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Abstract

Gold and tobacco have both been used as commodity money. One difference between the two is that gold yields utility, on account of its beauty, without diminishing its quantity. Tobacco yields utility when it is consumed. If this were the only difference, which would be the better money?

1 Introduction

Paper money is a relatively recent innovation. For much of history, commodities have played the role of money, serving both as medium of exchange and store of value. The characteristics that make a commodity suitable to use as money were discussed as far back as Aristotle [1], who emphasized portability and intrinsic value. Since that time many distinguished authors have also written on this subject ([2], [4], [5], [6], [7], [8], [9], [10], [11]), naming other virtues, such as divisibility and durability. In this paper we bring a new virtue to light, which seems curiously to have gone unnoticed.

Our point can be best put across by comparing gold and tobacco, both of which have had reputable runs as money. (See, in particular, the historical account in [5].) One difference between the two is that gold yields utility, on account of its beauty, without diminishing its quantity. Tobacco yields utility when it is consumed. If this were the only difference, which would be the better money?

Imagine a two-period economy with no uncertainty and with a single durable commodity. The agents agree on this commodity as money; all trades take place between this money and each of the other perishable commodities. Suppose equilibrium is achieved under conditions of perfect foresight and perfect competition. Suppose further that in the first period each agent spends less than his endowment of money, in fact inventorying a positive amount into the second period, where once again he spends only a part of it. Thus the constraint that purchases be financed out of money on hand is not binding on anyone: there is perfect liquidity. Is the resulting allocation necessarily efficient, or does the efficiency of equilibrium depend on any of the properties of the commodity money?

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We show that if money is “gold-like,” i.e., it yields a stream of utility (say as jewelry) without getting dissipated, then equilibrium is inefficient. In particular, gold itself is an inefficient money. This is so even if its utility services are not diminished when used as money, e.g., even if gold coins are just as beautiful to look at per ounce as gold necklaces. To the extent that its use as money also reduces its services, the inefficiency of gold is still worse. On the other hand, if the money is “tobacco-like,” and yields utility only at the point of consumption (when it disappears — into smoke!), then equilibrium under the above conditions is efficient.

The question of the relative efficiency of gold and tobacco money cannot be posed in the standard general equilibrium model because there is no money explicitly in that model. Expenditures and savings are recorded in imaginary units of account. (Implicitly the model allows for fiat money with unlimited credit.) If the one-period general equilibrium model is modified so that payments must be made in a specific commodity, then efficiency is achieved once it is sufficiently abundant and valuable throughout the economy. (See [3].) But in a multiperiod setting, something more is needed.

The role of money as a medium of exchange is perhaps less important than its role as a store of value. To capture this dual role of money we must turn to a multiperiod setting. Then it becomes necessary to distinguish gold-like money from tobacco-like money. We begin by postulating that one commodity is the sole medium of exchange (meaning any purchase is via the commodity money), and that the same money is the only durable good (store of value).

Our first theorem is that under these conditions, gold is (generically) an inefficient money. The reason is that it is desired on two counts, to save and to enjoy. Typically those who most desire to save do not also get the most aesthetic enjoyment out of looking at gold. But, in the absence of a rental market, they cannot decouple their savings of gold from their consumption of gold.

The lesson of this paper is not that the best commodity money is the one that yields the least utility. On the contrary, as long as it is divisible, and as long as its role as money does not reduce its use value, the more valuable it is the better, since that ensures that agents will tend not to spend all their money and hence the liquidity constraints will not be binding.¹ Durable commodities are usually a mix of gold-like and tobacco-like. They have both some consumption value (i.e., utility yielded when they are destroyed), and some service value (i.e., utility yielded without the commodities’ destruction). The lesson we draw is that, other things being equal, the higher the ratio of consumption value to service value, the more suitable is the durable commodity to be money.

Our second theorem shows that tobacco-like money *is* efficient, provided that it is sufficiently abundant and valuable throughout the economy. This is so even if all transactions and savings are via the one money. Fixing any economy, then adding more and more tobacco to the endowment of every agent at time 1, eventually leads

¹A caveat: if the value of money relative to other commodities becomes too high, then the divisibility assumption becomes implausible. The amount of money needed for small purchases will become so infinitesimal as to be impossible to use in practice — hence the use of copper, rather than gold, coins for small transactions.

to economies which have only efficient equilibria. With enough tobacco at time 1, every agent will want to defer some tobacco consumption until later (because of the diminishing marginal utility of tobacco), and this savings of tobacco will be sufficiently valuable to carry out all desired transactions in the future (since the marginal utility of tobacco is bounded from below). The initial distribution of resources among the perishable commodities does not affect the conclusion.

Section 2 of the paper introduces the formal model, with gold-like or tobacco-like money, and a cash-in-advance constraint. In Section 3, we prove Theorem 1 and in Section 4 we prove Theorem 2.

In Section 5 we show that adding more gold-like monies, such as silver and other precious metals, does not change the inefficiency result of Theorem 1. On the other hand, even in the presence of gold-like durables, if there is sufficient tobacco-like money, then efficiency of equilibrium is guaranteed. (But this will typically require that there be much more tobacco than gold.) Thus the allocation of art or gold coins among private homes, under the stylized conditions of this paper, is efficient in equilibrium, even when owners regard the art or coins partly as an investment, if and only if there is a separate tobacco-like commodity money in sufficient supply.

In Section 5 we also show that our results have nothing to do with the cash-in-advance constraint. The generic inefficiency of gold-like money is due solely to its role as a store of value, and the absence of credit, i.e. the requirement that all exchange be current value for current value.

2 The Formal Model

Let $H = \{1, \dots, h\}$ be the set of agents; $L = \{1, \dots, \ell + 1\}$ the set of commodities; and $T = \{1, 2\}$ the two² time periods, where $h \geq 2$ and $\ell \geq 1$. All commodities are perishable, except $\ell + 1$, which is perfectly durable and is the stipulated medium of exchange. We call it “money.”

