COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
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New Haven, Connecticut 06520

COWLES FOUNDATION DISCUSSION PAPER NO. 1076

INSURANCE MARKET GAMES: SCALE EFFECTS AND PUBLIC POLICY

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August 1994

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Abstract

We propose a game-theoretic model to study various effects of scale in an insurance market. After reviewing a simple static model, we present a one-period game in which both the buyers and sellers of insurance make strategic bids, and show that, under reasonably broad conditions, market equilibrium exists. For a special case, we then consider how both the price and quantity of insurance, as well as other quantities of interest to public policy decision makers, are affected by the number of insurance firms, the number of customers, and the total amount of capital provided by investors.

Keywords: Insurance Market Games, Scale Effects, Public Policy.

1. Introduction

In recent years, several trends have emerged which threaten, in the long run, to reduce the number of insurers in the United States property-liability market:

- First, escalating costs associated with litigation and medical services have led to greater uncertainty concerning the magnitude of future losses for insurers in many lines, especially automobile and workers' compensation insurance (see, for example, Insurance Information Institute, 1993). This increased risk may eventually cause less well-capitalized and less efficient firms to leave these markets.

- Second, political concerns at both the federal and state levels regarding the insurance industry's exemption from federal anti-trust legislation have prompted the elimination of advisory rates set by industry rating bureaus in many states. Although these bureaus continue to provide calculations of expected losses on which final rates can be based, the step of computing final rates from expected losses may ultimately prove too burdensome for some smaller insurers with little actuarial support.

- Finally, federal concerns about the ability of state regulators to monitor the solvency of insurers has caused many regulators to adopt more stringent standards, including higher levels of capitalization necessary to enter the market as a licensed insurer (see, for example, Cummins, et al., 1992).

The impact of a reduced number of property-liability insurers on the price and availability of insurance is difficult to assess, partly because the literature on economies of scale in the property-liability insurance industry is rather mixed. Hensley (1962), Hammond, Melander, and Shilling (1971), Cummins and VanDerhei (1979), Appel, Worrall, and Butler (1985), and Cummins and Weiss (1993) have found evidence of significant scale economies, whereas Joskow (1973), Quirin, et al. (1974), and C. G. Lee (1989) have not. (J-H. Lee, 1994, provides a survey of results in this area.) In addition, empirical studies of scale economies can only evaluate the relative efficiencies of firms of different sizes under a fixed market configuration, and cannot be directly consulted for predicting the effects on competition and solvency of reducing the number of insurers and/or the total amount of capital available in the insurance market.

In this article, we propose a game-theoretic model to study various effects of scale in an insurance market. We begin by reviewing a simple static model of the insurance firm, both to develop certain basic intuitions as well as to underscore the necessity of incorporating strategic components into the analysis. We then present a one-period game in which both the buyers and sellers of insurance make strategic bids, and show that, under reasonably broad conditions, market equilibrium exists. For a special case, we consider how both the price and quantity of insurance, as well as other quantities of interest to public policy decision makers, are affected by three separate measures of scale: (1) the number of insurance firms, (2) the number of customers, and (3) the total amount of capital provided by investors.

Game-theoretic models are not new in the insurance literature (see, for example, Borch, 1962a, 1962b, Baton and Lemaire, 1981a, 1981b, Kihlstrom and Roth, 1982, Schlesinger, 1984, and Kunreuther and Pauly, 1985). However, our approach differs from previous research in its use of the full process structure of a strategic market game to study equilibrium effects of scale and their public policy implications.

2. A Static Model

We first consider a simple static model involving one insurance firm that provides insurance to m individual customers, $i=1,2,\ldots,m$. At time 0, let the insurer have initial net worth (free reserves) R. During the policy period $[0,t]$, let the insurer incur a random loss amount $X_i \geq 0$ for each customer i, and receive a premium payment Π_i from each i.

The following well-known result shows that if one ignores the effect of the insurer's expenses and investment income,[1] then, for large m, the probability of insolvency during the policy period, ρ, may be expressed using the normal approximation.

Proposition 1: If $X_1,\ldots,X_m$ are i.i.d. random variables with mean θ and variance $\sigma^2 < +\infty$, and if $\Pi_i = (1+\xi)\theta$ for all i, where $\xi > 0$ denotes the insurer's risk loading expressed as a proportion of expected losses, then, as $m \to \infty$,

$$\rho \to \Phi\left(-\frac{R + m\xi\theta}{\sqrt{m}\sigma}\right).$$

Proof: See, for example, Smith and Kane (1994). ■

This result shows that, for large m and $R \propto \sqrt{m}\sigma$, the probability of insolvency tends to decrease in m, and that $\rho \to 0$ as $m \to \infty$. (In fact, $\lim_{m\to\infty} \rho = 0$ even for fixed R.) Furthermore, the proposition shows that, for large fixed values of m, ρ tends to decrease in R, and that $\rho \to 0$ as $R \to \infty$. Thus, for the static insurance model described above, insurer solvency can be enhanced both by increasing the number of customers paying the standard premium $\Pi_i > E[X_i]$ (with $R \propto \sqrt{m}\sigma$) and by increasing the insurer's invested capital.

A reasonable way to study the effect of increasing the number of insurance firms for fixed values of m and R is to divide both the number of customers and the total invested capital equally among n insurers, $j = 1,2,\ldots,n$. Let ρ_j denote the probability of insolvency for insurer j. Shaffer (1989) and Smith and Kane (1994) have observed that for $n = 2$,

[1]In a more detailed, operations-oriented study, one would have to account for taxes, as well as various administrative, marketing, and loss adjustment expenses paid by the insurer, and investment income received by the insurer. Taxes and other expenses typically amount to 30-40 percent of direct insurance premiums in the personal property-liability (automobile and homeowners) insurance lines, whereas investment income typically amounts to 10-12 percent of direct premiums in these lines (see National Association of Insurance Commissioners, 1993).

$$\rho \geq \rho_1 \rho_2 .$$

This inequality can easily be generalized for arbitrary n as

$$\rho \geq \prod_{j=1}^{n} \rho_j ,$$

which, in terms of the normal approximation, may be expressed as

$$\Phi\left(-\frac{R+m\xi\theta}{\sqrt{m}\sigma}\right) \geq \left[\Phi\left(-\frac{R/n+(m/n)\xi\theta}{\sqrt{m/n}\sigma}\right)\right]^n = \left[\Phi\left(-n^{-1/2}\frac{R+m\xi\theta}{\sqrt{m}\sigma}\right)\right]^n .$$

Although the above inequalities seem to suggest that dividing one large insurer into many smaller insurers is beneficial because the joint probability of insolvency declines, this would be an incomplete analysis. After all, the joint probability of insolvency provides only a worst-case comparison; a more appropriate solvency measure would be the probability that a given customer is affected by insolvency. Using the normal approximation, it can be seen that, for arbitrary n, this probability is given by

$$\Phi\left(-\frac{R/n+(m/n)\xi\theta}{\sqrt{m/n}\sigma}\right) = \Phi\left(-n^{-1/2}\frac{R+m\xi\theta}{\sqrt{m}\sigma}\right),$$

which is strictly increasing in n, approaching $\frac{1}{2}$ as $n \to \infty$. Thus, for the static model, insurer solvency is actually diminished by increasing the number of insurers.

