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# TRANSFORMATIONS OF THE COMMODITY SPACE, BEHAVIORAL HETEROGENEITY, AND THE AGGREGATION PROBLEM

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**ABSTRACT** : The aggregation problem in demand analysis and exchange equilibrium is studied by putting restrictions on the shape of the distribution of the agents' characteristics. This is done by exploiting the finite dimensional linear structure induced on demand functions by affine transformations of the commodity space (or household equivalence scales). Increasing the degree of behavioral heterogeneity in the household sector or more specifically, making the conditional distributions in each equivalence class of demand functions flat enough, has an important regularizing influence on aggregate budget shares : market demand has a negative dominant diagonal Jacobian matrix, aggregate excess demand has the gross substitutability property, on a large set of prices. These facts have strong consequences for the unicity and stability of equilibrium as well as for the prevalence of the weak axiom of revealed preference in the aggregate in a private ownership Walrasian exchange model. These strong macroeconomic regularities are obtained essentially through distributional restrictions, as the analysis does not rely upon any hypothesis about the "rationality" of individual demand functions other than homogeneity (absence of money illusion) and Walras's Law (compliance with individual budget constraints). While such results are in a sense comforting since they show that general Walrasian equilibrium theory may be more robust than previously thought, they also suggest that individual "rationality" postulates (utility maximization) might not be as necessary as some would like to believe to the construction of a sound quantitative macroeconomics.

JEL Code : C62, D10, D51, E10

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# TRANSFORMATIONS OF THE COMMODITY SPACE, BEHAVIORAL HETEROGENEITY AND THE AGGREGATION PROBLEM *

by

Jean-Michel Grandmont **

There are still quite a few unpleasant gaps nowadays between macroanalysis and microanalysis of socioeconomic systems. A leading instance, drawn from social choice theory, is Arrow's celebrated impossibility theorem (1951, 1963), which states that any attempt to aggregate individual preferences into a consistent social ordering, or a collective choice procedure satisfying a few appealing requirements, is bound to fail. The impossibility arises however only if one insists that the procedure should work for every possible environment, or in other words, *for all distributions of individual preferences*. A possible resolution of the problem is therefore to identify broad classes of distributions of preferences for which aggregation is indeed feasible, and to assess whether the particular world we happen to live in is more or less likely to meet the corresponding distributional requirements. Ideally, one would like not so much to put constraints on the range of allowable preferences, but rather to impose restrictions on the *shape* of their distribution. To implement such a program, however, one needs to have an appropriate *linear structure* on the space of preferences. A structure of this sort was indeed introduced some time ago (Grandmont (1978)), by defining linear operations on the set of allowable preferences, that are compatible with its lattice structure (*intermediate preferences*). Aggregation was then shown to be feasible through simple majority voting when the distribution of preferences in the society has nice *symmetry* properties, along the lines laid down by Tullock (1967), Arrow (1969), Davis, De Groot and Hinich (1972) in "spatial" voting models with Euclidean preferences. More recently, Caplin and Nalebuff (1988, 1991) have been able, by using this linear intermediate preferences structure, to show among other things the striking result that when the distribution of preferences in the society has a logconcave density, the proposal most preferred by the mean voter is unbeatable under 64 % - majority rule.

Another case (which will be the focus of the present paper) is drawn from the field of demand analysis and more generally from equilibrium theory, in which the situation is currently somewhat less satisfactory. The troublesome result here is Sonnenschein's indeterminacy theorem (1971, 1973, 1974) as refined by McFadden, Mantel, Mas-Colell, Richter, Debreu and others (see e.g. the survey by Shafer and Sonnenschein (1982)) : individual optimization ("microeconomic rationality") does not place any restriction on aggregate excess demand, on any given compact set of prices, other than homogeneity and Walras's Law. This would be true even if we required all traders to have homothetic preferences ! Whereas such findings do not threaten the usual results about existence and efficiency of market equilibrium, they appear to jeopardize two important ingredients of the neoclassical paradigm. There is apparently little chance indeed to get gross substitutability, diagonal dominance or the weak axiom of revealed preferences in the aggregate, or any other properties that have long been known to be needed for *uniqueness* and *stability* of competitive equilibrium (Wald (1936, 1951), Arrow and Hurwicz (1958), Arrow, Block and Hurwicz (1959), Negishi (1958, 1962), McKenzie (1960), Arrow and Hahn (1971, Ch. 9-12), Hahn (1982)). The implications of such results are unfortunately rarely discussed by specialists of applied demand analysis (see e.g. Deaton (1986)), although they raise delicate theoretical and empirical issues. In view of these considerations, in particular, quite a few recent efforts to provide systematic theoretical "microfoundations" to quantitative dynamic macroeconomics through models involving a single optimizing representative agent, appear at first sight extraordinarily naive, and might in fact be quite misleading.

