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A STRATEGIC MARKET GAME
WITH SEIGNIORAGE COSTS OF FIAT MONEY

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A Strategic Market Game with Seigniorage Costs of Fiat Money*

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Summary. A model that includes the cost of producing money is presented and the nature of the inefficient equilibria in the model are examined. It is suggested that if one acknowledges that transactions are a form of production, which requires the consumption of resources, then the concept of Pareto optimality is inappropriate for assessing efficiency. Instead it becomes necessary to provide an appropriate comparative analysis of alternative transactions mechanisms in the appropriate context.

Keywords and Phrases: Strategic market games, seigniorage costs, inefficiency

JEL Classification Numbers: D51, E51

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1 Introduction

Although the general equilibrium exchange model, especially in its simple form with two traders and two commodities, is of considerable use in the exposition of the basic properties of a price system and in the discussion of efficiency, it can easily be misleading. It may provide a false view of transactions costs and technology. The acts of exchange in most societies involve a considerable consumption of real resources. The percentage of resources consumed in running the exchange technology may run anywhere from a fraction of a percent to the order of ten percent of the resources. Furthermore the legal structure required to police commerce and to provide for the currency must be supplied by government, whose role and goals are different from those of ordinary agents.

In this note we examine a highly simplified form of transactions costs, specifically we consider an economy that has to pay for the issue and maintenance of its currency and that uses currency for trade.

The control of coinage to some extent depends upon the technology. Until the sixteenth century coins were "hammer-struck." A metal blank was placed between two dies and then hammered. Both speed and accuracy were relatively poor. Leonardo da Vinci had suggested means for mechanizing coin production and in 1580[1] Eloye Mestrelle a French moneyer produced coins using a screw press in England for Elizabeth, (Carson, 1970, p. 241). He was later hung as a forger, and the Royal Mint did not adopt machinery until 1662. In 1790 Matthew Boulton and James Watt built a self-powered coining machine. In 1839 a reducing machine that could copy a design in miniature was utilized in die making. In 1800 three million coins were struck in the U.S.; in 1850, 25 million at the Royal Mint. In 1966 the Royal mint produced 1.4 billion coins and in 1967 the U.S. mints produced 25 billion (Becker, 1969, Ch. 8). The stamping machine at the Philadelphia mint can produce 10,000 coins per minute (Becker, 1969, p. 157).

The power to mint[2] was in general part of the power of the monarch. In republican Rome the power was vested with the Roman Senate, and Roman generals could obtain the right to mint to pay their troops.

Kindleberger (1985) notes that the sovereign either owned the mint directly or farmed out the production of coins. A profit was available in seigniorage charges, although if the issuer priced the metal content too low in international trade the coins might be melted down and exported as bullion rather than increase the local means of exchange.

The United States Mint was established by an act of law of April 2, 1792. Free coinage was provided. Individuals could bring metals to the mint and have them returned as coins with the same bullion value, with the mint bearing the coinage expense (Hobson, 1965, p. 73). The free minting of silver was omitted from the Coinage Act of 1873 (Wemple in Butts and Coxe, 1967, p. 36). By the Bland Allison Act of 1878 the Treasury was required to buy $24 million

[1] Becker (1969, p. 121) gives this date as 1561 and the name is Mestrel.

[2] The Roman mint was in the Temple of Juno Moneta (Marx, 1978, p. 204) (from whence money? in contrast with pecuniary from "pecus").

of silver which produced 28 million silver dollars leaving a sizeable seigniorage spread (Hobson, 1965, p. 105). In March 1964 the U.S. Treasury discontinued the issue of silver coinage. Between 1961 and 1964 its stock of silver dollars was depleted from 130,000,000 to 3,000,000 (Wemple in Butts and Coxe, 1967, p. 48). Government revenue due to seigniorage has accounted for about 0.5% of G.N.P. in low inflation industrialized economies recently (Blanchard and Fischer, 1989, p. 195).

Recently the cost per coin for United States coinage including manufacturing and metal costs was:

Table 1. The Cost of Coins

Cents	0.61 cents
Nickels	2.32 cents
Dimes	1.22 cents
Quarters	2.64 cents
Halves	5.45 cents
Eisenhower Dollars	8.32 cents
S.B. Anthony Dollars	2.97 cents

Coins wear out or are lost. The estimated attrition rates for cents are 6.0–7.5% per annum, for nickels 5.2%; for dimes 4.6% and for quarters 2.5%.[3]

The production, distribution and supervision of the coinage of a country is part of the production and supervision process called for in the running of a monetized economy. Logically there is nothing wrong with considering the government as only involved in providing the supervision and legal power to verify the coinage standards and to leave the mining and minting of gold and other precious metals to private enterprise. When the money used (metal or paper or officially registered ciphers) has little intrinsic value or even no physical manifestation beyond an accounting number, the need for government production and supervision becomes larger. The printing of notes in the United States is performed by a government agency, but the actual work could be farmed out to private enterprise such as the Walter De La Rue company which has printed many currencies.