The greatest limitation of the static model is its inability to capture the effects of scale on competitive market forces. To provide a more realistic view of the insurance market, we now turn to a strategic model in which customers and insurers are able to bid on price and quantity in the insurance transaction.

3. A One-Period Strategic Game

Consider an insurance market game with players consisting of m homogeneous customers and n homogeneous insurance firms. At time 0, let each customer (buyer) i have initial endowment $B_i(0) = A + V$ consisting of A dollars of net worth and 1 unit of property with replacement value V. Furthermore, let each insurer (seller) j have initial endowment $S_j(0) = R/n$ dollars of net worth, where R is the total amount of capital supplied by investors to the insurance market.

During the policy period $[0,t]$, each customer's property is subject to one or more random perils that may damage it. For simplicity, we will assume that any damaged property is deprived of all value, and is therefore a total loss. Let δ_i be a random variable that equals 1 if customer i suffers a property loss during $[0,t]$, and equals 0 otherwise, where the δ_i ~ i.i.d. Bernoulli(π).

3.1. Strategies

To insure against a potential property loss in $[0,t]$, each customer i may purchase insurance from some insurer by making a strategic bid, $x_i \in [0, Min\{A,V\}]$, that represents the amount that he or she is willing to pay for insurance. Simultaneously, each insurer j may offer to sell insurance by making a strategic bid, $y_j \in [0, kR/n]$, that represents the total dollar amount of risk that j is willing to assume, where $k > 1$ is a solvency constraint imposed by government regulators.

For simplicity, we assume that the bids of all customers and insurers are submitted to a central clearinghouse[2] that performs the following three important functions:

(1) the clearinghouse calculates an average market price of insurance per unit exposure,

$$P(\mathbf{x},\mathbf{y}) = \sum_{i'=1}^{m} x_{i'} \Big/ \sum_{j'=1}^{n} y_{j'};^3$$

[2]Although the clearinghouse that we describe is not institutionally supported in traditional insurance markets, this mechanism provides the simplest analytical model of price formation in terms of a given amount of money chasing a given amount of product. Moreover, this mechanism is defined independently of equilibrium. (It should be noted that, in conventional equilibrium analysis models, no price formation mechanism is provided.)

[3]Given that $y_j = 0$ is a permissible bid, it is theoretically possible--although highly unlikely in a

(2) the clearinghouse collects all premium bids, x_i, and distributes them to the n insurers in proportion to the insurers' respective exposure bids, y_j (i.e., insurer j receives the premium amount $y_j P(\mathbf{x},\mathbf{y})$); and

(3) the clearinghouse randomly assigns each customer i to an insurer $j(i)$ so that each insurer ends up with the same number of customers, μ (i.e., it is assumed that n divides m exactly and that $\mu = m/n$).[4]

Let M_j denote the set of μ customers associated with insurer j, and note that if customer $i \in M_j$ suffers a loss in $[0,t]$, then he or she will receive a loss payment in the amount $y_j\left(x_i \Big/ \sum_{h \in M_j} x_h\right)$, which is proportional not only to i's premium bid, x_i, but also to j's exposure bid, y_j. (In all analytical work, this loss payment will be bounded above by V to reduce problems of moral hazard.)[5] To recognize the possibility of an insurer insolvency during $[0,t]$, let η_j be a Bernoulli random variable that equals 1 if insurer j becomes insolvent, and equals 0 otherwise. If there is an insolvency, it is assumed that government guaranty funds will pay a

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real insurance market--that $\sum_{j'=1}^{n} y_{j'} = 0$, causing $P(\mathbf{x},\mathbf{y})$ to be undefined. To avoid this problem, as well as similar problems associated with $\sum_{i'=1}^{m} x_{i'} = 0$, we take the approach of Dubey and Shubik (1978) and assume that the clearinghouse furnishes at least one insurer, and one customer per insurer, that must make non-zero bids.

[4]The assumption that μ is an integer is made to avoid complex integer-programming problems that contribute little to an understanding of the insurance market. Note that the assignment of equal numbers of customers to each insurer does not imply that all insurers provide the same amount of coverage (y_j) out of equilibrium. By intrinsic symmetry, however, one can show that if the set of non-cooperative equilibria is non-empty, then there must exist a type-symmetric equilibrium point (see Proposition 3).

[5]In reality, problems of moral hazard caused by insuring a property for more than its market value are rare because most insurance contracts are written to exclude this possibility. Paul Apilungo, director of special investigative services for the CIGNA Property & Casualty Companies, has indicated in personal communication that claims from overinsured properties typically account for only 1 to 2 percent of fraudulent claims.

fixed proportion $g \in [0,1]$ of all insurance claims made against the insolvent insurer.[6]

3.2. Payoffs

Given the preceding development, we see that at time t customer i's wealth will consist of

$$B_i(t) = (1-\delta_i)(A+V-x_i) + \delta_i(1-\eta_{j(i)})\left[A - x_i + y_{j(i)}\left(x_i \Big/ \sum_{h \in M_{j(i)}} x_h\right)\right]$$
$$+ \delta_i \eta_{j(i)}\left[A - x_i + g y_{j(i)}\left(x_i \Big/ \sum_{h \in M_{j(i)}} x_h\right)\right],$$

and insurer j's wealth will consist of

$$S_j(t) = R/n + y_j P(\mathbf{x},\mathbf{y}) - y_j\left(\sum_{h \in M_j} \delta_h x_h \Big/ \sum_{h \in M_j} x_h\right).$$

Note that both $B_i(t)$ and $S_j(t)$ can take on negative, as well as positive, values.

Let $u_B(.) : \Re \to \Re$ denote the utility function of customer i, for all i, and $u_S(.) : \Re \to \Re$ denote the utility function of insurer j, for all j. It then follows that the payoffs to customer i and insurer j are given by

$$E[u_B(B_i(t))] = (1-\pi)u_B(A+V-x_i) + \pi(1-\rho_{j(i)})u_B\left(A - x_i + y_{j(i)}\left(x_i \Big/ \sum_{h \in M_{j(i)}} x_h\right)\right)$$
$$+ \pi\rho_{j(i)}u_B\left(A - x_i + g y_{j(i)}\left(x_i \Big/ \sum_{h \in M_{j(i)}} x_h\right)\right)$$

and

$$E[u_S(S_j(t))] = \sum_{r=0}^{\mu} \sum_{H_r \subseteq M_j} \pi^r (1-\pi)^{\mu - r} u_S\left(R/n + y_j P(\mathbf{x},\mathbf{y}) - y_j\left(\sum_{h \in H_r} x_h \Big/ \sum_{h \in M_j} x_h\right)\right),$$

respectively, where $\rho_{j(i)} = Pr\{\eta_{j(i)} = 1 \mid \delta_i = 1, \mathbf{x}, \mathbf{y}\}$ and $H_r = \{h_1, h_2, \ldots, h_r\}$.