Underlying Sonnenschein's indeterminacy theorem is the assumption that the distribution of the agents' characteristics can be chosen arbitrarily. This fact suggests the same kind of research agenda as in the case of social choice theory discussed above, namely to look for broad classes of distributions of the agents' characteristics that would generate interesting qualitative and/or quantitative properties of aggregate demand (or excess demand). Here again, one would like not so much to put constraints on the range of the agents' allowable characteristics, but rather to impose qualitative or quantitative restrictions on the *shape* of their distribution. In contrast to the naive representative agent viewpoint alluded to above, this distributional approach might (perhaps with the help of additional assumptions) generate specific hypotheses not only about aggregate demand but also about the *distribution* of the agents' actual observable choices, that could then be tested on cross-sectional or panel data.

The principle of a possible resolution of the aggregation problem along these lines, in demand analysis or equilibrium theory, has been talked about for quite some time, but it is only recently that clear progress was made. A decisive step was indeed taken by W. Hildenbrand (1983). In Hildenbrand's formulation, an individual agent's characteristics are described by a preference relation, or more specifically a demand function satisfying the weak axiom of revealed preference (WARP), and by income, or expenditure. There are no constraints on the *support* of the distribution of these characteristics in the society. The marginal distribution over individual preferences or demand functions (i.e. along the horizontal axis in Fig. 1.*a*) is arbitrary, but there is a qualitative restriction on the *shape* of the conditional distributions of income, or expenditure, for a given preference or demand function (i.e. along any vertical line in Fig. 1.*a*) : each such conditional income distribution is assumed to have a *continuous nonincreasing density* with a bounded support. The striking result is then that competitive market demand has a negative quasi-definite Jacobian matrix everywhere. This implies in particular that market demand for a particular commodity is a decreasing function of its own price, and that the weak axiom of revealed preference is satisfied in the aggregate.

Hildenbrand's result is an outstanding methodological achievement in that it shows by way of example the applicability of the distributional viewpoint to demand analysis. Hildenbrand (1989) has further argued (relatively convincingly) that, although his main assumption about conditional income distributions is contradicted by the facts, the aggregate "Law of Demand" is nevertheless likely to be satisfied in practice. Yet his result applies only to the case in which income is independent of prices. It is apparently difficult to get uniqueness and stability of a pure exchange competitive equilibrium beyond the restrictive situation in which individual endowments are colinear (Hildenbrand, 1983, Section 3, Hildenbrand and Kirman, 1988, Chapter 6). There are good reasons to think that a better understanding of the aggregation problem, which is clearly multidimensional, presumably requires distributional assumptions not only on income but also on the agents' other characteristics, i.e. on preferences or individual demand functions (along the horizontal axis in Fig. 1.*a*). Theoretical progress along these lines might also pave the way toward generating testable hypotheses not only on aggregate demand, but also on the distribution of the agents' observable choices.

Fig. 1.*a* Fig. 1.*b*

In order to implement this program, we need also here a *linear structure* on preferences, or on the corresponding demand functions, that is appropriate to the task at hand (of course, a linear structure that is "appropriate" for demand aggregation need not bear any relationship with a linear structure, such as intermediate preferences, that is "appropriate" to deal with a social choice problem). We shall argue that such a structure is indeed already available : it is in fact generated by the group of *affine transformations* of the commodity space employed in a related context by A. Mas-Colell and W. Neuefind (1977, p. 597), and by E. Dierker, H. Dierker and W. Trockel (1984). Such affine transformations are closely related to the concept of *household equivalence scales* that has been introduced long ago in applied demand analysis by Prais and Houthakker (1955) and by Barten (1964), and that has been subsequently much used in econometric studies (Deaton and Muellbauer (1980), Muellbauer (1980), Jorgenson and Slesnick (1987), Deaton (1986)). The work of E. Dierker, H. Dierker and W. Trockel (1984) was concerned with showing that suitably dispersed distributions over the space of consumers' characteristics led to a nice "smoothing" of competitive aggregate demand – for an extensive review of that line of work, see Trockel (1984). Particular instances of such affine transformations i.e. *homothetic transformations*, have been already employed (Grandmont (1987)) to demonstrate that Hildenbrand's theorem (1983) could be obtained through distributional assumptions on preferences rather than on income. Similar homothetic transformations also appeared under the name of *replicas* in the work of M. Jerison (1982, Remark