Let us refer to money as “tobacco-like” or “gold-like,” depending on its characteristic.

Accordingly, a typical consumption plan is denoted $(x, \tilde{x}) \in \mathbb{R}_+^L \times \mathbb{R}_+^L$ where x, \tilde{x} is the consumption in period 1, 2 respectively; and

$$x_{\ell+1} = \begin{cases} \text{money consumed in period 1, if money is tobacco-like} \\ \text{money “consumed” and inventoried from period 1 to 2 if money is gold-like} \end{cases}.$$

The initial endowment of agent is $(e^\alpha, \tilde{e}^\alpha)$ in $\mathbb{R}_+^L \times \mathbb{R}_+^L$ where we assume, for all $\alpha \in H$:

- (i) $e_j^\alpha > 0$ for $1 \leq j \leq \ell + 1$
- (ii) $\tilde{e}_j^\alpha > 0$ for $1 \leq j \leq \ell$.

²This is for ease of notation. Our results easily extend to any finite number of periods.

Let \square be a hypercube in $\mathbb{R}_+^L \times \mathbb{R}_+^L$ with side at least $M > \max\{\max_{1 \leq j \leq \ell} \{\sum_{\alpha \in H} e_j^\alpha, \sum_{\alpha \in H} \tilde{e}_j^\alpha\}, \sum_{\alpha \in H} (e_{\ell+1}^\alpha + \tilde{e}_{\ell+1}^\alpha)\}$, i.e.,

$$\square = \{(x, \tilde{x}) \in \mathbb{R}_+^L \times \mathbb{R}_+^L : x_j \leq M, \tilde{x}_j \leq M \text{ for } j \in L\}.$$

Then, for any feasible trade, the consumption of each agent will lie in \square .

Let $\underline{u}^\alpha : \mathbb{R}_+^L \times \mathbb{R}_+^L \rightarrow \mathbb{R}$ give the utility of consumption of agent $\alpha \in H$. We assume each \underline{u}^α is smooth, strictly concave and strictly monotonic. It will be necessary for us to linearly perturb these utilities. To this end let U^α be an open set in $\mathbb{R}_+^L \times \mathbb{R}_+^L$. Any vector $(v, \tilde{v}) \in U^\alpha$ perturbs \underline{u}^α to $u^\alpha : \square \rightarrow \mathbb{R}$, where $u^\alpha(y, \tilde{y}) = \underline{u}^\alpha(y, \tilde{y}) + (v, \tilde{v}) \cdot (y, \tilde{y})$. We naturally assume that $0 \in U^\alpha$ and that U^α is small enough to ensure that the utility remains monotonic after the perturbation. Let $U = U^1 \times \dots \times U^h$. We shall keep endowments fixed and view U as the space of economies.

We shall also assume that for any $u^\alpha \in U^\alpha$, agent α will do better not to trade than to accept zero consumption in any good, i.e., introducing the symbol

$$\tau = \begin{cases} 0, & \text{if money is tobacco-like} \\ 1, & \text{if money is gold-like} \end{cases}$$

we assume (for all $u^\alpha \in U^\alpha$ and $\alpha \in H$): if (y, \tilde{y}) is not strictly positive in every coordinate, then

$$(iii) \quad u^\alpha(y, \tilde{y}) < \max_{0 \leq \gamma \leq 1} u^\alpha(e_1^\alpha, \dots, e_\ell^\alpha, \gamma e_{\ell+1}^\alpha, \tilde{e}_1^\alpha, \dots, \tilde{e}_\ell^\alpha, \tilde{e}_{\ell+1}^\alpha + (1 - \gamma)e_{\ell+1}^\alpha + \gamma \tau e_{\ell+1}^\alpha)$$

The assumption can be dropped, but it makes for a cleaner analysis by keeping equilibrium consumptions in the interior, and enabling us to take derivatives.

Let $p = (p_1, \dots, p_\ell)$ and $\tilde{p} = (\tilde{p}_1, \dots, \tilde{p}_\ell)$ be the prices of commodities 1, ..., ℓ , in periods 1 and 2 (denoted in terms of money). We always take $p_{\ell+1} = \tilde{p}_{\ell+1} = 1$. We now describe the set of consumption bundles $(x^\alpha, \tilde{x}^\alpha) \in \mathbb{R}_+^L \times \mathbb{R}_+^L$ that are affordable by agent α at the given prices (p, \tilde{p}) . Let $c_+ \equiv \max\{0, c\}$ for any real number c . For each of the following constraints (k), denote by $\Delta(k)$ the difference between the RHS and the LSH. Then the budget set $B^\alpha(p, \tilde{p})$ consists of all $(x^\alpha, \tilde{x}^\alpha) \in \mathbb{R}_+^L \times \mathbb{R}_+^L$ which satisfy:

$$\sum_{j=1}^{\ell} p_j (x_j^\alpha - e_j^\alpha)_+ \leq e_{\ell+1}^\alpha \quad (I)$$

$$0 \leq x_{\ell+1}^\alpha \leq \Delta(I) + \sum_{j=1}^{\ell} p_j (e_j^\alpha - x_j^\alpha)_+ \quad (II)$$

$$\sum_{j=1}^{\ell} \tilde{p}_j (\tilde{x}_j^\alpha - \tilde{e}_j^\alpha)_+ \leq \tilde{e}_{\ell+1}^\alpha + \Delta(II) + \tau x_{\ell+1}^\alpha \quad (III)$$

$$\tilde{x}_{\ell+1}^\alpha = \Delta(III) + \sum_{j=1}^{\ell} \tilde{p}_j (\tilde{e}_j^\alpha - \tilde{x}_j^\alpha)_+ \quad (IV)$$