Although $\rho_{j(i)}$ is functionally dependent on x_i, the actual effect of x_i on the

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[6]This assumption is made to facilitate the mathematics. In the United States, the large majority of state guaranty funds have been set up in accordance with the National Association of Insurance Commissioner's Model Act, which provides for the payment of losses up to a dollar limit (often $300,000); see Duncan (1984).

ruin probability will generally be insignificant. This is because

$$\rho_{j(i)} = Pr\{S_{j(i)}(t) \le 0 \mid \delta_i = 1, \mathbf{x}, \mathbf{y}\}$$

$$= Pr\left\{ R/n + y_{j(i)}P(\mathbf{x},\mathbf{y}) - y_{j(i)}\left(x_i \Big/ \sum_{h \in M_{j(i)}} x_h \right) - y_{j(i)}\left(\sum_{h \in M_{j(i)}, h \ne i} \delta_h x_h \Big/ \sum_{h \in M_{j(i)}} x_h \right) \le 0 \right\}, \qquad (1)$$

and the loss payment $y_{j(i)}\left(x_i \Big/ \sum_{h \in M_{j(i)}} x_h \right)$ must be less than or equal to V (to reduce moral hazard), which in turn is substantially less than $S_{j(i)}(0) = R/n$. In the Appendix, the normal approximation to $\sum_{h \in M_{j(i)}, h \ne i} \delta_h x_h$ is used to show that, for fixed values of $\mathbf{x}_{-i} = [x_1, x_2, \ldots, x_{i-1}, x_i, \ldots, x_m]$ and $\mathbf{y}$ and for $x_i \in [0, Min\{A, V\}]$,

$$\left| \frac{\partial \rho_{j(i)}}{\partial x_i} \right| \approx O\left(Re^{-\frac{(R/n - V)^2}{D}} \right) \text{ as } R \to \infty,$$

where $D = 2V^2(\mu - 1)\pi(1 - \pi)$. Thus, it is generally reasonable to assume that $\frac{\partial \rho_{j(i)}}{\partial x_i} = 0$ for all i, as is done below.

3.3. Existence of Equilibria

Let G denote the one-period game described above. The following proposition shows that, under reasonably broad conditions, G possesses a pure strategy equilibrium.

Proposition 2: If $u_B(.)$ and $u_S(.)$ are both twice-differentiable functions on $\Re$, with $u_B{}'(.), u_S{}'(.) > 0$ and $u_B{}''(.), u_S{}''(.) \le 0$, and if $\frac{\partial \rho_{j(i)}}{\partial x_i} = 0$ for all i, then the set of pure strategy non-cooperative equilibria $[x_1^*, x_2^*, \ldots, x_m^*, y_1^*, y_2^*, \ldots, y_n^*]$ for G is non-empty.

Proof:

Given that $\frac{\partial \rho_{j(i)}}{\partial x_i} = 0$, the concavity of $E[u_B(B_i(t))]$ is easily established for all i by noting that this payoff is a convex combination of $u_B(A + V - x_i)$,

$u_B\left(A - x_i + y_{j(i)}\left(x_i \Big/ \sum_{h\in M_{j(i)}} x_h\right)\right)$, and $u_B\left(A - x_i + gy_{j(i)}\left(x_i \Big/ \sum_{h\in M_{j(i)}} x_h\right)\right)$, each of which is a concave function of x_i. The concavity of $E\left[u_S\left(S_j(t)\right)\right]$ is likewise established for all j by noting that this payoff is a convex combination of the functions $u_S\left(R/n + y_j P(\mathbf{x},\mathbf{y}) - y_j\left(\sum_{h\in H_r} x_h \Big/ \sum_{h\in M_j} x_h\right)\right)$, which are concave in y_j for each pair (r, H_r).

The proposition then follows from Nash (1951). ■

4. A Special Case

To develop a better understanding of the properties of the non-cooperative equilibria described in Proposition 2, we consider a special case of G in which the customers possess the exponential utility function

$$u_B(w) = \frac{1 - e^{-\beta w}}{\beta} \tag{2}$$

for some $\beta > 0$, and the insurers possess the exponential utility function

$$u_S(w) = \frac{1 - e^{-\sigma w}}{\sigma} \tag{3}$$

for some $\sigma > 0$. (Note that these functional forms admit the special limiting case of linear utility as $\beta \to 0$ and $\sigma \to 0$, respectively.) For this game, the following proposition reveals that a unique symmetric pure strategy equilibrium exists for which it is possible to obtain explicit analytical expressions.

<u>**Proposition 3**</u>: If the utility functions $u_B(.)$ and $u_S(.)$ are as given by Equations (2) and (3), respectively, and if $\frac{\partial \rho_{j(i)}}{\partial x_i} = 0$ for all i, then there exists a unique non-cooperative equilibrium $[x_1^*, x_2^*, \ldots, x_m^*, y_1^*, y_2^*, \ldots, y_n^*]$ for G in which $x_i^* = x^*$ for all i and $y_j^* = y^*$ for all j, where x^* and y^* are defined implicitly by the equations

$$\begin{aligned}(1-\pi)e^{-\beta V} + \pi(1-\rho^*)e^{-\beta x^*/P(\mathbf{x}^*,\mathbf{y}^*)}\left[1 - \frac{\mu - 1}{\mu P(\mathbf{x}^*,\mathbf{y}^*)}\right] \\ + \pi\rho^* e^{-\beta g x^*/P(\mathbf{x}^*,\mathbf{y}^*)}\left[1 - \frac{g(\mu-1)}{\mu P(\mathbf{x}^*,\mathbf{y}^*)}\right] = 0\end{aligned} \tag{4}$$

and

$$\left(\frac{n-1}{n}\right)P(\mathbf{x}^*,\mathbf{y}^*)\left(\pi e^{\sigma y^*/\mu}+1-\pi\right)-\pi e^{\sigma y^*/\mu}=0, \tag{5}$$

with $\rho^* = Pr\left\{\eta_{j(i)}=1 \mid \delta_i = 1, \mathbf{x}^*, \mathbf{y}^*\right\}$ for all i and $P(\mathbf{x}^*,\mathbf{y}^*)=\mu x^*/y^*$.

