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COWLES FOUNDATION DISCUSSION PAPER NO. 1037

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Some Dynamics of a Strategic Market Game
With a Large Number of Agents

by

John H. Miller

and

Martin Shubik

November 1992

Abstract

This paper is designed to combine the game theoretic investigation of the static or equilibrium properties of large strategic market games together with the investigation of some very simple dynamics, which nevertheless are sufficient to show differences between two related games, one in which both borrowing and trade take place. The role of banking reserves emerges as relevant and sensitive to the transient state dynamics.

Several 100,000 player games are simulated and the behavior is constructed with the analytical prediction for the games with a continuum of agents.

Some Dynamics of a Strategic Market Game with a Large Number of Agents

John H. Miller^{*} and Martin Shubik^{}**

1. Introduction

This paper combines game-theoretic investigation of the static or equilibrium properties of large strategic market games with investigation of some very simple dynamics, which nevertheless are sufficient to show differences between two related games, one in which trade uses fiat money, but there is no borrowing, and the other in which both borrowing and trade take place. The role of banking reserves emerges as relevant and sensitive to the transient state dynamics.

Keynes (1936) in his discussion of the demand for money notes the transactions, precautionary, and speculative demands for money. The requirement that individuals use fiat for bidding is a formalization of the transactions use of money. The motivation to hold money illustrates the precautionary demand, which appears in the models below.

We use a mixture of results obtained from the mathematical analysis of the equilibrium conditions for a process model and simple "experimental computational methods" to provide insight into three problems. They are:

- (1) The relationship between the equilibria of finite and the continuum games.

^{*}Department of Social and Decision Sciences, Carnegie Mellon University, Pittsburgh, Pennsylvania 15213 and Santa Fe Institute, 1660 Old Pecos Trail, Santa Fe, New Mexico 87501.

^{**}Cowles Foundation for Research in Economics, Yale University, New Haven, Connecticut 06520 and Santa Fe Institute, 1660 Old Pecos Trail, Santa Fe, New Mexico 87501.

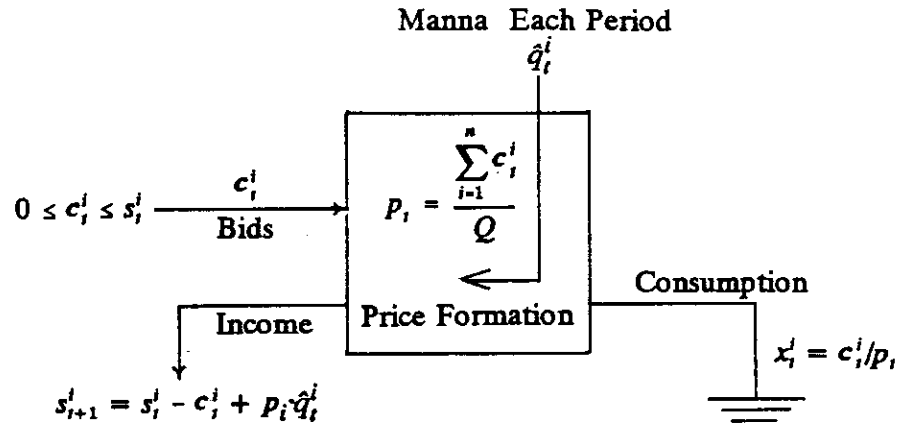
- (2) The relationship among the equilibria of the games with and without borrowing. In the first, the wealth distribution and the price level are both unique once the amount of money in the system has been specified. In the second, the price level will be determined (within certain bounds) by the initial conditions, but the shape of the equilibrium wealth distribution is preserved (with a changed scale).
- (3) The stability or instability and convergence to equilibrium of the system with and without borrowing.

We stress the interlinkage between an analytical approach and simulation and gaming. The mathematical analysis of equilibrium, though tractable, obscures the critical role of reserves and forecasting in banking. The finite player game by itself in both the equilibrium and dynamic versions is difficult to analyze, but can be explored with simulation using the analytical equilibrium results as a benchmark.

Karatzas, Shubik and Sudderth (KSS, 1992) have established the existence of a stationary wealth distribution for a class of strategic market games where there is a single perishable good, say, manna, which is put up for sale each period. There is a continuum of traders each of whom starts with an initial endowment of fiat money (which can be viewed physically as "blue chips"). They each bid some amount of their money to buy a share of the manna. Price is then determined, and the manna is distributed in proportion to money bid. The total money income obtained from the sale of the manna is then distributed among the traders in proportion to their (randomly determined) ownership claims.

Figure 1 shows the market mechanism. The bids of the agents (denoted by a^i for agent i) constrained by the agent's wealth, s^i , are all aggregated by the market; Q units of the manna drop into the market; price, p , is formed as indicated; individual consumption is determined by dividing the amount bid by the market price and then income is distributed to the agents in accordance with a random variable q^i reflecting the ownership claims of i to a portion of Q and returning to i the income derived from its sale ($\hat{q}_t^i p_t$), where \hat{q}_t^i is the realization of q_t^i .

A specific example was studied in detail and for this example an analytical solution is obtainable. The proofs of the general theorem and the optimal policy for the special case considered



Bids, Consumption and Income
Figure 1

here are given in the paper noted above. In that paper, however, no analysis of any dynamics was provided. In this note we consider a game that is a finite approximation of the game with a continuum of traders. We observe that for the market without lending there is a simple updating rule which yields an almost stationary price to an approximation which varies as the number of traders. We also consider the evolution of the wealth distribution as the game is played many times. In a subsequent section a bank or money market for loans is introduced together with a bankruptcy law and a default pool which enables all loans to be repaid in full. We consider the behavior of price and the wealth distribution for this instance as well.

Although it is conjectured that the strategic market game with a finite number of traders has a solution which converges smoothly to the solution of the game with a continuum, it is not even known that the finite player game has a pure strategy solution. It is of some interest at least to be in a position to utilize a computation to explore the dynamics of games up to the size of 200,000 agents.

2. A Simple Strategic Market Game

From the point of view of the economist concerned with conventional models of supply and demand the model built here is radically oversimplified. The supply of manna is totally inelastic. Individuals want at most one unit of the good each period. Even more "unrealistic" from some points of view is that they do not know their current ownership claims as they bid before they know the outcome of the random variable which will be used to determine the income size. But many economic decisions are made without knowledge of future income, this merely emphasises this phenomenon.

2.1. Some Modeling Considerations

Instead of having the consumer good fall into the market, it is possible to construct a slightly different model where the random variable determines the distribution of the good to the traders, they then decide whether to offer part of the good to the market or to bid for the good offered. This market may be regarded as "more realistic" but it has the disadvantage that the strategy sets of the individuals each period have been doubled. We could even have them bid and offer personal prices (*a la* Bertrand-Edgeworth) and this would increase the market mechanism to four dimensions each period. Our attitude however is that we prefer to consider the most simple of mechanisms and hence allow for the complications in numbers and dynamics and in being able to look at a loan market. In part 3 we consider the extension to borrowing, leaving the modeling of buying and selling for future analysis.