Define the space of allocations

$$\mathcal{A} = \left\{ (x^\alpha, \tilde{x}^\alpha)_{\alpha \in H} \in (\mathbb{R}_+^L \times \mathbb{R}_+^L) : \sum_{\alpha \in H} x_j^\alpha = \sum_{\alpha \in H} e_j^\alpha \text{ and } \sum_{\alpha \in H} \tilde{x}_j^\alpha = \sum_{\alpha \in H} \tilde{e}_j^\alpha \text{ for } 1 \leq j \leq \ell; \right. \\ \left. \sum_{\alpha \in H} x_{\ell+1}^\alpha \leq \sum_{\alpha \in H} e_{\ell+1}^\alpha \text{ and } \sum_{\alpha \in H} \tilde{x}_{\ell+1}^\alpha = \sum_{\alpha \in H} \tilde{e}_{\ell+1}^\alpha + \sum_{\alpha \in H} e_{\ell+1}^\alpha - (1 - \tau) \sum_{\alpha \in H} x_{\ell+1}^\alpha \right\}.$$

An equilibrium of $u \in U$ is a triple $(p, \tilde{p}, (x^\alpha, \tilde{x}^\alpha)_{\alpha \in H})$ such that:

$$(x^\alpha, \tilde{x}^\alpha) \text{ maximizes } u^\alpha \text{ on } B^\alpha(p, \tilde{p}) \text{ for } \alpha \in H,$$

and

$$(x^\alpha, \tilde{x}^\alpha)_{\alpha \in H} \in \mathcal{A}.$$

Recall that an allocation $(x^\alpha, \tilde{x}^\alpha)_{\alpha \in H}$ is *efficient* if there does not exist any $(y^\alpha, \tilde{y}^\alpha)_{\alpha \in H} \in \mathcal{A}$ such that

$$u^\alpha(x^\alpha, \tilde{x}^\alpha) \leq u^\alpha(y^\alpha, \tilde{y}^\alpha)$$

for all α , with strict inequality for at least one α .

We say that an equilibrium is *inefficient* if the allocation yielded by it is *not* efficient.

Finally, we say that a property holds for “generic $u \in U$,” if there is an open set $U^* \subset U$, whose complement has zero Lebesgue measure in U , such that the property holds for all $u \in U^*$.

3 Gold

Theorem 1 *Let money be gold-like. Then, for generic $u \in U$, every equilibrium of u is inefficient.*

3.1 Example

Before giving the proof, we illustrate the theorem with an example. Let there be two agents α and β , with identical utilities for food (x_1, \tilde{x}_1) , and gold services (x_2, \tilde{x}_2) , given by $u(x_1, x_2, \tilde{x}_1, \tilde{x}_2) = \log(x_1 x_2 \tilde{x}_1 \tilde{x}_2)$. Let the endowment of agent α of food and gold, in periods 1 and 2, be $(2, 4, 6, 0)$ and that of agent β be $(6, 4, 2, 0)$. Evidently there is a unique Walrasian (competitive) equilibrium with consumption $(4, 4, 4, 4)$ for each agent (recall that gold is durable). This allocation is unachievable as a monetary equilibrium, for α would need to give up money to get more x_1 .

We can compute a monetary equilibrium by solving the following 10 equations in the 10 unknowns $((p_1, \tilde{p}_1), (x_1^\alpha, x_2^\alpha, \tilde{x}_1^\alpha, \tilde{x}_2^\alpha), (x_1^\beta, x_2^\beta, \tilde{x}_1^\beta, \tilde{x}_2^\beta))$:

$$\frac{1}{x_1^\alpha} / p_1 = \frac{1}{x_2^\alpha} + \frac{1}{\tilde{x}_2^\alpha} \tag{1}$$

$$\frac{1}{\tilde{x}_1^\alpha}/\tilde{p}_1 = \frac{1}{\tilde{x}_2^\alpha} \quad (2)$$

$$p_1(x_1^\alpha - e_1^\alpha) + (x_2^\alpha - e_2^\alpha) = 0 \quad (3)$$

$$\tilde{p}_1(\tilde{x}_1^\alpha - \tilde{e}_1^\alpha) + (\tilde{x}_2^\alpha - \tilde{e}_2^\alpha) = x_2^\alpha \quad (4)$$

$$\frac{1}{x_1^\beta}/p_1 = \frac{1}{x_2^\beta} + \frac{1}{\tilde{x}_2^\beta} \quad (5)$$

$$\frac{1}{\tilde{x}_1^\beta}/\tilde{p}_1 = \frac{1}{\tilde{x}_2^\beta} \quad (6)$$

$$p_1(x_1^\beta - e_1^\beta) + (x_2^\beta - e_2^\beta) = 0 \quad (7)$$

$$\tilde{p}_1(\tilde{x}_1^\beta - \tilde{e}_1^\beta) + (\tilde{x}_2^\beta - \tilde{e}_2^\beta) = x_2^\beta \quad (8)$$

$$x_1^\alpha + x_1^\beta = e_1^\alpha + e_1^\beta \quad (9)$$

$$\tilde{x}_1^\alpha + \tilde{x}_1^\beta = \tilde{e}_1^\alpha + \tilde{e}_1^\beta. \quad (10)$$

Equations (1) and (5) say that any agent who consumes food in period 1 should not prefer to sell (a little of) it and buy gold, and then enjoy the pleasure of admiring the gold in periods 1 and 2; or vice versa. Equations (2) and (6) say that an agent who consumes food in period 2 should not prefer to sell (a little of) it and buy gold to admire in period 2; or vice versa. Equations (3) and (7) say that households trade value for value in period 1, and equations (4) and (8) say that households trade value for value in period 2, after inventorying their gold. Equations (9) and (10) say that the market for food clears in both periods.

There are two budget constraints for each agent ((3) and (4) for α , (7) and (8) for β). On account of these budget constraints, the gold market clears automatically in each period once the food markets clear.

Any solution $(p, \tilde{p}, (x^\alpha, \tilde{x}^\alpha), (x^\beta, \tilde{x}^\beta))$ to these equations must be an equilibrium, provided the variables are all strictly positive.