Proof:

Given Equation (2), it follows that

$$E\left[u_B(B_i(t))\right]=\frac{1}{\beta}-\frac{1-\pi}{\beta}e^{-\beta(A+V-x_i)}-\frac{\pi\left(1-\rho_{j(i)}\right)}{\beta}e^{-\beta\left[A-x_i+y_{j(i)}\left(x_i/\sum_{h\in M_{j(i)}}x_h\right)\right]}$$
$$-\frac{\pi\rho_{j(i)}}{\beta}e^{-\beta\left[A-x_i+gy_{j(i)}\left(x_i/\sum_{h\in M_{j(i)}}x_h\right)\right]}; \tag{6}$$

since $\frac{\partial\rho_{j(i)}}{\partial x_i}=0$, it follows further that

$$\frac{\partial E\left[u_B(B_i(t))\right]}{\partial x_i}=-(1-\pi)e^{-\beta(A+V-x_i)}$$
$$-\pi\left(1-\rho_{j(i)}\right)e^{-\beta\left[A-x_i+y_{j(i)}\left(x_i/\sum_{h\in M_{j(i)}}x_h\right)\right]}\left[1-y_{j(i)}\left(\sum_{h\in M_{j(i)}}x_h-x_i\right)\Big/\left(\sum_{h\in M_{j(i)}}x_h\right)^2\right]$$
$$-\pi\rho_{j(i)}e^{-\beta\left[A-x_i+gy_{j(i)}\left(x_i/\sum_{h\in M_{j(i)}}x_h\right)\right]}\left[1-gy_{j(i)}\left(\sum_{h\in M_{j(i)}}x_h-x_i\right)\Big/\left(\sum_{h\in M_{j(i)}}x_h\right)^2\right].$$

Setting this derivative equal to 0 and letting $x_i = x^*$ for all i and $y_j = y^*$ for all j yields Equation (4).

Now consider that Equation (3) implies

$$E\left[u_S(S_j(t))\right]=\frac{1}{\sigma}-\frac{1}{\sigma}\sum_{r=0}^{\mu}\sum_{H_r\subseteq M_j}\pi^r(1-\pi)^{\mu-r}e^{-\sigma\left[R/n+y_jP(\mathbf{x},\mathbf{y})-y_j\left(\sum_{h\in H_r}x_h/\sum_{h\in M_j}x_h\right)\right]}$$

and

$$\frac{\partial E\left[u_S(S_j(t))\right]}{\partial y_j}=$$
$$\sum_{r=0}^{\mu}\sum_{H_r\subseteq M_j}\pi^r(1-\pi)^{\mu-r}e^{-\sigma\left[R/n+y_jP(\mathbf{x},\mathbf{y})-y_j\left(\sum_{h\in H_r}x_h/\sum_{h\in M_j}x_h\right)\right]}\left\{\left[1-\left(y_j\Big/\sum_{j'=1}^{n}y_{j'}\right)\right]P(\mathbf{x},\mathbf{y})-\left(\sum_{h\in H_r}x_h\Big/\sum_{h\in M_j}x_h\right)\right\}.$$

Setting the above derivative equal to 0 and substituting x^* for all x_i and y^* for all y_j, we see that

$$\sum_{r=0}^{\mu}\binom{\mu}{r}\pi^r(1-\pi)^{\mu-r}\left[\left(\frac{n-1}{n}\right)P(\mathbf{x}^*,\mathbf{y}^*)e^{\sigma y^* r/\mu}-\frac{r}{\mu}e^{\sigma y^* r/\mu}\right]=0,$$

or equivalently,

$$\left(\frac{n}{n-1}\right)P(\mathbf{x}^*,\mathbf{y}^*)E\left[e^{\frac{\sigma y^*}{\mu}W}\right]-\frac{1}{\mu}E\left[We^{\frac{\sigma y^*}{\mu}W}\right]=0,$$

where $W \sim Binomial(\mu,\pi)$. It then follows that

$$\left(\frac{n}{n-1}\right)P(\mathbf{x}^*,\mathbf{y}^*)\varphi_W\left(\frac{\sigma y^*}{\mu}\right)-\frac{1}{\mu}\varphi_W'\left(\frac{\sigma y^*}{\mu}\right)=0,$$

where $\varphi_W(\cdot)$ denotes the moment generating function of W, which implies Equation (5). ■

In the remainder of this section, we consider the effects of the number of insurers, the number of customers, and the total amount of capital provided by investors on various quantities of interest. To simplify calculations, we restrict attention to the case in which $\sigma \to 0$ and the insurers have the linear utility function, $u_S(w)=w$. For this case, Equation (5) reveals that the equilibrium price of insurance is given by

$$P(\mathbf{x}^*,\mathbf{y}^*)=\left(\frac{n}{n-1}\right)\pi,$$

so that the equilibrium quantity is

$$Q(\mathbf{x}^*,\mathbf{y}^*)=ny^*=\frac{n\mu x^*}{P(\mathbf{x}^*,\mathbf{y}^*)}=\left[\frac{\mu(n-1)}{\pi}\right]x^*.$$

4.1. Number of Insurers

To study the effects of the number of insurers, we first consider how the number of insurers affects the expressions for $P^*=P(\mathbf{x}^*,\mathbf{y}^*)$ and $Q^*=Q(\mathbf{x}^*,\mathbf{y}^*)$ given above. Since $\frac{\partial P^*}{\partial n}=-\pi/(n-1)^2<0$ and $\frac{\partial^2 P^*}{\partial n^2}=2\pi/(n-1)^3>0$, we see that the equilibrium price is a strictly decreasing concave upward function of n. This shows that competitive forces cause the price of insurance to approach its marginal cost as

n increases, while the risk neutrality of the insurers obviates the need to increase prices to compensate for the increased risk of insolvency associated with each of the numerous smaller firms.

Substituting $\left(\frac{n}{n-1}\right)\pi$ for $P(\mathbf{x}^*,\mathbf{y}^*)$ and $\left[\frac{\pi}{\mu(n-1)}\right]Q^*$ for x^* in Equation (4), we find that

$$
\begin{aligned}
&(1-\pi)e^{-\beta V} + (1-\rho^*)e^{-\beta Q^*/m}\left[\pi - \frac{(\mu-1)(n-1)}{m}\right] \\
&\quad + \rho^* e^{-\beta g Q^*/m}\left[\pi - \frac{g(\mu-1)(n-1)}{m}\right] = 0 ,
\end{aligned}
\tag{7}
$$

from which it is possible to obtain an expression for $\frac{\partial Q^*}{\partial n}$ by implicit differentiation. Because the sign of this derivative depends on the relative sizes of n and the various other parameter values, we consider the behavior of Q^* over n (as well as over m and R) for certain typical parameter values in the tables of Exhibit 1.[7]

These tables show very clearly and consistently that, for a fixed number of customers, m, and a fixed amount of invested capital, R, the equilibrium quantity Q^* tends to increase over n for smaller values of n, and then tends to decrease for larger values.[8] Thus, there is a unique maximum Q^* for the parameter values considered. This result reveals that the decreasing price of insurance at first encourages the purchase of greater amounts until a certain point at which the increased risk of insolvency (caused by a decrease in the net worth, R/n; cf. the static model of Section 2) offsets this effect. This behavior is confirmed by looking at the quantity of insurance purchased by one customer in equilibrium, Q^*/m, presented

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[7]For Exhibits 1 through 4, the following parameter values are used: $A = 20{,}000$, $V = 10{,}000$, $k = 1.2$, $\pi = 0.10$, $g = 0.75$, and $\beta = 1$. In addition, the normal approximation to $\sum_{h \in M_{j(i)}, h \neq i} \delta_h x_h$ is used to estimate $\rho_{j(i)}$.