In Sections 2 and 3 we consider some simple variations of an economy with the production and use of money and the definition of efficient exchange. The ideal model would be multistage with no specific end and with all individuals holding an initial supply of money. In fact, in the United States, the replacement of money is borne directly by the government which will replace old bills and coins by new and raises the funds to cover costs from Treasury profits. We can, nevertheless, capture the basic phenomenon of currency which must be produced, paid for, distributed and recycled by imagining a simple one period playable game where the individuals begin with no money and are charged a

[3] Personal correspondence from A. Cosgarea, Associate Director for Operations, United States Mint, November 18, 1986. The figures are for Fiscal Year 1985, except for the dollar coins which are 1979 figures.

seigniorage cost or rate of interest for its use for one period, after which they are required to return the money. We have two fairly straightforward theorems in Section 2. The first is the existence of an active noncooperative equilibrium, and the second shows that if the government charges individuals in proportion to their use of money efficiency cannot be achieved. In contrast, in Section 3.1, we consider the Associated Resource Reduced Competitive economy where no direct charge is made for the use of money but a lump sum tax will preserve efficiency with the production of the currency either produced by a government production facility or farmed out to a firm. Finally, in Section 3.2 we modify the model of Section 2 so that seigniorage costs are financed via a poll-tax.

2 The Model

We construct a playable exchange game in which the government (or referee) is able to extract seigniorage costs from the players in the form of interest rate payments for the amount of fiat money utilized in exchange. The government participates in exchange and bids to provide for its inputs of production. It attempts to minimizes interest rate subject to the requirement to replace worn out fiat used in exchange and it is the interest rate which in turn determines its revenues which is its strategic choice variable. At the end of the game the government replaces depreciated money used in exchange.

We utilize a Strategic Market Game with the offer-for-sale price formation mechanism in which money depreciates when used in exchange and its replacement is costly. We assume that initial money supply enters exogenously and that is big but finite. Figure 1 shows the extensive form of the game. At the first move the government specifies its strategic variable, i.e. determines the interest rate based on interest rate minimization. At the second move, individuals $P_1, ..., P_N$ obtain fiat money in the money market at the predetermined interest rate. At the third move individuals bid and offer commodities and the government bids for inputs of production to be used for the replacement of depreciated money. We avoid the complex functional forms of strategies which may arise when traders have perfect information by assuming a continuum of traders, simultaneous moves and a minimum of information at the second and the third stage. Moreover, we implicitly assume a one-period lag in the interest rate payments so that to be used by the government for the purchase of inputs of production. Then traders pay back their loans. Finally, the government replaces depreciated money.

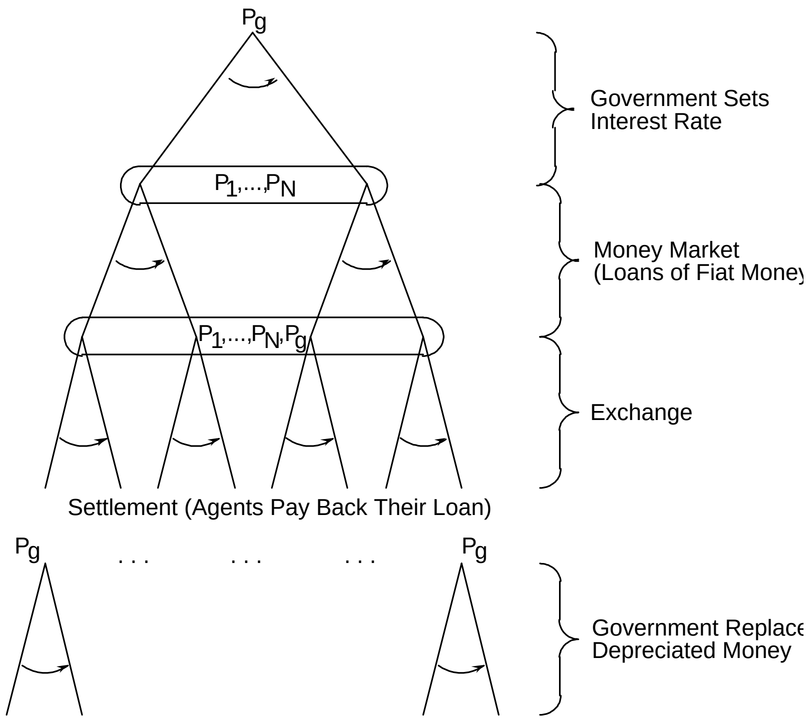

Figure 1. Trade with Seigniorage Cost for Fiat Money
(Note that the labeling $P_1, ..., P_N, P_g$ indicates that all agents move simultaneously)

2.1 The Strategic Market Game with Seigniorage Costs

There is a continuum of traders $T = [0, n)$ which consists of n types of traders $T_i = [i1, i)$ where $a^t = a^i$, $\phi^t = \phi^i$ for $t \in T_i$. The endowments of traders are $a^i = (a^i_1, ..., a^i_m) \in \mathbb{R}^m_+$. For $1 \leq i \leq n$ the utility functions are, $\phi^i : \mathbb{R}^m_+ \times \mathbb{R} \mapsto \mathbb{R}$. We assume,

(i) $\sum_{i=1}^n a^i \gg 0$,

(ii) $a^i \neq 0$, for all $i = 1, ..., n$,

(iii) ϕ^i is continuous, concave and strongly monotonic for all $i = 1, ..., n$.