Solving the equations, we get

$$\begin{aligned} & ((p_1, \tilde{p}_1), ((x_1^\alpha, x_2^\alpha), (\tilde{x}_1^\alpha, \tilde{x}_2^\alpha)), (x_1^\beta, x_2^\beta), (\tilde{x}_1^\beta, \tilde{x}_2^\beta))) \\ & = ((.484, .999), ((3.83, 3.12), (4.56, 4.56)), ((4.17, 4.88), (3.44, 3.44))). \end{aligned}$$

The allocation is inefficient, because agent β is holding more gold than food in period 1, while α is doing the reverse. The only reason β holds so much gold (which at the margin gives him very little pleasure) is because he wants to save, and holding gold is the only way to do it. The agents could both be made better off if β were permitted to rent some of his gold to α for the first period, in exchange for food, and the absence of this rental market is another way of explaining why the monetary equilibrium is inefficient. A rental is not equivalent to a sale in period 1 and a repurchase at the beginning of period 2, because by that time the original food has spoiled, and the repurchase must be with period 2 food. In the rental, period 2 food is not involved. Note finally that neither agent is liquidity constrained, and both save, i.e. both inventory positive amounts of money into period 2. ■

Theorem 1 generalizes this example. With three or more commodities the analysis becomes more difficult because positive consumption of all commodities does not imply the interiority of choices. An agent might sell all the gold on hand to buy food, but still end up with a positive amount of gold after trade, on account of the sale of his labor. Furthermore, the theorem holds only generically; if in the example the initial endowments corresponded to the unique Walrasian equilibrium, there would be no trade and no inefficiency.

3.2 The Proof

To prove Theorem 1 we first establish

Lemma 1 *Let money be gold-like. And let $(p, \tilde{p}, (x^\alpha, \tilde{x}^\alpha)_{\alpha \in H})$ be an equilibrium of the economy $\{u^\alpha\}_{\alpha \in H} \in U$. Put*

$$\begin{aligned}\nabla_j^\alpha &\equiv \frac{\partial u^\alpha}{\partial x_j}(x^\alpha, \tilde{x}^\alpha) \\ \tilde{\nabla}_j^\alpha &\equiv \frac{\partial u^\alpha}{\partial \tilde{x}_j}(x^\alpha, \tilde{x}^\alpha)\end{aligned}$$

for $j \in L$. Then the equilibrium is efficient only if

$$\begin{aligned}\text{(iv)} \quad \frac{\nabla_j^\alpha}{p_j} &= \nabla_{\ell+1}^\alpha + \tilde{\nabla}_{\ell+1}^\alpha \\ \text{(v)} \quad \frac{\tilde{\nabla}_j^\alpha}{\tilde{p}_j} &= \tilde{\nabla}_{\ell+1}^\alpha\end{aligned}$$

for $\alpha \in H$ and $1 \leq j \leq \ell$.

Proof of Lemma 1 By condition (iii) on u^α , e^α , \tilde{e}^α we must have $(x^\alpha, \tilde{x}^\alpha) \gg 0$, otherwise α would do better by not trading and inventorying all his gold from period 1 to 2. Thus we can take derivatives of agents' utilities. Note that conditions (iv) and (v) hold by virtue of being in equilibrium (efficient or not) when every agent spends less than the money on hand in both periods, i.e., when the equilibrium is not liquidity-constrained. When liquidity constraints are binding, (iv) and (v) may not hold in equilibrium, but the lemma says they must if equilibrium is efficient.

Case I Suppose $\tilde{x}_j^\alpha - \tilde{e}_j^\alpha < 0$, i.e., α is a net seller of j . Then since $\tilde{x}_j^\alpha > 0$, α must be selling strictly less than his endowment (or else selling all of \tilde{e}_j^α and buying back $\tilde{x}_j^\alpha > 0$). If α sells ε/p_j more of j (or buys ε/\tilde{p}_j less), his increase in utility is

$$\left(\tilde{\nabla}_{\ell+1}^\alpha - \frac{\tilde{\nabla}_j^\alpha}{\tilde{p}_j} \right) \varepsilon;$$

if he sells ε/p_j less of j , this is

$$\left(\frac{\tilde{V}_j^\alpha}{\tilde{p}_j} - \tilde{V}_{\ell+1}^\alpha \right) \varepsilon.$$

Both terms must be nonpositive at an equilibrium, so (v) holds by virtue of being in equilibrium.

Case II Next suppose $\tilde{x}_j^\alpha - \tilde{e}_j^\alpha > 0$, i.e., α is a net buyer of j . If budget constraint (III) is not binding on α , i.e., if $\sum_{i=1}^\ell \tilde{p}_i [\tilde{x}_i^\alpha - \tilde{e}_i^\alpha]_+ < M^\alpha$ (where M^α is the money on hand for α , prior to trade in period 2), then it is clear that $\tilde{V}_j^\alpha / \tilde{p}_j - \tilde{V}_{\ell+1}^\alpha = 0$. (For if the term is positive (negative) α should spend a little more (less) on commodity j , improving his utility.) If $\sum_{i=1}^\ell \tilde{p}_i [\tilde{x}_i^\alpha - \tilde{e}_i^\alpha]_+ = M^\alpha$, then $\tilde{V}_j^\alpha / \tilde{p}_j \geq \tilde{V}_{\ell+1}^\alpha$ or else α would spend less on j . Suppose $>$ holds. Then consider β who is a net seller of j . (Clearly such a β exists since the market for j clears in equilibrium.) We must have $\tilde{V}_j^\beta / \tilde{p}_j = \tilde{V}_{\ell+1}^\beta$ (as shown in Case I). Let β give ε/\tilde{p}_j units of j to α and take ε' units of gold from α . For $\varepsilon' > \varepsilon$ and sufficiently close to ε , and ε' and ε small, both α and β improve, a contradiction. We conclude again that (v) holds, this time by virtue of being an efficient equilibrium.