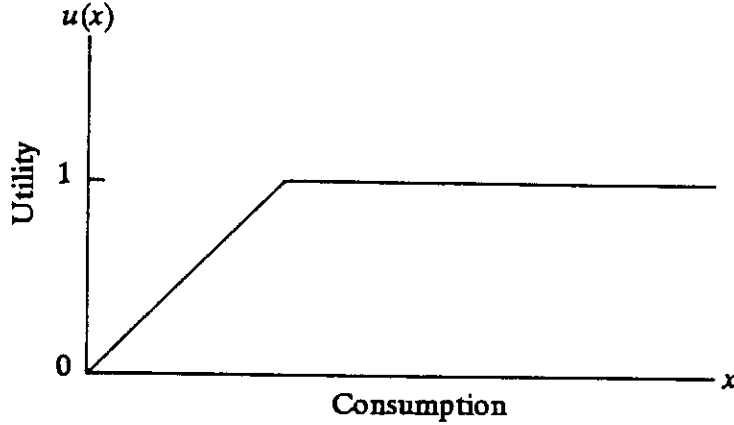
2.2. Equilibrium with a Continuum

Before we specialize to the specific utility function the general problem can be posed as follows:

A bid by each trader is made simultaneously without knowledge of each trader's ownership claim to current assets.

After all traders have moved, the random variable determines the ownership claims. After this the individuals all receive their income from the sale of all assets and utility from consumption, the game then goes into the next period.

A policy is described by $\pi = (\pi_0, \pi_1, \dots)$ where π_n chooses the action a_n on the n^{th} day, based on the history $H_n = (s_0, a_0, s_1, a_1, \dots, s_{n-1}, a_{n-1}, s_n)$, where s_t is the state and a_t the action at t .



A Simple Utility Function Saturating at One
Figure 2

A stationary policy takes $a_n = \phi(s_n)$ ($\forall n \in \mathbb{N}_0$), where $\phi : S \rightarrow A$ is a given measurable function.

Once a_n has been selected based on H_n , the system moves to the new state $s_{n+1} = s_n(1 - a_n) + y_n$ ($n \in \mathbb{N}_0$). In other words, the *law of motion* is given as $q(\cdot | s_n, a_n) = F_{s_n(1-a_n)}(\cdot)$.

According to standard theory (e.g., Maitra (1968)) there exists an optimal stationary policy π^* (with $\pi^* = a^*(s_n)$ for a suitable measurable function $a^* : S \rightarrow A$, $\forall n \in \mathbb{N}_0$), and the optimal reward

$$(1) \quad V(s) := \sup_{\pi} E_{s_0=s}^{\pi} \sum_{n=0}^{\infty} \beta^n r(s_n, a_n)$$

for a given $0 < \beta < 1$, is u.s.c. on $S = [0, \infty)$ and satisfies the *optimality equation*

$$(2) \quad \begin{aligned} v(s) &= \max_{0 \leq sa \leq 1} [U(sa) + \beta \cdot EV(s - sa + y)] \\ &= U(sa^*(s)) + \beta \cdot EV(s - sa^*(s) + y) , \end{aligned}$$

or equivalently, with $c^*(s) = sa^*(s)$:

$$(3) \quad \begin{aligned} v(s) &= \max_{0 \leq c \leq 1} [U(c) + \beta \cdot EV(s - c + y)] \\ &= U(c^*(s)) + \beta \cdot EV(s - c^*(s) + y) . \end{aligned}$$

Following Karatzas, Shubik and Sudderth (1982) we focus in this paper on the (very) *special case*

$$(4) \quad U(c) = \begin{cases} c; & 0 \leq c \leq 1 \\ 1; & c \geq 1 \end{cases} .$$

The utility function in a single period is illustrated in Figure 2.

The optimal (stationary) policy of equation (3) has the very simple form

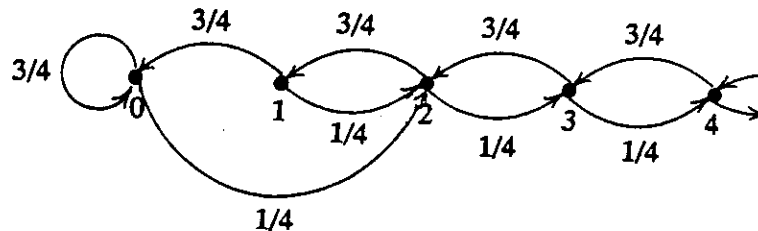
$$(5) \quad c^*(s) = \begin{cases} s; & 0 \leq s \leq 1 \\ 1; & s \geq 1 \end{cases} = u(s) .$$

We are able to compute explicitly the value function V on \mathbb{N}_0 as well as the unique invariant measure of the Markov chain as shown in Figure 3:

$$(6) \quad s_{n+1} = s_n - c^*(s_n) + y_n, \quad s_0 \in \mathbb{N}_0$$

on \mathbb{N}_0 , when the distribution F has the particularly simple form

$$(7) \quad P(y = 2) = \gamma, \quad P(y = 0) = 1 - \gamma \quad \text{with } 0 < \gamma < \frac{1}{2} .$$



The Markov Chain
Figure 3

Suppose that the random variable y has the simple distribution (7). Then the value function $V(\cdot)$ can be computed explicitly on the integers:¹

$$(8) \quad V(0) = A\theta + \frac{\beta}{1-\beta} \quad \text{and} \quad V(s) = A\theta^s + \frac{1}{1-\beta}, \quad s \in \mathbb{N}$$

where

$$(9) \quad \theta = \frac{1 - \sqrt{1 - 4\beta^2\gamma(1-\gamma)}}{2\beta\gamma}, \quad A = \frac{1-\gamma}{\gamma\theta\left(1 - \theta + \frac{1-\beta}{\beta\gamma}\right)}.$$

Suppose, furthermore, that $s_0 \in \mathbb{N}_0$; then the resulting random sequence in equation (6) is an ergodic Markov chain on \mathbb{N}_0 , with invariant measure $\mu = (\mu_0, \mu_1, \dots)$ given by

$$(10) \quad \mu_0 = c(1-\gamma), \quad \mu_1 = c\gamma, \quad \mu_s = c\left(\frac{\gamma}{1-\gamma}\right)^{s-1} \quad (s \geq 2), \quad \text{where } c = \frac{1-2\gamma}{1-\gamma}.$$

Suppose for specificity $\gamma = 1/4$ and $\beta = 1/2$.

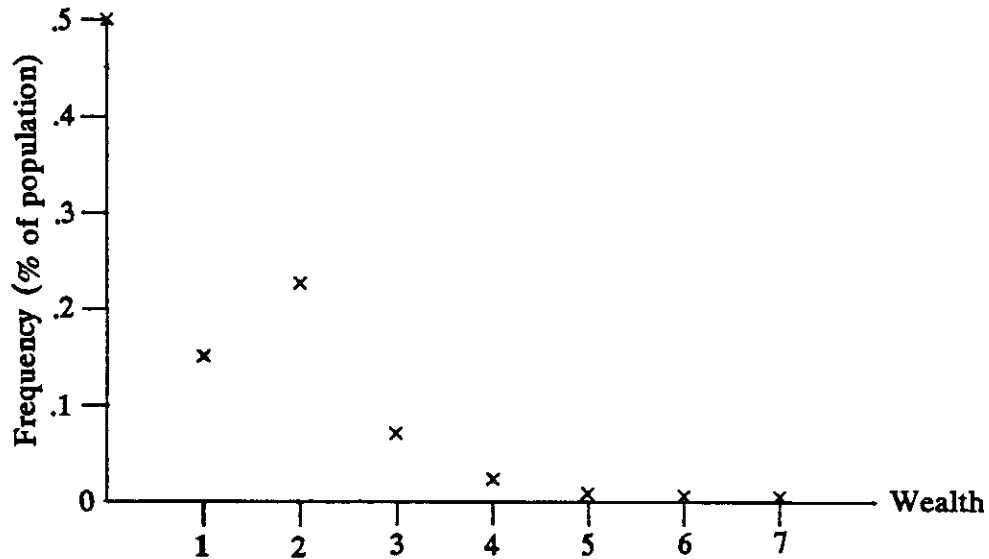
The stationary wealth distribution is as illustrated in Figure 4, where

