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Approximately Median-Unbiased Estimation of Autoregressive Models with Applications to U.S. Macroeconomic and Financial Time Series

by

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APPROXIMATELY MEDIAN-UNBIASED ESTIMATION OF AUTOREGRESSIVE MODELS WITH APPLICATIONS TO U.S. MACROECONOMIC AND FINANCIAL TIME SERIES

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ABSTRACT

This paper introduces approximately median-unbiased estimators for univariate AR(p) models with time trends. Confidence intervals also are considered. The methods are applied to the Nelson-Plosser macroeconomic data series, the extended Nelson-Plosser macroeconomic data series, and some annual stock dividend and price series. The results show that most of the series exhibit substantially greater persistence than least squares estimates and some Bayesian estimates suggest. For example, for the extended Nelson-Plosser data set, eight of the fourteen series are estimated to have a unit root, while six are estimated to be trend stationary. In contrast, the least squares estimates indicate trend stationarity for all of the series.

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1. INTRODUCTION

This paper focuses on methods for, and applications of, median-unbiased estimation and confidence interval construction in univariate p-th order autoregressive (AR(p)) models with time trends. This focus reflects our interest in assessing the degree of persistence exhibited by various economic time series. In particular, the time series we analyze here include the fourteen Nelson-Plosser macroeconomic time series, the fourteen extended Nelson-Plosser time series, and six stock dividend and price series that have received considerable attention in the literature. Our interest in persistence of economic time series is in common with many recent papers in empirical macroeconomics, which focus on the question of whether economic time series possess a unit root or are trend stationary.

There are two motivations for this paper. The first is the emphasis placed in the unit root literature on hypothesis testing. A problem that arises with hypothesis tests in the unit root context is that tests have low power in many scenarios of empirical interest, including those analyzed here. In such cases, point and interval estimators can be used to provide more information than that given by unit root tests.

The second motivation is the recent growth in Bayesian estimation methods for the models considered here (e.g., see the fourth issue of the 1991 edition of the *Journal of Applied Econometrics*, which is devoted entirely to this subject). This development of Bayesian methods is very useful. We feel, however, that a corresponding development of classical estimation methods also is likely to provide useful tools and applied results. In particular, classical estimation methods that exhibit unbiasedness properties can provide results that exhibit a degree of impartiality that may not be attainable via Bayesian methods. This is especially true in the present context in which the specification of objective (i.e., non-informative) priors is particularly difficult.

The problem with utilizing the standard classical estimators (i.e., the least squares (LS) estimator) in the AR(p) model with time trend is that of bias. The LS estimators of key parameters exhibit substantial biases. In particular, for estimating the sum of the AR coefficients, α, the bias tends to be downward and quite large. For estimating the coefficient on the time trend, β, the bias is upward and quite large. In consequence, the LS estimator is a misleading indicator of the true values of α and β.

To deal with the problem of bias, this paper introduces a bias correction for the LS estimator. The proposed method is an extension to AR(p) models of the exactly median-unbiased estimation method introduced in Andrews (1993) for AR(1) models. The extended method yields only approximately, not exactly, median-unbiased estimators.

The first question that arises when considering a higher order AR model with time trend is the question of what are the key parameters of interest. To answer this question we note that our interest is in the long run persistence properties of the time series under study. These properties are exhibited by the impulse response function (IRF) of the series. For a series with a unit root, the IRF never dies out (i.e., it does not converge to zero as the time horizon goes to infinity). For a trend stationary series, on the other hand, the IRF does die out. In either case, the magnitude of the IRF across different time horizons indicates how much persistence is present in the series.

It is often useful to focus attention on a scalar measure of persistence rather than consider the whole IRF, which is an unwieldy infinite dimensional vector. In this paper, we develop a median-unbiased estimator and confidence intervals for such a measure. The measure we focus on is the cumulative impulse response (CIR), i.e., the sum of the IRF over all time horizons. This measure has the attribute that its relation to the persistence of the series is immediate -- it is a simple function of the IRF. In addition, the CIR is a monotone transformation of the spectral density function at zero frequency (in the model under consideration), which also is a measure of persistence of a time series.

In p-th order AR models (with or without a time trend), the CIR equals $1/(1-\alpha)$, where α is the sum of the AR coefficients (equivalently, α is the coefficient on the lagged variable in an "augmented Dickey-Fuller regression" in which the right-hand side variables include a constant, a time trend, a single lag of the series, plus $p-1$ lags of the changes in the series). Thus, our analysis can focus on the parameter α (since median-unbiasedness and confidence interval coverage probabilities are invariant under monotone transformations).

The essence of our bias correction method for the LS estimator of α is as follows: If the LS estimate of α equals .8, say, one does not use .8 as the estimate of α, but rather, one uses the value of α which yields the LS estimator to have a median of .8. If the distribution of the LS estimator of α depends only on α and is monotone in α, as in the AR(1) case, then the resultant estimator is exactly median-unbiased. If the distribution of the LS estimator of α depends on some nuisance parameters as well as on α, as in the AR(p) case for $p > 1$, then we use an iterative procedure that jointly estimates α and the nuisance parameters and yields only an approximately median-unbiased estimate of α. In fact, simulations reported below show that the approximation is very good -- the proposed estimator is essentially median-unbiased.

Once we obtain the approximately median-unbiased estimator of α, we impose this estimate on the model and run an augmented Dickey-Fuller LS regression to obtain estimates of the other parameters in the model. Again, simulations show that the resultant procedure leads to estimators that are essentially median-unbiased. We obtain standard error estimates for all of the parameter estimates via simulation.

The method for obtaining a median-unbiased estimator of α can be extended to generate confidence intervals for α. In addition, it leads to an approximately unbiased model selection procedure for determining whether a data series has a unit root or is trend stationary.

An alternative scalar parameter that has been considered in the literature to be a parameter of interest in AR(p) models is the magnitude of the largest root of the model. For example, Stock (1991) develops asymptotic confidence intervals for this parameter and DeJong and Whiteman (1991a, b) consider Bayes estimators of this parameter.

When considering tests of a unit root, it does not matter whether one focusses on the sum of the AR coefficients or the magnitude of the largest root, since the null hypothesis obtains if and only if both equal one. In contrast, if one is interested in point or interval estimation, one clearly wants to be able to interpret the magnitude of the parameter in a meaningful way regardless of its value. In particular, if the model is trend stationary we want our parameter to index the degree of persistence of the time series. For the sum of the AR coefficients, there is no problem -- it is a simple monotone transform of the CIR and of the spectrum at zero. For the dominant root, however, a problem occurs. We show below that depending upon the values of the other roots, the persistence properties of two series with the same dominant root can be very different. In consequence, knowledge of the magnitude of the largest root is quite useful when one is interested in persistence if the magnitude is one. But, if it is less than one, then knowledge of its value is not sufficient for conveying the series' persistence properties -- the other roots need to be known. In consequence the empirical results of Stock (1991) and DeJong and Whiteman (1991a, b) are not as informative as is desirable.

Although Stock (1991) focusses on what we view to be the wrong parameter, his method requires considerably less computational effort than is required by our method outlined above. For this reason, we provide a trivial extension of his method to the estimation of the parameter α, which we consider to be of main interest. Using his tables and a simple iterative scheme, one can compute asymptotically median-unbiased estimates of α and corresponding confidence intervals for it. Since these procedures are based on local-to-unity asymptotics, they are reliable only if the sample size is large enough and α is close enough to one. In contrast, the procedure outlined above is less reliant on a large sample and does not require α to be near one.

The approximately median-unbiased estimation procedures described above are applied here to three different data sets. The first is the Nelson-Plosser (NP) data set compiled by Nelson and Plosser (1982). (See the latter paper for a description of the data. All series except the interest rate are logged.) Nelson and Plosser (1982) applied tests for unit roots on these data series and found that they could reject the null hypothesis of a unit root for only one of the fourteen series, viz., the unemployment rate. Their results were initially perceived as establishing that many economic time series possess a unit root. Subsequently, this interpretation was challenged by several authors, because it became apparent that the tests employed by Nelson and Plosser have relatively low power against relevant trend stationary alternatives given the sample sizes employed, e.g., see DeJong, Nankervis, Savin and Whiteman (1992).

As an alternative to classical hypothesis tests, Bayesian estimation methods have now been applied to the NP data series, see DeJong and Whiteman (1991a), Phillips (1991a), and Zivot and Phillips (1991). The results are mixed with respect to the degree of persistence found, depending on the priors employed and the parameters considered. DeJong and Whiteman (1991a), for instance, obtain estimates of the magnitude of the largest root that are substantially less than one for most series. They conclude that most of the NP series are better represented as trend stationary series than as unit root series. Phillips (1991a) finds more evidence of unit root behavior, but still the evidence for it is not strong.

The median-unbiased estimates of α obtained here show considerably more persistence in the NP data than the LS, DeJong and Whiteman, or Phillips estimates show. For three series (out of fourteen) the estimates equal 1.0, for seven series the estimates are .96 or larger, and for thirteen series the estimates are .89 or larger.

The second data set we consider is an extension of the NP data set, which terminates in 1970, to include observations up to 1988. This extension was compiled by Schotman and van Dijk (1991). The median-unbiased estimates for the extended Nelson-Plosser (ENP) data set show very high levels of persistence for many of the series. Eight of the fourteen series, including all of the nominal variables except money stock, have α estimates equal to 1.0. Most of the real variables, including real GNP, real per capita GNP, industrial production, and employment have α estimates in the range .86 to .91, which corresponds to considerable persistence, although less than unit root-like behavior. In fact, for the former two series, as well as for the unemployment rate, the null hypothesis of a unit root can be rejected at the 5% level using the ENP data. Nevertheless, the overall picture obtained from the ENP data set is one of noticeably greater persistence than with the NP data set.

The third data set we analyze consists of annual series for Dow-Jones (DJ) dividends and prices (1928-1978), NYSE dividends and prices (1926-1981), and Standard and Poor's (S&P) dividends and prices. We use the same data as DeJong and Whiteman (1991b), some of which was compiled by Shiller (1981). (See these papers for descriptions of the data. All series are logged.) Interest in the unit root versus trend stationarity question for these data series arises because of their implications for volatility tests of the perfect markets hypothesis as initiated by Shiller (1981). See DeJong and Whiteman (1991b) for details and references.

DeJong and Whiteman (1991b) have presented some Bayesian estimates of the magnitude Λ of the largest root of AR(3) models with time trend fitted to the above data series (also see DeJong and Whiteman (1992)). Their estimates for Λ are quite low: .72, .76, .77, .84, .72, and .87 respectively. As noted above and argued below in Section 2, the magnitude of the largest root does not provide a good measure of the closeness of a series to a unit root model in terms of persistence of the series. In addition, DeJong and Whiteman's (1991b) estimates of Λ are likely to be biased away from 1.0 in a sampling theoretic sense, since Bayesian methods generally bias parameter estimates towards the middle of the parameter space, which in their case is specified to be [.55, 1.055] for Λ. For these reasons, it is of interest to see what sort of parameter estimates and CIs are obtained using the approximately median-unbiased methods described above.

In short, the median-unbiased parameter estimates show considerably greater persistence than DeJong and Whiteman's (1991b) estimates show. The confidence intervals obtained are very wide, however, so a key feature of our results is that for most of the series it is not possible to make definite statements one way or another regarding the unit root/trend stationary question. Our parameter estimates for α for these series are: .79, .91, .90, 1.0, .82, and .94 respectively. Our corresponding estimates for Λ for these series are nearly the same: .79, .92, .90, 1.0, .77, and .94. Thus, the difference between our estimates and those of DeJong and Whiteman (1991b) is not due simply to a difference in the estimand. It is due to the bias properties of the estimators. Our estimators are essentially median-unbiased, whereas DeJong and Whiteman's (1991b) Bayesian procedure appears to have a substantial downward bias.

Next, we discuss several related papers that have not been referenced above. First, a method similar to that considered here has been considered recently by Rudebusch (1992). In fact, Rudebusch (1992) applies his method to the NP data and his implied estimates for the sum of the AR coefficients α are quite similar to those obtained with the method considered here. Rudebusch (1992) obtains a median-bias correction to the LS estimates of each of the individual AR(p) parameters by searching for a vector of AR(p) parameters such that the medians of each of the LS estimators are equal to the observed LS estimates when this vector of parameters is taken to be the truth.

Rudebusch's procedure differs from that considered here in that he aims for median-unbiased estimators of each of the AR(p) parameters, whereas we focus on the main parameter of interest here, viz., the sum of the AR(p) parameters. (Note that the property of median-unbiasedness is not inherited by linear combinations of median-unbiased estimators.) Rudebusch's (1992) procedure is subject to the criticism that the existence and uniqueness of a vector that satisfies the requisite properties outlined above is an open question (although he reports no difficulties in finding such a vector with his algorithm). The method considered here is not subject to this criticism. Lastly, Rudebusch obtains estimates of the AR parameters, but does not provide any measure of the variability of these estimates. This significantly reduces the value of the estimates themselves. In this paper, we provide measures of variability of the median-unbiased parameter estimates via standard error estimates and via confidence intervals for the parameter.

Another recent paper that is related to the present paper is Fair (1992). Earlier papers that are related include Quenouille (1949, 1956), Hurwicz (1950), Marriott and Pope (1954), Kendall (1954), and Orcutt and Winokur (1969).

The remainder of this paper is organized as follows. Section 2 defines the model to be considered and provides a discussion of the parameters of interest. Section 3 introduces approximately median-unbiased estimators and confidence intervals for the main parameter of interest. It also describes an approximately unbiased model selection procedure for determining whether a series has a unit root or is trend stationary. Section 4 extends the local-to-unity asymptotic results of Stock (1991) to generate asymptotically median-unbiased estimators and asymptotically valid confidence intervals for α, the main parameter of interest in this paper. Sections 5-7 provide the empirical results. Section 5 gives the results for the Nelson-Plosser data series. It also provides comparisons of several estimates that have appeared in the literature for these series. Section 6 presents results for the extended Nelson-Plosser data series. Section 7 does likewise for the stock dividend and price series.

2. THE MODEL AND PARAMETERS OF INTEREST

2.1. *Definition of the model*

The model we consider is an AR(p) model with intercept and time trend. It can be written in an unobserved components form and in a regression form. In unobserved components form, it is given by

$$
\begin{aligned}
&Y_t = \mu^* + \beta^* t + Y_t^* \quad \text{for } t = -p+1, \ldots, T, \\
(2.1) \qquad &Y_t^* = \alpha Y_{t-1}^* + \psi_1 \Delta Y_{t-1}^* + \cdots + \psi_{p-1} \Delta Y_{t-1}^* + U_t \quad \text{for } t = 1, \ldots, T, \\
&U_t \sim \text{iid } N(0, \sigma^2) \quad \text{for } t = 1, \ldots, T,
\end{aligned}
$$

where $\{Y_t : t = -p+1, \ldots, T\}$ is the observed series. The variable ΔY_t^* denotes $Y_t^* - Y_{t-1}^*$. The parameters $(\mu^*, \beta^*, \sigma^2, \alpha)$ satisfy $\mu^* \in R$, $\beta^* \in R$, $\sigma^2 > 0$, and $\alpha \in (-1, 1]$. When $\alpha = 1$ the model is nonstationary. The parameters $(\psi_1, \ldots, \psi_{p-1})$ are such that the AR model for Y_t^* is stationary when $\alpha \in (-1, 1)$ and the AR model for ΔY_t^* is stationary when $\alpha = 1$. The initial values of Y_t^*, i.e., $(Y_{-p+1}^*, \ldots, Y_0^*)$, are taken to be such that $\{Y_t^* : t \geq -p+1\}$ is stationary when $\alpha \in (-1, 1)$ and $\{\Delta Y_t^* : t \geq -p+2\}$ is stationary when $\alpha = 1$. The level of the ΔY_t^* series is arbitrary when $\alpha = 1$. (That is, when $\alpha = 1$, the initial rv Y_{-p+1}^* can be fixed or can have any distribution provided the subsequent Y_t^* values are such that ΔY_t^* is stationary.)