[8]Note that values of Q^* (and other variables) that are found along boundary constraints are presented in boldface throughout Exhibits 1 to 4.

in the tables of Exhibit 2.

From a public policy perspective, it is important to consider the effects of the number of insurers on the payoffs of both the customers and insurers. To this end, let

$$B^* = E\left[u_B(B_i(t)) \mid \mathbf{x}^*, \mathbf{y}^*\right]$$

denote the expected utility of one customer in equilibrium, and let

$$S^* = \left(\frac{1}{R/n}\right)E\left[u_S(S_j(t)) \mid \mathbf{x}^*, \mathbf{y}^*\right] - 1 = \frac{R/n + y^* P^* - y^* \pi}{R/n} - 1 = \frac{\pi Q^*}{(n-1)R}$$

denote the expected total rate of return for the investors associated with one insurer in equilibrium.

From Equation (6), we see that

$$B^* = \frac{1}{\beta} - \frac{1-\pi}{\beta}e^{-\beta(A+V-x^*)} - \frac{\pi(1-\rho^*)}{\beta}e^{-\beta\left(A-x^*+\frac{x^*}{P(\mathbf{x}^*,\mathbf{y}^*)}\right)} - \frac{\pi\rho^*}{\beta}e^{-\beta\left(A-x^*+\frac{gx^*}{P(\mathbf{x}^*,\mathbf{y}^*)}\right)};$$

substituting $\left(\frac{n}{n-1}\right)\pi$ for $P(\mathbf{x}^*, \mathbf{y}^*)$ and $\left[\frac{\pi}{\mu(n-1)}\right]Q^*$ for x^* in this equation, we find that

$$B^* = \frac{1}{\beta} - \frac{1-\pi}{\beta}e^{-\beta\left(A+V-\frac{\pi Q^*}{\mu(n-1)}\right)} - \frac{\pi(1-\rho^*)}{\beta}e^{-\beta\left[A+\left(1-\frac{n\pi}{n-1}\right)\frac{Q^*}{m}\right]} - \frac{\pi\rho^*}{\beta}e^{-\beta\left[A+\left(g-\frac{n\pi}{n-1}\right)\frac{Q^*}{m}\right]},$$

and so $\frac{\partial B^*}{\partial n}$ may be expressed in terms of $\frac{\partial Q^*}{\partial n}$. To study the behavior of B^* over n for the typical parameter values selected above, we consider the tables of Exhibit 3. These tables reveal that the behavior of B^* is similar to that of Q^* in that, for fixed values of m and R, there is a unique maximum over n. Again, this shows that the decreasing price of insurance is beneficial to the customers until a certain point is reached at which the increased risk of insolvency offsets this effect.

From the definition of S^* above, we find that

$$\frac{\partial S^*}{\partial n} = \left[\frac{\pi}{(n-1)R}\right]\frac{\partial Q^*}{\partial n} - \frac{\pi Q^*}{(n-1)^2 R}.$$

The tables of Exhibit 4 allow us to study the behavior of S^* over n for the same parameter values used before. From these tables, we see that, for fixed values of m and R, there is a unique maximum over n. In this case, the insurers benefit from the increased income associated with increases in the equilibrium quantity of insurance until the reduced price of insurance counters this effect. It is instructive to note that the maximum value of S^* occurs at a substantially smaller value of n than does the maximum value of B^*. Thus, there is a natural conflict between the buyers and sellers of insurance in terms of the optimal number of sellers in the market.

4.2. *Number of Customers*

The effects of the number of customers in the market are not nearly as striking as those associated with the number of insurers. Since $\frac{\partial P^*}{\partial m} = 0$, we see that the number of customers has no impact on the equilibrium price of insurance. This is because the risk neutral insurers require no compensation for the additional risk of writing more exposures.

As with the partial derivative $\frac{\partial Q^*}{\partial n}$, it is possible to obtain an expression for $\frac{\partial Q^*}{\partial m}$ by implicit differentiation of Equation (7); furthermore, it is also possible to express both $\frac{\partial B^*}{\partial m}$ and $\frac{\partial S^*}{\partial m}$ in terms of $\frac{\partial Q^*}{\partial m}$. However, to study the behavior of Q^* (as well as Q^*/m, B^*, and S^*) over m, we must turn again to the tables of Exhibits 1 through 4.

These tables reveal that, for a fixed number of insurers, n, and a fixed amount of invested capital, R, the equilibrium values Q^*, Q^*/m, B^*, and S^* are all strictly increasing over m for the selected parameter values. This is because the risk of insolvency in equilibrium tends to decrease as the number of customers increases (as in the static model of Section 2 for sufficiently large m); hence, more insurance is purchased, both in the aggregate and on an individual basis. The decreased risk of insolvency is beneficial to customers but irrelevant to the risk neutral insurers. However, the insurers benefit from the increased income associated with larger equilibrium quantities of insurance.

4.3. *Total Invested Capital*

We now turn to the effects of total invested capital, i.e., the combined initial net worth of all insurers, $\sum_{j'=1}^{n} S_{j'}(0) = R$. Clearly, $\frac{\partial P^*}{\partial R} = 0$, which shows that the equilibrium price of insurance is not affected by the capitalization of the insurers. This is not surprising because the risk neutral insurers do not require any profit loading to compensate for the increased risk of insolvency associated with smaller values of R.

By implicit differentiation of Equation (7), it is possible to derive an expression for $\frac{\partial Q^*}{\partial R}$, and therefore also possible to derive expressions for both $\frac{\partial B^*}{\partial R}$ and $\frac{\partial S^*}{\partial R}$. To study the behavior of Q^*, Q^*/m, B^*, and S^* over R, we again consider the tables of Exhibits 1 through 4. These tables show that for a fixed number of insurers, n, and a fixed number of customers, m, the equilibrium values Q^*, Q^*/m, B^*, and S^* are all strictly increasing over R for the selected parameter values. This is because the risk of insolvency tends to decrease as the amount of invested capital increases (as in the static model of Section 2), and so more insurance is purchased, both in the aggregate and on an individual basis.

5. Conclusions

In this article, we have used a game-theoretic model of an insurance market to study various effects of scale and their public policy implications. After showing that equilibrium exists, we considered a special case in which the customers have exponential utility functions and the insurers have linear utility functions. For this special case we analyzed the equilibrium effects on price, quantity, and the payoffs of both customers and insurers of: (1) the number of insurance firms, (2) the number of customers, and (3) the total amount of capital provided by investors. Our findings include:

- The price of insurance is strictly decreasing as the number of insurers increases, but is unaffected by either the number of customers or the total invested capital.

- The quantity of insurance sold (whether measured in the aggregate or on an individual basis) increases and then decreases as the number of insurers increases, but is strictly increasing as both the number of customers and the total invested capital increase.

- Both the expected utility of an individual customer and the expected total rate of return for the investors associated with one insurer behave qualitatively like the quantity of insurance (increasing and then decreasing as the number of insurers increases, but strictly increasing as both the number of customers and the total invested capital increase); however, the number of firms that maximizes a customer's payoff is substantially greater than the number of firms maximizing the insurer's rate of return.