$$\begin{aligned} \mu_0 &= c(1-\gamma) = \frac{1}{2} = .50000 \\ \mu_1 &= c\gamma = 1/6 = .16667 \\ \mu_2 &= c\left(\frac{\gamma}{1-\gamma}\right) = 2/9 = .22222 \\ \mu_3 &= c\left(\frac{\gamma}{1-\gamma}\right)^2 = .07407 \\ \mu_4 &= c\left(\frac{\gamma}{1-\gamma}\right)^3 = .02469 \\ \mu_s &= c\left(\frac{\gamma}{1-\gamma}\right)^{s-1} \quad \text{for } s \geq 3. \end{aligned}$$

¹Outside the lattice $V(\cdot)$ is determined by linear interpolation

$$V(s) = \frac{1}{1-p} = \left[\frac{1}{1-\theta} - (s - [s])\theta^{[s]} \right] \theta^{[s]} \quad 0 \leq s \leq \infty$$

where $[s]$ is the integer part of s .



Wealth Distribution
Figure 4

2.3. Testing for Equilibrium

In each period in the market a move of an individual player involves the selection of an amount of money c where $0 < c < s$, i.e. where the bid c is greater than or equal to zero and less than or equal to the wealth s of the individual. A strategy in general is a policy based on the total known history to date. In a repeated game with a continuum of players and a structure as is indicated the optimal policy depends only on the individual's wealth and his estimate of future price. Formally we can imagine the logically well defined, but behaviorally implausible, proposition that all individuals predict the existence of a sequence of constant future market prices and optimize on this assumption. If all of their speculations are mutually consistent we have verified the existence of a noncooperative equilibrium.

We can replace this construction by a rather simplistic behavioral pattern which should also yield the same equilibrium. Suppose that instead of predicting the infinite price series each trader were considered to be composed of two agents, one fairly simplistic--the forecaster and the other a sophisticated calculator, but otherwise naive. The forecaster bases his forecast on the previous market price, the calculator believes the forecast and solves the appropriate dynamic program. Neither learn from their previous errors. In a projected further investigation we propose to use a learning dynamics, but for this investigation, even with such a simple

rule, there appears to be some worthwhile inferences in connecting the statics with the dynamics.

We now make a "leap of faith" with an appropriate *caveat* attached to it. We will guess that for very large finite games there is a pure strategy equilibrium and the strategies are of the same structure as those for the game with a continuum of traders. There is the possibility that the equilibrium for the finite game might be in mixed strategies and that the convergence might be to the pure strategy solution of the continuum game but through mixed strategies.

After we have used this simple forecasting mechanism to consider static equilibria, as we have a fully defined process model and the capacity to examine large games we can at least perform a "myopic sensitivity analysis," i.e. examine the motion of the system if started away from equilibrium.

2.4. Dynamics with a Finite Number of Traders

In the defining of the game with a continuum of agents it was necessary to be explicit about the updating procedure. In particular, an easy convention to use is that individuals condition their choice upon the belief that the previously formed market price will prevail. Given this prediction, we may then consider the optimal policy myopically applied to this price.

We study the dynamics of the system with all traders following the policy of bidding $\min\{\text{previous price, current wealth}\}$. As we are forced to use a finite number of traders we know that we can only at best obtain an approximation to the stationary distribution as there will always be a certain amount of noise coming through even though numbers are large.

We consider four sizes of games in order to obtain some inkling into the dynamics of the system. In the first game there are 100 traders, in the second 1,000, in the third 10,000 and in the fourth game there are 100,000.² As income could be any real number we face a problem

²We also ran a simulation with 200,000 traders which showed a further shrinkage in the variance, however for aesthetics we compare $n = 100, 1,000, 10,000$ and 100,000 with the continuum.

in displaying wealth after payment to individuals, this is handled by rounding the income category in our statistics up or down to the nearest integer.³

The data below were collected at the start of the 20th iteration. The process was simulated 50 times (each with a different random seed), and the values displayed represent the average and sample standard deviation of the price and wealth distributions.

The following data are for the case where the predicted price for this period is last period's price, in symbols $\hat{p}_t = p_{t-1}$, there is no borrowing, gamma = 0.25, initial predicted price = 1.0, and initial wealth = 1.0.

Table 1

Agents	Wealth Range					
	Price	[-.5, .5)	[.5, 1.5)	[1.5, 2.5)	[2.5, 3.5)	[3.5, +)
100	0.9981 (0.0720)	50.68 (2.65)	16.10 (3.03)	22.18 (3.68)	7.60 (2.54)	3.44 (1.54)
1000	0.9962 (0.0208)	49.94 (0.99)	16.55 (1.13)	22.45 (1.09)	7.37 (0.65)	3.69 (0.44)
10000	0.9992 (0.0067)	49.93 (0.25)	16.68 (0.28)	22.40 (0.29)	7.29 (0.22)	3.72 (0.18)
100000	1.0008 (0.0024)	50.05 (0.09)	16.60 (0.11)	22.30 (0.11)	7.33 (0.07)	3.71 (0.07)
∞	1.0000	50.00	16.67	22.22	7.41	2.47

We observe that even though the system was started away from the equilibrium wealth distribution, the wealth distribution and the market price for the set of 100 player games is fairly close to the continuum game and as we progress to bigger games the fit improves.

It is important to note that in the system with no lending, not only is the wealth distribution unique, but as soon as the amount of money in the system has been specified the price system is uniquely determined. This can be seen from equation 11

³It requires some care in presentation of the boxes in order to compare distributions with different price levels and wealth levels in a useful manner.

$$(11) \quad \sum_{i=1}^{\infty} p_i \mu(p_i) = M$$

where p_i is the level of wealth and $\mu(p_i)$ is the percentage of the population at wealth p_i and M is the total money supply. Given that $\mu(p_i)$ is unique and M is specified then p is unique. Thus the dynamic system has only one attractor. Our numerical results indicate that the system rapidly converges to this attractor.

An immediate question to ask of any numerical analysis is how robust are the observed results. We performed a variety of sensitivity analyses in order to explore this issue.

An important item to check is the influence of changes in n . We have illustrated above in Table 1 the influence on convergence of games with 100, 1,000, 10,000 and 100,000 traders. There are clear indications of convergence to the continuum distribution.

We also need to check the influence of the initial prediction on the process. In this case it appears that the initial prediction may influence the first few iterations and then dies away. We found little influence of initial price in the range 1 to 2 on the system.

Another aspect requiring checking is the influence of the initial wealth distribution on the dynamics of the system.