The regression form of model (2.1) is given by

$$(2.2)\quad \begin{aligned} &Y_t = \mu + \beta t + \alpha Y_{t-1} + \psi_1 \Delta Y_{t-1} + \cdots + \psi_{p-1}\Delta Y_{t-p+1} + U_t \quad \text{for} \quad t = 1, \ldots, T, \\ &\mu = \mu^*(1-\alpha) + (\alpha - \psi_1 - \psi_2)\beta^* , \quad \text{and} \quad \beta = \beta^*(1-\alpha) , \end{aligned}$$

where $(Y_{-p+1}, \ldots, Y_0)$ and $\{U_t : t = 1, \ldots, T\}$ are as defined in (2.1). The AR(p) model for Y_t in (2.2) is written in "augmented Dickey-Fuller" regression form. It can be written in standard AR(p) regression form as

$$(2.3)\quad Y_t = \mu + \beta t + \gamma_1 Y_{t-1} + \gamma_2 Y_{t-2} + \cdots + \gamma_p Y_{t-p} + U_t .$$

The parameter α in the augmented Dickey-Fuller form equals the "sum of the AR coefficients" $(\gamma_1, \ldots, \gamma_p)$. As argued below, the augmented Dickey-Fuller parameterization is the most useful for the purposes of the present paper. The parameters $(\psi_1, \ldots, \psi_{p-1})$ and $(\gamma_1, \ldots, \gamma_p)$ are related via $\psi_j = -(\gamma_{j+1} + \cdots + \gamma_p)$ for $j = 1, \ldots, p-1$.

Note that the parameter β on the time trend is necessarily 0 when $\alpha = 1$ in (2.2) and (2.3). This is a desirable feature of the model because it implies that the mean of Y_t is a linear function of t for all $\alpha \in (-1, 1]$. If $\beta \neq 0$ was allowed when $\alpha = 1$, then the mean of Y_t would be a linear function of Y_t when $\alpha \in (-1, 1)$, but a quadratic function of t when $\alpha = 1$. This discontinuity is naturally avoided in the model above.

2.2. *Scalar Measures of Persistence*

In this paper, as in many empirical papers in the macroeconomic and financial literature, we are interested in assessing the persistence of a time series Y_t. In particular, we are interested in the persistence of shocks to the series. The impulse response function (IRF) is a suitable measure of such persistence. The IRF traces out the effect of a change in the innovation U_t by a unit quantity on the current and subsequent values of Y_t. In particular, if Y_t is the series based on the innovations $\{U_1, U_2, \ldots\}$ and $\hat{Y}_t$ is the series based on $\{U_1, \ldots, U_{t-1}, U_t+1, U_{t+1}, U_{t+2}, \ldots\}$, then

$$(2.4)\quad \text{IRF}(h) = \hat{Y}_{t+h} - Y_{t+h} \quad \text{for} \quad h = 0, 1, \ldots .$$

By linearity, the IRF does not depend on t, on the values $\{U_1, U_2, \ldots\}$, or on the parameters (μ^*, β^*). It only depends on $(\alpha, \psi_1, \ldots, \psi_{p-1})$. The IRF can be computed by supposing $\mu^* = \beta^* = 0$ and then by calculating the (infinite-order) moving average representation of Y_t. The coefficient on U_{t-h} in this representation is IRF(h). That is, when $\mu^* = \beta^* = 0$, we can write

(2.5) $$Y_t = \left(1 - \gamma_1 L - \cdots - \gamma_p L^p\right)^{-1} U_t = \sum_{h=0}^{\infty} c_h U_{t-h} \quad \text{and} \quad \text{IRF}(h) = c_h ,$$

for $h = 0, 1, \ldots,$ where L is the lag operator.

Being an infinite vector of numbers, the IRF can be a rather unwieldy measure of persistence. In consequence, it is often convenient to have a scalar measure of persistence that summarizes the information contained in the IRF. One such measure is the cumulative impulse response (CIR). It is defined by

(2.6) $$\text{CIR} = \sum_{h=0}^{\infty} \text{IRF}(h) .$$

The CIR yields an especially useful summary of the IRF if one is dealing with different series for which the IRFs are of the same basic shape. This is the case for the data series considered here. With few exceptions, the series considered below have IRFs that start at 1, increase monotonically and smoothly for several periods, and then decrease monotonically and smoothly to 0. In some cases, there is no increase in the IRF over the first few periods. In a small number of cases, the IRF becomes negative for some large values of h, but the magnitudes of such negative values are always small (.03 or less). Also, in a small number of cases, the decrease to 0 is not completely monotone, but exhibits some small wiggles.

If one is considering several series whose IRFs are of quite different shapes, then the CIR may not be sufficiently informative about the difference in their IRFs. Consider the following two examples. The first example is the case where one series has an everywhere positive IRF and another series has an IRF that oscillates between positive and negative values. The two series could have equal CIRs, but quite different IRFs, due to the cancellation of positive and negative terms in the computation of the CIR for the second series.

The second example is the case where one series is given by $Y_t = \alpha Y_{t-1} + U_t$ and another series is given by $Y_t = \alpha Y_{t-k} + U_t$ for some $k > 1$. The IRF functions of these series are (i) $\text{IRF}(h) = \alpha^h$ for $h = 0, 1, \ldots$ and (ii) $\text{IRF}(h) = \alpha^b$ for $h = bk$ for $b = 0, 1, \ldots$ and $\text{IRF}(h) = 0$ otherwise respectively. The CIRs of these two series are identical, but their IRFs are noticeably different with the latter exhibiting more persistence as k is increased. (We thank Chris Sims for suggesting this example.)

Fortunately, neither of the two examples mentioned above, where the CIR is noticeably deficient, are of real concern for the economic applications we consider below. In no cases are there IRFs with substantial positive and negative terms. In no cases are there IRFs with the non-monotone and non-smooth behavior of that of the model $Y_t = \alpha Y_{t-k} + U_t$ for $k > 1$. The one feature of the IRFs that appears empirically, but is not captured by the CIR, is the difference between a relatively large initial increase and subsequent quick decrease in the IRF and a relatively small initial increase and subsequent slow decrease in the IRF. Two series can have the same CIR but somewhat differently shaped IRFs due to such differences. In the empirical applications, differences of this sort arise but they are not extreme.

Based on the above discussion, we conclude that the CIR yields a fairly good scalar summary of the IRF, at least for the type of data series that are of interest here. In addition, the CIR is a simple function of the parameters of the model:

$$\text{CIR} = \frac{1}{1-\alpha} . \tag{2.7}$$

The fact that the CIR is directly related to α in such a simple way means that one can rely on α as a measure of the persistence of a series. Different values of α can be interpreted easily in terms of persistence since they correspond straightforwardly to different values of the CIR. It is for this reason that we utilize the augmented Dickey-Fuller parameterization of the AR(p) model in (2.2) rather than the standard AR parameterization in (2.3).

The parameter α can be interpreted as a measure of persistence in a second way, viz., via the spectrum of Y_t. This interpretation is discussed by Phillips (1991b). The spectrum at zero frequency is a measure of the low frequency autocovariance of the series. For the model (2.1)-(2.3), it is given by

$$\text{spectrum at zero} = \frac{\sigma^2}{(1-\alpha)^2} . \tag{2.8}$$

Thus, by this measure too, persistence of Y_t depends directly on the magnitude of the parameter α.

Before deciding to emphasize the parameter α as an appropriate scalar measure of persistence, we need to consider another possibility, viz., the magnitude of the largest root of the AR(p) model. The latter parameter is relied on by DeJong and Whiteman (1991a, 1991b) and Stock (1991), among others.

The magnitude of the largest root of the AR(p) model turns out to be a very poor summary measure of the IRF. The reason is simply that the shape and height of the IRF depends on more than just the magnitude of the largest root. Depending on the values of the other roots, one can observe a very wide range of different persistence properties from series that have the same magnitude of largest root. This is illustrated by Table 1. Table 1 considers four pairs of models. Each model corresponds to an estimated model (estimated via the approximately median-unbiased method described below) using the Nelson-Plosser or extended Nelson-Plosser data. The reason for considering estimated models is to ensure empirical relevance of the results. We are not considering pathological cases in Table 1.

In each of the first two pairs of models in Table 1, the magnitudes of the largest roots are the same (or almost the same) for the two models, but the IRFs are quite different. For each pair, the IRF of the second model is much larger, almost uniformly twice as large or more, than that of the first. These differences are reflected in the α values of the model, but not in the magnitudes of the largest roots. The α values for the first pair yield a CIR that is twice as large for the second model as for the first. The α values for the second pair yield a CIR that is five times as large for the second model as for the first!

In the third pair of models in Table 1, the first model is chosen to have a noticeably larger largest root than the second model (by .08), but the corresponding α values are reversed in relative magnitude (with a difference of .10). In this case, it is clear from the IRFs of the two series that the second model (with the smaller largest root) exhibits noticeably more persistence than the first model. In fact, the second model's CIR is approximately 50% larger than that of the first, as is reflected by its larger α value.

The fourth pair of models in Table 4 illustrates the case of two models with identical α values, but quite different magnitudes of their largest root. The largest root of the first series is .17 larger than that of the second series. The IRFs of the two models exhibit somewhat different shapes. The first declines monotonically and slowly to zero from $h = 0$ whereas the second increases from $h = 0$ to $h = 1$ and then declines monotonically and relatively quickly to zero. The first model has greater persistence over horizons $h > 4$, but less persistence over horizons $h < 4$. The CIRs of the two models are identical, since their α values are. Overall, the persistence exhibited by the two models is similar, though many may argue that the first exhibits somewhat more persistence. The latter view is supported by the observation that the two models have the same area under their IRFs, but the IRF of the first model is distributed to the right of that of the second model. The CIR measure fails to pick up this difference. On the other hand, the very large difference in the magnitudes of the largest roots of the two models does not reflect the closeness of the IRFs of the two models.

In sum, we find that the magnitude of the largest root does not provide an adequate summary measure of the IRF. The other roots have too great an effect on the persistence of the series to rely solely on the magnitude of the largest root. The parameter α, on the other hand, is a fairly reliable measure of the persistence of a series because it alone determines both the CIR and the spectrum at zero of the series. For these reasons we focus attention below primarily on the estimation of α and secondarily on the estimation of the other parameters.

3. APPROXIMATELY MEDIAN-UNBIASED ESTIMATORS

3.1. Definition of the Approximately Median-unbiased Estimators

In this section, we describe a method for obtaining approximately median-unbiased estimators of the parameters of the augmented Dickey-Fuller model (2.2). The method is an extension of an exactly median-unbiased estimation procedure introduced in Andrews (1993) for the AR(1) version of model (2.2).

We start by defining median-unbiasedness and comparing it to the more standard property of mean-unbiasedness. By definition, a number m is a *median* of a rv X if

(3.1) $$P(X \geq m) \geq 1/2 \quad \text{and} \quad P(X \leq m) \geq 1/2 .$$

This definition of a median allows for non-uniqueness, but all of the medians considered here are unique. The definition also allows for the median of X to be a probability mass point of X. This feature of the definition is used here. If a median m of X is not a probability mass point, then $P(X > m) = P(X < m) = 1/2$.

Let $\hat{\alpha}$ be an estimator of the parameter α. By definition, $\hat{\alpha}$ is *median-unbiased* for α if the true parameter α is a median of $\hat{\alpha}$ for each α in the parameter space. The condition of median-unbiasedness has the intuitive impartiality property that the probability of under-estimation equals the probability of over-estimation. This holds unless the true parameter value is estimated with positive probability and in this case the probabilities of under-estimation and over-estimation are each less than one half. In scenarios where the magnitude of a parameter is a contentious issue, such as in the (trend) stationary versus unit root debate, this impartiality property is quite useful. Advocates of one view are not likely to accept estimates that are biased towards a different view. Median-unbiased estimators are more likely to be acceptable to a broad audience than biased estimators, because they do not favor any particular outcome.

The condition of median-unbiasedness is often more useful than that of mean-unbiasedness when the parameter space is bounded or when the distributions of estimators are skewed and/or kurtotic. When the parameter space is bounded and closed, it is impossible to have a mean-unbiased estimator because all estimators are biased at extreme boundary points. Boundary points do not present problems, however, for the condition of median-unbiasedness. If an estimator is median-unbiased for a parameter space $A \subset R$, then the estimator restricted to a closed subset A^* of A is median-unbiased for the restricted parameter space A^*. (The method of restricting the estimator, say $\hat{\alpha}$, to A^* is to set $\hat{\alpha}$ equal to the nearest element of A^* that is larger or smaller than $\hat{\alpha}$.) Next, when estimators have asymmetric distributions, there is no unambiguous measure of the centers of their distributions. In this case, the median may be a preferred measure to the mean, especially in kurtotic cases, because the median is less sensitive to the tails of the distribution.

We note that in the classical normal linear regression model with fixed regressors the LS estimator is median-unbiased. In fact, it is the best median-unbiased estimator for a wide variety of loss functions (see Andrews and Phillips (1987)). In the AR(p) model (2.2), on the other hand, the LS estimator is not median-unbiased, and hence, does not possess the same optimality properties.

Next, we describe the method used in Andrews (1993) for obtaining exactly median-unbiased estimators of α in the AR(1) version of model (2.2). Suppose $\hat{\alpha}$ is an estimator whose median function $m(\alpha)$ ($= m_T(\alpha)$) is uniquely defined, depends only on α, and is strictly increasing on the parameter space $(-1, 1]$. Then, a median-unbiased estimator, $\hat{\alpha}_U$, of α is given by

$$\hat{\alpha}_U = \begin{cases} 1 & \text{if } \hat{\alpha} > m(1) \\ m^{-1}(\hat{\alpha}) & \text{if } m(-1) < \hat{\alpha} \leq m(1) \\ -1 & \text{if } \hat{\alpha} \leq m(-1) \end{cases}, \tag{3.2}$$

where $m(-1) = \lim_{\alpha \to -1} m(\alpha)$ and $m^{-1} : (m(-1), m(1)] \to (-1, 1]$ is the inverse function of $m(\cdot)$ that satisfies $m^{-1}(m(\alpha)) = \alpha$ for $\alpha \in (-1, 1]$. Thus, if the observed value of $\hat{\alpha}$ is .8, say, one does not use .8 as the estimate of α, but rather, one uses the value of α that yields the estimator $\hat{\alpha}$ to have a median of .8. This method was applied in Andrews (1993) with $\hat{\alpha}$ equal to the LS estimator of α for model (2.2) with $p = 1$. (The general method is not due to Andrews (1993), e.g., it more or less corresponds to the method discussed by Lehmann (1959, Sec. 3.5, p. 83).)

For higher-order versions of model (2.2) (i.e., $p > 1$), the LS estimator of α has a distribution that depends on more parameters than just α. In consequence, the exact bias correction method outlined above cannot be applied. In fact, the LS estimator of α, $\hat{\alpha}_{LS}$, has distribution that depends on (α, ψ_1, ..., ψ_{p-1}). It does not depend on μ^*, β^*, or σ^2, and when $\alpha = 1$ it does not depend on the value or distribution of the initial rv Y^*_{-p+1}, see the Appendix. (Similar invariance properties in the AR(1) model have been pointed out by several authors. For references, see Andrews (1993).) In consequence, if (ψ_1, ..., ψ_{p-1}) were known, the bias correction method of (3.2) could be applied.

Since (ψ_1, ..., ψ_{p-1}) are unknown in practice, we suggest a simple iterative procedure that yields an approximately median-unbiased estimator. First, compute the LS estimator of (α, ψ_1, ..., ψ_{p-1}, μ, β) by regressing Y_t on (Y_{t-1}, ΔY_{t-1}, ..., ΔY_{t-p+1}, 1, t), call it ($\hat{\alpha}_{LS1}$, $\hat{\psi}_{1,LS1}$, ..., $\hat{\psi}_{p-1,LS1}$, $\hat{\mu}_{LS1}$, $\hat{\beta}_{LS1}$). Second, treat ($\hat{\psi}_{1,LS1}$, ..., $\hat{\psi}_{p-1,LS1}$) as though they were the true values of (ψ_1, ..., ψ_{p-1}) and compute the bias-corrected estimator of α, $\hat{\alpha}_{U1}$, using (3.2). Third, treat $\hat{\alpha}_{U1}$ as though it was the true value of α and compute a second round set of LS estimates of (ψ_1, ..., ψ_{p-1}), call them ($\hat{\psi}_{1,LS2}$, ..., $\hat{\psi}_{p-1,LS2}$), by regressing $Y_t - \hat{\alpha}_{LS1} Y_{t-1}$ on (ΔY_{t-1}, ..., ΔY_{t-p+1}, 1, t). When $\hat{\alpha}_{LS1} = 1$, exclude the regressor t in the latter regression in order to impose the constraint that $\beta = 0$. Next, treat ($\hat{\psi}_{1,LS2}$, ..., $\hat{\psi}_{p-1,LS2}$) as though they were the true values of (ψ_1, ..., ψ_{p-1}) to generate a second round bias-corrected estimator of α, $\hat{\alpha}_{U2}$. Continue this procedure either for a fixed number of iterations or until convergence. For the empirical results below, we specify a maximum of ten iterations. For most of the series, convergence is obtained in two iterations, while one series took four.