These results show that, other things being equal, government policies that reduce the number of insurers in the property-liability market (without reducing the total invested capital) will probably have the effect of increasing price, if only modestly. The effect of such policies on the quantity of insurance sold is less predictable because it depends on the current status of the market--i.e., whether the current quantity is above or below its maximum value. If one assumes that the number of insurers in the market is somewhere between the optimal number for insurers (at the low end) and the optimal number for customers (at the high end), then a decrease in the number of insurers will tend to benefit insurers and hurt customers.

If the total capital invested in the market is reduced along with the number of insurers, this will have little effect on price, but will have a negative effect on the quantity of insurance sold, as well as on the payoffs of both customers and insurers. One significant aspect of capital formation that has not been taken into account in our model is the role of reinsurance. This additional source of financial security, whether in the form of simple pooling or more complicated arrangements, should be considered in further research.

Appendix

Proof that $\left|\frac{\partial \rho_{j(i)}}{\partial x_i}\right| \approx O\left(Re^{-\frac{(R/n-V)^2}{D}}\right)$ as $R \to \infty$:

First, recall from Equation (1) that

$$\rho_{j(i)} = Pr\left\{S_{j(i)}(t) \le 0 \mid \delta_i = 1, \mathbf{x}, \mathbf{y}\right\}$$

$$= Pr\left\{R/n + y_{j(i)}P(\mathbf{x},\mathbf{y}) - y_{j(i)}\left(x_i \Big/ \sum_{h \in M_{j(i)}} x_h\right) - y_{j(i)}\left(\sum_{h \in M_{j(i)}, h \ne i} \delta_h x_h \Big/ \sum_{h \in M_{j(i)}} x_h\right) \le 0\right\}.$$

Using the normal approximation to $\sum_{h \in M_{j(i)}, h \ne i} \delta_h x_h$ yields

$$\rho_{j(i)} \approx \Phi(\zeta),$$

where

$$\zeta = \frac{-R/n - y_{j(i)}P(\mathbf{x},\mathbf{y}) + y_{j(i)}\left(x_i \Big/ \sum_{h \in M_{j(i)}} x_h\right) + \pi y_{j(i)}\left(\sum_{h \in M_{j(i)}, h \ne i} x_h \Big/ \sum_{h \in M_{j(i)}} x_h\right)}{\sqrt{\left(y_{j(i)} \Big/ \sum_{h \in M_{j(i)}} x_h\right)^2 \pi(1-\pi) \sum_{h \in M_{j(i)}, h \ne i} x_h^2}}$$

$$= \frac{-R/n + \left[1 - P(\mathbf{x},\mathbf{y})\right]y_{j(i)}\left(x_i \Big/ \sum_{h \in M_{j(i)}} x_h\right) + \left[\pi - P(\mathbf{x},\mathbf{y})\right]y_{j(i)}\left(\sum_{h \in M_{j(i)}, h \ne i} x_h \Big/ \sum_{h \in M_{j(i)}} x_h\right)}{\sqrt{\left(y_{j(i)} \Big/ \sum_{h \in M_{j(i)}} x_h\right)^2 \pi(1-\pi) \sum_{h \in M_{j(i)}, h \ne i} x_h^2}}.$$

Assuming that no insurers j are risk prone, it follows that $P(\mathbf{x},\mathbf{y})$ must be greater than or equal to π. Then, since the loss payment $y_{j(i)}\left(x_i \Big/ \sum_{h \in M_{j(i)}} x_h\right)$ must be less than or equal to V (to reduce moral hazard) for all i, and since $V < R/n$, it follows that

$$\zeta \le \frac{-R/n + V}{V\sqrt{(\mu - 1)\pi(1-\pi)}}.$$

Thus,

$$\frac{\partial \rho_{j(i)}}{\partial \zeta} \approx \Phi'(\zeta) = O\left(e^{-\frac{(R/n-V)^2}{D}}\right) \text{ as } R \to \infty, \tag{A1}$$

where $D = 2V^2(\mu - 1)\pi(1-\pi)$.

Now consider

$$\frac{\partial \zeta}{\partial x_i} = \frac{-R/n - y_{j(i)}P(\mathbf{x},\mathbf{y}) - y_{j(i)}\left(\sum_{h \in M_{j(i)}} x_h \Big/ \sum_{j'=1}^{n} y_{j'}\right) + y_{j(i)}}{y_{j(i)}\sqrt{\pi(1-\pi)\sum_{h \in M_{j(i)}, h \neq i} x_h^2}}$$

$$= \frac{-R/n}{y_{j(i)}\sqrt{\pi(1-\pi)\sum_{h \in M_{j(i)}, h \neq i} x_h^2}} + \frac{-P(\mathbf{x},\mathbf{y}) - \left(\sum_{h \in M_{j(i)}} x_h \Big/ \sum_{j'=1}^{n} y_{j'}\right) + 1}{\sqrt{\pi(1-\pi)\sum_{h \in M_{j(i)}, h \neq i} x_h^2}}.$$

Assuming that no customers i have decreasing utility, it follows that $P(\mathbf{x},\mathbf{y})$ must be less than or equal to 1, and therefore that

$$\sum_{h \in M_{j(i)}} x_h \Big/ \sum_{j'=1}^{n} y_{j'} \leq P(\mathbf{x},\mathbf{y}) \leq 1.$$

It then follows that

$$\frac{-R/n}{y_{j(i)}\sqrt{\pi(1-\pi)\sum_{h \in M_{j(i)}, h \neq i} x_h^2}} - \frac{1}{\sqrt{\pi(1-\pi)\sum_{h \in M_{j(i)}, h \neq i} x_h^2}} \leq \frac{\partial \zeta}{\partial x_i}$$

$$\leq \frac{-R/n}{y_{j(i)}\sqrt{\pi(1-\pi)\sum_{h \in M_{j(i)}, h \neq i} x_h^2}} + \frac{1}{\sqrt{\pi(1-\pi)\sum_{h \in M_{j(i)}, h \neq i} x_h^2}},$$

and so

$$\left|\frac{\partial \zeta}{\partial x_i}\right| = O(R) \text{ as } R \to \infty. \tag{A2}$$