A little elementary game theory for finite numbers indicates that the distribution of wealth can initially be selected to influence any convergence to an equilibrium for an arbitrarily finite length of time, for example, in a million person game with a million units of money, all one has to do is give it all to one millionaire then the trickle down will take at least a million periods.

Examples of some long transients are given when we consider the loan market.

In virtually all dynamical models in economics without overlapping generations it is desirable to check the influence of changes in β . In this simple example however the optimal program is not influenced by changes in β although the value of the program changes. Thus there is no need for a sensitivity analysis on β .

We might wish to check the influence of a change in γ . This change will give a different stationary distribution as indicated by the solution for the continuum case. An example for

$\gamma = .1$ has been run as a check, but is not displayed as nothing particularly new can be gleaned from this variation.

2.5. Some Interpretation

The simulations of the special large finite games with the simple price prediction mechanism and the optimal policy based on the predicted price clearly do not provide a proof of the open conjecture that the large finite games approach the continuum game in general, but they do offer some indication of the plausibility of the conjecture. Because of the extremely low dimensionality of the problem (given type symmetric behavior) we had a chance (in KSS) to establish uniqueness of the active equilibrium. As soon as we consider two or more goods this is bound to be false.

The cases considered also indicate that there are zones of convergence for the fairly simple dynamics proposed, but that there exist initial conditions which can influence the speed of convergence in an arbitrarily long manner. We are unable to establish if there are zones where no convergence takes place. But as is shown below considerable differences emerge in the dynamics between markets with and without loans.

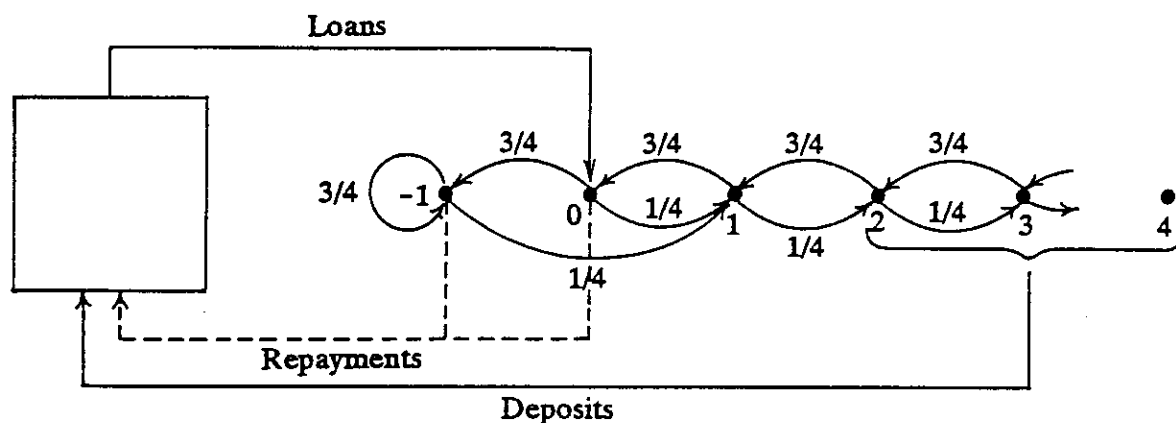
3. A Money Market and Default Reserves

3.1. Borrowing and Lending with a Continuum of Traders

The particular example explored in Section 2 has the interesting property that there is a relatively straightforward extension of the model to include borrowing and lending with an extreme form of the handling of default, i.e. the inability of a debtor to repay his loans in a timely manner results in the debtor being unable to borrow more until he has paid back his debt. If the rate of interest does not accrue on his debt or is zero and his expected income is positive he will eventually be able to emerge from debt after the garnishing of his income has covered the debt.

Figure 5 shows the Markov chain with the bank reserve pool included. This has to be added to account for all of the money in the system. In the original chain shown in Figure 3, when we sum the amount of money held by those with wealth levels of 0 and above we

account for all of the money in the system. If we do the same summation with the new wealth frequencies we obtain only half of the money. The remainder has to be somewhere. It can be accounted for by a loan reserve or bank pool whose size equals or is greater than the total outstanding debt in the economy. This debt can be represented by IOU notes or green chips which at some point must be destroyed by being converted into blue chips -- i.e. by having debtors pay money to retire their IOU notes outstanding.



The System with a Bank or Loan Pool
Figure 5

Figure 5 shows the system where any money of the rich not needed for immediate transactions is swept into the bank or loan pool. The borrowers borrow from the bank. At the start of every period collections are made from debtors whose incomes are garnished or borrowers who are in a position to pay back. At the start of any period depositors may increase or withdraw deposits.

At equilibrium Tables 2 and 3 show the location and distribution of all resources for all individuals and the bank for the economy with a continuum of traders. The nature of the bank as a buffer guarantees that the depositors are paid back in full if reserves are large enough. The fluctuations in repayment by the debtors are absorbed by the pool.

Table 2
Wealth and Financial Flows Without Lending

	Level of Wealth							
	0	1	2	3	4	5	B	ΣP
Frequency	1/2	1/6	2/9	2/27	2/81	2/243	--	--
Total Wealth	0	1/6	4/9	2/9	8/81	10/243	0	1
Total Spending	0	1/6	2/9	2/27	2/81	2/243	0	1/2
Loanable Funds = Hoarding	--	0	2/9	4/27	6/81	8/243	0	1/2

Table 3
Wealth and Financial Flows With Lending

	Level of Wealth								
	-1	0	1	2	3	4	5	B	ΣP
Frequency	1/2	1/6	2/9	2/27	2/81	2/243		--	1
Total Wealth	-1/2	0	2/9	4/27	6/81	8/243		1/2	1/2
Total Spending	0	1/6	2/9	2/27	2/81	2/243			1/2
Borrowing	0	1/6	0	0	0	0		[1/6]**	1/6
Lending	0	0	0	2/27	4/81	6/243			1/6
Repay	1/8	1/24	0	0	0	0		[1/6]*	--
Deposits	0	0	0	2/27	4/81	6/243		4/6	

*Collections

**Borrowing

The Markov chain shown in Figure 5 is for the example with one period borrowing and lending, where the following optimal equilibrium policy applies:

If $x < 0$ the individual cannot spend or borrow

If $0 \leq x \leq 1$ the individual borrows $1-x$ and spends 1

If $x = 1$ the individual spends 1

If $x > 1$ the individual spends 1 and lends $x-1$

Assertion: There exists a stationary distribution with lending with the following properties:

$$\rho = \text{the rate of interest} = 0$$

$$p = \text{price} = 1$$

and there is no hoarding. The amount of loanable funds = $1/6$. The bankruptcy pool = $1/2$ and the total number of blue chips = 1.

Under the policy above the Markov chain is as before except for a left shift to $-1, 0, 1, 2, \dots$. Thus the fraction of the population at -1 is $M_{-1} = 1/2$ similarly $\mu_0 = 1/6$, $\mu_1 = 2/9$ and $\mu_s = 2/3(1/3)^s$ for $n \geq 2$.

The demand for loans by those at wealth = 0 is $D = 1/6$.