If $\hat{\alpha}_{LSj}$ is the final estimate of α, then ($\hat{\psi}_{1,LSj+1}$, ..., $\hat{\psi}_{p-1,LSj+1}$, $\hat{\mu}_{LSj+1}$, $\hat{\beta}_{LSj+1}$) are the final estimates of (ψ_1, ..., ψ_{p-1}, μ, β). Let

(3.3) $$(\hat{\alpha}_{MU}, \hat{\psi}_{1,MU}, \ldots, \hat{\psi}_{p-1,MU}, \hat{\mu}_{MU}, \hat{\beta}_{MU}) = (\hat{\alpha}_{LSj}, \hat{\psi}_{1,LSj+1}, \ldots, \hat{\psi}_{p-1,LSj+1}, \hat{\mu}_{LSj+1}, \hat{\beta}_{LSj+1})$$

denote the final round approximately median-unbiased estimators. We refer to these estimators as the MU estimators.

Simulation methods can be used to compute the bias-corrected estimator defined in (3.2) given a vector $(\psi_1, \ldots, \psi_{p-1})$. More specifically, for a given value of α and fixed $(\psi_1, \ldots, \psi_{p-1})$, set $\mu^* = \beta^* = 0$ and $\sigma^2 = 1$ in (2.2) and simulate a data set $\{Y_{r,t} : t = -p+1, \ldots, T\}$ according to the model. Regress the simulated $Y_{r,t}$ on $(Y_{r,t-1}, \Delta Y_{r,t-1}, \ldots, \Delta Y_{r,t-p+1}, 1, t)$ to obtain a single random draw of the LS estimator $\hat{\alpha}_{LS}$. Repeat this procedure R times, i.e., $r = 1, \ldots, R$ ($R = 1{,}000$ is used in the empirical results below) and take the sample median of the simulated LS estimates of α to be a Monte Carlo estimate of the median of $\hat{\alpha}_{LS}$ when the true parameters are $(\alpha, \psi_1, \ldots, \psi_{p-1})$. Given the observed value of $\hat{\alpha}_{LS}$, say .8, iteratively determine the value of α such that the Monte Carlo estimate of the median of the estimator $\hat{\alpha}_{LS}$ equals .8. This yields the desired estimator $\hat{\alpha}_U$. Monotonicity of the median function of $\hat{\alpha}_{LS}$ for given $(\psi_1, \ldots, \psi_{p-1})$ generally makes the iterative procedure converge quickly.

In keeping with the computer-intensive methods employed above, one can generate standard errors for all of the MU estimators $(\hat{\alpha}_{MU}, \hat{\psi}_{1,MU}, \ldots, \hat{\psi}_{p-1,MU}, \hat{\mu}_{MU}, \hat{\beta}_{MU})$ as follows. Treat the observed estimates, say $(\hat{\alpha}^0_{MU}, \hat{\psi}^0_{1,MU}, \ldots, \hat{\beta}^0_{MU})$, as though they were the true values and perform a simulation study of the estimators $(\hat{\alpha}_{MU}, \ldots, \hat{\beta}_{MU})$ with these true values. For each repetition of the simulation, one generates a simulated data set, computes the LS estimates for this data set, and then computes the corresponding approximately median-unbiased estimators of $(\alpha, \psi_1, \ldots, \psi_{p-1}, \mu, \beta)$ for this data set. Having completed the desired numbers of repetitions R^* ($R^* = 1{,}000$ for the cases reported below), one has R^* realizations from the distribution of $(\hat{\alpha}_{MU}, \ldots, \hat{\beta}_{MU})$ (up to simulation error) when the true parameters are $(\hat{\alpha}^0_{MU}, \ldots, \hat{\beta}^0_{MU})$. The sample standard errors from these R^* realizations are used as estimates of the standard errors of $(\hat{\alpha}_{MU}, \ldots, \hat{\beta}_{MU})$ for the original data series.

The above method of simulating standard errors is straightforward, but computer-intensive, since each of the R^* repetitions involves computing bias-corrected estimates $(\hat{\alpha}_{MU}, \ldots, \hat{\beta}_{MU})$ which by themselves requires a simulation procedure. Using a 486 33MHz PC, it took 50 hours to generate the parameter estimates and corresponding standard errors for each of the data series analyzed below. Although slow, this performance shows the proposed method to be quite feasible. Hopefully, within a few years, the required time will be reduced to a few hours on the fastest PCs.

3.2. Confidence Intervals for α

Approximate CIs for α can be obtained in a similar way to that of the approximately median-unbiased estimator of α. Suppose $\hat{\alpha}$ is an estimator whose p_1 and p_2 quantiles are uniquely defined, depend only on α, and are strictly increasing in α on the parameter space $(-1, 1]$. Let $q_{p_1}(\alpha)$ and $q_{p_2}(\alpha)$ denote these quantile functions. Then, an exact level $100(1 - p_1 - p_2)\%$ CI for α is given by $[\hat{c}_L, \hat{c}_U]$, where

$$(3.4)\ \hat{c}_L = \begin{cases} > 1 & \text{if } \hat{\alpha} > q_{p_2}(1) \\ q_{p_2}^{-1}(\hat{\alpha}) & \text{if } q_{p_2}(-1) < \hat{\alpha} \le q_{p_2}(1) \\ -1 & \text{if } \hat{\alpha} \le q_{p_2}(-1) \end{cases} \quad \text{and} \quad \hat{c}_U = \begin{cases} 1 & \text{if } \hat{\alpha} > q_{p_1}(1) \\ q_{p_1}^{-1}(\hat{\alpha}) & \text{if } q_{p_1}(-1) < \hat{\alpha} \le q_{p_1}(1) \\ -1 & \text{if } \hat{\alpha} \le q_{p_1}(-1) \end{cases}.$$

In (3.4), for $i = 1, 2$, $q_{p_i}(-1) = \lim_{\alpha \to -1} q_{p_i}(\alpha)$ and $q_{p_i}^{-1} : (q_{p_i}(-1), q_{p_i}(1)] \to (-1, 1]$ is the inverse function of $q_{p_i}(\cdot)$ that satisfies $q_{p_i}^{-1}(q_{p_i}(\alpha)) = \alpha$ for $\alpha \in (-1, 1]$. In Andrews (1993), this method was used to construct exact CIs for α for the first-order AR version of model (2.2). (Note that this method of constructing CIs is time honored, only the application of it in the present context is original.)

Letting $\hat{\alpha}$ of (3.4) be the LS estimator of α from the regression in (2.2), one finds that its distribution depends on $(\alpha, \psi_1, \ldots, \psi_{p-1})$ rather than just α. Hence, one cannot obtain an exact confidence interval for α using the method of (3.4). One can obtain an approximate one, however, by taking the final bias-corrected estimates of $(\psi_1, \ldots, \psi_{p-1})$ defined above and treating them as though they were the true values. Given these values, $\hat{c}_L$ and $\hat{c}_U$ can be computed by simulation using an analogous procedure to that described above for computing $\hat{\alpha}_U$.

3.3. An Unbiased Model Selection Procedure

The approximately median-unbiased estimator introduced above can be used to construct approximately unbiased model selection procedures. By definition, a model selection procedure is *unbiased* if for any correct model the probability of selecting the correct model is at least as large as the probability of selecting each incorrect model. For example, one might want to select between the (trend) stationary model for which $\alpha \in (-1, 1)$ and the unit root (with drift) model for which $\alpha = 1$. An unbiased selection procedure in this case has the property that if $\alpha = 1$ the probability of selecting the unit root model is $\geq$ the probability of selecting the (trend) stationary model and if $\alpha \in (-1, 1)$ the P_α-probability of selecting the (trend) stationary model is $\geq$ the P_α-probability of selecting the unit root model for each $\alpha \in (-1, 1)$. Unbiased selection procedures exhibit an intuitive impartiality property that may be useful if the selection of one model or another is a contentious issue.

The concept of unbiased selection procedures is a special case of that of risk-unbiased decision rules, see Lehmann (1959, p. 12). For selection procedures, the space of actions is finite -- one chooses one model from a finite set of models. If the loss function equals zero when the correct model is chosen and one otherwise, then a risk-unbiased decision rule for this problem is an unbiased selection procedure.

Consider the problem of selecting one of two models defined by $\alpha \in I_a$ and $\alpha \in I_b$, where I_a and I_b are intervals that partition the parameter space $(-1, 1]$ for α. For example, one might have $I_a = (-1, 1)$ and $I_b = \{1\}$ or $I_a = (-1, .975)$ and $I_b = [.975, 1]$. (The latter are considered in DeJong and Whiteman (1991a) and Phillips (1991a).)

The selection procedure we consider here is

(3.5) $$\text{"}\textit{choose } I_k \text{ if } \hat{\alpha}_{MU} \in I_k \quad \text{for} \quad k = a, b \text{ ."}$$

This procedure is exactly unbiased, if $\hat{\alpha}_{MU}$ is exactly median-unbiased. To see this, suppose I_a lies to the left of I_b, and $\hat{\alpha}_{MU}$ is exactly median-unbiased. Then, for all $\alpha \in I_a$,

(3.6) $$P_\alpha(\hat{\alpha}_{MU} \in I_b) \leq P_\alpha(\hat{\alpha}_{MU} > \alpha) \leq \frac{1}{2} \leq P_\alpha(\hat{\alpha}_{MU} \leq \alpha) \leq P_\alpha(\hat{\alpha}_{MU} \in I_a) \ ,$$

where the second and third inequalities use the median-unbiasedness of $\hat{\alpha}_{MU}$. For $\alpha \in I_b$, the argument is analogous, so the selection procedure of (3.5) is unbiased. We note that the selection procedure of (3.5) is also a valid level .5 (unbiased) test in this case of $H_0 : \alpha \in I_a$ versus $H_1 : \alpha \in I_b$ and of $H_0 : \alpha \in I_b$ versus $H_1 : \alpha \in I_a$.

Since $\hat{\alpha}_{MU}$ is only approximately median-unbiased when $p > 1$, the model selection procedure of (3.5) is correspondingly only approximately unbiased. In fact, simulations reported in the next section show that $\hat{\alpha}_{MU}$ is very close to being median-unbiased for several scenarios of empirical relevance. In consequence, the model selection procedure also is very close to being unbiased at least in these scenarios.

3.4. Properties of the Approximately Median-unbiased Estimators

We now state several asymptotic properties of the MU estimators. Since we have not written down a formal proof of these properties, the reader may wish to treat them as conjectures. Consider first the model (2.2) (with normal errors). All of the MU estimators $(\hat{\alpha}_{MU}, \ldots, \hat{\beta}_{MU})$ and their standard error estimators are consistent and the coverage probability of the CI $[\hat{c}_L, \hat{c}_U]$ is asymptotically correct (because the initial LS estimators are consistent). In addition, $\hat{\alpha}_{MU}$ is asymptotically median-unbiased (because it would be exactly median-unbiased if the true values $(\psi_1, \ldots, \psi_{p-1})$ were used in the bias-correction step and the values actually used differ from the true values by an asymptotically negligible amount). We conjecture that the estimators $(\hat{\psi}_{1,MU}, \ldots, \hat{\beta}_{MU})$ are close to being median-unbiased, because of the removal of the bias of the estimator of α.

Next, consider the asymptotic behavior of $(\hat{\alpha}_{MU}, \ldots, \hat{\beta}_{MU})$ when the errors in (2.2) have mean 0 and variance σ^2 but are not necessarily normally distributed. The MU estimators $(\hat{\alpha}_{MU}, \ldots, \hat{\beta}_{MU})$ and their standard error estimators (appropriately normalized) are still consistent in this case and the CI $[\hat{c}_L, \hat{c}_U]$ has the correct coverage probability asymptotically (because (i) the LS estimators are consistent, (ii) the magnitude of the bias-correction declines to 0 as $T \to \infty$, and (iii) the asymptotic distribution of the LS estimators is the same for errors with any mean 0 variance σ^2 distribution as for normal errors). In addition, $\hat{\alpha}_{MU}$ is asymptotically median-unbiased for any mean 0 variance σ^2 error distribution (for the third reason listed immediately above). These results hold whether one considers asymptotics with a fixed value of α for all T or with a sample-size dependent value of α, as in local-to-unity asymptotics.

Consider, now, the finite-sample properties of the MU estimators $(\hat{\alpha}_{MU}, \ldots, \hat{\beta}_{MU})$. The main features of these estimators that are of interest are their median-bias properties and their variability relative to the LS estimators. These properties can be assessed using the same simulation procedure as is used to generate the standard error estimates. In particular, given a data series and the corresponding observed MU estimates $(\hat{\alpha}^0_{MU}, \ldots, \hat{\beta}^0_{MU})$, the simulation procedure generates R^* random draws from the distribution of the LS and MU estimators of $(\alpha, \psi_1, \ldots, \psi_{p-1}, \mu, \beta)$. In addition, one can compute estimates of the IRF at different time horizons and of the magnitudes of the roots of the AR(p) model corresponding to both the LS and MU estimates for each repetition.

The difference between the sample median of any of these parameter estimates over the R^* repetitions and the true value gives a Monte Carlo estimate of the median-bias of the LS and MU estimators when the true parameters are $(\hat{\alpha}^0_{MU}, \ldots, \hat{\beta}^0_{MU})$. Corresponding Monte Carlo estimates of the standard deviation, root mean squared error (MSE), and interquartile range of the LS and MU estimators can be computed analogously.

Table 2 provides the results of the above simulation procedure when the true parameter values are taken to mimic those of three different series that exhibit varying degrees of persistence. The series mimicked are the Nelson-Plosser series for real GNP, GNP deflator, and consumer prices whose α values are .88, .96, and 1.0, respectively (as estimated by $\hat{\alpha}_{MU}$).

The results for the $\alpha = .88$ (real GNP) and $\alpha = .96$ (GNP deflator) cases with $p = 2$ show the following. The median-bias of the MU estimator for all estimands is essentially zero. The median-bias of the LS estimator, on the other hand, is substantial for α, μ, β, the magnitudes of the two roots, and the IRF at most time horizons. The standard deviation of the MU estimator is the same or somewhat larger than that of the LS estimator for all estimands except the IRF at long time horizons for which it is substantially larger. The root MSE of the MU estimator is noticeably smaller than that of the LS estimator for estimation of α, μ, β, and the IRF at short time horizons. It is approximately equal for ψ_1, σ^2, and the magnitudes of the roots, and is substantially larger for the IRF at long time horizons. The interquartile range of the LS estimator does not include the true value for the estimands α, μ, β, and IRF(h) $\forall h \geq 3$. On the other hand, the interquartile range of the MU estimator includes the true value and is symmetrically centered about it for these estimands. For the other estimands, the interquartile range results for the two estimators are more comparable.

Next we describe the results for the $\alpha = 1.0$ (consumer prices) case with $p = 4$. The MU estimators of $(\alpha, \psi_2, \psi_3, \mu, \beta, \sigma^2)$ are essentially median-unbiased, while that of ψ_1 has a small downward median-bias. In contrast, the LS estimators of $(\alpha, \psi_1, \psi_2, \psi_3, \mu, \beta)$ are all significantly median-biased. The MU and LS estimators of the magnitudes of the roots each have median-biases. Those of the MU estimator are smaller. The MU estimator of the IRF is downward median-biased, especially at long time horizons. Its downward bias is quite small, however, in comparison to that of the LS estimator, which is huge, especially for long time horizons. The standard deviations of the MU and LS estimators are approximately equal for all estimands except the IRF at long time horizons, for which the MU estimator has considerably larger standard deviations. The root MSE of the MU estimator is substantially smaller than that of the LS estimator for the estimands α, μ, β, the magnitudes of the two largest roots, and IRF(h) for all h. For the other estimands, the MU and LS estimators have comparable root MSEs. The length and location of the interquartile ranges of the MU and LS estimators corroborate the results based on the standard deviations and median-biases.

Central 90% CIs for α calculated as described in Section 3.2 are found to have simulated confidence levels of 88.9%, 89.7%, and 86.9% for the α = .88, .96, and 1.0 case respectively. These simulated confidence levels have standard errors of approximately .7% each. Thus, there appears to be a tendency for the CIs' coverage probabilities to be somewhat too low.

In conclusion, we find that the MU estimator achieves a substantial reduction in median-bias over the LS estimator for almost all of the estimands considered. The MU estimator is essentially median-unbiased for most of the estimands with the greatest exception being the IRF when $\alpha = 1$. The MU estimator pays a negligible to small price in terms of increased standard deviation for its improved median-bias properties, except when estimating the IRF at long time horizons, in which case the price is large. In consequence, the root MSE of the MU estimator is noticeably smaller than that of the LS estimator for many estimands, with the main exception being the IRF at long time horizons when $\alpha \leq .96$.