Case III Finally, suppose $\tilde{x}_j^\alpha - \tilde{e}_j^\alpha = 0$. Then $\tilde{V}_j^\alpha / \tilde{p}_j \geq \tilde{V}_{\ell+1}^\alpha$, as just argued (since $e_j^\alpha > 0$ and α can sell j). Suppose

$$\frac{\tilde{V}_j^\alpha}{\tilde{p}_j} > \tilde{V}_{\ell+1}^\alpha.$$

We claim that there exists a β with

$$\frac{\tilde{V}_j^\beta}{\tilde{p}_j} = \tilde{V}_{\ell+1}^\beta. \quad (11)$$

If there is a net trade in j at the equilibrium then this is obvious, for (11) will hold for any β who is a net seller of j (by Case I). Suppose there is no net trade in j . If there is a net trade in some other commodity $k \neq j$, let β be a net buyer of k , and then $\tilde{V}_k^\beta / \tilde{p}_k = \tilde{V}_{\ell+1}^\beta$, as shown in the preceding Case II. But then $\tilde{V}_j^\beta / \tilde{p}_j \leq \tilde{V}_k^\beta / \tilde{p}_k = \tilde{V}_{\ell+1}^\beta$, otherwise if $>$ holds instead of \leq , β would do better to spend ε less on k and ε more on j . At the same time β could certainly sell j , and the fact that he does not do so implies $\tilde{V}_j^\beta / \tilde{p}_j \geq \tilde{V}_{\ell+1}^\beta$, again proving (11). This leaves the possibility that there is no net trade in any commodity $1 \leq k \leq \ell$ in period 2. Then agent α is consuming his initial endowment of commodities 1 till ℓ in period 2. Since $x_{\ell+1}^\alpha > 0$, he is also inventorying gold into period 2. Hence $\tilde{V}_j^\alpha / \tilde{p}_j \leq \tilde{V}_{\ell+1}^\alpha$, otherwise α could buy j with his gold and improve his utility. But this contradicts our assumption that $\tilde{V}_j^\alpha / \tilde{p}_j > \tilde{V}_{\ell+1}^\alpha$. We conclude that if $\tilde{V}_j^\alpha / \tilde{p}_j > \tilde{V}_{\ell+1}^\alpha$, then (11) holds.

Now let β give ε/p_j units of j to α and take $\varepsilon' > \varepsilon$ units of gold from α . For ε' sufficiently close to ε (and ε' small), both α and β improve, a contradiction. So (v) holds in Case III as well.

To prove (iv), repeat the above argument, noting that the marginal utility of a unit of gold in period 1 is $\nabla_{\ell+1}^\alpha + \tilde{\nabla}_{\ell+1}^\alpha$, since gold yields utility $\nabla_{\ell+1}^\alpha$ (as services) in period 1, and can be inventoried into period 2 where it yields $\tilde{\nabla}_{\ell+1}^\alpha$. ■

Proof of Theorem 1 As is well-known, if the equilibrium is efficient, then a necessary and sufficient condition is that the gradient vectors

$$(\nabla^\alpha, \tilde{\nabla}^\alpha) = (\nabla_1^\alpha, \dots, \nabla_{\ell+1}^\alpha, \tilde{\nabla}_1^\alpha, \dots, \tilde{\nabla}_{\ell+1}^\alpha)$$

for $\alpha \in H$, must all point in the same direction. In the light of Lemma 1, we see that this will be so only if

$$(vi) \quad \frac{\nabla_{\ell+1}^\alpha}{\tilde{\nabla}_{\ell+1}^\alpha} \text{ is invariant of } \alpha \in H.$$

Note that, since $(x^\alpha, \tilde{x}^\alpha) \in B^\alpha(p, \tilde{p})$, the choice of $x_1^\alpha, \dots, x_\ell^\alpha$ and $\tilde{x}_1^\alpha, \dots, \tilde{x}_\ell^\alpha$ automatically determines $x_{\ell+1}^\alpha$ and $\tilde{x}_{\ell+1}^\alpha$ via the equations:

$$\begin{aligned} x_{\ell+1}^\alpha &= e_{\ell+1}^\alpha - \sum_{j=1}^{\ell} p_j (x_j^\alpha - e_j^\alpha) \\ \tilde{x}_{\ell+1}^\alpha &= x_{\ell+1}^\alpha + \tilde{e}_{\ell+1}^\alpha - \sum_{j=1}^{\ell} \tilde{p}_j (\tilde{x}_j^\alpha - \tilde{e}_j^\alpha). \end{aligned}$$

Also, once $(x^\alpha, \tilde{x}^\alpha)$ is chosen for agents $1, \dots, h-1$, then (x^h, \tilde{x}^h) is also automatically determined at equilibrium by the requirement that $(x^\alpha, \tilde{x}^\alpha)_{\alpha \in H} \in \mathcal{A}$.

Thus the number of independent variables among $p, \tilde{p}, (x^\alpha, \tilde{x}^\alpha)_{\alpha \in H}$ to solve for equilibrium is 2ℓ (for prices) + $(h-1)2\ell$ (for allocations) = $2h\ell$.

But as we have just seen (recall (iv), (v), (vi)) efficiency implies that the following $2h+1$ equations must be satisfied (where we have fixed two distinct agents $\bar{\alpha}$ and $\bar{\beta}$ arbitrarily):

$$\left. \begin{aligned} \frac{\nabla_j^\alpha}{p_j} - (\nabla_{\ell+1}^\alpha + \tilde{\nabla}_{\ell+1}^\alpha) &= 0 \\ \frac{\tilde{\nabla}_j^\alpha}{\tilde{p}_j} - \tilde{\nabla}_{\ell+1}^\alpha &= 0 \end{aligned} \right\} \text{ for } \alpha \in H, 1 \leq j \leq \ell$$

$$\frac{\nabla_{\ell+1}^{\bar{\alpha}}}{\tilde{\nabla}_{\ell+1}^{\bar{\alpha}}} - \frac{\nabla_{\ell+1}^{\bar{\beta}}}{\tilde{\nabla}_{\ell+1}^{\bar{\beta}}} = 0.$$