The supply of loans is given by

$$S = \frac{2}{3} \sum_{s=2}^{\infty} \left(\frac{1}{3}\right)^s (s-1) = \frac{1}{6}.$$

The repayment from those garnished and those borrowers who can repay is:

$$\frac{1}{4} \left(\frac{1}{2}\right) + \frac{1}{4} \left(\frac{1}{6}\right) = \frac{1}{4} \left(\frac{4}{6}\right) = \frac{1}{6}.$$

At a rate of interest $\rho = 0$ the lenders are indifferent between hoarding or lending as they run no risk of not being paid.

3.2. The Multiplicity of Equilibria

Before we consider the simulations an important theoretical distinction between the game with and without loans must be made. In the game without loans the equilibrium price is unique as can be seen from equation 11, but from equation 12 described below we observe that the equilibrium price may be anywhere in the range between 0 and 2.

$$(12) \quad \sum_{s=0}^{\infty} ps\mu(ps) + B = M.$$

This is because the bank reserves B act as a buffer which can be moved up and down to compensate for changes in p in the open range from 0 to 2 and still have equation 12 hold. In the actual dynamics our model requires further specification to indicate what happens if a fluctua-

tion reduces bank reserves to zero. There will be a difference in the dynamics if 100% reserve banking is considered, i.e. if the bank is not permitted to lend more than its capital, in contrast with allowing it to lend some fraction of its deposits as well as its own capital.

For the purposes of this paper however we do not wish to digress into the many specific interesting problems concerning reserve banking in disequilibrium, but to touch on an important aspect of dynamics arising from the difference between the no loans and the loans model, thus we assume the bank always has sufficient reserves.

We have just shown that there is a continuum of equilibrium points for this model thus the dynamical system no longer has a single target point to aim at but if it is to reach equilibrium it is aimed at a continuum in the form of an open set of different prices. In disequilibrium there is thus no clear price level at which to aim. This can have considerable consequences for the dynamics.

3.3. Dynamics with a Finite Number of Depositors and Borrowers

In this section the same treatment is given for the finite games with depositing and borrowing as was given for the games without borrowing. Before discussing the results some observations need to be made concerning the differences between the dynamics and the statics for the two models. The static theory for the continuum predicts the same distribution shifted one unit to the negative.

In the economy without borrowing all of the money supply is held by the individuals and half of it is always in hoard. In the economy with lending and the same price level much of the money must be held in a pool as the sum of all wealth held by individuals with positive wealth adds to only one half of the money supply.

A new phenomenon appears in the form of the behavior of bank or pool reserves and the size of deposits. If, as we suggest here, it is useful as a conceptual device to think of money as some form of physical object such as blue chips issued in finite supply, then the total supply must be somewhere, thus we must divide the supply between individuals and the bank. When there are no loans and hence no default, all of the supply is in the hands of individuals. When there are one period loans we find that in this model we are required to have 50% of

the supply in the bank, or loan pool if the same price is to be maintained, but by varying the bank reserves different equilibria with different prices are feasible.

If we were to start the economy with a random distribution of wealth, the model suggested here would not be fully defined if the pool were too small as the initial borrowing might be larger than the pool which is not feasible. Even if the pool is large enough to absorb fluctuations we expect that even if the system converges, the level of the pool of reserves will have been permanently influenced by the activity during disequilibrium and this in turn influences price.

Once more we consider four sizes of games in order to obtain some inkling into the dynamics of the system. In the first game there are 100 traders, in the second 1,000, in the third there are 10,000 and in the fourth 100,000. As before income could be any real number, and now borrowing could be any real number up to \hat{p} , the expected price. We face a problem in displaying the wealth categories. This is handled by displaying bins for intervals $[-1, 0)$, $[0, 1)$, ... and showing the frequency of the occurrence of individuals in each category. Table 4 shows information on the loan model which is comparable with the data for the games without loans.

The following data are for the case of $\hat{p}_t = p_{t-1}$, borrowing, gamma = 0.25, initial price = 1.0, and initial wealth = 0.5.

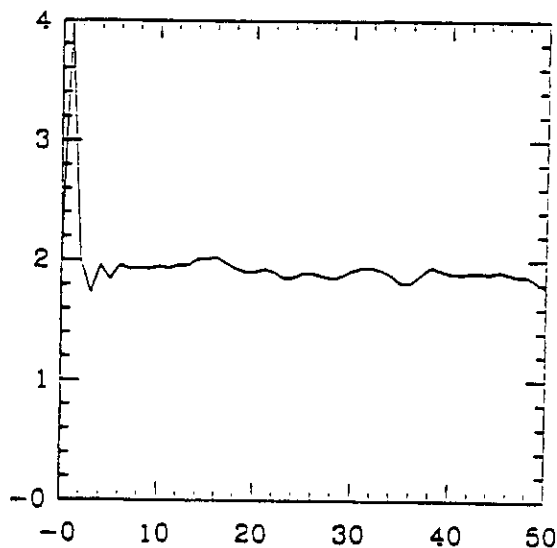
Table 4

Agents	Wealth Range						
	Price	[-1, 0)	[0, 1)	[1, 2)	[2, 3)	[3, 4)	[4, +)
100	0.9868 (0.0907)	50.02 (2.68)	17.64 (3.31)	21.56 (2.38)	7.16 (2.13)	2.42 (1.30)	1.20 (0.93)
1000	0.9781 (0.0347)	50.00 (0.69)	16.66 (0.81)	22.41 (1.15)	7.21 (0.76)	2.50 (0.54)	1.22 (0.34)
10000	0.9761 (0.0081)	49.97 (0.24)	16.64 (0.27)	22.32 (0.21)	7.34 (0.21)	2.52 (0.16)	1.20 (0.12)
100000	0.9788 (0.0026)	50.04 (0.09)	16.60 (0.11)	22.32 (0.08)	7.33 (0.08)	2.51 (0.05)	1.20 (0.04)
∞	1.0000	50.00	16.67	22.22	7.41	2.47	.82

We observe, as in Table 1 that the wealth distributions obtained are close to but somewhat lower than that predicted by the continuum game without loans and that the variances decrease in size as n increases. Furthermore we cannot conclude that price is converging to 1, but it is possible that it is converging to a different attractor somewhat below 1.

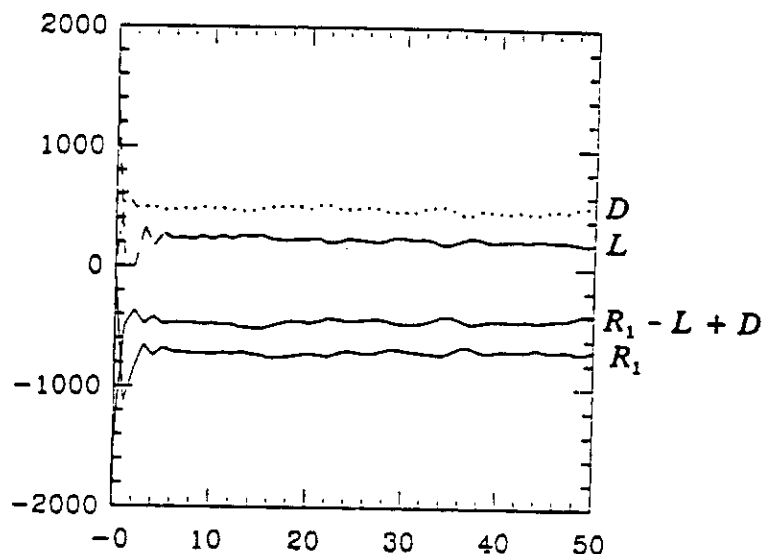
3.4. Transient Behavior

We now consider transient behavior with and without borrowing. Graphs 1 and 2 illustrate the long transients inherent in the banking case. The underlying parameters are set so that there is borrowing with 1000 agents, γ equals 0.25, initial price is set at 2.0, and all individuals have an initial wealth of 0.5. Graphs 1a and 1b illustrate the respective price and banking paths over the first 50 iterations. Graphs 2a and 2b show the similar information, but for 10000 iterations (using the identical random seed).