3.5. Properties of the Approximately Unbiased Model Selection Procedure

Here we briefly investigate the properties of the approximately unbiased model selection procedure introduced in Section 3.3. We consider the two models defined by $I_a = (-1, 1)$ and $I_b = \{1\}$. The selection rule is to choose the unit root model I_b if $\hat{\alpha}_{MU} = 1$ and otherwise to choose the trend stationary model.

Table 3 shows how the probability of selecting a unit root model varies as a function of the true parameter α for a number of AR(3) models. This probability also depends on the parameters ψ_1 and ψ_2 and on the sample size $T+p$. The α, ψ_1, ψ_2, and $T+p$ combinations considered were chosen to mimic different Nelson-Plosser data series. (That is, the true parameters listed correspond to the MU estimates for the data series listed.) The probabilities of selecting a model with $\alpha = 1$ were calculated by simulation using 1,000 repetitions. The simulation standard errors for these probabilities range from .0062 for the $\alpha = .81$ case to .017 for the $\alpha = 1.0$ case.

The first eight rows of Table 3 show the probabilities of erroneously choosing a unit root model for different α values less than 1.0. When the value of α is $\leq .95$ the probabilities are small ($\leq .20$) for the sample sizes considered. For α values closer to 1.0, the probabilities are larger. For example, for $\alpha = .97$, the probability is .44 when $T+p = 100$. The last four rows of Table 3 show the probabilities of correctly selecting a unit root model when $\alpha = 1$ for several different sample sizes. These probabilities are just above .5. They are much lower than the corresponding probabilities for a level .05 test of a unit root null hypothesis, because the unbiasedness condition precludes giving the unit root model favorable status a priori.

4. AN ASYMPTOTICALLY MEDIAN-UNBIASED ESTIMATOR OF α

A recent paper by Stock (1991) uses local-to-unity asymptotics to obtain confidence intervals for the magnitude of the largest root in model (2.2). His work builds on the local-to-unity testing results of Bobkoski (1983), Cavanagh (1985), Phillips (1987, 1988), Chan and Wei (1987), and Chan (1988) and especially on the local-to-unity CI results of Cavanagh (1985). Cavanagh (1985) considers asymptotic CIs for α in an AR(1) model without intercept or time trend. Stock (1991) extends these results to the empirically relevant case of AR(p) models with intercept and time trend.

As argued in Section 2, point or interval estimates for the magnitude of the largest root of the model (2.2) are not very useful summary measures of the persistence of a series as measured by its IRF or its spectrum at zero. In consequence, it seems worthwhile to introduce a trivial extension to Stock's methods that focusses on point and interval estimation of the parameter α, the sum of the AR coefficients, rather than on the magnitude of the largest root. The method is based on local-to-unity asymptotics and yields estimators and CIs that are easy to compute given the tables provided by Stock (1991).

In comparison with the computer-intensive methods described in Section 3, the methods considered here are very quick to compute. On the other hand, they are probably less accurate, especially when the sample size is small or α is not close to one. In addition, they do not yield estimates and standard error estimates for the wide range of estimands considered in Table 2, as does the method of Section 3. As noted in Section 3, the methods there can be given asymptotic justifications even if the errors are non-normal, just as the methods here can. Thus, there is no inherent advantage of either method with respect to robustness against non-normal errors (with several moments finite).

Our asymptotically median-unbiased estimator $\hat{\alpha}_{AMU}$ of α and central CI $[\hat{L}, \hat{U}]$ for α of asymptotic confidence level $100(1 - p_0)\%$ are defined by

$$\begin{aligned} \hat{\alpha}_{AMU} &= 1 + c_{med}\hat{b}(1)/T \quad \text{and} \\ [\hat{L}, \hat{U}] &= [1 + c_0\hat{b}(1)/T,\ 1 + c_1\hat{b}(1)/T] \ , \end{aligned} \tag{4.1}$$

where $\hat{b}(1)$ is a consistent estimator (defined below) of $b(1) = 1 - \sum_{j=1}^{p-1} \psi_j$.

The rv's c_{med}, c_0, and c_1 are determined using Stock's (1991) Table A.1 Part B as follows. Let $\hat{\tau}^{\tau}$ denote the t-statistic for testing $H_0 : \alpha = 1$ in the regression of Y_t on $(Y_{t-1}, \Delta Y_{t-1}, \ldots, \Delta Y_{t-p+1}, 1, t)$, where α is the coefficient on Y_{t-1}. (It is often convenient for computing $\hat{\tau}^{\tau}$ to note that it equals the t-statistic for testing whether the coefficient on Y_{t-1} is zero in the regression of ΔY_t on $(Y_{t-1}, \Delta Y_{t-1}, \ldots, \Delta Y_{t-p+1}, 1, t)$.) In the column labelled "Stat" in Stock's Table A.1 Part B, one finds the row corresponding to the observed value of $\hat{\tau}^{\tau}$. The value c_{med} in (4.1) is the number in the column labelled "Median" that is in the aforementioned row. The values c_0 and c_1 in (4.1) are the numbers in the columns labelled c_0 and c_1 (corresponding to the desired confidence level $100(1 - p_0)\%$ being equal to 95%, 90%, 80%, or 70%) that are in the aforementioned row. (If the model (2.2) of interest does not contain a time trend, then one computes c_{med}, c_0, and c_1 from Stock's Table A.1 Part A and one omits the time trend in the regressions used to calculate $\hat{\tau}^{\tau}$ and in the regressions described below used to calculate $\hat{b}(1)$.)

Our suggested estimator of $b(1)$ is an iterative one. Let $\hat{b}_1(1) = 1 - \sum_{j=1}^{p-1} \hat{\psi}_{j,LS1}$, where $\hat{\psi}_{j,LS1}$ is the LS estimator of ψ_j (the coefficient on ΔY_{t-j}) from the regression of Y_t on $(Y_{t-1}, \Delta Y_{t-1}, \ldots, \Delta Y_{t-p+1}, 1, t)$. Let $\hat{\alpha}_{AMU1} = 1 + c_{med}\hat{b}_1(1)/T$. Let $\hat{b}_2(1) = 1 - \sum_{j=1}^{p-1} \hat{\psi}_{j,LS2}$, where $\hat{\psi}_{j,LS2}$ is the LS estimator of ψ_j (the coefficient on ΔY_{t-j}) from the regression of $Y_t - \hat{\alpha}_{AMU1}Y_{t-1}$ on $(\Delta Y_{t-1}, \ldots, \Delta Y_{t-p+1}, 1, t)$. Let $\hat{\alpha}_{AMU2} = 1 + c_{med}\hat{b}_2(1)/T$. The estimators $\hat{b}_3(1)$, $\hat{b}_4(1)$, ... and $\hat{\alpha}_{AMU3}$, $\hat{\alpha}_{AMU4}$, ... are defined analogously to $\hat{b}_2(1)$ and $\hat{\alpha}_{AMU2}$. The estimator $\hat{b}(1)$ is then defined to equal either $\hat{b}_k(1)$ for some fixed integer

k or the limiting value of $\hat{b}_1(1)$, $\hat{b}_2(1)$, ... (provided convergence occurs). In practice, we find that $k = 2$ is sufficient to achieve convergence to within two decimal places for $\hat{b}_k(1)$ and $\hat{\alpha}_{AMUk}$ for many series, although $k = 6$ is required for one series reported below.

The asymptotic justification for $\hat{\alpha}_{AMU}$ and $[\hat{L}, \hat{U}]$ is sketched in the Appendix. It is a straightforward extension of Stock's results. Note that the use of $\hat{\alpha}_{AMU}$ and $[\hat{L}, \hat{U}]$ is appropriate only when the sample size is not "too" small and α is "near" one.

Lastly, we briefly mention a theoretical issue concerning the CI $[\hat{L}, \hat{U}]$. The CI $[\hat{L}, \hat{U}]$ for α can be used to obtain a CI for the parameter c, where $\alpha = 1 + cb(1)/T$. Since c cannot be estimated consistently, it may seem odd that one can construct a CI for c. In fact, the fact that c cannot be estimated consistently means that the length of the CI for c does not go to zero (in some probabilistic sense) as $T \rightarrow \infty$, but it does not preclude the construction of a CI for c whose coverage probability is correct asymptotically.

5. RESULTS BASED ON THE NELSON-PLOSSER DATA

In this section we apply the approximately median-unbiased (MU) estimation method to the Nelson-Plosser (NP) data series. The first subsection gives the main empirical results. The second subsection compares MU estimates to other estimates in the literature.

5.1. Empirical Results for the Nelson-Plosser Data Series

Table 4 provides the MU and LS estimates of a variety of different estimands for the fourteen NP data series. The simulated biases and standard deviations of the estimates, taking the MU estimates to be the truth, are given in parentheses below each estimate. The lag lengths (p) of the AR(p) models that are estimated are taken to be the same as in Nelson and Plosser (1982).

The results can be summarized as follows. First, consider the estimates of the key parameter α. The MU estimates of α show a dichotomy between real variables and nominal variables. The MU estimates of α for the real variables are all $\leq .92$, with all but the unemployment rate being between .88 and .92. The corresponding estimates for the nominal variables are all $\geq .94$, with three series having estimates equal to one. These results indicate strong persistence, though less than unit root persistence, for all real series except the unemployment rate. They indicate very strong persistence, equal to or bordering on unit root persistence, for each of the nominal series. The 90% CIs for α and the standard deviation estimates reveal considerable variability in the MU estimates. The unit root null hypothesis can be rejected only for the unemployment rate series. At the same time, the lower bound of the 90% CI for α is as low as .76 for a few of the series (and it is lower for the unemployment rate series).

The MU estimates of α are uniformly closer to one than the LS estimates. The differences vary but many are quite large -- around .06. The smallest difference, .026, is for money stock, but even this translates into a 45% larger CIR estimate for MU than for LS. Turning to the IRF estimates, one sees a corresponding large difference between the MU and LS estimates. The MU estimates are much larger for all series except the interest rate, for which the LS estimate exceeds one. The estimates of the median-biases of the MU and LS estimators of α and the IRF show the MU estimators to be quite close to being unbiased, while the LS estimators are strongly biased towards zero.

As argued in Section 2, it is not wise to focus too much attention on the magnitude of the largest root. Nevertheless, since this estimand has received considerable attention in the literature, we briefly summarize the results for it here. The differences between the MU and LS estimates of the magnitude of the largest root are larger and more varied than for α. For example, the differences for nominal GNP and the unemployment rate are .22 and −.02 respectively. Except for the unemployment rate, all MU estimates are closer to one than the LS estimates, often substantially so. The estimates of the bias show LS to be substantially biased away from one in many cases.

The MU estimates of β in Table 4 are quite small. They range from .0000 to .0035. The LS estimates are uniformly larger, with the exception of the unemployment rate. The LS estimates range from −.0005 to .0067. The estimates of the bias show the LS estimator to be biased upward and the MU estimator to be essentially unbiased.

Our main conclusions from Table 4 are the following. The nominal variables exhibit very high persistence. The real variables exhibit less persistence, but it is still substantial. The precision attached to these conclusions is not great, since the CIs for α are wide. The MU and LS estimates differ substantially in terms of the amount of persistence they indicate. The LS estimators of α and the IRF are strongly biased towards zero, whereas the MU estimators are essentially unbiased (except for the case of the IRF when $\alpha = 1$, in which case it too is biased towards zero but much less so than LS).

5.2. Comparison of Different Estimates Using the Nelson-Plosser Data

Here we compare the MU and LS estimates for the NP data series with other estimates given in the literature including those of Rudebusch (1992), DeJong and Whiteman (1991a), and Phillips (1991a). In addition, we make comparisons with estimates given by Stock's (1991) asymptotically median-unbiased estimator of the magnitude of the largest root (although not with results actually reported in Stock (1991)). We also make comparisons with the asymptotically median-unbiased (AMU) estimator of α considered in Section 4 above. Since different authors use different lag lengths p, comparisons across all methods are not always possible. Rudebusch (1992) uses the Nelson and Plosser (1982) choice of p. DeJong and Whiteman (1991a) and Phillips (1991a) use $p = 3$. Also, different authors choose to report different estimands. Rudebusch (1992) gives estimates of $\gamma_1, \ldots, \gamma_p$, from which an estimate of $\alpha = \sum_{j=1}^{p} \gamma_j$ can be obtained. DeJong and Whiteman (1991a) report only estimates of the magnitude of the largest root, Λ, and the time trend parameter β. Phillips (1991a) reports only estimates of α. First we compare the MU estimates with those of Rudebusch for models with the Nelson-Plosser choices of p. Next, we consider all models with $p = 3$ and compare the MU estimates with those of LS, DeJong and Whiteman (1991a) (DW), Phillips (1991a) (Ph), and AMU.

Rudebusch's (1992) estimation method has been described briefly in the Introduction. DeJong and Whiteman's (1991a) estimators of Λ and β are Bayesian posterior means where the prior is chosen to be uniform over the AR coefficients $\gamma_1, \ldots, \gamma_3$ and over the time trend parameter β subject to the restriction that $\Lambda \in [.55, 1.055]$ and $\beta \in [.000, .016]$. Phillips' (1991a) estimator is defined here to be the posterior median of his posterior distributions for α obtained using the Jefferies prior and some analytic approximations. (Phillips does not report posterior medians. The Ph estimates reported in Table 5 are obtained by eyeballing Phillips' posterior distributions given in his Figure 4. In consequence, these estimates are subject to (our own) computational error.)

Summing Rudebusch's (1992) estimates of $\gamma_1, \ldots, \gamma_p$ yields the following estimates of α. We give the Rudebusch estimate first and the MU estimates second for the series as ordered in Table 4 (with the

NP choice of p): (.898, .885), (.946, .958), (.882, .875), (.919, .919), (.900, .914), (.773, .765), (.968, .960), (.985, 1.00), (.974, .970), (.913, .896), (.947, .942), (.995, 1.00), (.984, 1.00), and (.984, .970). (Rudebusch does not provide any measure of the variability of his estimates, so none can be given here.) Overall, the differences are small. They vary from .000 for industrial production to .017 for real wages. Thus, the Rudebusch estimates are much closer to the MU estimates than to the LS estimates. This is to be expected, because the MU and Rudebusch methods are quite similar.

Next, we turn to comparisons of MU, LS, DW, Ph, and AMU estimates for AR(3) models, see Table 5. Results are reported for estimates of α, β, and the magnitude of the largest root Λ. Bias and standard deviation estimates are provided in parentheses beside each of the MU and LS estimates. These were obtained by the simulation method outlined in Section 3.1. The standard deviation of the posterior distribution of Λ is provided in parentheses beside the DW estimates. Asymptotic 90% central CIs for α and Λ are provided in square brackets beside each estimate for the AMU estimator. The CI for Λ is as defined by Stock (1991); that for α is as defined in Section 4.

First, we summarize the results for the main parameter of interest α. The comparison between the MU and LS estimates is quite similar to that in Table 4. The MU estimates are uniformly closer to one than the LS estimates. The differences between the two estimates range from .02 to .09. These differences correspond to MU estimates of the CIR that are from 38% to ∞% larger than those of the LS estimates.

The Ph estimates are slightly larger (i.e., larger by .01 or .02) than the LS estimates for all series except the industrial production, velocity, and interest rate series. For the latter two, the Ph estimates are much larger. The latter two are the series with the largest LS estimates. For these series, the Ph estimates are much larger than the LS estimates because of the large weight that the Jefferies prior puts on $\alpha > 1$. Since the Ph estimates of α are just slightly larger than the LS estimates for most series, the MU estimates are noticeably larger than the Ph estimates for most series.

The MU and AMU estimates of α are quite similar. The differences are between .00 and .02. The differences in the lower bounds of the MU and AMU CIs for α also are fairly small in most cases, although they differ by .03 for real wages.

Next, we compare estimates of the magnitude of the largest root Λ. Although Λ is not a parameter of great interest by itself, as argued in Section 2, these comparisons indicate whether the differences between the MU estimates and other estimates in the literature, such as those of DeJong and Whiteman (1991a), are due to differences in the methods employed or just to the choice of estimand considered. The differences between the MU and DW estimates of Λ are very large. They range from $-.10$ to $.14$, with most being in the .07 to .10 range. The MU estimates are usually significantly closer to one than the DW estimates, but not always. For many cases the bias of the MU estimator is small, though for a few cases it is large. In each case where it is large, the DW estimate is in the direction of the bias relative to the MU estimate, which suggests that the DW estimator is more biased than the MU estimator. The LS and DW estimates of Λ are closer together than the MU and DW estimates are, but there still are noticeable differences. Unlike the estimates of α, the MU and AMU estimates of Λ differ noticeably for a few series.