The strategy set of trader i is of the form

$$S^i = \{v^i; b^i_1, q^i_1, ..., b^i_m, q^i_m\} \tag{1}$$

where the arguments appearing in S^i are constrained as follows:

$$\left.\begin{array}{ll} \text{(a)} & 0 \leq q_j^i \leq a_j^i, \text{ for all } j = 1, ..., m \\ \text{(b)} & 0 \leq \sum_{j=1}^{m} b_j^i \leq 1, \text{ for all } i = 1, ..., n \\ \text{(c)} & 0 \leq v^i \leq \bar{M} \end{array}\right\} \quad (2)$$

with

v^i = amount of fiat money borrowed at the money market;

$\bar{M}$ = an exogenously determined amount of fiat money, big but finite, is supplied by the referee at the beginning of the game.

It should be noted that one may naturally expect that traders could not offer to repay more money than the total amount of money supply present in the economy. Moreover, the presence of a sufficiently harsh default or bankruptcy penalty provides for a motivation against unreasonably high borrowing, but the physical bound provided by 2(c) suffices. If the total amount required for loans by the traders, i.e., $\sum_{i \in T} v^i > M$, is higher than the total money supply then loans are given to traders in a *pro rata* form. This rule is needed so that the game can be specified under all contingencies even though such a circumstance does not arise at equilibrium. Using the restrictions (2) on S^i, we can specify the strategy set as follows,

$$\Sigma^i = \{s^i \in S^i \text{ subject to } (2)\}. \quad (3)$$

It is easily seen that i is compact and convex.

The government (or referee) enters the model as a strategic player as contrasted with a strategic dummy (i.e., strategy sets are exogenously fixed and given as data to the model). The part of the money supply used for exchange by the traders and the government depreciates at a rate η. For example, if the amount borrowed by the traders is sum from $\sum_i v^i = \bar{v}$ and the amount bid by the government for the purchase of inputs of production is $\bar{g}$ then $\eta(\bar{v} + \bar{g})$ is the amount of money worn out at the end of each period. We *assume* that the initial cost of production of $\bar{M}$ is zero.[4] The government's production function for money is decreasing (or constant) returns to scale,[5]

$$z_{m+1} = F(x_1^g, ..., x_m^g) \quad (4)$$

with

z_{m+1} = amount of fiat produced,

x_j^g = inputs of production, i.e., final allocations of commodities held by the government after exchange has taken place, for all $j = 1, ..., m$.

We impose the standard assumptions on the production set, $Y^g \in \mathbb{R}_+^m$,

[4] More accurately charged to a previous period.

[5] For example, a Leontief production technology with coefficients γ_i, for all $i = 1, ..., m$,

$$z_{m+1} = \min[\gamma_1 x_1^g, ..., \gamma_m x_m^g].$$

(iv) $0 \in Y^g$,

(v) Y^g is convex and closed.

(vi) There exists $B > 0$ such that if $(x_1^g, ..., x_m^g; z_{m+1}) \in Y^g$ then $x_j^g \in B$, $\forall j = 1, ..., m$ and $z_{m+1} \leq B$.

The monetary strategic variable controlled by the government is the interest rate, ρ, which is exogenous for the traders.

The strategy set for the government is of the form,

$$S^g = \{\rho; b_1^g, ..., b_m^g\}. \tag{5}$$

where the arguments appearing in S^g are constrained as follows,

$$\sum_{j=1}^{m} b_j^g \leq 1. \tag{6}$$

Using the restriction (6) on S^g, we can specify the strategy set as,

$$\Sigma^g = \{s^g \in S^g \text{ subject to (6)}\}. \tag{7}$$

The government's optimization problem becomes,[6]

$$\max_{\rho, b_j^g, j=1,...,m} -\rho \quad \text{s.t.} \quad z_{m+1} = \eta \left[\sum_{i=1}^{n} v^i + \sum_{j=1}^{m} b_j^g V \right] \text{ and } \sum_{j=1}^{m} b_j^g \leq 1, \tag{8}$$

where $V = \rho \sum_{i=1}^{n} v^i$. Since production is financed by interest rate payments, $\rho \geq 0$ and the convexity of the production set bounds from above ρ. Hence Σ^g is compact and convex.

We will employ the Dubey–Shubik offer-for-sale price formation mechanism. Thus

$$p_j = \begin{cases} \frac{\int_T b_j^t v^t dt + b_j^g V}{\int_T q_j^t dt} & \text{, if } \int_T b_j^t v^t dt + b_j^g V, \ \int_T q_j^t dt > 0 \\ 0 & \text{, otherwise.} \end{cases} \tag{9}$$

The final allocations of the goods for the traders is given by,

$$x_j^i = a_j^i - q_j^i + \frac{b_j^i v^i}{p_j}, \ \forall j = 1, ..., m. \tag{10}$$

And similarly for the government,

$$x_j^g = \frac{b_j V}{p_j}, \ \forall j = 1, ..., m. \tag{11}$$

[6] Governments' purchases are *all* used in the production process, i.e., government is *not* endowed with a utility of consumption.