Theorem 1 will follow, by the transversal density and openness theorems, if we can show that by linear perturbations of the u^α it is possible to arbitrarily vary these equations one at a time, leaving the others unaffected. For the first 2ℓ equations, note we can vary ∇_j^α (or $\tilde{\nabla}_j^\alpha$) by adding to $u^\alpha(y, \tilde{y})$ the linear term $\delta r y_j$ (or $\delta r \tilde{y}_j$) for any real number r , where δ is the perturbation parameter. For the last equation add the linear term $\delta(\sum_{j=1}^{\ell} r p_j y_j + r y_{\ell+1})$ to agent $\bar{\alpha}$'s utility. ■

4 Tobacco

Next let us turn to the use of tobacco as money. Here it is possible to get efficient equilibria, provided there is “enough” tobacco in the endowment of each trader and certain conditions obtain for the utility of tobacco. We will give these conditions in their simplest (rather than their most refined or tight) form.

It will first help to make a general definition. Let $J \subset L$, and define

$$\square_J = \left\{ (x, \tilde{x}) \in \mathbb{R}_+^L \times \mathbb{R}_+^L : x_k, \tilde{x}_k \leq \sum_{\alpha \in H} (e_k^\alpha + \tilde{e}_k^\alpha) \text{ for } k \in L \setminus J \right\}.$$

In other words, \square_J bounds the consumption of all commodities that are *not* in J .

We will say that “ α likes J ” if conditions I and II below hold.

Condition I There exists a $B > 0$ such that, if $j \in J$,

$$\begin{aligned} \frac{\partial u^\alpha}{\partial x_j}(x, \tilde{x}) &> B \frac{\partial u^\alpha}{\partial x_k}(x, \tilde{x}) \\ \frac{\partial u^\alpha}{\partial \tilde{x}_j}(x, \tilde{x}) &> B \frac{\partial u^\alpha}{\partial \tilde{x}_k}(x, \tilde{x}) \end{aligned}$$

for all $k \in L \setminus J$ and $(x, \tilde{x}) \in \square_J$.

Condition II For any $m > 0$, there exists $M(m) > 0$ such that, if $j \in J$,

$$\begin{aligned} \left. \begin{array}{l} (x, \tilde{x}) \in \square_J \\ x_j < m \\ \tilde{x}_j > M(m) \end{array} \right\} &\Rightarrow \frac{\partial u^\alpha}{\partial x_j}(x, \tilde{x}) > \frac{\partial u^\alpha}{\partial \tilde{x}_j}(x, \tilde{x}) \\ \left. \begin{array}{l} (x, \tilde{x}) \in \square_J \\ \tilde{x}_j < m \\ x_j > M(m) \end{array} \right\} &\Rightarrow \frac{\partial u^\alpha}{\partial x_j}(x, \tilde{x}) < \frac{\partial u^\alpha}{\partial \tilde{x}_j}(x, \tilde{x}) \end{aligned}$$

Condition I says that (on the margin) j remains desirable for α relative to commodities in $L \setminus J$, even when it is consumed in large quantities, provided the consumption in $L \setminus J$ is bounded.

Condition II says that α prefers to distribute the consumption of j between the two periods in a not-too-skewed manner.

When “ α likes $\{j\}$,” we will simply write “ α likes j .” (Note that if α likes j , Condition II rules out the possibility that the utility function of α has a separable linear term in the consumption of x_j or \tilde{x}_j .)

We shall examine the effect of pouring in money in period 1 while the rest of the data of the economy is held fixed, i.e., $e_1^\alpha, \dots, e_\ell^\alpha, \tilde{e}_1^\alpha, \dots, \tilde{e}_\ell^\alpha, \tilde{e}_{\ell+1}^\alpha$; u^α are all held fixed for $\alpha \in H$; while the money endowments $\{e_{\ell+1}^\alpha\}_{\alpha \in H}$ are varied.

Theorem 2 Suppose the money commodity $\ell+1$ is tobacco-like, and α likes $\ell+1$, for all $\alpha \in H$. There exists an M^* such that, if $\min_{\alpha \in H} \{e_{\ell+1}^\alpha\} > M^*$, every equilibrium is efficient.

Proof First we establish two claims. Both refer to arbitrary equilibria of the economy, and the first is independent of the levels $\{e_{\ell+1}^\alpha\}_{\alpha \in H}$.

Claim 1 Let $m^* \equiv \ell E/B$, where $E = \max_{1 \leq j \leq \ell} \{\max\{\sum_{\alpha \in H} e_j^\alpha, \sum_{\alpha \in H} \tilde{e}_j^\alpha\}\}$ and B is as in Condition I. Then no agent α spends more than m^* on purchases in either period 1 or period 2.

Proof of Claim 1 Suppose α spends $m > 0$ in (w.l.o.g.) period 1. Then α must have spent at least m/ℓ on at least one commodity $1 \leq j \leq \ell$. Consequently

$$p_j \geq \frac{m}{\ell E}. \quad (12)$$

Denote the consumption of α by $(x^\alpha, \tilde{x}^\alpha)$. Since α could buy less of j and does not, we must have

$$p_j \leq \frac{\frac{\partial u^\alpha}{\partial x_j}(x^\alpha, \tilde{x}^\alpha)}{\frac{\partial u^\alpha}{\partial x_{\ell+1}}(x^\alpha, \tilde{x}^\alpha)} \leq \frac{1}{B} \quad (13)$$

where the second inequality comes from Condition I. Together (12) and (13) imply

$$m \leq \frac{\ell E}{B}.$$

The same inequalities hold in period 2 replacing $x_j, x_{\ell+1}$ by $\tilde{x}_j, \tilde{x}_{\ell+1}$ throughout, verifying the claim.

Claim 2 Put $M^* > M(m^*) + 2m^*$, where $M(m^*)$ is as in condition II, and suppose

$$\min_{\alpha \in H} e_{\ell+1}^\alpha > M^*.$$

Then (a) each agent has positive amounts of money on hand for spending in each period; (b) no agent spends all the money at hand in either period.