Combining the results of lines (A1) and (A2), we find that, for fixed values of $\mathbf{x}_{-i}$ and $\mathbf{y}$,

$$\left|\frac{\partial \rho_{j(i)}}{\partial x_i}\right| = \frac{\partial \rho_{j(i)}}{\partial \zeta}\left|\frac{\partial \zeta}{\partial x_i}\right| \approx O\left(Re^{-\frac{(R/n-V)^2}{D}}\right)$$

for $x_i \in [0, Min\{A, V\}]$. ■

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Exhibit 1 Q^*

$R =$ 1E+10

m = (in 1,000)	n = 15	30	150	300	1500	3000
1	6.096238E+05	2.822549E+06				
5	9.836651E+06	2.777244E+07	2.931887E+07	1.363500E+07		
10	2.135755E+07	5.890791E+07	7.574525E+07	6.237505E+07		
50	1.135145E+08	3.079511E+08	4.461181E+08	4.479897E+08	3.267744E+08	1.531702E+08
100	2.287094E+08	6.192500E+08	9.089572E+08	9.294928E+08	8.245061E+08	6.572673E+08
500	1.150268E+09	3.109637E+09	4.611569E+09	4.781114E+09	4.795992E+09	4.647016E+09
1,000	2.302216E+09	6.222621E+09	9.239822E+09	9.595591E+09	9.759080E+09	9.629054E+09
5,000	1.151780E+10	***1.200000E+10***	***1.200000E+10***	***1.200000E+10***	***1.200000E+10***	***1.200000E+10***
10,000	***1.200000E+10***	***1.200000E+10***	***1.200000E+10***	***1.200000E+10***	***1.200000E+10***	***1.200000E+10***
50,000	***1.200000E+10***	***1.200000E+10***	***1.200000E+10***	***1.200000E+10***	***1.200000E+10***	***1.200000E+10***

$R =$ 1E+11

m = (in 1,000)	n = 15	30	150	300	1500	3000
1	6.096238E+05	2.822549E+06				
5	9.836651E+06	2.777244E+07	2.931887E+07	1.363500E+07		
10	2.135755E+07	5.890791E+07	7.574525E+07	6.237505E+07		
50	1.135145E+08	3.079511E+08	4.461181E+08	4.479897E+08	3.267744E+08	1.531702E+08
100	2.287094E+08	6.192500E+08	9.089572E+08	9.294928E+08	8.245061E+08	6.572673E+08
500	1.150268E+09	3.109637E+09	4.611569E+09	4.781114E+09	4.795992E+09	4.647016E+09
1,000	2.302216E+09	6.222621E+09	9.239822E+09	9.595591E+09	9.759080E+09	9.629054E+09
5,000	1.151780E+10	3.112649E+10	4.626583E+10	4.811136E+10	4.946279E+10	4.948132E+10
10,000	2.303728E+10	6.225633E+10	9.254834E+10	9.625607E+10	9.909229E+10	9.929616E+10
50,000	1.151931E+11	***1.200000E+11***	***1.200000E+11***	***1.200000E+11***	***1.200000E+11***	***1.200000E+11***

$R =$ 1E+12

m = (in 1,000)	n = 15	30	150	300	1500	3000
1	6.096238E+05	2.822549E+06				
5	9.836651E+06	2.777244E+07	2.931887E+07	1.363500E+07		
10	2.135755E+07	5.890791E+07	7.574525E+07	6.237505E+07		
50	1.135145E+08	3.079511E+08	4.461181E+08	4.479897E+08	3.267744E+08	1.531702E+08
100	2.287094E+08	6.192500E+08	9.089572E+08	9.294928E+08	8.245061E+08	6.572673E+08
500	1.150268E+09	3.109637E+09	4.611569E+09	4.781114E+09	4.795992E+09	4.647016E+09
1,000	2.302216E+09	6.222621E+09	9.239822E+09	9.595591E+09	9.759080E+09	9.629054E+09
5,000	1.151780E+10	3.112649E+10	4.626583E+10	4.811136E+10	4.946279E+10	4.948132E+10
10,000	2.303728E+10	6.225633E+10	9.254834E+10	9.625607E+10	9.909229E+10	9.929616E+10
50,000	1.151931E+11	3.112950E+11	4.628084E+11	4.814138E+11	4.961282E+11	4.978144E+11

Exhibit 2 Q^*/m

$R =$ 1E+10

$m =$ (in 1,000)	$n =$ 15	30	150	300	1500	3000
1	609.6238	2822.5486				
5	1967.3301	5554.4870	5863.7736	2726.9995		
10	2135.7546	5890.7907	7574.5254	6237.5050		
50	2270.2902	6159.0217	8922.3624	8959.7938	6535.4877	3063.4043
100	2287.0945	6192.5001	9089.5719	9294.9276	8245.0610	6572.6732
500	2300.5358	6219.2747	9223.1384	9562.2284	9591.9836	9294.0324
1,000	2302.2159	6222.6210	9239.8217	9595.5908	9759.0804	9629.0536
5,000	2303.5599	***2400.0000***	***2400.0000***	***2400.0000***	***2400.0000***	***2400.0000***
10,000	***1200.0000***	***1200.0000***	***1200.0000***	***1200.0000***	***1200.0000***	***1200.0000***
50,000	***240.0000***	***240.0000***	***240.0000***	***240.0000***	***240.0000***	***240.0000***

$R =$ 1E+11

$m =$ (in 1,000)	$n =$ 15	30	150	300	1500	3000
1	609.6238	2822.5486				
5	1967.3301	5554.4870	5863.7736	2726.9995		
10	2135.7546	5890.7907	7574.5254	6237.5050		
50	2270.2902	6159.0217	8922.3624	8959.7938	6535.4877	3063.4043
100	2287.0945	6192.5001	9089.5719	9294.9276	8245.0610	6572.6732
500	2300.5358	6219.2747	9223.1384	9562.2284	9591.9836	9294.0324
1,000	2302.2159	6222.6210	9239.8217	9595.5908	9759.0804	9629.0536
5,000	2303.5599	6225.2980	9253.1663	9622.2727	9892.5571	9896.2647
10,000	2303.7279	6225.6326	9254.8342	9625.6074	9909.2292	9929.6160
50,000	2303.8623	***2400.0000***	***2400.0000***	***2400.0000***	***2400.0000***	***2400.0000***

$R =$ 1E+12

$m =$ (in 1,000)	$n =$ 15	30	150	300	1500	3000
1	609.6238	2822.5486				
5	1967.3301	5554.4870	5863.7736	2726.9995		
10	2135.7546	5890.7907	7574.5254	6237.5050		
50	2270.2902	6159.0217	8922.3624	8959.7938	6535.4877	3063.4043
100	2287.0945	6192.5001	9089.5719	9294.9276	8245.0610	6572.6732
500	2300.5358	6219.2747	9223.1384	9562.2284	9591.9836	9294.0324
1,000	2302.2159	6222.6210	9239.8217	9595.5908	9759.0804	9629.0536
5,000	2303.5599	6225.2980	9253.1663	9622.2727	9892.5571	9896.2647
10,000	2303.7279	6225.6326	9254.8342	9625.6074	9909.2292	9929.6160
50,000	2303.8623	6225.9003	9256.1686	9628.2751	9922.5648	9956.2890