The final amount of monetary holding of fiat which also indicates the trader's debt position at the end of exchange is given by,

$$D^i = \sum_{j=1}^{m} p_j q_j^i - \sum_{j=1}^{m} b_j^i v^i - (1+\rho)v^i + v^i. \tag{12}$$

Let $D_-^i(s^i) = \min[D^i, 0]$. Thus, the final payoff to a trader i is given by,

$$\Pi^i(s^i) = \phi^i(x^i(s)) + \mu D_-^i(s) \tag{13}$$

where μ is the bankruptcy penalty imposed in the case of default.[7] Finally, the final payoff to the government is given by,

$$\Pi^g(s^g) = -\rho. \tag{14}$$

Before we prove existence, we need to define:

A *Nash Equilibrium* (N.E.) of $\Gamma(n, \phi^i, a^i, \eta, \bar{M}, \mu, s^i, s^g)$ is an integrable choice $s = (s^i, s^g)$ such that $\forall t \in T$ and the government and $\alpha = (\alpha^t, \alpha^g) \in \Sigma = \mathsf{X}_{i=1}^{N} \Sigma^i \times \Sigma^g$,

$$\Pi(s|\alpha) \leq \Pi(s)$$

where $(s|\alpha)$ is s with either s^t or s^g replaced by α^t or α^g. If $s^t = s^{t'}$ for every $t, t' \in T_i$ and for all $i = 1, ..., n$ then s is a type symmetric play and that N.E. is *type-symmetric* (T.S.N.E.).

Finally, an *active* T.S.N.E. of $\Gamma(n, \phi^i, a^i, \eta, \bar{M}, \mu, s^i, s^g)$ is a T.S.N.E. with $p_j > 0, \forall j = 1, ..., m$ and $\rho \geq 0$.

2.2 Existence and Inefficiency

Theorem 1 *The strategic market game $\Gamma(n, \phi^i, a^i, \eta, \bar{M}, \mu, s^i, s^g)$ with a continuum of traders of n types and seigniorage costs for any finite non-zero $\bar{M}$ and any bankruptcy penalty μ has an active* T.S.N.E.

Proof

Step 0. (i) Let an outside agency place a fixed bid and a fixed offer $\varepsilon > 0$ in each trading post. So, let the modified strategy space be,

$$(\Sigma^i)^\varepsilon \times (\Sigma^g)^\varepsilon = \{(s^i, s^g) \in \Sigma^i \times \Sigma^g : s = (s^i, \tilde{s}^g) \geq \varepsilon,\ \forall (s^i, \tilde{s}^g) \in \Sigma^i \times \Sigma^g\}$$

where $\tilde{s}^g = (b_1^g, ..., b_m^g)$.

Note that we do not need to place upper bounds because by construction we placed bounds on the strategy sets of the traders and since $\bar{M} < \infty$ this imposes a bound on ρ and (6) bounds b_j^g's.

[7] Bankruptcy can be avoided by selecting a sufficiently harsh penalty so that nobody will be induced to strategically bankrupt. This can be achieved by setting $\mu = \max\{\lambda_1, ..., \lambda_n\}$ where the λ's reflect the marginal utility of wealth at equilibrium of the resource reduced economy (see Section 3.1). For more on this see Shubik and Wilson (1977).

(ii) Γ^ε is the ε-modified game where we restricted the strategies of the traders and government to $(\Sigma^i)^\varepsilon$ and $(\Sigma^g)^\varepsilon$ respectively. Then for $n+1$-tuple s, the map from strategies to outcomes, $(s/\alpha) \to (x^t, D^t, -\rho)$ is a *linear* function where $\alpha \in (\Sigma^i)^\varepsilon \times (\Sigma^g)^\varepsilon$, since α^t for $t \in T$ cannot affect either p or ρ in the continuum and α^g affects linearly ρ, through setting ρ or affecting p_j via b_j^g in a linear manner. Thus,

$$\Pi^t(s/\alpha^t) = \phi^t(x^t(s)) + \mu D_-^t(s).$$

is continuous and concave. Furthermore, $\Pi^g(s/\alpha^g) = -\rho$ is also linear and continuous. Note that 0 is trivially concave and $D_-^t(s)$ is concave (linear) too. Therefore, $\min[0, D^t]$ is concave. Also,

$$(BR^i)^\varepsilon : \Sigma^\varepsilon \times (\Sigma^g)^\varepsilon \mapsto (\Sigma^i)^\varepsilon \text{ is convex}$$

and

$$(BR^g)^\varepsilon : \Sigma^\varepsilon \times (\Sigma^g)^\varepsilon \mapsto (\Sigma^g)^\varepsilon \text{ is linear}$$

where

$$(BR^i)^\varepsilon(s) = \arg\max\{\Pi^t(s/\alpha^t) : \alpha^t \in \Sigma^\varepsilon,\ t \in T_i\}$$

and

$$(BR^g)^\varepsilon(s) = \arg\max\{\Pi^t(s/s^g) : s^g \in (\Sigma^g)^\varepsilon \times \Sigma^\varepsilon\}$$

and

$$\Sigma^\varepsilon = \mathop{\times}_{i \in T} (\Sigma^i)^\varepsilon.$$

Hence, both $(BR^i)^\varepsilon$, $(BR^g)^\varepsilon$ are compact valued and upper-semicontinuous, by the "Maximum Theorem" (Berge, 1963, p. 116).