(a)

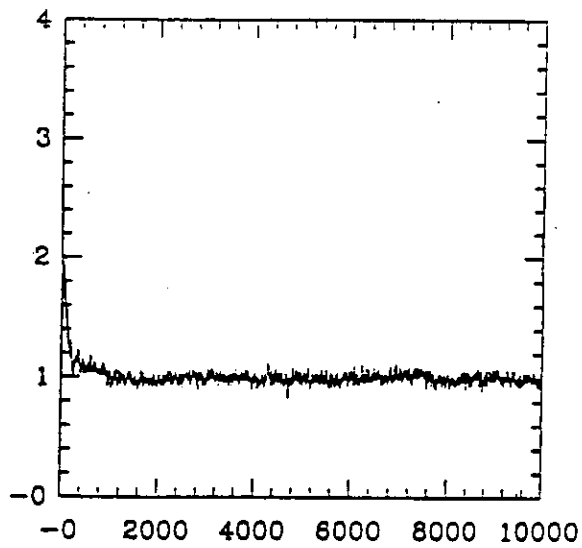
Price Path with Banking



(b)

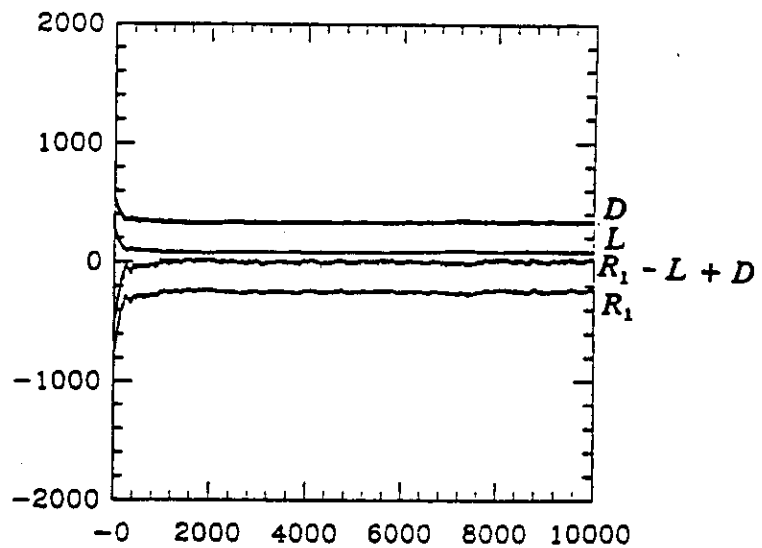
Bank Reserves (R_1), Deposits (D) and Loans (L)

Graph 1



(a)

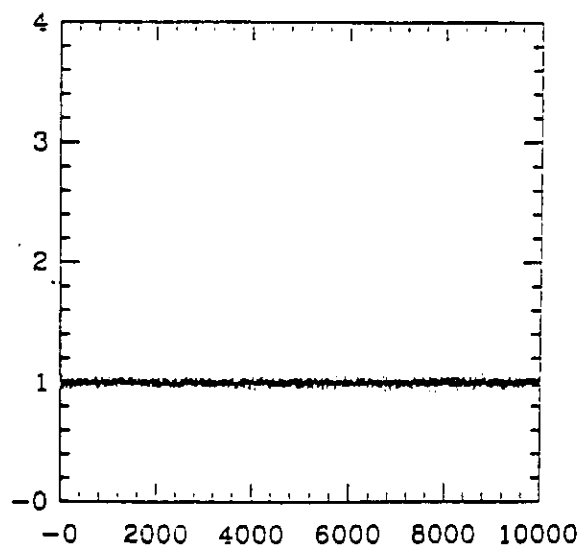
Price over 10,000 Periods
with Banking



(b)

Bank Reserves (R_1), Deposits (D)
and Loans (L) over 10,000 periods

Graph 2



Price over 10,000 Periods Without Banking

Graph 3

Graph 3 shows a similar run for a case without banking. Here, the parameters were identical to those above, except for initial wealth was set to 1.0. Note that relatively little long term transient behavior is apparent over the 10000 iterations in the model without banking, as compared to the model with banking.

We also ran simulations with initial wealth of .25 and an initial prediction of price of .5 and obtained similar results with prices essentially cut in half (see Table 5).

Table 5

Agents	Wealth Range						
	Price	[-.5, 0)	[0, .5)	[.5, 1)	[1, 1.5)	[1.5, 2)	[2, +)
1000	0.4904 0.0175	50.01 0.75	16.75 0.91	22.28 0.96	7.25 0.63	2.51 0.38	1.20 0.34

3.5. Crashes and Variations in Initial Wealth Distribution

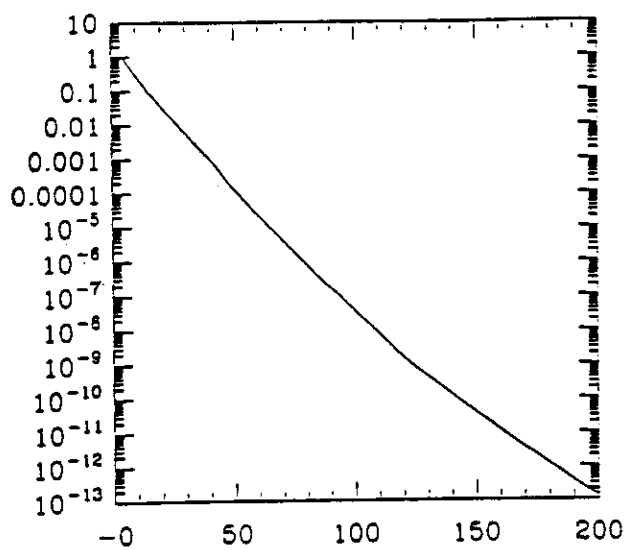
The sensitivity analysis of the dynamics to the selection of initial wealth is now considered and the economies with and without a loan market are contrasted.

The set of graphs below illustrate the potential for price crashes in the system with banking. We begin by considering an economy with banking and with 1000 agents, gamma equal 0.25, initial predicted price is 1.0, and the initial wealth distribution is set equal to the predicted stable state distribution for a continuum of traders.

Graph 4 is the LOG of price plotted over the first 200 iterations. The graph is typical of such crashes, with an initial rapid exponential decay of price, eventually slowing to another regime of exponential decay at a slower rate (apparently constant to at least 800 iterations). The wealth distribution during this crash is shown in Table 6 at intervals of 100 periods.