Exhibit 3 B^*

$R =$ 1E+10

$m =$ (in 1,000)	$n =$ 15	30	150	300	1500	3000
1	-0.7486006771	-0.7485132547				
5	-0.7485907588	-0.7484742534	-0.7483512663	-0.7484520187		
10	-0.7485904542	-0.7484730759	-0.7483215458	-0.7483273280		
50	-0.7485903569	-0.7484727012	-0.7483122980	-0.7482896439	-0.7483076223	-0.7484293938
100	-0.7485903539	-0.7484726895	-0.7483120139	-0.7482885061	-0.7482781024	-0.7483051248
500	-0.7485903529	-0.7484726858	-0.7483119232	-0.7482881440	-0.7482689168	-0.7482675678
1,000	-0.7485903529	-0.7484726856	-0.7483119204	-0.7482881328	-0.7482686347	-0.7482664339
5,000	-0.7485903529	***-0.7485240114***	***-0.7484741230***	***-0.7484680749***	***-0.7484632655***	***-0.7484626661***
10,000	***-0.7485947288***	***-0.7485615406***	***-0.7485365948***	***-0.7485335705***	***-0.7485311655***	***-0.7485308658***
50,000	***-0.7486056880***	***-0.7485990502***	***-0.7485940607***	***-0.7485934558***	***-0.7485929747***	***-0.7485929148***

$R =$ 1E+11

$m =$ (in 1,000)	$n =$ 15	30	150	300	1500	3000
1	-0.7486006771	-0.7485132547				
5	-0.7485907588	-0.7484742534	-0.7483512663	-0.7484520187		
10	-0.7485904542	-0.7484730759	-0.7483215458	-0.7483273280		
50	-0.7485903569	-0.7484727012	-0.7483122980	-0.7482896439	-0.7483076223	-0.7484293938
100	-0.7485903539	-0.7484726895	-0.7483120139	-0.7482885061	-0.7482781024	-0.7483051248
500	-0.7485903529	-0.7484726858	-0.7483119232	-0.7482881440	-0.7482689168	-0.7482675678
1,000	-0.7485903529	-0.7484726856	-0.7483119204	-0.7482881328	-0.7482686347	-0.7482664339
5,000	-0.7485903529	-0.7484726856	-0.7483119195	-0.7482881291	-0.7482685446	-0.7482660730
10,000	-0.7485903529	-0.7484726856	-0.7483119194	-0.7482881290	-0.7482685418	-0.7482660617
50,000	-0.7485903529	***-0.7485240114***	***-0.7484741230***	***-0.7484680749***	***-0.7484632655***	***-0.7484626661***

$R =$ 1E+12

$m =$ (in 1,000)	$n =$ 15	30	150	300	1500	3000
1	-0.7486006771	-0.7485132547				
5	-0.7485907588	-0.7484742534	-0.7483512663	-0.7484520187		
10	-0.7485904542	-0.7484730759	-0.7483215458	-0.7483273280		
50	-0.7485903569	-0.7484727012	-0.7483122980	-0.7482896439	-0.7483076223	-0.7484293938
100	-0.7485903539	-0.7484726895	-0.7483120139	-0.7482885061	-0.7482781024	-0.7483051248
500	-0.7485903529	-0.7484726858	-0.7483119232	-0.7482881440	-0.7482689168	-0.7482675678
1,000	-0.7485903529	-0.7484726856	-0.7483119204	-0.7482881328	-0.7482686347	-0.7482664339
5,000	-0.7485903529	-0.7484726856	-0.7483119195	-0.7482881291	-0.7482685446	-0.7482660730
10,000	-0.7485903529	-0.7484726856	-0.7483119194	-0.7482881290	-0.7482685418	-0.7482660617
50,000	-0.7485903529	-0.7484726856	-0.7483119194	-0.7482881290	-0.7482685409	-0.7482660582

Exhibit 4 S^*

$R =$ 1E+10

$m =$ (in 1,000)	$n =$ 15	30	150	300	1500	3000
1	4.35446E-07	9.73293E-07				
5	7.02618E-06	9.57670E-06	1.96771E-06	4.56020E-07		
10	1.52554E-05	2.03131E-05	5.08357E-06	2.08612E-06		
50	8.10818E-05	1.06190E-04	2.99408E-05	1.49829E-05	2.17995E-06	5.10738E-07
100	1.63364E-04	2.13534E-04	6.10038E-05	3.10867E-05	5.50037E-06	2.19162E-06
500	8.21620E-04	1.07229E-03	3.09501E-04	1.59903E-04	3.19946E-05	1.54952E-05
1,000	1.64444E-03	2.14573E-03	6.20122E-04	3.20923E-04	6.51039E-05	3.21075E-05
5,000	8.22700E-03	***4.13793E-03***	***8.05369E-04***	***4.01338E-04***	***8.00534E-05***	***4.00133E-05***
10,000	***8.57143E-03***	***4.13793E-03***	***8.05369E-04***	***4.01338E-04***	***8.00534E-05***	***4.00133E-05***
50,000	***8.57143E-03***	***4.13793E-03***	***8.05369E-04***	***4.01338E-04***	***8.00534E-05***	***4.00133E-05***

$R =$ 1E+11

$m =$ (in 1,000)	$n =$ 15	30	150	300	1500	3000
1	4.354456E-08	9.732926E-08				
5	7.026179E-07	9.576702E-07	1.967709E-07	4.560200E-08		
10	1.525539E-06	2.031307E-06	5.083574E-07	2.086122E-07		
50	8.108179E-06	1.061900E-05	2.994081E-06	1.498293E-06	2.179949E-07	5.107376E-08
100	1.633639E-05	2.135345E-05	6.100384E-06	3.108671E-06	5.500374E-07	2.191622E-07
500	8.216199E-05	1.072289E-04	3.095013E-05	1.599035E-05	3.199461E-06	1.549522E-06
1,000	1.644440E-04	2.145731E-04	6.201223E-05	3.209228E-05	6.510394E-06	3.210755E-06
5,000	8.227000E-04	1.073327E-03	3.105089E-04	1.609076E-04	3.299719E-05	1.649927E-05
10,000	1.645520E-03	2.146770E-03	6.211298E-04	3.219267E-04	6.610560E-05	3.310976E-05
50,000	8.228080E-03	***4.137931E-03***	***8.053691E-04***	***4.013378E-04***	***8.005337E-05***	***4.001334E-05***

$R =$ 1E+12

$m =$ (in 1,000)	$n =$ 15	30	150	300	1500	3000
1	4.354456E-09	9.732926E-09				
5	7.026179E-08	9.576702E-08	1.967709E-08	4.560200E-09		
10	1.525539E-07	2.031307E-07	5.083574E-08	2.086122E-08		
50	8.108179E-07	1.061900E-06	2.994081E-07	1.498293E-07	2.179949E-08	5.107376E-09
100	1.633639E-06	2.135345E-06	6.100384E-07	3.108671E-07	5.500374E-08	2.191622E-08
500	8.216199E-06	1.072289E-05	3.095013E-06	1.599035E-06	3.199461E-07	1.549522E-07
1,000	1.644440E-05	2.145731E-05	6.201223E-06	3.209228E-06	6.510394E-07	3.210755E-07
5,000	8.227000E-05	1.073327E-04	3.105089E-05	1.609076E-05	3.299719E-06	1.649927E-06
10,000	1.645520E-04	2.146770E-04	6.211298E-05	3.219267E-05	6.610560E-06	3.310976E-06
50,000	8.228080E-04	1.073431E-03	3.106097E-04	1.610079E-04	3.309728E-05	1.659935E-05