Step 1: has at least one T.S.N.E.

The correspondence,

$$BR^\varepsilon(s) = \mathop{\times}_{i \in T} (BR^i)^\varepsilon \times (BR^g)^\varepsilon : \Sigma^\varepsilon \times (\Sigma^g)^\varepsilon \mapsto \Sigma^\varepsilon \times (\Sigma^g)^\varepsilon$$

where $s = (s^t, s^g)$ is a type symmetric play, satisfies all the conditions of Kakutani fixed point theorem, and therefore admits a fixed point $BR^\varepsilon(\hat{s}^\varepsilon) \ni \hat{s}^\varepsilon$. It is easily verified that $\hat{s}^\varepsilon$ is a T.S.N.E. Thereafter s^ε, p^ε, ρ indicates a T.S.N.E.

Step 2: $\rho \geq 0$ at any T.S.N.E.

Recall the government's optimization,

$$\mathop{\arg\max}_{\rho, b_j^g, j=1,\ldots,m} \left\{ -\rho : z_{m+1} = \eta \left(\sum_{i=1}^{n} v^i + \sum_{j=1}^{m} b_j^g V \right), \sum_{j=1}^{m} b_j^g \leq 1 \right\}.$$

Since both v^i, $b_j^g \geq 0$, $-\rho \in \mathbb{R}_-^m$ which in turn implies $\rho \in \mathbb{R}_+^m$.

Step 3: There exists $c > 0$ such that $\forall \varepsilon > 0$ sufficiently small $p^\varepsilon > ce$, where $e = (1, ..., 1)$ of suitable dimensionality.

Suppose $\lim_{\varepsilon \to 0} p_j^\varepsilon \to 0$ for some j. Recall, that at equilibrium,

$$p_j^\varepsilon = \frac{\int_T b_j^t v^t dt + b_j^g V + \varepsilon}{(\int_T q_j^t dt + \varepsilon)}.$$

If for a commodity $\ell \neq j$, $p_\ell^\varepsilon \to 0$ when trader t could have made a bid smaller by Δ or otherwise if all $p_j^\varepsilon \to 0$, $\forall j = 1, ..., m$ then there definitely exists a trader of type i with $\Delta > 0$ of fiat money.

Alternatively, choose player t who borrows Δ and bids at the jth trading post. This transaction would have him incur a bankruptcy penalty $\mu\Delta < \infty$ since both μ or Δ are bounded from above. Now let,

$$\left\{ \frac{\partial \phi^t}{\partial x_j^t}(y) : y \leq \bar{G} \right\},$$

where $\bar{G} \leq \bar{M}$, i.e. the total money supply available.

Since nobody can acquire more $\bar{G}$ than at equilibrium we see that because of strict monotonicity of utilities, the net gain in utility of trader $t \in T_i$ as a result of his transaction is at least

$$\left\{ \frac{\partial \phi^t / \partial x_j^t(y)}{p_j^\varepsilon} - \mu\rho \right\} \Delta,$$

for a Δ sufficiently small. Note that by Step 2, $\rho \geq 0$. In addition, since $\eta < \infty$ and $\sum_{i=1}^n v_j^i + \sum_{j=1}^m b_j^g V < \infty$ then $\rho < \infty$. Since $\int_T v^t(1+\rho)dt$ has to be repaid at the end of trade, consequently some trader t has to pay a bankruptcy penalty equal to at least $\mu(\int_T v^t \rho dt/n)$. Therefore, if $\rho \to \infty$ then t could have been less worse off than he is now at our assumed equilibrium by bidding less. So, the expression in the brackets has to be non-positive,

$$p_j^\varepsilon \geq \frac{\partial \phi^t / \partial x_j^t(y)}{\mu\rho}.$$

We construct the T.S.N.E. s and the corresponding vector of prices p and interest rate ρ of Γ as a limit of Γ^ε and its associated equilibrium. For this purpose take a sequence ε which defines a limiting pair s.

Step 4: Select a sequence of ε and subsequences of subsequences so that:

(i) $\lim_{\varepsilon \to 0} \hat{s}^\varepsilon = s$. This is possible since strategy spaces are bounded.

(ii) Choose a subsequence so that $E^\varepsilon = \sum_{j=1}^m p_j^\varepsilon$ is convergent to either $\psi < \infty$ or ∞.

(iii) $p_j^\varepsilon / \sum_{j=1}^m p_j^\varepsilon$ converges $\forall j = 1, ..., m$.