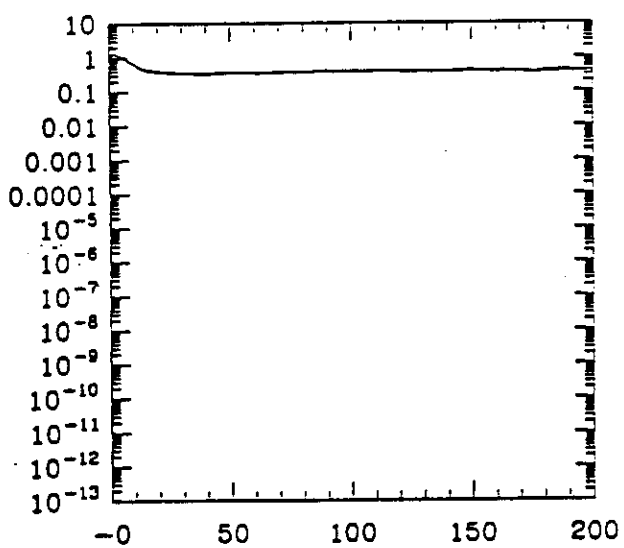
Table 6

Time	[-1, 0)	[0, 1)	[1, 2)	[2, 3)	[3, 4)	[4, +)
0	50.0	16.6	22.2	7.5	2.4	1.3
100	58.2	40.1	1.3	0.3	0.0	0.1
200	54.9	43.4	1.3	0.3	0.0	0.1
300	48.1	50.2	1.3	0.3	0.0	0.1

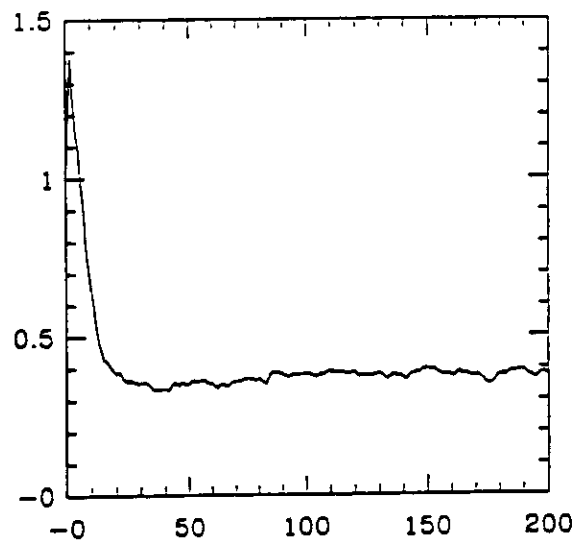


Price Decay in a Crash

Graph 4



(a)



(b)

Price Decay with Modified Initial Wealth Distribution

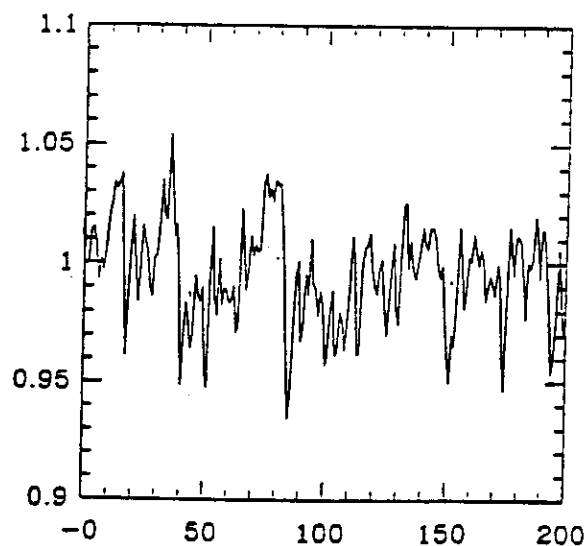
Graph 5

The next runs are identical to the above, except we modify the initial wealth distribution. Here, 10% of the individuals with -1 initial wealth are transferred to the initial wealth equal to 1 category (thus, maintaining a constant amount of wealth in the system). Graphs 5a, b (5a is on a log scale) show the price over the first 200 iterations. Note that in this case, the initial decline quickly moderates and levels off at around 0.38. The associated wealth distributions are given in Table 7.

Table 7

Time	[-1, 0)	[0, 1)	[1, 2)	[2, 3)	[3, 4)	[4, +)
0	40.0	16.6	32.2	7.5	2.4	1.3
100	49.6	43.8	6.1	0.5	0.0	0.0
200	50.2	42.2	7.4	0.1	0.1	0.0
300	49.7	43.5	6.4	0.4	0.0	0.0

Finally, we consider the system with no banking but otherwise with the same parameters as above and with the continuum stationary wealth distribution. Graph 6 shows the price over the 200 iterations. Note that the price remained very close to 1.00 throughout all of the first 200 iterations (with a sample standard deviation of 0.02). The wealth distribution also remained close to the stable state distribution throughout the entire run.



Graph 6

At first glance the crashing of the prices in the system with banking given initial conditions near the stable wealth distribution appears to be paradoxical, however on closer analysis it is easy to set up a situation in which initial price drops cut the income of the debtors in a manner which prevents them from getting out of debt as fast as they would otherwise, which in turn contributes to the price dropping further and trapping more new debtors. In more detail the explanation is as follows:

The observed crash is caused by the initial distribution being exactly on the integers (whereas, after many generations the wealth distribution has been spread across the reals -- thus, explaining why we can see the "identical" distribution and see a crash in one case and not another.

Consider the following exact wealth distribution:

-1	0	1
507	160	333

This distribution is very close to the stable state distribution with two modifications. The first is that we have aggregated all individuals with wealth above 1 into the one category (this eases exposition and does not alter the fundamental result). The second is that there are slightly more individuals in the -1 category than predicted (507 verses 500). This can easily happen due to stochastic influences.

Let the expected price be equal to 1.0. Given the above distribution, there will be $1.0 \times 493 = 493$ of spending, yielding a price of $493/500$ or .986. After spending, the distribution is given by:

-1	0	1
667	333	0

Now, we add $0.986 \times 2 = 1.972$ income to a quarter of the population. (Thus, of the 667 individuals with -1 , 168 of them will have a new income of 0.972 (1.972 minus the 1 they owe). This leaves the following distribution:

-1	0	0.972	1.972
499	250	168	83

Total spending in the next period will be equal to $0.986 \times 501 = 494$, yielding a price of 0.988 ($494/500$). After spending the distribution is:

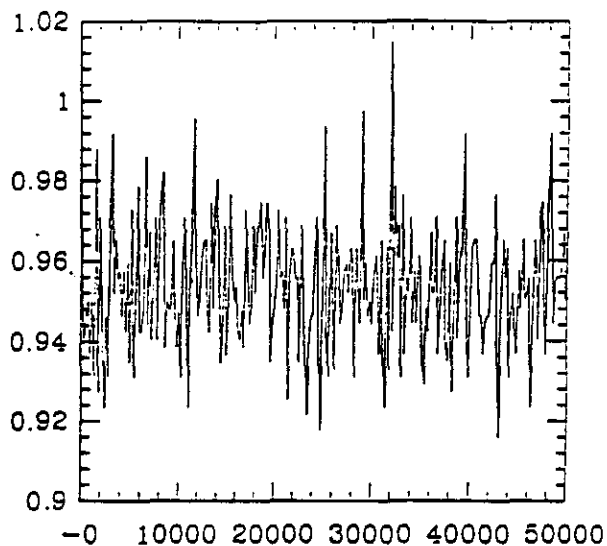
-1	-0.986	-0.014	0.986
499	250	168	83

It is at this point that the stage has been set for the crash. Notice that there are now 168 individuals who have a slight negative wealth. Also note that if the initial price drops below one, this will always happen, as the individuals in this category started with -1 , received $2 \times p$ income leaving them with $2p-1$, and then spent p , leaving $p-1$, which is of course negative since $p < 1$. (Given the model, the income received at the end of period t is based on the same price used to determine spending in period $t+1$.) The system now has 168 individuals in a slightly negative wealth category, which, implies that they can not participate in the market (since any level of negative wealth prevents borrowing). After adding income, 75% of the 917 individuals will remain in debt, and therefore only around 313 individuals will be spending. Thus, the price will experience a sharp decline. Once the price begins its decline, more individuals are trapped. The above sensitivity to price fluctuations will not happen if the distribution is spread out across the reals.