Lastly, we compare estimates of the time trend parameter β. The LS and DW estimates of β are almost the same. The MU estimates are noticeably closer to zero than the LS and DW estimates. The difference between the MU and LS estimates of β are approximately the same as the upward bias of the LS estimator. The MU estimator of β is essentially unbiased. One might conjecture based on these results that the DW estimates of β have an upward bias roughly equal to that of the LS estimates. Bias-correction of the LS and DW estimates, then, would yield estimates approximately equal to the MU estimates.

Overall, the results of Table 5 lead to the following conclusions. There are noticeable differences between the MU and AMU estimates on one hand and the LS, DW, and Ph estimates on the other. The former show considerably greater persistence for most of the series than the latter. The differences can be attributed to the fact that the MU and AMU estimators of α and Λ are not biased towards zero and those of β are not biased away from zero.

6. EMPIRICAL RESULTS FOR THE EXTENDED NELSON-PLOSSER DATA

Table 6 presents MU and LS estimates of numerous parameters for each of the series in the extended Nelson-Plosser (ENP) data set compiled by Schotman and van Dijk (1991). In the table, simulated estimates of the biases and standard deviations of the estimators, computed using the MU estimates as the truth, are given in parentheses below each estimate. As in Section 5.1, the lag lengths (p) of the AR(p) models that are estimated are taken to be the same as in Nelson and Plosser (1982). This choice is made because it facilitates comparison with the results of Section 5.1 and because an analysis of the residuals of the estimated models did not provide evidence that the NP lag lengths are inappropriate. (The only exception is some weak evidence that a longer lag length than $p = 1$ may be appropriate for velocity.) Of course, a data dependent method of choosing p may very well choose different lag lengths.

Eight of the fourteen MU estimates of α equal 1.0. All of the nominal variables have an MU estimate of α equal to 1.0 except money stock, whose estimate is .96. Real wages is the only real variable for which the MU estimate of α is 1.0. The other real variables, except the unemployment rate, have MU estimates of α in the range of .86 to .91. The unemployment rate has the lowest estimate of α; it is .76. The 90% CIs for α for the nominal variables are relatively short with the lower bound ranging from .91 to 1.0. The 90% CIs for α for the real variables are noticeably longer ranging in length from .18 to .25. The null hypothesis of a unit root ($\alpha = 1$) can be rejected at a 5% level using a one-sided test for three of the series: real GNP, real per capita GNP, and the unemployment rate.

The MU estimates of α are substantially closer to 1.0 than are the LS estimates. The range of differences is .02 to .07. These differences are due to the downward median-bias of the LS estimator.

The MU estimates of the time trend parameter β are fairly small. The LS estimates of β are larger than the MU estimates for every series except the unemployment rate. The bias of the MU estimator of β is essentially zero. In contrast, the LS estimators of β are upward biased by approximately the amount that the LS estimates exceed the MU estimates.

The unbiased model selection rule introduced in Section 3.3 says to choose a unit root model if the MU estimator is 1.0 and to choose a trend stationary model if the MU estimator is less than 1.0. This rule selects eight series as being unit root models and six as being trend stationary. The unit root models include all nominal variables except money stock, plus real wages.

Now we compare the MU and LS estimates for the ENP series with those for the NP series. For convenience in making these comparisons, the last two columns of Table 6 provide the estimates of α for the NP data. The biggest changes occur with the real wage series. The LS and MU estimates increase enormously when the new data is added from .83 and .89 to .93 and 1.0 respectively. The 90% CI for α shrinkages in length from .22 to .09. The graph of real wages is flat over the period of new data 1971-1988, whereas it increases throughout the period of the NP data 1900-1970. The next largest changes occur for the nominal GNP, GNP deflator, nominal wages, and common stock price series. The LS and MU estimates for each of these series increased by .03 or .04 with the MU estimates going from .96 or .97 to 1.0 in each case. The lengths of the 90% CIs for α for these series shrink from .14 to .07, .11 to .03, .12 to .08, and .12 to .09. For the first three of these series, the graphs of the series show a steeper slope (presumably due to increased inflation) over the new period of data 1971-1988 than previously.

Next, the interest rate series shows a large drop in the LS estimate of α from 1.03 to .95 with addition of the new data, but the MU estimate stays constant at 1.0. The real GNP and real per capita GNP series show little or no change in the LS and MU estimates of α, but the increased precision due to the addition of data allows one to reject the null hypothesis that $\alpha = 1$ with the GNP data, whereas one cannot reject this hypothesis with the NP data. The consumer price and velocity series exhibit a small increase in the LS estimates of α by .02, but no change in the MU estimates of α which equal 1.0 with both data sets. The industrial production, employment, and unemployment rate series show little or no change in the parameter estimates between the two data sets.

7. EMPIRICAL RESULTS FOR THE STOCK DIVIDEND AND PRICE DATA

In this section, we present empirical results for the stock dividend and price data referred to in the Introduction. We use an AR(3) model for each series, as in DeJong and Whiteman (1991b). This choice is made for comparative purposes and because residual analysis did not indicate that this choice is inappropriate.

Table 7 presents MU and LS estimates of a variety of estimands for the stock market data series. In addition, the DW posterior mean estimates of the magnitude Λ of the largest root and the coefficient β on the time trend are provided. Bias and standard deviation estimates for the MU and LS estimators (computed using the simulation method outlined in Section 3.1 and taking the MU estimates as the truth) are given in parentheses below each estimate. The standard deviations of the posterior distributions of Λ and β are given in parentheses below the DW estimates of these parameters.

We now discuss the results of Table 7. Four of the six series show considerable persistence; two show noticeably less persistence. In particular, DJ prices, NYSE dividends and prices, and S&P prices all have MU estimates of α equal to .90 or greater, whereas DJ dividends and S&P dividends have MU estimates of α equal to .79 and .82 respectively. Only NYSE prices have an MU estimate of α equal to 1.0. Thus, the unbiased model selection procedure of Section 3.3 chooses a unit root model for NYSE prices and trend stationary models for all other series.

The 90% central CIs for α are extremely wide for the DJ dividend and price and S&P dividend series with widths of .45, .31, and .28 respectively. The CIs for α for the NYSE price and S&P price and dividend series are also wide, but much less so, with widths of .19, .21, and .16. The principal explanation for the excessively wide CIs is the small number of observations ($T+p$) for the DJ and NYSE series, viz., 51 and 55, respectively.

The LS estimates of α and of the IRF are much smaller than the MU estimates, especially for the DJ and NYSE series. The differences in LS and MU estimates of α for these series range from .10 to .21, which are very large. In all cases, the LS estimates are closer to zero than the MU estimates. This is due to the downward bias of the LS estimators, which is particularly large for small sample sizes. Given these biases, we do not believe the LS estimates give impartial estimates of the amount of persistence in the series, as measured by α or by the IRF. The MU estimates of α and the IRF, on the other hand, are essentially median-unbiased in most cases. Hence, they provide a more objective estimate of the amount of persistence.

The MU estimates of α and of the magnitude Λ of the largest root are approximately the same for each series except S&P dividends. The same is true of the LS estimates. In consequence, for five of these series, the magnitude of the largest root can be given an interpretation related to the persistence of the series.

Comparing the DW estimates of Λ with those of LS, we find that the DW and LS estimates are approximately equal for all series except DJ dividends and S&P dividends. Comparing the DW estimates of Λ with the MU estimates, we find that the DW estimates are uniformly smaller than the MU estimates. The differences for the six series are .07, .16, .13, .16, .05, and .07, which are substantial. Thus, the MU estimates indicate considerably greater persistence in the series than the DW estimates do. The explanation for the differences is the difference in the bias properties of the MU and DW estimators.

We conclude that the MU estimates differ noticeably from the LS and DW estimates. Of the point estimates given, we believe the MU estimates of α and the IRF to be the most informative regarding persistence, since they are approximately median-unbiased. The interval estimates for α also are quite informative, since they make clear that the level of uncertainty about the "true" values of α is quite high. The MU estimates of α indicate a high degree of persistence for four of the six series and a lesser degree for two series. One of the six series is estimated to have a unit root and five are estimated to be trend stationary.

APPENDIX

A.1. Invariance of $\hat{\alpha}_{LS}$

First we establish the claim made in Section 3.1 that $\hat{\alpha}_{LS}$ has distribution that does not depend on $(\mu^*, \beta^*, \sigma^2)$ or when $\alpha = 1$ on Y^*_{-p+1}. In fact, we will show that these invariance properties hold for the distribution of $(\hat{\alpha}_{LS}, \hat{\psi}_{1,LS}, \ldots, \hat{\psi}_{p-1,LS})$.

Consider successive regressions of $Y_t, Y_{t-1}, \Delta Y_{t-1}, \ldots, \Delta Y_{t-p+1}$ on $(1, t)$ for $t = 1, \ldots, T$. Then $(\hat{\alpha}_{LS}, \hat{\psi}_{1,LS}, \ldots, \hat{\psi}_{p-1,LS})$ equals the LS estimator from the regression of the residuals from the regression with Y_t as dependent variable on the vector of residuals from the regressions with $Y_{t-1}, \Delta Y_{t-1}, \ldots, \Delta Y_{t-p+1}$ as dependent variables. Since $Y_t = \mu^* + \beta^* t + Y^*_t$ by (2.1), all of the residuals above are invariant with respect to (μ^*, β^*). In consequence, the distribution of $(\hat{\alpha}_{LS}, \ldots, \hat{\psi}_{p-1,LS})$ is invariant with respect to (μ^*, β^*).

Given this invariance, we can suppose $\mu^* = \beta^* = 0$ and $Y_t = Y^*_t$ in the remainder of the proof. Multiplication of σ^2 by a positive constant c in (2.1) causes Y^*_t and Y_t to be multiplied by the same constant c for $t = -p+1, \ldots, T$ when $\alpha \in (-1, 1)$ (using the fact that stationarity of $\{Y_t : t \geq -p+1\}$ requires that the initial rv's $Y^*_{-p+1}, \ldots, Y^*_0$ are scaled by the same constant c). In consequence, the residuals from the regressions of $Y_t, \ldots, \Delta Y_{t-p+1}$ on $(1, t)$ are multiplied by the same constant. This constant cancels out in the expression for the LS estimator $(\hat{\alpha}_{LS}, \ldots, \hat{\psi}_{p-1,LS})$ given by the regression of the residuals from Y_t on those from $(Y_{t-1}, \ldots, \Delta Y_{t-p+1})$. Thus, the distribution of $\hat{\alpha}_{LS}$ is invariant with respect to σ^2 when $\alpha \in (-1, 1)$.

Now suppose $\alpha = 1$. We can always write $Y^*_t = Y^*_{-p+1} + \sum_{s=-p+2}^{t} \Delta Y^*_s$. By assumption, when $\alpha = 1$, $\{\Delta Y^*_t : t \geq -p+2\}$ is stationary with level that is arbitrary. That is, a change in Y^*_{-p+1} has no effect on $\{\Delta Y^*_t : t \geq -p+2\}$. In consequence, since $Y_t = Y^*_t$, the residuals from the regressions of $Y_t, \ldots, \Delta Y_{t-p+1}$ on $(1, t)$ are invariant with respect to the value of Y^*_{-p+1} and $(\hat{\alpha}_{LS}, \ldots, \hat{\psi}_{p-1,LS})$ is likewise. Given this invariance, suppose $Y^*_{-p+1} = 0$. Then, the multiplication of σ^2 by a constant c causes ΔY^*_t, Y^*_t, ΔY_t, and Y_t to be scaled by the same constant. As above, this leaves $(\hat{\alpha}_{LS}, \ldots, \hat{\psi}_{p-1,LS})$ unchanged. The proof is now complete.

A.2. Asymptotic Properties of $\hat{\alpha}_{AMU}$ and $[\hat{L}, \hat{U}]$

Next we consider the asymptotic justification for $\hat{\alpha}_{AMU}$ and $[\hat{L}, \hat{U}]$. We use the same model and assumptions as Stock (1991). The parameter α in his notation is $\alpha(1) = 1 + cb(1)/T$, where c is a constant and $b(1) = 1 - \sum_{j=1}^{p-1} \psi_j$. Equation (5) of Stock (1991) gives the local-to-unity asymptotic distribution of the statistic $\hat{\tau}^{\tau}$. This distribution depends only on c. Let $f_{\ell,p}(c)$ and $f_{u,p}(c)$ denote the lower and upper p quantiles of this distribution.

Consider the following CI for α:

$$\text{(A.1)} \qquad \hat{CI} = \{\alpha : \alpha = 1 + \tilde{c}\hat{b}(1)/T \text{ and } f_{\ell,p_\ell}(\tilde{c}) \le \hat{\tau}^{\tau} \le f_{u,p_u}(\tilde{c})\} .$$

This CI has asymptotic confidence level $100(1 - p_\ell - p_u)\%$:

$$\text{(A.2)} \qquad \begin{aligned} & P_{\alpha_T}(\alpha_T \in \hat{CI}) \\ &= P_{\alpha_T}[f_{\ell,p_\ell}(\tilde{c}) \le \hat{\tau}^{\tau} \le f_{u,p_u}(\tilde{c}) \text{ for } \tilde{c} \text{ defined by } \alpha_T = 1 + \tilde{c}\hat{b}(1)/T] \\ &= P_{\alpha_T}[f_{\ell,p_\ell}(cb(1)/\hat{b}(1)) \le \hat{\tau}^{\tau} \le f_{u,p_u}(cb(1)/\hat{b}(1))] \\ &\to 1 - p_\ell - p_u \text{ as } T \to \infty , \end{aligned}$$

where $\alpha_T = 1 + cb(1)/T$ and $P_{\alpha_T}(\cdot)$ denotes the probability measure when α_T is the true value of α. The convergence to $1 - p_\ell - p_u$ above uses the fact that $f_{\ell,p}(c)$ and $f_{u,p}(c)$ are continuous functions of c and $\hat{\tau}^{\tau}$ has absolutely continuous limit distribution.

Let $f_{\ell,p}^{-1}(y) = \sup\{c : f_{\ell,p}(c) \le y\}$ and $f_{u,p}^{-1}(y) = \inf\{c : f_{u,p}(c) \ge y\}$. If $f_{\ell,p_\ell}(c)$ and $f_{u,p_u}(c)$ are monotone increasing functions of c, then $f_{\ell,p_\ell}(\tilde{c}) \le \hat{\tau}^{\tau} \le f_{u,p_u}(\tilde{c})$ iff $1 + f_{u,p_u}^{-1}(\hat{\tau}^{\tau})\hat{b}(1)/T \le 1 + \tilde{c}\hat{b}(1)/T \le 1 + f_{\ell,p_\ell}^{-1}(\hat{\tau}^{\tau})\hat{b}(1)/T$. In this case, $\hat{CI} = [\hat{L}, \hat{U}]$ with $c_0 = f_{u,p_u}^{-1}(\hat{\tau}^{\tau})$ and $c_1 = f_{\ell,p_\ell}^{-1}(\hat{\tau}^{\tau})$ and $[\hat{L}, \hat{U}]$ is an asymptotically valid $100(1 - p_\ell - p_u)\%$ CI for α. If $f_{\ell,p_\ell}(c)$ or $f_{u,p_u}(c)$ is not everywhere monotone increasing in c, then $\hat{CI} \subset [\hat{L}, \hat{U}]$ and $[\hat{L}, \hat{U}]$ is an asymptotically valid CI for α with confidence level $\ge 100(1 - p_\ell - p_u)\%$. In fact, $f_{\ell,p_\ell}(c)$ and $f_{u,p_u}(c)$ are almost, but not quite, monotone in c, see Stock's (1991) Figure 2. In consequence, $[\hat{L}, \hat{U}]$ has asymptotic significance level just slightly above $100(1 - p_\ell - p_u)\%$. (To obtain a CI with asymptotic confidence level *exactly* $100(1 - p_\ell - p_u)\%$, if this precision is deemed necessary for some reason, one can use $\hat{CI}$ defined above in conjunction with Stock's Figure 2.)

Furthermore, if $f_{\ell,\,1/2}(c)$ is monotone increasing in c, then for (p_ℓ, p_u) equal to (0, 1/2) and (1/2, 1) the two corresponding $\hat{CI}$ CIs are of the form $[\hat{\alpha}_{AMU}, \infty)$ and $(-\infty, \hat{\alpha}_{AMU}]$, respectively. These CIs have the property that their probabilities of covering the true α are both 1/2 asymptotically. In consequence, $\hat{\alpha}_{AMU}$ is asymptotically median-unbiased. In fact, $f_{\ell,\,1/2}(c)$ is not quite monotone increasing in c, see Stock's (1991) Figure 2. The extent of non-monotonicity is sufficiently small that $\hat{\alpha}_{AMU}$ is very close to being asymptotically median-unbiased (close enough for practical purposes), although it is not exactly so. Furthermore, the small region where non-monotonicity occurs is just above $\alpha = 1$, so if one restricts the parameter space to be $(-1, 1]$, then this problem disappears.