Also,

- if $E^\varepsilon < \infty$ define $\lim_{\varepsilon \to 0} p_j^\varepsilon = p_j$ and $P_F = 1$, where P_F = price of fiat money.

- if $E^\varepsilon = \infty$ define $\lim_{\varepsilon\to 0}[(p_j/E)^\varepsilon] = p_j$ and $p_F = 0$.

Step 5: If $E^\varepsilon \to \infty$ then
(i) $p_j^e \to \infty$, $\forall j = 1, ..., m$.

Clearly, $p_j^\varepsilon \to \infty$ for some j. Let p_ℓ^ε be bounded for some $\ell \neq j$. Consider a trader who has a positive amount of j. Then this trader is in a position to borrow a small amount Δ of fiat and offer for sale $(1+\rho)\Delta/p_j$ of commodity j, and afterwards buy $\Delta(b_\ell^t)^\varepsilon/p_\ell^\varepsilon$ where in this case $(b_\ell^t)^\varepsilon = 1$. Consequently, he can return $(1+\rho)\Delta$: Therefore, his net gain in utility is positive and we get a contradiction.

(ii) $(q_j^t)^\varepsilon \to 0$ and $(b_j^t)^\varepsilon v^t/p_j^\varepsilon \to 0$, $\forall t \in T_i$ and $i = 1, ..., n$:

$$p_j^\varepsilon \leq \frac{(\bar{M}+1)\rho + \varepsilon}{\sum_{i=1}^{n}(q_j^i)^\varepsilon + \varepsilon} \text{ and } \sum_{j=1}^{m}(b_j^i)^\varepsilon v^i \leq \bar{M}.$$

So, by (i) $(q_j^t)^\varepsilon \to 0$ and $(b_j^t)^\varepsilon v^t/p_j^t \to 0$.

Step 6: If $E^\varepsilon < \infty$ and $p_F = 1$ then by Step 3, $p^\varepsilon > 0$ $\forall j = 1, ..., m$. Moreover, s is a T.S.N.E. It follows from Step 4 that $\lim_{\varepsilon\to 0}(BR^i)^\varepsilon = BR^i$. From Step 5, we know that $s_i = \lim_{\varepsilon\to 0}(s^i)^\varepsilon$ $\forall i = 1, ..., n$ except when $p_F = 0$. So s maximizes $\phi^t(x_j^t)$ on Γ and therefore s is a T.S.N.E.

Step 7: Any limit s of T.S.N.E. for Γ^s is an active T.S.N.E.

By Step 2, $\rho > -1$ and by Steps 5 and 6, $p_j > 0$. ∎

Theorem 2 *The strategic market game $\Gamma(n, \phi^i, a^i, \eta, \bar{M}, \mu, s^i, s^g)$ with a continuum of traders of n types and seigniorage costs for any $\bar{M}$ and a sufficiently high bankruptcy penalty μ results into* T.S.N.E.*'s which do* not *involve bankruptcy, produce $\rho > 0$, and are inefficient, provided that initial endowments are not Pareto optimal.*

Proof So long as initial endowments are not Pareto optimal, traders will borrow to finance their exchange and the subsequent depreciation of money used in transactions induces the government to set $\rho > 0$, to replace worn out money.[8]

As $\bar{M} \to \infty$ no agent bankrupts whenever $E^\varepsilon = \infty$, given a bankruptcy penalty $\mu > 0$. If $D^t < 0$ then trader t could have borrowed Δ less amount of fiat money and *not* bid for a commodity j whose $(p_j^\varepsilon) \to \infty$ and finally obtain

$$(\hat{x}_j^t)^\varepsilon = (x_j^t)^\varepsilon - \frac{\Delta}{p_j^\varepsilon}.$$

Therefore, he is saving the bankruptcy penalty $(1+\rho)\Delta\mu$ without reducing his utility for a sufficiently small Δ as $p_j^\varepsilon \to \infty$. Similarly, no agent bankrupts whenever $E^\varepsilon < \infty$, since $\rho > 0$, for a sufficiently high μ.

[8]For a more general discussion on the condition for sufficient gains to trade to exist so that markets to be active, see Dubey and Geanakoplos (1992).

From Step 3 of Theorem 1, $p_j > c$, $\forall j = 1, ..., m$. Also let

$$\bar{Q} = 1 + \max\left\{\frac{\nabla\Pi_j^t(x^t)}{\nabla\Pi_i^t(x^t)} : t \in T,\ i, j = 1, ..., m,\ x^i \in \diamondsuit\right\},$$

where

$$\diamondsuit = \left\{x^t \in \mathbb{R}_+^m : x_j^t \leq 1 + \max_{1\leq j\leq m} \sum_{t\in T} a_j^t\right\}.$$

If t goes bankrupt then he can reduce his bid on a commodity j by ε and use this amount to defray his loan. His gain in utility will be $\varepsilon\mu$ whereas his loss will be at most

$$\frac{\varepsilon\nabla\Pi_j^t(x^t)}{p_j} \leq \frac{\varepsilon\nabla\Pi_j^t(x^t)\bar{Q}}{c}.$$

Thus, set $\mu^* = (\bar{Q}/c)\max\{\nabla\Pi_j^t(x^t),\ t \in T,\ j = 1, ..., m,\ x^t \in \diamondsuit\}$. As long as $\mu > \mu^*$ no agent will go bankrupt.