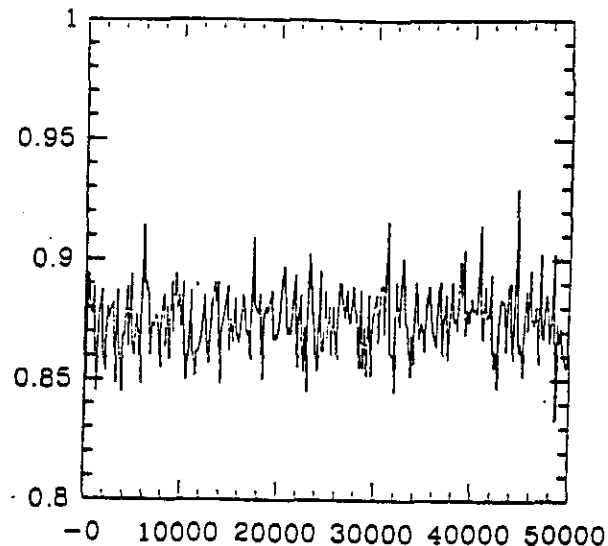
3.6. Preconceived Prediction

What happens when individuals stick with a preconceived nonadaptive estimate of price? For example, we might suppose that all wish to believe that price "should be" some given number? This appears to be a trivial example of a totally nonadaptive behavior but it will turn out that for checking certain equilibria, especially those with borrowing, this simple nonadaptive rule will have interesting properties. We also note that the proposition that because one's prediction has been disconfirmed some k times in a row previously does not constitute a proof that it will be disconfirmed again, but it is a reasonable axiom to accept that a long series of disconfirmation lowers the probability that it will occur and a Bayesian update may provide a better decision procedure than a religious belief that: this time we must win the lottery. This point deals with one of the basic questions concerning subjective probability and is not discussed further here.

In this section we select two fixed predictions and examine them without and with a loan market. When there is no loan market there is only one attractor in the system. We observe that for individuals with the same preconceived predictions in a mass market without loans their predictions are always disconfirmed unless by chance they have predicted precisely the right price. Thus we might argue that this nonlearning behavior is unacceptable and unrealistic. The situation is different when we consider the economy with borrowing. We encounter self-fulfilling prophecy. Borrowing is highly dependent on the prediction and this in turn influences bank reserves which influences the final price equilibrium. As there is a continuum of equilibria the different predictions become close to self-fulfilling.



Graph 7



Graph 8

Figures 7 and 8 show the effect of total lack of adaptation when there is a loan market. We consider 1,000 agents with $\gamma = .25$, initial wealth of 1 each and expectations respectively set at .95 and .85 regardless of what happens. We obtain wealth distributions close to the predicted (they are not shown here) and prices varying around .956 and .875 over 50,000 runs. In equilibrium for the continuum the self-predicting prophecy for the price level is precise, but the influence of the path to equilibrium on the distribution of resources during disequilibrium depends on the magnitude of price.

4. Concluding Remarks

4.1. Game Theory, Behavioral Economics and Simulation

In any attempt to deal with infinite horizon games in a formal rigorous manner it is clear that the formal game theoretic definition of the strategy set permits enormous complexity in the selection of strategies. In attempting to consider dynamics it appears to be a reasonable research strategy to confine the selection of strategies to a limited set. An interpretation of the updating rule used here is that we are restricting the price forecasting so radically that it implicitly covers the essential aspects of competition with short term forecasting. Essentially,

virtually any behavioral rule can be interpreted as a strategy in the formal game theoretic analysis. One may wish to accept or reject its use based on its optimizing properties in the context of having selected a solution concept.

Once one contemplates the complexity of multiperiod interaction the apparently clear distinction between behavioral and game theoretic solutions blurs.

One simple utility function used as the basis for our simulations does not constitute a general proof, but given both the conceptual difficulties in even defining acceptable solution conventions and in obtaining analytic results we suggest that general insights can be obtained and new open questions can be generated by the type of exploration illustrated here. But even more generally a special example may illustrate otherwise overlooked phenomenon. Specifically, here, this is so for the role of bank reserves and the possibility for a range of self fulfilling expectations.

It is desirable to be able to extend our investigations to less restrictive models and the structure of the optimal policy for piecewise linear utility function is known but for more general shapes considerably more computation is required.

4.2. Comments on the Program and Learning Extensions

A natural extension of the approach adopted here is to replace the combination of the simple price prediction with dynamic programming by some form of adaptive learning process. There are two basic reasons for doing this. We can use our simple example where we have analytical results to check the effectiveness of the learning program against a known analysis. If this test is met then we can attempt to use the program to provide "solutions" to games with arbitrarily selected utility functions and then attempt to verify if the solutions obtained are consistent with the analytical approach (Holland and Miller, 1991).

4.3. The Comparative Stability of Credit Systems

It would be dangerous to claim too much from the analysis of a simple special example with a simple behavioral rule. Yet there are several specific lessons to be learned. There is a distinct increase in instability in the system introduced by the credit mechanism that appears

to be caused primarily by the distinction between a unique price equilibrium level and a continuum of prices.

It is clear that an individual with no current liquidity, benefits from the presence of a loan market as can be seen when β is small. But the introduction of a loan market changes the wealth distribution, here by an artifact of the special example, by a left shift of the Markov chain as shown in Figure 5, but more generally by a change in distribution and range. In this change the necessity for bank reserves emerges in order to account for the location of the fixed supply of money, but as soon as bank reserves are larger than zero a continuum of equilibria in the price system appear. This in turn shows that so called "rational expectations" are not well defined in the sense that many expectations can be self fulfilling in the same system.

Another feature that emerges both from the simulations and the analysis is that if the disequilibrium initial conditions are radically far from an equilibrium the resultant shocks to the system may cause arbitrarily long periods of instability and this is clearly a function of the nature of the banking system---such as can one lend more than capital, more than capital plus deposits and what are the rules of default and garnishing. All of these phenomena are intrinsically associated with dynamics and do not appear in the static analysis.

Our last comment concerns the size of the bank reserves and their relationship to the Fort Knox paradox. In equilibrium with a continuum of traders there is no apparent need for bank reserves. In disequilibrium with a finite number of traders the size of reserves and the rules of lending become critical in determining the bank's ability to extend credit. Just because some of the reserves may apparently never be loaned does not mean that they are not used. They are in principle an inventory whose presence lowers the probability of a stockout.

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