*, 15 (1990) 364–380.

[5] L. Lovász, "Geometry of numbers and integer programming". In: Mathematical programming: Recent developments and applications, M. Iri and K. Tanabe, eds, Norwell, MA: Kluwer Acad, 1989, 177–210.

[6] H. E. Scarf, "An observation on the structure of production sets with indivisibilities", *Proceedings of the National Academy of Sciences USA*, 74 (1977) 3637–3641.

[7] H. E. Scarf, "Production sets with indivisibilities. Part I. Generalities", *Econometrica*, 49 (1981) 1–32.

[8] H. E. Scarf, "Production sets with indivisibilities. Part II. The case of two activities", *Econometrica*, 49 (1981) 395–423.

[9] P. White, "Discrete activity analysis", Ph.D Thesis, Yale University, Department of Economics, 1983.