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Table 1. Comparison of Impulse Response Functions for Several Pairs of Models

Data Series Mimicked	Order of AR Model	Magnitude of Largest Root	α	Magnitudes of Other Roots	$\psi_1, \psi_2, \ldots$	Impulse Response Function										
						1	2	3	4	5	7	10	15	20	25	30
Industrial Production-NP	6	.95	.92	.80,.80,.78, .78,.70	.05,-.08,.01, -.08,-.26	1.0	.8	.7	.6	.3	.3	.4	.2	.2	.2	.1
Nominal Wages-NP	3	.94	.96	.36,.36	.53,-.12	1.5	1.6	1.5	1.4	1.3	1.2	.9	.7	.5	.4	.3
Unemployment Rate-ENP	3	.81	.80	.50,.50	.22,-.20	1.0	.6	.4	.4	.3	.2	.1	.04	.01	.01	.00
Money Stock-ENP	2	.81	.96	.81	.65	1.6	1.9	2.1	2.1	2.0	1.7	1.1	.5	.2	.06	.02
Unemployment Rate-ENP	4	.72	.76	.72,.65,.65	.36,-.23,.22	1.1	.7	.5	.5	.3	.07	-.03	-.01	.00	.00	.00
Real GNP-ENP	2	.64	.86	.62	.39	1.3	1.2	1.0	.8	.6	.3	.1	.02	.00	.00	.00
						1	3	5	7	9	11	13	15	17	19	21
Industrial Production-NP	3	.86	.87	.24,.24	.10,-.05	1.0	.7	.5	.4	.3	.2	.2	.1	.1	.06	.05
Real GNP-NP	3	.69	.87	.60,.02	.39,.01	1.3	1.0	.6	.4	.2	.1	.05	.02	.01	.01	.00

Table 2. Properties of the Median-unbiased (MU) Estimators for Various Parameters

	α	ψ_1	ψ_2	ψ_3	μ	$100\times\beta$	$100\times\sigma^2$	Magnitude of Roots in Descending Order				Impulse Response Function									
								1	2	3	4	1	2	3	4	5	10	15	20	25	30
Data Series Mimicked: Real GNP–NP $T+p = 62$																					
True Parameter Values	.88	.39			.54	.39	.35	.75	.52			1.27	1.22	1.06	.87	.69	.18	.04	.01	.00	.00
Median Bias of MU	−.001	−.004			.00	.00	.00	.01	.00			−.01	−.01	.00	.00	−.01	−.02	−.01	.00	.00	.00
Median Bias of LS	−.061	.026			.27	.19	−.01	−.05	.12			−.04	−.13	−.21	−.26	−.30	−.18	−.04	−.01	.00	.00
Standard Deviation of MU	.079	.12			.35	.26	.06	.12	.19			.12	.22	.29	.36	.41	.45	.42	.41	.41	.41
Standard Deviation of LS	.069	.12			.31	.24	.06	.08	.16			.13	.22	.27	.28	.28	.15	.06	.04	.02	.02
Root MSE of MU	.079	.12			.35	.26	.06	.13	.19			.12	.22	.29	.36	.41	.47	.47	.46	.45	.44
Root MSE of LS	.10	.12			.44	.31	.06	.10	.17			.14	.25	.34	.39	.41	.20	.07	.03	.03	.00
Interquartile Range of MU	.83,.93	.29,.47			.32,.78	.23,.57	.31,.39	.69,.88	.35,.67			1.18,1.34	1.06,1.36	.86,1.24	.63,1.12	.42,1.0	.01,.51	.00,.26	.00,.14	.00,.08	.00,.04
Interquartile Range of LS	.77,.86	.33,.49			.63,1.03	.45,.74	.30,.38	.65,.76	.49,.70			1.14,1.31	.95,1.25	.68,1.02	.42,.78	.19,.58	−.04,.10	.00,.03	.00,.01	.00,.00	.00,.00
Data Series Mimicked: GNP Deflator–NP $T+p = 82$																					
True Parameter Values	.96	.44			.12	.11	.21	.92	.47			1.40	1.52	1.51	1.45	1.36	.93	.63	.42	.28	.19
Median Bias of MU	.000	−.01			.00	−.01	.00	−.001	−.003			−.01	−.02	−.02	−.01	−.01	.01	.00	−.01	−.01	−.01
Median Bias of LS	−.047	.01			.12	.13	.00	−.12	.10			−.04	−.13	−.22	−.31	−.39	−.58	−.52	−.39	−.27	−.19
Standard Deviation of MU	.048	.10			.13	.13	.04	.10	.14			.11	.21	.29	.36	.42	.63	.71	.74	.76	.78
Standard Deviation of LS	.044	.10			.12	.13	.03	.088	.15			.11	.21	.28	.32	.34	.33	.24	.18	.15	.12
Root MSE of MU	.048	.10			.13	.13	.04	.11	.14			.11	.21	.29	.36	.42	.63	.74	.82	.88	.92
Root MSE of LS	.070	.10			.19	.20	.03	.15	.17			.12	.25	.35	.44	.52	.62	.50	.36	.26	.18
Interquartile Range of MU	.93,1.0	.36,.49			.02,.21	.00,.20	.19,.24	.83,1.0	.38,.58			1.31,1.46	1.36,1.63	1.29,1.67	1.18,1.67	1.04,1.65	.47,1.54	.19,1.52	.07,1.52	.03,1.52	.01,1.52
Interquartile Range of LS	.88,.94	.38,.51			.17,.33	.17,.32	.19,.23	.73,.87	.45,.70			1.28,1.43	1.26,1.54	1.11,1.48	.92,1.36	.73,1.22	.14,.61	.01,.30	.00,.15	.00,.08	.00,.04
Data Series Mimicked: Consumer Prices–NP $T+p = 111$																					
True Parameter Values	1.0	.71	−.29	.08	.005	.00	.19	1.0	.45	.42	.42	1.71	1.92	1.94	1.95	1.97	1.98	1.98	1.98	1.98	1.98
Median Bias of MU	.000	−.02	.00	.00	.000	.00	.00	.00	.13	.09	−.04	−.04	−.09	−.15	−.21	−.25	−.40	−.52	−.55	−.56	−.56
Median Bias of LS	−.047	−.02	.02	.02	.000	.04	−.01	−.11	.19	.09	.00	−.08	−.19	−.32	−.44	−.57	−1.13	−1.53	−1.73	−1.85	−1.91
Standard Deviation of MU	.030	.09	.12	.10	.013	.04	.03	.07	.11	.10	.18	.10	.20	.29	.37	.44	.66	.79	.85	.89	.92
Standard Deviation of LS	.031	.09	.12	.10	.024	.06	.03	.07	.13	.11	.17	.10	.20	.28	.36	.41	.50	.48	.43	.40	.39
Root MSE of MU	.037	.09	.12	.10	.013	.05	.03	.09	.18	.13	.19	.11	.22	.33	.43	.51	.82	1.00	1.10	1.17	1.21
Root MSE of LS	.061	.10	.12	.10	.024	.08	.03	.14	.24	.13	.18	.12	.28	.43	.56	.69	1.22	1.51	1.66	1.75	1.80
Interquartile Range of MU	.96,1.0	.63,.75	−.37,−.21	.01,.14	.000,.010	.00,.03	.17,.20	.92,1.0	.51,.65	.43,.57	.19,.49	1.61,1.74	1.69,1.96	1.59,1.99	1.48,1.99	1.40,2.00	.99,1.97	.66,1.97	.44,1.97	.28,1.96	.18,1.96
Interquartile Range of LS	.93,.97	.63,.75	−.35,−.19	.03,.16	−.01,.02	.01,.08	.16,.20	.83,.93	.54,.76	.43,.57	.24,.51	1.57,1.70	1.59,1.86	1.42,1.82	1.28,1.76	1.13,1.69	.50,1.18	.18,.85	.05,.58	.02,.41	.01,.28

Table 3. Properties of the Approximately Unbiased Model Selection Procedure

True Parameters				Probability of Selecting a Model with $\alpha = 1$	Data Series Mimicked (using NP data)
α	ψ_1	ψ_2	$T+p$		
.81	.21	−.20	81	.04	Unemployment Rate
.87	.10	−.05	111	.04	Industrial Production
.87	.39	.01	62	.12	Real GNP
.89	.23	−.02	71	.20	Real Wages
.92	.39	−.11	81	.19	Employment
.95	.70	−.08	82	.19	Money Stock
.96	.42	.05	82	.34	GNP Deflator
.97	.27	−.18	100	.44	Common Stock Prices
1.0	.74	−.27	111	.56	Consumer Prices
1.0	.10	−.05	102	.55	Velocity
1.0	.18	.37	71	.59	Interest Rate
1.0	.50	−.14	62	.54	Nominal GNP

Table 4. Median-unbiased (MU) and Least Squares (LS) Estimates for the Nelson-Plosser Data Series

Data Series	Esti-mator	α	90% CI for α	ψ_1	ψ_2	ψ_3	μ	$100\times\beta$	$100\times\sigma^2$	Magnitude of Roots in Descending Order: 1	2	3	4
Real GNP (1909-1970, $T+p=62$)	MU	.885 (.00,.08)	[.77,1.0]	.39 (.00,.12)			.54 (.00,.35)	.38 (.00,.26)	.35 (.00,.06)	.77 (.01,.12)	.31 (.00,.19)		
	LS	.825 (−.06,.07)		.42 (.03,.12)			.82 (.27,.31)	.56 (.19,.24)	.34 (−.01,.06)	.65 (−.05,.08)	.65 (.12,.16)		
Nominal GNP (1909-1970, $T+p=62$)	MU	.958 (.00,.06)	[.86,1.0]	.46 (−.01,.12)			.45 (−.05,.61)	.25 (−.04,.38)	.78 (.01,.15)	.92 (.01,.12)	.50 (−.01,.16)		
	LS	.899 (−.06,.06)		.49 (.01,.12)			1.06 (.62,.58)	.56 (.38,.38)	.76 (−.02,.14)	.70 (−.15,.09)	.70 (.14,.16)		
Real per Capita GNP (1909-1970, $T+p=62$)	MU	.875 (.00,.08)	[.76,1.0]	.38 (.00,.12)			.88 (.00,.54)	.25 (.00,.16)	.35 (.00,.07)	.74 (.02,.12)	.52 (.00,.19)		
	LS	.818 (−.06,.07)		.41 (.03,.12)			1.28 (.43,.47)	.35 (.12,.15)	.35 (−.01,.07)	.64 (−.06,.08)	.64 (.13,.16)		
Industrial Production* (1860-1970, $T+p=111$)	MU	.919 (.00,.10)	[.78,1.0]	.05 (.00,.11)	−.08 (−.01,.11)	.01 (.01,.10)	.09 (−.01,.03)	.32 (.00,.40)	.96 (.00,.14)	.94 (.00,.07)	.80 (.01,.06)	.80 (.01,.06)	.78 (−.01,.07)
	LS	.835 (−.09,.09)		.10 (.05,.10)	−.04 (.04,.10)	.04 (.05,.10)	.10 (.03,.05)	.67 (.36,.38)	.95 (−.02,.13)	.88 (−.07,.06)	.79 (.01,.06)	.79 (−.01,.07)	.77 (−.02,.07)
Employment (1890-1970, $T+p=81$)	MU	.914 (.00,.07)	[.81,1.0]	.39 (.00,.11)	−.11 (.00,.11)		.88 (.00,.74)	.13 (.00,.11)	.13 (.00,.02)	.88 (−.01,.12)	.35 (.09,.14)	.33 (.00,.17)	
	LS	.861 (−.06,.06)		.41 (.01,.10)	−.07 (.05,.11)		1.42 (.42,.65)	.21 (.09,.10)	.12 (.00,.02)	.76 (−.12,.11)	.31 (.14,.15)	.31 (−.07,.18)	
Unemploy-ment Rate (1890-1970, $T+p=81$)	MU	.765 (.00,.10)	[.62,.89]	.35 (.00,.12)	−.23 (.00,.11)	.22 (−.01,.12)	.40 (.00,.21)	.02 (.0,.2)	16.7 (.1,2.9)	.72 (.08,.10)	.72 (.04,.10)	.65 (−.03,.11)	.65 (−.07,.20)
	LS	.706 (−.06,.10)		.38 (.03,.11)	−.20 (.03,.11)	.24 (.02,.11)	.51 (.10,.21)	−.05 (.0,.3)	16.5 (.1,2.9)	.74 (.08,.08)	.74 (.07,.09)	.63 (−.02,.11)	.60 (−.04,.16)
GNP Deflator (1889-1970, $T+p=82$)	MU	.960 (.00,.05)	[.89,1.0]	.44 (−.01,.10)			.12 (.00,.13)	.11 (−.01,.13)	.22 (.00,.04)	.92 (.00,.10)	.47 (.00,.14)		
	LS	.915 (−.05,.04)		.45 (.01,.10)			.26 (.12,.12)	.21 (.00,.13)	.21 (.00,.03)	.80 (−.12,.09)	.56 (.10,.15)		

Data Series	Esti-mator	Impulse Response Function: 1	2	3	4	5	10	15	20	25
Real GNP (1909-1970, $T+p=62$)	MU	1.27 (−.01,.12)	1.24 (−.01,.22)	1.08 (.00,.29)	.89 (.00,.36)	.72 (−.01,.41)	.21 (−.02,.45)	.06 (−.01,.42)	.02 (.00,.41)	.00 (.00,.41)
	LS	1.24 (−.04,.13)	1.13 (−.13,.22)	.88 (−.21,.27)	.62 (−.26,.28)	.41 (−.30,.28)	.00 (−.18,.15)	−.01 (−.04,.06)	.00 (−.01,.04)	.00 (.00,.02)
Nominal GNP (1909-1970, $T+p=62$)	MU	1.42 (−.02,.13)	1.55 (−.05,.25)	1.54 (−.06,.36)	1.48 (−.04,.45)	1.39 (−.02,.54)	.92 (.06,.75)	.60 (.06,.82)	.39 (.05,.85)	.25 (.06,.87)
	LS	1.39 (−.07,.13)	1.44 (−.18,.25)	1.32 (−.31,.34)	1.13 (−.43,.39)	.92 (−.52,.42)	.23 (−.71,.34)	.04 (−.55,.24)	.00 (−.37,.18)	.00 (−.24,.14)
Real per Capita GNP (1909-1970, $T+p=62$)	MU	1.26 (−.01,.12)	1.20 (−.02,.21)	1.03 (−.01,.29)	.83 (−.01,.34)	.65 (.00,.38)	.16 (−.01,.40)	.04 (.00,.38)	.01 (.00,.37)	.00 (.00,.36)
	LS	1.23 (−.05,.13)	1.10 (−.13,.22)	.84 (−.21,.26)	.59 (−.27,.27)	.37 (−.30,.26)	.00 (−.16,.13)	.00 (−.04,.06)	.00 (−.01,.03)	.00 (.00,.02)
Industrial Production* (1860-1970, $T+p=111$)	MU	.97 (−.01,.10)	.80 (−.03,.15)	.74 (−.02,.18)	.61 (−.02,.21)	.30 (−.01,.23)	.42 (.01,.24)	.24 (.00,.25)	.21 (.00,.27)	.15 (.00,.27)
	LS	.93 (−.05,.10)	.73 (−.10,.15)	.64 (−.12,.17)	.48 (−.16,.18)	.16 (−.16,.20)	.22 (−.21,.15)	.07 (−.19,.10)	.05 (−.16,.08)	.02 (−.13,.07)
Employment (1890-1970, $T+p=81$)	MU	1.31 (−.02,.12)	1.21 (−.03,.20)	1.33 (−.02,.27)	.88 (−.01,.31)	.77 (.00,.36)	.40 (−.01,.44)	.21 (−.01,.46)	.11 (−.02,.47)	.05 (−.01,.47)
	LS	1.27 (−.06,.11)	1.13 (−.11,.20)	.90 (−.18,.25)	.69 (−.24,.27)	.52 (−.29,.27)	.14 (−.31,.20)	.04 (−.21,.12)	.01 (−.12,.09)	.00 (−.07,.07)
Unemploy-ment Rate (1890-1970, $T+p=81$)	MU	1.12 (−.02,.12)	.66 (−.01,.19)	.54 (−.02,.21)	.50 (−.02,.23)	.30 (.01,.24)	−.02 (.02,.18)	−.01 (.02,.13)	.00 (.00,.11)	.00 (.00,.10)
	LS	1.09 (−.05,.12)	.60 (−.07,.18)	.46 (−.10,.20)	.39 (−.13,.21)	.16 (−.14,.20)	−.07 (−.02,.10)	.00 (.02,.04)	.00 (.00,.02)	.00 (.00,.01)
GNP Deflator (1889-1970, $T+p=82$)	MU	1.40 (−.01,.11)	1.51 (−.02,.21)	1.50 (−.02,.29)	1.44 (−.01,.36)	1.35 (−.01,.42)	.93 (.01,.63)	.63 (.00,.71)	.42 (−.01,.74)	.28 (−.01,.76)
	LS	1.37 (−.04,.11)	1.42 (−.13,.21)	1.32 (−.22,.28)	1.16 (−.31,.32)	.99 (−.39,.34)	.37 (−.58,.33)	.13 (−.52,.24)	.04 (−.39,.18)	.01 (−.27,.15)