Proof of Claim 2 Since, by Claim 1, α spends at most m^* in period 1, there is nothing to check in period 1. Suppose any $\alpha \in H$ spends all his money (which may be zero) in period 2 on purchases. Let $m^\alpha =$ money inventoried by α from period 1 into 2. Then, since he spends all of m^α , we see from Claim 1 that

$$m^\alpha \leq m^*.$$

Hence the most that α could be consuming of money in period 2 is:

$$\begin{aligned} \tilde{x}_{\ell+1}^\alpha &\leq \sum_{j=1}^{\ell} \tilde{p}_j \tilde{e}_j^\alpha \\ &\leq \frac{\ell E}{B} \equiv m^* \end{aligned}$$

(since $\tilde{e}_j^\alpha \leq E$ for all j and $\tilde{p}_j \leq 1/B$ for all j , by the analogue of (13) for period 2).

Also the money consumed by α in period 1 is at least

$$\begin{aligned} x_{\ell+1}^\alpha &\geq e_{\ell+1}^\alpha - m^* - m^\alpha \\ &> M(m^*) + 2m^* - m^* - m^* \\ &= M(m^*). \end{aligned}$$

By Condition II, α would be better off inventorying more money from period 1 into period 2, and consuming it in period 2. This verifies the Claim 2. ■

But then at equilibrium we must have:

$$\frac{\nabla_j^\alpha}{p_j} = \nabla_{\ell+1}^\alpha, \quad \frac{\tilde{\nabla}_j^\alpha}{\tilde{p}_j} = \tilde{\nabla}_{\ell+1}^\alpha \quad (14)$$

for $\alpha \in H$ and $1 \leq j \leq \ell$ (otherwise α would either spend more or less on the purchases of j , both of which choices are available to him in light of the Claim 3).

Further, it must be that

$$\nabla_{\ell+1}^\alpha = \tilde{\nabla}_{\ell+1}^\alpha \quad (15)$$

for all $\alpha \in H$. For if $<$ (or $>$), α would do better to consume ε less (or more) in period 1, inventory ε more (or less) into period 2, and consuming ε more (or less) in period 2.

But (14) and (15) together imply that all the $(\nabla^\alpha, \tilde{\nabla}^\alpha)$ point in the same direction, so the equilibrium is efficient proving Theorem 2. ■

5 Generalizations

Suppose we drop the “liquidity constraint” in the definition of the budget set, i.e., replace (I)–(IV), by

$$\sum_{j=1}^{\ell+1} p_j (x_j^\alpha - e_j^\alpha) \leq 0 \quad (\text{I}^*)$$

$$\sum_{j=1}^{\ell+1} \tilde{p}_j (\tilde{x}_j^\alpha - \tilde{e}_j^\alpha) \leq \Delta(\text{I}^*) + \tau x_{\ell+1}^\alpha. \quad (\text{II}^*)$$

This is tantamount to the assumption that α can use the receipts from his sales for purchases, so that the “cash-in-advance” constraint is no longer binding. (Implicitly, α can borrow money unlimitedly within each period at zero rate of interest but must pay it back after trade and prior to consumption.)

Denote this enhanced budget set by $B_*^\alpha(p, \tilde{p})$, and define equilibrium as before, replacing $B^\alpha(p, \tilde{p})$ by $B_*^\alpha(p, \tilde{p})$ for each $\alpha \in H$.

We shall call this an equilibrium of the *liquidity-free model*. (Our previous equilibrium was of the liquidity-constrained model.)

We so far focused attention on the case of a single durable (which we took to be money) to bring out the results with minimal notation. But our results in fact carry over to a broader setting. Let us now suppose that the economy has several durable commodities of either variety whose initial endowment in period 2 is zero. Further, while money is a commodity, we will no longer insist that it be durable. Equilibria (of the liquidity-free or liquidity-constrained models) is defined as before, with the obvious amendments in the definition of the budget sets.

The analogue of condition (iii) is now: there exist $0 < \alpha_j < 1$ for each durable j such that $u^h(\underline{e}^h, \tilde{e}^h) > u^h(y, \tilde{y})$ if $(y, \tilde{y}) \in \square$ and $y_i = 0$ or $\tilde{y}_i = 0$ for any $i \in L$. Here

$$\underline{e}_j^h = \begin{cases} e_j^h & \text{if } j \text{ is perishable or gold-like} \\ \alpha_j e_j^h & \text{if } j \text{ is tobacco-like} \end{cases}$$

$$\tilde{e}_j^h = \begin{cases} \tilde{e}_j^h & \text{if } j \text{ is perishable} \\ e_j^h + \tilde{e}_j^h & \text{if } j \text{ is gold-like} \\ (1 - \alpha_j)e_j^h + \tilde{e}_j^h & \text{if } j \text{ is tobacco-like} \end{cases}$$

This simply says that each agent would do better not trading (and inventorying his durables in some manner) than accept zero consumption anywhere.

To generalize our results it will first help to enunciate some lemmas.

Lemma 2 *Consider any model (above). Put*

$$\tau^* = \begin{cases} 1, & \text{if commodity } \ell + 1 \text{ is gold-like} \\ 0, & \text{if commodity } \ell + 1 \text{ is tobacco-like or perishable} \end{cases}.$$

Then, if the equilibrium is efficient, the following differential conditions must hold for $\alpha \in H$ and $1 \leq j \leq \ell$.