Now, consider a trader t, w.l.o.g., who buys commodity j and sells commodity ℓ and a trader t' which is involved in the reverse transaction. From the existence argument all markets are active, i.e., $p_j > 0$, $\forall j = 1, ..., m$. Let $\forall t \in T$, $J^t = \{j = 1, ..., m : b_j^t > 0\}$ and $L^t = \{j = 1, ..., m : q_j^t > 0\}$. In other words, partition the set of commodities each trader buys from the ones he sells. Since all markets are active and strategy sets are bounded below by ε, $\exists t, t'\ \forall j = 1, ..., m \ni J^t \cap L^{t'} \neq \emptyset$ (or equivalently $J^{t'} \cap L^t \neq \emptyset$). Otherwise, all traders would be buying or seller the same commodities and the markets would be lopsided. Thus, there exists always a pair of traders involved in reverse transactions when oneconsiders trade in *all* commodities. Otherwise, either one trader would violate his budget constraint or be left with unused cash. The optimization conditions give,

$$\frac{\nabla\Pi_j^t(\vec{x}, D^t)}{p_j(1+\rho)} = \frac{\nabla\Pi_\ell^t(\vec{x}, D^t)}{p_\ell}$$

and

$$\frac{\nabla\Pi_j^{t'}(\vec{x}, D^t)}{p_j} = \frac{\nabla\Pi_\ell^{t'}(\vec{x}, D^t)}{p_\ell(1+\rho)}.$$

These imply,[9]

$$\frac{\nabla\Pi_j^t(\vec{x}, D^t)}{\nabla\Pi_\ell^t(\vec{x}, D^t)} \neq \frac{\nabla\Pi_j^{t'}(\vec{x}, D^t)}{\nabla\Pi_\ell^{t'}(\vec{x}, D^t)},$$

where

$$\nabla\Pi_j^t(\vec{x}, D^t) = \frac{\partial\Pi_t(\vec{x}, D^t)}{\partial x_j},\ \forall t \in T,\ j = 1, ..., m.$$

■

[9] As far as the "inefficiency" argument is concerned, the "wedge" of size $(1+\rho)$ results in gradients of t and t' tilting away from the price ratio p_j/p_i in *opposite* directions. Therefore, it is possible for t and t' to trade for further utility improvements. However, this possibility is blocked by the transactions cost caused by $\rho > 0$.

3 Alternative Methods of Seigniorage

The model described before has a convenient taxation interpretation. In particular, the government is fundamentally charging an income tax in proportion to the amount of value guarantee (i.e., amount of money each trader uses in exchange). However, since a production technology of exchange is involved to compare the efficiency of different institutional setups is inappropriate. More refined definitions of societal and institutional efficiency are called for.

However, we construct two variants which accommodate the need for seigniorage. In the first, we generate a resource reduced competitive economy in which the resource reduction is equivalent to the cost of surrogated trust. We encountered a resource depletion in the $\Gamma(\cdot)$ of Section 2 because of depreciation. Consequently, we need to define the associated resource reduced competitive economy. The resource reduction will be represented via *income taxation*.

Alternatively, we can consider the game of Section 2 in which fiat money is lent at $\rho = 0$. However, we impose a *poll-tax* on each agent such that the total revenue will be equal to interest payments made in the game with $\rho > 0$.

Both alternative models will finance the seigniorage cost needed for the production of money. However, they indicate that a different concept of institutional efficiency is needed if we are to be able to draw comparisons and equivalences among the three. The economy runs using mechanisms. These mechanisms are formal and informal financial institutions reflecting a complex intermix of law and custom. When we consider the many variations available where each institution consumes resources in the running of its transactions process we are confronted with a need for a criterion of efficiency which takes into account the resources needed to run the process. Given that all processes consume resources the analysis of efficiency becomes an investigation into comparative resource consumption by different mechanism for providing the same set of services. A comparison of institutional efficiency leaves us with a partial ordering. For any specific level of operation it is possible to state that A is better than B, B better than A, they are equivalent or neither is better than the other. A procedure that may be efficient for a small market may not be so for a large one.

3.1 Associated Resource Reduced Competitive Economy $E(n, \phi^i, a^i, T^i)$

For each trader type i, $i = 1, ..., n+1$ in $E(n, \phi^i, a^i, T^i)$ let,

$$T^i = \begin{cases} \dfrac{\max\left[\sum_{j=1}^{m} p_j(x_j^i - a_j^i), 0\right]}{\sum_{i=1}^{n} \max\left[\sum_{j=1}^{m} p_j(x_j^i - a_j^i), 0\right]} \cdot z_{m+1}, & \text{, if } \sum_{j=1}^{m} p_j(x_j^i - a_j^i) > 0 \text{ and } \sum_{i=1}^{n} \max\left[\sum_{j=1}^{m} p_j(x_j^i - a_j^i), 0\right] > 0 \\ 0 & \text{, otherwise} \end{cases}$$

Note that T^i, which represents an income tax levied from traders from their initial endowments using the data of $\Gamma(\cdot)$ of Section 2, will generate total tax revenue equal to the amount of depreciated fiat money in the $\Gamma(n, \phi^i, a^i, \eta, \bar{M}, s^i, s^g)$. In other words,

$$\sum_{i=1}^{n} T_i = z_{m+1} \geq 0.$$

Thus income taxation corresponds to the resource reduction caused by depreciation of money. Remark that if $z_{m+1} = 0$, i.e., $\eta = 0$, then we are operating in the standard Walrasian economy.