Table 4. (continued)

Data Series	Estimator	α	90% CI for α	ψ_1	ψ_2	ψ_3	μ	$100\times\beta$	$100\times\sigma^2$	Magnitude of Roots in Descending Order: 1	2	3	4	Impulse Response Function: 1	2	3	4	5	10	15	20	25
Consumer	MU	1.00	[.96,1.0]	.71	−.29	.08	.005	.00	.19	1.0	.45	.42	.42	1.71	1.92	1.94	1.95	1.97	1.98	1.98	1.98	1.98
Prices		(.00,.03)		(−.02,.09)	(.00,.12)	(.00,.10)	(.00,.01)	(.00,.04)	(.00,.03)	(.00,.07)	(.13,.11)	(.09,.10)	(−.04,.18)	(−.04,.10)	(−.09,.20)	(−.15,.29)	(−.21,.37)	(−.25,.44)	(−.40,.66)	(−.52,.79)	(−.55.,.85)	(−.56,.89)
(1860-1970,	LS	.969		.69	−.27	.09	.09	.06	.18	.93	.54	.43	.43	1.66	1.79	1.74	1.67	1.60	1.16	.81	.57	.40
$T+p$ = 111)		(−.05,.03)		(−.02,.09)	(.02,.12)	(.02,.10)	(.00,.02)	(.04,.06)	(−.01,.03)	(−.11,.07)	(.19,.13)	(.09,.11)	(.00,.17)	(−.08,.10)	(−.19,.20)	(−.32,.28)	(−.44,.36)	(−.57,.41)	(−1.13,.50)	(−1.53,.48)	(−1.73,.43)	(−1.85,.40)
Nominal	MU	.970	[.88,1.0]	.53	−.12		.20	.14	.37	.95	.36	.36		1.50	1.60	1.54	1.45	1.37	1.04	.80	.61	.47
Wages		(.01,.06)		(−.02,.11)	(.00,.12)		(−.04.34)	(−.06,.28)	(.00,.07)	(.02,.10)	(.11,.14)	(.00,.18)		(−.04,.13)	(−.07,.24)	(−.09,.34)	(−.10,−.41)	(−.08,.47)	(.07,.66)	(.16,.71)	(.16,.74)	(.18,.76)
(1900-1970,	LS	.910		.54	−.07		.57	.38	.36	.79	.47	.18		1.46	1.49	1.34	1.14	.94	.31	.10	.03	.01
$T+p$ = 71)		(−.06,.06)		(−.02,.11)	(.05,.12)		(.37,.33)	(.28,.28)	(−.02,.07)	(−.13,.09)	(.17,.16)	(−.09,.19)		(−.09,.13)	(−.19,.23)	(−.30,.32)	(−.42,.37)	(−.50,.39)	(−.75,.35)	(−.71,.25)	(−.58,.19)	(−.46,.16)
Real Wages	MU	.896	[.78,1.0]	.23			.30	.23	.12	.86	.26			1.12	1.03	.90	.78	.67	.31	.15	.07	.03
(1900-1970,		(.00,.09)		(−.01,.12)			(.00,.23)	(.00,.19)	(.00,.02)	(.00,.14)	(.00,.17)			(−.01,.13)	(−.01,.20)	(.00,.20)	(.00,.31)	(−.01,.35)	(.00,.42)	(.00,.43)	(.00,.44)	(.00,.44)
$T+p$ = 71)	LS	.831		.25			.49	.35	.12	.74	.34			1.08	.92	.73	.55	.42	.10	.02	.01	.00
		(−.07,.08)		(.03,.11)			(.19,.21)	(.15,.17)	(.00,.02)	(−.12,.12)	(.09,.18)			(−.05,.13)	(−.13,.19)	(−.19,.22)	(−.25,.24)	(−.28,.23)	(−.24,.16)	(−.14,.12)	(−.08,.12)	(−.04,.15)
Money Stock	MU	.942	[.89,1.0]	.65			.099	.34	.22	.81	.81			1.59	1.88	1.96	1.90	1.75	.71	.13	−.02	−.03
(1889-1970,		(.00,.04)		(−.02,.09)			(.00,.05)	(.01,.21)	(.00,.04)	(.05,.08)	(−.04,.12)			(−.02,.09)	(−.04,.20)	(−.04,.31)	(−.04,.41)	(−.03,.51)	(.06,.86)	(.08,.93)	(.07,.92)	(.05,.92)
$T+p$ = 82)	LS	.916		.66			.13	.49	.22	.81	.81			1.58	1.83	1.84	1.70	1.46	.21	−.14	−.06	.00
		(−.03,.03)		(.00,.09)			(.03,.04)	(.15,.18)	(−.01,.04)	(.01,.05)	(.00,.09)			(−.04,.10)	(−.10,.21)	(−.19,.32)	(−.30,.40)	(−.39,.46)	(−.56,.47)	(−.31,.29)	(−.06,.16)	(.01,.09)
Velocity	MU	1.00	[.93,1.0]				−.012	.00	.48	1.0				1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
(1869-1970,		(.00,.05)					(.00,.02)	(.00,.07)	(−.01,.07)	(.00,.05)				(.00,.05)	(.00,.10)	(.00,.14)	(.00,.17)	(.00,.20)	(.00,.29)	(.00,.35)	(.00,.38)	(.00,.41)
$T+p$ = 102)	LS	.941					.052	−.03	.46	.941				.94	.89	.83	.78	.74	.54	.40	.30	.22
		(−.09,.05)					(.00,.03)	(−.09,.10)	(−.02,.06)	(−.09,.05)				(−.09,.05)	(−.17,.09)	(−.24,.12)	(−.31,.15)	(−.37,.16)	(−.60,.20)	(−.75,.19)	(−.84,.18)	(−.90,.17)
Interest	MU	1.00	[1.0,1.0]	.18	.37		.04	.00	8.32	1.00	.70	.52		1.18	1.58	1.72	1.89	1.97	2.17	2.20	2.21	2.21
Rate		(.00,.05)		(−.02,.11)	(−.03,.11)		(.00,.01)	(.00,.55)	(−.20,1.58)	(.00,.08)	(−.01,.13)	(−.01,.12)		(−.05,.12)	(−.10,.19)	(−.18,.29)	(−.25,.38)	(−.32,.47)	(−.56,.81)	(−.66,.98)	(−.69,1.05)	(−.69,1.08)
(1900-1970,	LS	1.03		.10	.30		−.19	.32	8.04	1.05	.58	.50		1.13	1.49	1.61	1.78	1.90	2.49	3.20	4.10	5.26
$T+p$ = 71)		(−.07,.05)		(−.02,.11)	(−.02,.12)		(.00,.22)	(.53,.86)	(−.65,1.45)	(−.14,.06)	(.09,.14)	(.00,.12)		(−.10,.12)	(−.21,.19)	(−.38,.28)	(−.54,.36)	(−.74,.42)	(−1.52,.53)	(−1.97,.48)	(−2.16,.42)	(−2.20,.39)
Common	MU	.970	[.88,1.0]	.27	−.18		.030	.15	2.43	.97	.43	.43		1.23	1.08	.97	.93	.92	.78	.65	.55	.47
Stock Prices		(.00,.06)		(.00,.10)	(.00,.10)		(.01,.06)	(−.02,.31)	(−.02,.36)	(.00,.08)	(.01,.10)	(.01,.13)		(−.02,.11)	(−.05,.18)	(−.05,.21)	(−.03,.23)	(−.02,.26)	(.02,.36)	(.03,.42)	(.05,.45)	(.06,.47)
(1871-1970,	LS	.908		.29	−.13		.096	.32	2.38	.89	.39	.39		1.20	1.01	.84	.74	.66	.37	.21	.12	.07
$T+p$ = 100)		(−.07,.06)		(.02,.10)	(.03,.10)		(−.04,.09)	(.34,.32)	(−.09,.35)	(−.08,.09)	(−.01,.11)	(−.02,.14)		(−.07,.11)	(−.13,.18)	(−.21,.20)	(−.26,.20)	(−.30,.21)	(−.44,.21)	(−.47,.19)	(−.45,.16)	(−.41,.14)

*For the industrial production series, the MU estimates of ψ_4 and ψ_5 are −.08 (−.01,.10) and −.25 (.02,.10), the LS estimates of ψ_4 and ψ_5 are −.05 (.04,.10) and −.22 (.05,.10), the MU estimates of the magnitudes of the 5-th and 6-th largest roots are .80 (−.02,.08) and .70 (−.02,.13), the LS estimates of the 5-th and 6-th largest roots are .77 (−.03,.09) and .68 (−.04,.15).

Table 5. A Comparison of Different Estimates for the Nelson-Plosser Data Series Using AR(3) Models

Data Series	Estimator	α	90% CI for α	$100\times\beta$	Magnitude of Largest Root
Real GNP	MU	.87 (.00,.09)	[.75,1.0]	.45 (.00,.31)	.69 (.12,.12)
(1909-1970,	LS	.81 (−.06,.08)		.61 (.21,.27)	.70 (.07,.09)
$T+p$ = 62)	DW			.6 (.2)	.76 (.09)
	Ph	~.82			
	AMU	.88	[.76,1.03]		.81 [.61,1.04]
Nominal GNP	MU	1.0 (.00,.07)	[.88,1.0]	.00 (.00,.45)	.96 (.00,.11)
(1909-1970,	LS	.91 (−.11,.07)		.52 (.59,.52)	.78 (−.19,.11)
$T+p$ = 62)	DW			.5 (.2)	.82 (.09)
	Ph	~.92			
	AMU	1.02	[.87,1.04]		1.03 [.81,1.04]
Real per	MU	.86 (.00,.09)	[.74,1.0]	.28 (.00,.19)	.65 (.09,.12)
Capita GNP	LS	.80 (−.07,.08)		.38 (.13,.17)	.70 (.04,.09)
(1909-1970,	DW			.4 (.1)	.75 (.09)
$T+p$ = 62)	Ph	~.82			
	AMU	.87	[.75,1.03]		.79 [.60,1.04]
Industrial	MU	.87 (.00,.07)	[.76,1.0]	.53 (.01,.30)	.86 (.00,.10)
Production	LS	.82 (−.05,.07)		.75 (.22,.28)	.80 (−.07,.10)
(1860-1970,	DW			.7 (.2)	.78 (.09)
$T+p$ = 111)	Ph	~.86			
	AMU	.87	[.76,1.02]		.86 [.75,1.02]
Employment	MU	.91 (.00,.07)	[.81,1.0]	.13 (.00,.11)	.88 (−.01,.12)
(1890-1970,	LS	.86 (−.06,.06)		.21 (.09,.10)	.76 (−.12,.11)
$T+p$ = 81)	DW			.2 (.0)	.78 (.10)
	Ph	~.88			
	AMU	.93	[.82,1.03]		.90 [.76,1.04]
Unemploy-	MU	.81 (.00,.11)	[.65,.97]	−.17 (.03,.27)	.82 (−.01,.13)
ment Rate*	LS	.73 (−.08,.10)		−.20 (−.05,.33)	.72 (−.10,.12)
(1890-1970,	DW			.1 (.1)	.75 (.10)
$T+p$ = 81)	Ph	~.75			
	AMU	.82	[.67,1.02]		.82 [.67,1.03]
GNP	MU	.95 (.00,.05)	[.88,1.0]	.13 (.00,.13)	.89 (.00,.10)
Deflator	LS	.91 (−.05,.05)		.22 (.12,.13)	.75 (−.10,.07
(1889-1970,	DW			.2 (.0)	.82 (.07)
$T+p$ = 82)	Ph	~.92			
	AMU	.96	[.88,1.02]		.92 [.77,1.04]

Data Series	Estimator	α	90% CI for α	$100\times\beta$	Magnitude of Largest Root
Consumer	MU	1.0 (.00,.03)	[.97,1.0]	.00 (.00,.05)	1.0 (.00,.07)
Prices	LS	.98 (−.05,.03)		.05 (.05,.06)	.95 (−.10,.08)
(1860-1970,	DW			.0 (.0)	.95 (.04)
$T+p$ = 111)	Ph	~.98			
	AMU	1.01	[.97,1.02]		1.02 [.95,1.04]
Nominal	MU	.97 (.01,.06)	[.88,1.0]	.14 (−.06,.28)	.95 (.02,.10)
Wages	LS	.91 (−.06,.06)		.38 (.28,.28)	.79 (−.13,.09)
(1900-1970,	DW			.4 (.1)	.82 (.09)
$T+p$ = 71)	Ph	~.93			
	AMU	.99	[.88,1.03]		.98 [.80,1.05]
Real	MU	.89 (.01,.09)	[.76,1.0]	.24 (−.02,.21)	.85 (.01,.13)
Wages	LS	.82 (−.07,.08)		.38 (.15,.18)	.71 (−.11,.10)
(1900-1970,	DW			.4 (.1)	.76 (.09)
$T+p$ = 71)	Ph	~.83			
	AMU	.87	[.73,1.03]		.83 [.66,1.04]
Money	MU	.95 (.00,.04)	[.89,1.0]	.29 (−.01,.22)	.79 (.07,.09)
Stock	LS	.92 (−.03,.04)		.47 (.17,.20)	.80 (.02,.06)
(1889-1970,	DW			.5 (.1)	.83 (.07)
$T+p$ = 82)	Ph	~.93			
	AMU	.95	[.90,1.01]		.88 [.74,1.04]
Velocity	MU	1.0 (.00,.05)	[.93,1.0]	.00 (.00,.07)	1.0 (.00,.07)
1869-1970,	LS	.94 (−.09,.06)		−.04 (−.08,.10)	.94 (−.10,.08)
$T+p$ = 102)	DW			.0 (.0)	.99 (.02)
	Ph	~1.01			
	AMU	1.02	[.95,1.04]		1.02 [.95,1.04]
Interest	MU	1.0 (.00,.05)	[1.0,1.0]	.00 (.00,.55)	1.0 (.00,.08)
Rate	LS	1.03 (−.07,.05)		.32 (.53,.86)	1.05 (−.14,.06)
(1900-1970,	DW			.3 (.1)	.98 (.06)
$T+p$ = 71)	Ph	~1.15			
	AMU	1.03	[1.02,1.04]		1.05 [1.03,1.08]
Common	MU	.97 (.00,.06)	[.88,1.0]	.15 (−.02,.31)	.97 (.00,.08)
Stock	LS	.91 (−.07,.06)		.32 (.34,.32)	.89 (−.08,.09)
Prices	DW			.3 (.1)	.88 (.07)
(1871-1970,	Ph	~.93			
$T+p$ = 100)	AMU	1.01	[.88,1.04]		1.01 [.87,1.04]

*The Phillips estimate for this series is based on an AR(4) model.