Table 1

Period	j is a tobacco-like durable, or j is perishable	j is a gold-like durable
	$\frac{\nabla_j^\alpha}{p_j} = \nabla_{\ell+1}^\alpha + \tau^* \tilde{\nabla}_{\ell+1}^\alpha$	$\frac{\nabla_j^\alpha + \tilde{\nabla}_j^\alpha}{p_j} = \nabla_{\ell+1}^\alpha + \tau^* \tilde{\nabla}_{\ell+1}^\alpha$
1		
2	$\frac{\tilde{\nabla}_j^\alpha}{\tilde{p}_j} = \tilde{\nabla}_{\ell+1}^\alpha$	$\frac{\tilde{\nabla}_j^\alpha}{\tilde{p}_j} = \tilde{\nabla}_{\ell+1}^\alpha$

Proof Entirely analogous to the proof of Lemma 1. ■

Lemma 3 *Consider an equilibrium of any model. If the equalities of Table 1 hold, and if there exists at least one tobacco-like durable which is inventoried by every agent in equilibrium, then the equilibrium is efficient.*

Proof If j^* is inventoried by α (where j^* is a tobacco-like durable) then $\nabla_{j^*}^\alpha = \tilde{\nabla}_{j^*}^\alpha$. But then, using Claim 2, $(\nabla_{\ell+1}^\alpha + \tau^* \tilde{\nabla}_{\ell+1}^\alpha) / \tilde{\nabla}_{\ell+1}^\alpha = (\nabla_{j^*}^\alpha / p_{j^*}) / (\tilde{\nabla}_{j^*}^\alpha / \tilde{p}_{j^*}) = \tilde{p}_{j^*} / p_{j^*}$ is invariant of $\alpha \in H$. It immediately follows that the gradients $(\nabla^\alpha, \tilde{\nabla}^\alpha)$ all point in the same direction for $\alpha \in H$, hence the equilibrium is efficient. ■

We are now ready to state the general versions of Theorems 1 and 2.

Theorem 3 *Take any model. Suppose all the durables in the economy are gold-like. Then, generically in utilities, all the equilibria are inefficient.*

Proof In the light of Lemma 2, the proof follows that of Theorem 1 *mutatis mutandis*. ■

Theorem 4 *Consider the liquidity-free model. Suppose there is at least one tobacco-like durable j^* in the economy, and that α likes j^* for all $\alpha \in H$. Fix all the data of the economy except for $\{e_{j^*}^\alpha\}_{\alpha \in H}$. There exists $M^* > 0$ such that, if $\min_{\alpha \in H} e_{j^*}^\alpha > M^*$, then the equilibria are efficient.*

Proof Reread the proof of Theorem 2, replacing “ α spends $\ell + 1$ ” by “ α sells j^* ” (if $j^* \neq \ell + 1$), to obtain an M^* with the property: $e_{j^*}^\alpha > M^* \Rightarrow \alpha$ does not sell all of $e_{j^*}^\alpha$ in period 1, nor does he sell all of the j^* on hand in period 2. But then, since α can vary the amounts that he inventories, $\nabla_{j^*}^\alpha = \tilde{\nabla}_{j^*}^\alpha$. Also the other equalities of Table 1 now obtain (as is easily checked, remembering that we are in the liquidity-free model). Then the result follows from Lemma 3. ■

There are intuitively two reasons for inefficiency of equilibria. The first has to do with liquidity constraints, since agents must put up cash-in-advance for purchases, and may be stuck with an inefficient supply of money in their endowments. By turning to the liquidity-free model we, of course, wipe out this effect. But inefficiency may still persist if all durables are gold-like (Theorem 3). This is because agents are unable to decouple consumption (of the durables in period 1) from savings. Holding these durables is the only device whereby an agent can distribute purchasing power between the two periods, i.e., the durables serve as a “store-of-value.” The trouble is that agents are also forced to consume exactly the amount they hold of the durables.

The case left to be analyzed is that of a liquidity-constrained model with at least one durable that is tobacco-like. Here we need to ensure that not only the durable, but also money, is in plentiful supply with all agents, so that the liquidity constraints are not binding. More precisely, we have

Theorem 5 *Consider the liquidity-constrained model, with commodity $\ell + 1$ being money. Suppose that there is at least one tobacco-like durable j^* in the economy, and that α likes $\{j^*, \ell + 1\}$ for each $\alpha \in H$. Then there exists an M^* such that any*

equilibrium is efficient, provided

$$\min_{\alpha \in H} e_{j^*}^{\alpha} > M^*$$

and either

$$\min_{\alpha \in H} e_{\ell+1}^{\alpha} > M^* \text{ if money is durable}$$

or

$$\left. \begin{array}{l} \min_{\alpha \in H} e_{\ell+1}^{\alpha} > M^* \\ \min_{\alpha \in H} \tilde{e}_{\ell+1}^{\alpha} > M^* \end{array} \right\} \text{ if money is perishable}$$

Proof Repeat the proof of Theorem 2 to conclude that liquidity constraints are not binding if the endowment of money is above M^* for each agent. Then the proof proceeds exactly as that of Theorem 4. ■

Remark Existence of equilibria (in all our models) follows from standard arguments (and hence is omitted).

References

- [1] Aristotle, Politics
- [2] Del Mar, A.: A history of precious metals. London: G. Bell and Sons 1968
- [3] Dubey, P., L. S. Shapley: Noncooperative general exchange with a continuum of traders: Two models. *Journal of Mathematical Economics* **23**(3), 253–293 (1994)
- [4] Einzig, P.: Primitive money. London: Spottiswoode 1951
- [5] Galbraith, J. K.: Money : Whence it came, where it went. Houghton Mifflin Co. 1995
- [6] Jevons, W. S.: Money and the mechanism of exchange. London: MacMillan 1875
- [7] Keynes, J. M.: A treatise on money, vol. I. London: MacMillan 1958
- [8] Shapley, L. S., M. Shubik: Trade using one commodity as a means of payment. *The Journal of Political Economy* **85**(5), 937–968 (1977)
- [9] Smith, A.: The wealth of nations. New York: Random House 1937.
- [10] Vissering, W.: On Chinese currency, coin and paper money (quoting Henry Yule’s edition of Marco Polo’s journeys). Taipei: Ch’Eng Wen Publishing 1968
- [11] Wicksell, K.: Lectures on political economy, Vol Two: Money. London: Routledge & Kegan Paul 1935