An allocation (x, p, T) of the resource reduced economy $E(n, \phi^i, a^i, T^i)$ where $x^i, x^g \in \mathbb{R}^m_+$, $p \in \mathbb{R}^m_{++}$, $T^i \in \mathbb{R}_+$ is a *Resource Reduced Walras Equilibrium* (R.R.W.E.) if,

(i) $\sum_{i=1}^{n} T^i = z_{m+1}$

(ii) $\sum_{i=1}^{n} x^i = \sum_{i=1}^{n} a^i - x^g$

and

(iii) $x^i = \arg\max\{\phi^i(y) : y \in \mathbb{R}^m_+,\ px^i = pa^i - T^i\}$ for $1 \leq i \leq n$.

Existence of R.R.W.E. follows from standard general equilibrium arguments and therefore is omitted. Also, note that the resource reduced competitive economy via income taxation extracts the necessary seigniorage costs to replace the depreciated fiat money. Pareto optimality follows from the first welfare theorem.

3.2 The Poll-Tax Game $\Gamma^T(n, \phi^i, a^i, \eta, \bar{M}, T, s^i, s^g)$

Reconsidering the game $\Gamma(\cdot)$ as described in Section 2, recall that the depreciated fiat money that needs to be replaced by the government equals

$$T = \eta \left[\sum_{i=1}^{n} v^i + \sum_{j=1}^{m} b_j^g V \right].$$

Now, in an analogous manner as in the resource reduced competitive economy seigniorage will not be financed by interest rate payments but by a poll-tax.

Thus, to each trader type i a poll-tax is levied equal to,

$$T^i = \frac{T}{n},\ 1 \leq i \leq n.$$

The strategy set of trader i is as in (1) and the arguments are constrained as in (2). However, the final amount of monetary holdings of fiat which also indicates the trader's debt position at the end of exchange is given by

$$D^i = \sum_{j=1}^{m} p_j q_j^i - \sum_{j=1}^{m} b_j^i v^i - (1+\rho)v^i - T^i + v^i. \tag{12'}$$

The rest of the formulation is as in Section 2.

Theorem 3 *The strategic market game $\Gamma^T(n, \phi^i, a^i, \eta, \vec{M}, T, s^i, s^g)$ with a continuum of traders of n types and seigniorage costs for any finite non-zero $\bar{M}$, any bankruptcy penalty μ and $T \in \mathbb{R}^n_+$ such that $T^i \backslash T/n$ has an active* T.S.N.E.

Proof The proof follows *mutatis mutandis* from Theorem 1. ■

Remark that the allocations produced in $\Gamma^T(\cdot)$ do not necessarily Pareto dominate the one produced by $\Gamma(\cdot)$ or vice versa. It remains an open question to determine the ranges of the corresponding Pareto optimal surfaces at which the allocations $\Gamma(\cdot)$ Pareto dominate the ones of $\Gamma^T(\cdot)$ and vice versa for the class of $\Gamma(\cdot)$ and $\Gamma^T(\cdot)$ games.

4 Concluding Remarks

4.1 The Unique Minimal Cash-Flow Equilibrium for Efficient Trade

The modeling of government's objective function, i.e., minimizing the interest rate charged for borrowing, produces the minimum liquidity equilibrium for $E(\cdot)$. We have analyzed in our previous work, see M. Shubik and D.P. Tsomocos (1992), that the bankruptcy penalty bounds prices from below whereas the money supply bounds prices from above. Therefore, the minimization of interest rates leads to minimization of *effective* money supply. By effective money supply we understand the money required for exchange for the attainment of efficient trade in $E(\cdot)$. The remaining money supply is stored costlessly in a one-period model. Hence, the minimization of interest rates, provided that we have selected the appropriate bankruptcy penalty, results into the equalization of upper and lower bounds on prices. We, therefore, have selected endogenously the unique minimal cash-flow equilibrium. Finally, note the model prevents inflationary pressures and resolves the indeterminacy of the price level. Note also when we have exchange with fiat money there exists one extra degree of freedom to the model which can be reduced either by fixing money supply or interest rates or a strategic variable which determines any of the two.

When we view the process as a playable experimental game we see a general difficulty in comparing a static equilibrium with actual dynamics. The existence of an equilibrium requires that we believe that the government can announce *in advance* the correct interest rate and how it is going to spend revenues it has not yet received. This requires a forecasting ability which does not match with experience. At best, over time, we can hope for a convergent adjustment process.

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