Table 6. Median-unbiased (MU) and Least Squares (LS) Estimates for the Extended Nelson-Plosser Data Series

Data Series	Estimator	α	90% CI for α	ψ_1	ψ_2	ψ_3	μ	$100\times\beta$	$100\times\sigma^2$	Magnitude of Roots in Descending Order				NP Data	
										1	2	3	4	α	90% CI for α
Real GNP	MU	.864	[.77,.99]	.39			.64	.44	.27	.67	.58			.885	[.77,1.0]
(1909-1988,		(.00,.06)		(−.01,.10)			(−.01,.29)	(−.02,.21)	(.00,.05)	(.07,.10)	(−.05,.17)			(.00,.08)	
$T+p$ = 80)	LS	.824		.41			.82	.57	.27	.64	.64			.825	
		(−.04,.06)		(.01,.10)			(.19,.26)	(.14,.19)	(−.01,.04)	(.01,.07)	(.04,.14)			(−.06,.07)	
Nominal GNP	MU	1.00	[.93,1.0]	.45			.035	.00	.64	1.00	.45			.958	[.86,1.0]
(1909-1988,		(.00,.04)		(−.02,.10)			(.00,.03)	(.00,.27)	(−.01,.11)	(.00,.09)	(.00,.13)			(.00,.06)	
$T+p$ = 80)	LS	.939		.47			.64	.41	.61	.87	.54			.899	
		(−.07,.04)		(−.01,.11)			(.00,.05)	(.41,.31)	(−.04,.10)	(−.14,.09)	(.07,.15)			(−.06,.06)	
Real per Capita GNP	MU	.858	[.77,.97]	.38			.99	.28	.28	.67	.58			.875	[.76,1.0]
		(.01,.07)		(−.01,.11)			(−.03,.46)	(−.01,.14)	(.00,.05)	(.06,.10)	(−.01,.17)			(.00,.08)	
(1909-1988,	LS	.816		.40			1.29	.36	.28	.63	.63			.818	
$T+p$ = 80)		(−.04,.06)		(.02,.10)			(.28,.43)	(.08,.13)	(.00,.05)	(.02,.07)	(.05,.14)			(−.06,.07)	
Industrial Production*	MU	.910	[.79,1.0]	.06	−.10	.00	.10	.34	.86	.94	.79	.79	.76	.919	[.78,1.0]
		(.00,.09)		(.01,.10)	(−.01,.10)	(.00,.10)	(.00,.04)	(.01,.33)	(.00,.12)	(.00,.07)	(.02,.06)	(.01,.06)	(−.01,.07)		
(1860-1988,	LS	.841		.10	−.06	.03	.11	.63	.85	.88	.78	.78	.75	.835	
$T+p$ = 129)		(−.08,.08)		(.05,.10)	(.03,.09)	(.04,.09)	(.03,.05)	(.30,.30)	(−.01,.11)	(−.06,.06)	(.01,.06)	(.00,.07)	(−.01,.07)		
Employment	MU	.904	[.82,1.0]	.40	−.11		.98	.15	.11	.86	.36	.36		.914	[.81,1.0]
(1890-1988,		(.00,.06)		(−.01,.10)	(−.01,.10)		(−.04,.64)	(−.01,.10)	(.00,.02)	(.01,.11)	(.09,.13)	(.01,.16)		(.00,.07)	
$T+p$ = 99)	LS	.864		.41	−.08		1.38	.21	.10	.78	.32	.32		.861	
		(−.04,.06)		(.00,.10)	(.03,.10)		(.43,.57)	(.07,.09)	(.00,.02)	(−.08,.10)	(.10,.14)	(−.03,.17)		(−.06,.06)	
Unemploy-ment Rate	MU	.756	[.63,.88]	.36	−.23	.22	.40	.04	13.7	.72	.72	.65	.65	.765	[.62,.89]
		(.00,.09)		(.00,.10)	(.00,.10)	(−.01,.10)	(−.01,.16)	(.01,.15)	(−.07,2.08)	(.04,.07)	(.01,.09)	(.00,.09)	(−.02,.17)	(.00,.10)	
(1890-1988,	LS	.715		.38	−.21	.23	.47	.03	13.7	.74	.74	.66	.66	.706	
$T+p$ = 99)		(−.05,.08)		(.03,.10)	(.03,.10)	(.02,.10)	(.07,.17)	(.02,.17)	(−.17,2.07)	(.03,.07)	(.03,.08)	(.01,.09)	(.00,.14)	(−.06,.10)	
GNP Deflator	MU	1.00	[.97,1.0]	.50			.014	.00	.20	1.0	.50			.960	[.89,1.0]
(1889-1988,		(.00,.03)		(−.02,.09)			(.00,.01)	(.00,.10)	(.00,.03)	(.00,.07)	(.00,.10)			(.00,.05)	
$T+p$ = 100)	LS	.968		.47			.092	.11	.19	.94	.50			.915	
		(−.05,.03)		(−.02,.09)			(.00,.02)	(.14,.12)	(.00,.03)	(−.11,.08)	(.05,.13)			(−.05,.04)	

Table 6. (continued)

Data Series	Estimator	α	90% CI for α	ψ_1	ψ_2	ψ_3	μ	$100\times\beta$	$100\times\sigma^2$	Magnitude of Roots in Descending Order 1	2	3	4	NP Data α	NP Data 90% CI for α
Consumer Prices (1860-1988, $T+p$ = 129)	MU	1.00	[1.0,1.0]	.77	−.31	.13	.007	.00	.17	1.0	.60	.46	.46	1.00	[.97,1.0]
		(.00,.02)		(−.03,.09)	(.01,.11)	(−.01,.09)	(.00,.01)	(.00,.05)	(.00,.02)	(.00,.06)	(.02,.13)	(.03,.10)	(−.02,.16)		
	LS	.987		.71	−.29	.10	.026	.05	.16	.97	.52	.43	.43	.969	
		(−.03,.02)		(−.03,.09)	(.02,.10)	(.01,.09)	(.00,.02)	(.05,.06)	(−.01,.02)	(−.09,.06)	(.09,.14)	(.04,.10)	(.01,.15)		
Nominal Wages (1900-1988, $T+p$ = 89)	MU	1.00	[.92,1.0]	.54	−.13		.026	.00	.31	1.0	.36	.36		.970	[.88,1.0]
		(.00,.04)		(−.02,.11)	(.01,.11)		(.00,.02)	(.00,.20)	(−.01,.05)	(.00,.09)	(.08,.13)	(.00,.03)		(.01,.06)	
	LS	.939		.53	−.08		.38	.30	.29	.88	.32	.28		.910	
		(−.07,.05)		(−.03,.10)	(.04,.11)		(.00,.03)	(.30,.22)	(−.02,.05)	(−.13,.09)	(.12,.15)	(−.05,.18)		(−.06,.06)	
Real Wages (1900-1988, $T+p$ = 89)	MU	1.00	[.91,1.0]	.22			.01	.00	.13	1.0	.22			.896	[.78,1.0]
		(.00,.05)		(.00,.10)			(.00,.01)	(.00,.08)	(.00,.02)	(.00,.08)	(.01,.12)			(.00,.09)	
	LS	.929		.27			.22	.12	.12	.90	.30			.831	
		(−.08,.06)		(.02,.11)			(.00,.02)	(.11,.09)	(−.01,.02)	(−.11,.09)	(.05,.14)			(−.07,.08)	
Money Stock (1889-1988, $T+p$ = 100)	MU	.958	[.92,1.0]	.65			.071	.26	.20	.81	.81			.942	[.89,1.0]
		(.00,.03)		(−.01,.08)			(.00,.04)	(−.01,.19)	(.00,.03)	(.05,.08)	(−.05,.12)			(.00,.04)	
	LS	.937		.67			.097	.39	.19	.82	.81			.916	
		(−.02,.03)		(.00,.08)			(.02,.04)	(.13,.17)	(.00,.03)	(.02,.05)	(.00,.09)			(−.03,.03)	
Velocity (1869-1988, $T+p$ = 120)	MU	1.00	[.96,1.0]				−.009	.00	.41	1.0				1.00	[.93,1.0]
		(.00,.05)					(.00,.02)	(.00,.06)	(−.01,.05)	(.00,.05)				(.00,.05)	
	LS	.962					.018	.00	.40	.962				.941	
		(−.08,.05)					(.00,.03)	(−.06,.08)	(−.02,.05)	(−.08,.05)				(−.09,.05)	
Interest Rate (1900-1988, $T+p$ = 99)	MU	1.0	[.95,1.0]	.20	−.18		.068	.00	36.3	1.0	.43	.43		1.00	[1.0,1.0]
		(.00,.07)		(.00,.11)	(.01,.11)		(.00,.20)	(.00,.72)	(−.55,5.97)	(.00,.08)	(.01,.11)	(.00,.13)		(.00,.05)	
	LS	.953		.22	−.16		.078	.50	35.6	.95	.41	.41		1.03	
		(−.10,.07)		(.02,.10)	(.05,.11)		(−.03,.36)	(.65,1.07)	(−1.83,5.69)	(−.11,.09)	(−.01,.12)	(−.03,.14)		(−.07,.06)	
Common Stock Prices (1871-1988, $T+p$ = 118)	MU	1.00	[.91,1.0]	.23	−.19		.03	.00	2.40	1.0	.43	.43		.970	[.88,1.0]
		(.00,.05)		(.00,.09)	(.01,.10)		(.00,.04)	(.00,.21)	(−.08,.33)	(.00,.06)	(.00,.11)	(.00,.13)		(.00,.06)	
	LS	.932		.24	−.16		.06	.29	2.30	.93	.41	.41		.908	
		(−.07,.05)		(.02,.09)	(.03,.10)		(.00,.08)	(.24,.25)	(−.14,.32)	(−.08,.07)	(−.01,.11)	(−.02,.14)		(−.07,.06)	

*For the industrial production series, the MU estimates of ψ_4 and ψ_5 are −.07 (.00,.10) and −.23 (.01,.09), the LS estimates of ψ_4 and ψ_5 are −.05 (.03,.09) and −.20 (.04,.09), the MU estimates of the magnitudes of the 5-th and 6-th largest roots are .76 (−.01,.08) and .68 (−.02,.13), and the LS estimates of the magnitudes of the 5-th and 6-th largest roots are .75 (−.02,.09) and .67 (−.04,.14).

Table 7. Median-unbiased (MU), Least Squares (LS), and DeJong-Whiteman (DW) Estimates for Stock Dividend and Price Series

Data Series	Estimator	α	90% CI for α	ψ_1	ψ_2	μ	$100\times\beta$	$100\times\sigma^2$	Magnitude of Roots in Descending Order			Impulse Response Function								
									1	2	3	1	2	3	4	5	10	15	20	25
Dow Jones Dividends (1928-1978, $T+p=51$)	MU	.786 (−.01,.17)	[.55,1.0]	.30 (.00,.14)	−.25 (.00,.16)	.42 (.00,.31)	.41 (−.01,.37)	1.8 (−.01,.41)	.79 (−.02,.14)	.56 (.04,.12)	.56 (−.01,.18)	1.09 (−.02,.16)	.63 (−.01,.25)	.34 (.02,.29)	.29 (.00,.31)	.29 (.00,.31)	.08 (−.01,.29)	.03 (−.01,.29)	.01 (.00,.28)	.00 (.00,.28)
	LS	.675 (−.13,.14)		.34 (.05,.13)	−.18 (.08,.14)	.65 (.22,.27)	.61 (.24,.37)	1.8 (−.06,.39)	.64 (−.11,.11)	.52 (.04,.12)	.52 (−.07,.20)	1.02 (−.09,.15)	.52 (−.15,.24)	.18 (−.20,.24)	.09 (−.21,.21)	.09 (−.19,.18)	.01 (−.07,.06)	.00 (−.02,.03)	.00 (−.01,.02)	.00 (.00,.01)
	DW						.6 (.03)		.72 (.10)											
Dow Jones Prices (1928-1978, $T+p=51$)	MU	.912 (.00,.15)	[.69,1.0]	.00 (.01,.15)	−.10 (.00,.14)	.44 (−.02,.69)	.22 (−.15,.49)	5.3 (−.04,1.2)	.92 (.00,.14)	.33 (.07,.15)	.33 (.03,.16)	.91 (−.03,.16)	.73 (.05,.23)	.68 (−.02,.26)	.63 (.02,.29)	.58 (.01,.32)	.39 (.02,.37)	.26 (.01,.39)	.17 (.01,.40)	.11 (.01,.41)
	LS	.767 (−.16,.14)		.07 (.08,.14)	−.03 (.06,.14)	1.15 (.74,.66)	.56 (.38,.64)	5.1 (−.29,1.1)	.76 (−.18,.14)	.19 (.10,.16)	.19 (.00,.15)	.84 (−.11,.16)	.60 (−.17,.22)	.45 (−.27,.22)	.34 (−.31,.21)	.26 (−.35,.20)	.06 (−.34,.14)	.02 (−.25,.12)	.00 (−.17,.12)	.00 (−.11,.14)
	DW						.6 (.03)		.76 (.11)											
NYSE Dividends (1926-1981, $T+p=55$)	MU	.900 (−.01,.13)	[.72,1.0]	.30 (.01,.13)	−.24 (.00,.14)	1.52 (.13,1.93)	.39 (.03,.53)	1.4 (.02,.32)	.90 (−.01,.14)	.51 (.04,.12)	.51 (.01,.17)	1.20 (−.02,.15)	.91 (−.05,.25)	.68 (−.06,.31)	.61 (−.03,.34)	.58 (−.02,.36)	.34 (−.04,.40)	.19 (−.03,.41)	.11 (−.02,.42)	.06 (−.01,.43)
	LS	.795 (−.13,.12)		.34 (.04,.12)	−.17 (.07,.13)	3.08 (1.94,1.73)	.79 (.50,.51)	1.4 (−.06,.30)	.76 (−.15,.12)	.47 (.04,.02)	.47 (−.04,.19)	1.13 (−.10,.15)	.77 (−.19,.24)	.47 (−.28,.26)	.33 (−.34,.25)	.27 (−.34,.24)	.07 (−.29,.13)	.02 (−.18,.09)	.00 (−.11,.07)	.00 (−.06,.07)
	DW						.8 (.03)		.77 (.10)											
NYSE Prices (1926-1981, $T+p=55$)	MU	1.00 (−.02,.13)	[.81,1.0]	.06 (.03,.14)	−.22 (.02,.14)	.04 (−.01,.10)	.00 (.00,.53)	4.7 (−.12,.98)	1.0 (−.01,.13)	.47 (.02,.14)	.47 (.00,.02)	1.06 (−.06,.15)	.85 (−.11,.24)	.82 (−.16,.26)	.86 (−.12,.28)	.87 (−.15,.31)	.86 (−.19,.37)	.86 (−.23,.39)	.86 (−.24,.41)	.86 (−.24,.42)
	LS	.851 (−.20,.13)		.14 (.08,.13)	−.14 (.07,.14)	2.68 (−.03,.17)	.68 (.56,.75)	4.5 (−.35,.90)	.85 (−.19.13)	.40 (.01,.14)	.40 (−.03,.17)	.99 (−.14,.15)	.71 (−.26,.23)	.56 (−.39,.23)	.50 (−.47,.23)	.43 (−.54,.23)	.20 (−.76,.20)	.09 (−.83,.22)	.04 (−.85,.33)	.02 (−.86,.56)
	DW						.7 (.03)		.84 (.10)											
Standard and Poor's Dividends (1871-1985, $T+p=115$)	MU	.820 (.00,.07)	[.71,.92]	.25 (.00,.09)	−.07 (−.10,.10)	−.05 (.00,.04)	.21 (.00,.09)	1.4 (.00,.19)	.77 (.00,.11)	.31 (.09,.13)	.31 (.02,.15)	1.07 (−.10,.10)	.82 (−.02,.14)	.61 (−.02,.16)	.46 (.00,.18)	.36 (.01,.18)	.10 (.00,.14)	.03 (.00,.09)	.01 (.00,.07)	.00 (.00,.06)
	LS	.777 (−.04,.07)		.27 (.02,.09)	−.05 (.02,.09)	−.07 (−.01,.04)	.26 (.05,.09)	1.4 (−.01,.19)	.68 (−.07,.11)	.27 (.12,.14)	.27 (−.02,.15)	1.05 (−.04,.10)	.78 (−.06,.14)	.53 (−.09,.15)	.36 (−.10,.16)	.24 (−.10,.15)	.04 (−.06,.08)	.01 (−.02,.03)	.00 (−.01,.02)	.00 (.00,.01)
	DW						.3 (.01)		.72 (.09)											
Standard and Poor's Prices (1871-1985, $T+p=115$)	MU	.936 (−.01,.07)	[.84,1.0]	.10 (.00,.10)	−.14 (.00,.10)	.19 (.00,.19)	.07 (−.01,.11)	3.1 (.00,.44)	.94 (.00,.08)	.39 (.01,.12)	.39 (.00,.13)	1.04 (−.02,.10)	.84 (−.02,.15)	.76 (−.03,.17)	.73 (−.02,.18)	.69 (−.02,.21)	.51 (−.03,.29)	.37 (−.03,.32)	.27 (−.03,.34)	.20 (−.02,.35)
	LS	.881 (−.07,.06)		.13 (.03,.09)	−.11 (.03,.09)	.33 (.17,.18)	.15 (.07,.13)	3.0 (−.06,.43)	.88 (−.07,.09)	.36 (−.01,.12)	.36 (−.03,.14)	1.01 (−.05,.10)	.78 (−.09,.15)	.65 (−.14,.15)	.59 (−.17,.15)	.52 (−.20,.16)	.28 (−.26,.15)	.15 (−.25,.13)	.08 (−.21,.10)	.04 (−.17,.08)
	DW						.2 (.01)		.87 (.07)											