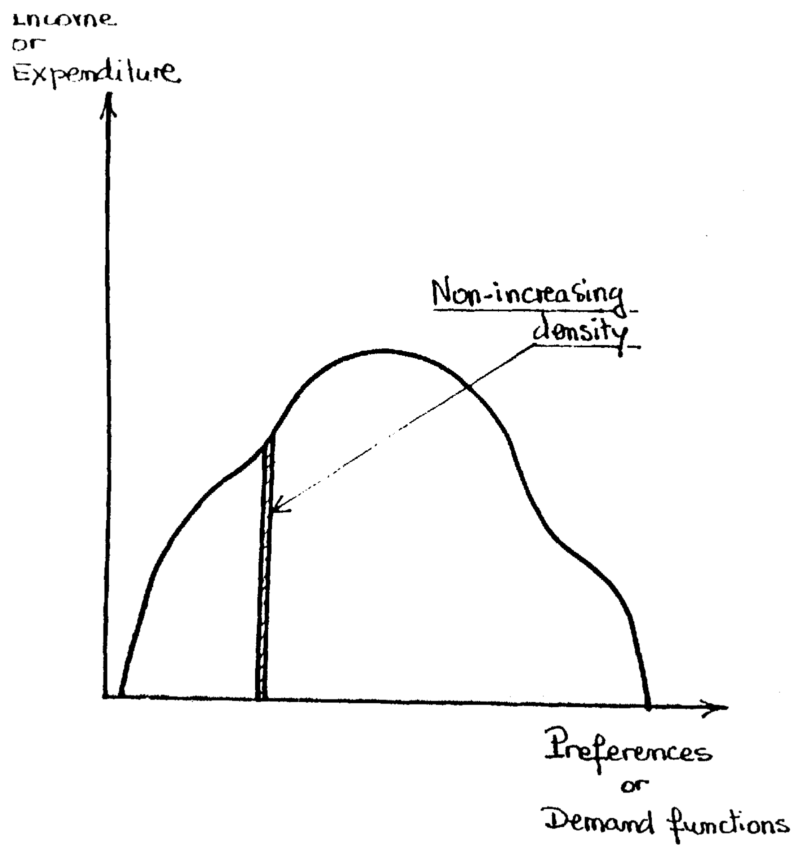

Income
or
Expenditure
Non-increasing
density
Preferences
or
Demand functions

Fig. 1.a

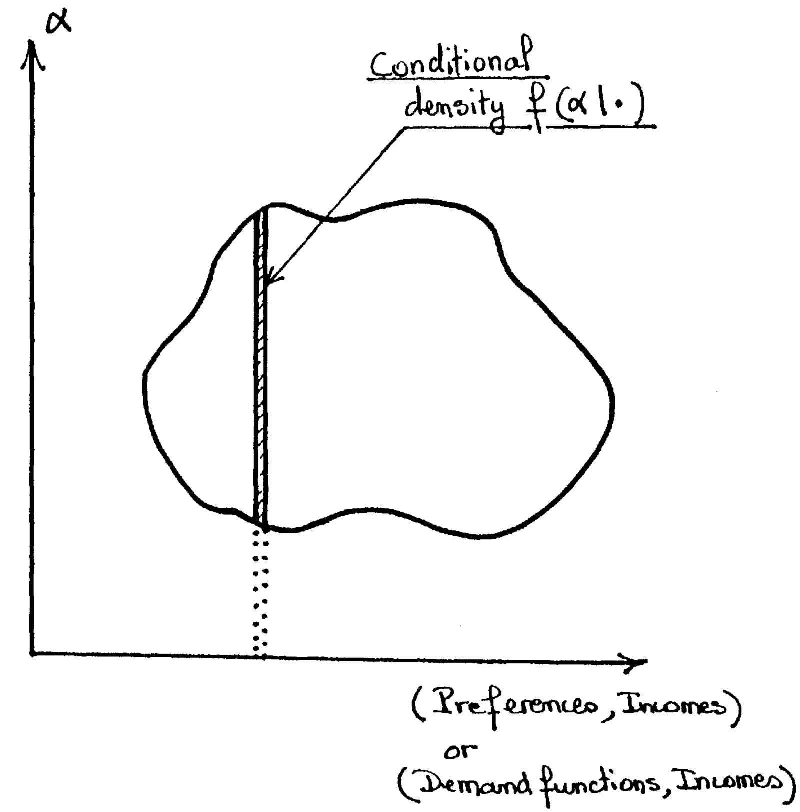

α
Conditional
density f(α|·)
(Preferences, Incomes)
or
(Demand functions, Incomes)

Fig. 1.b

5). More recently, H. Dierker (1989) and E. Dierker (1989) have used this linear structure to study how demand aggregation might help in ensuring existence of equilibrium in imperfect competition models.

The principle of the approach is rather straightforward. Let $\ell$ be the number of available commodities. For every preference relation $\succsim$ defined on the nonnegative orthant of the commodity space and, say, for every income $w$ , one generates a new pair ( $\succsim_\alpha$ , w) involving the same income, but in which the new preference $\succsim_\alpha$ is derived from the original one by stretching the axis corresponding to every commodity $h$ by a factor $e^{\alpha_h}$ . This transformation is thus indexed by an $\ell$-dimensional vector $\alpha = (\alpha_1,\ldots,\alpha_\ell)$ and the composition of two transformations indexed by $\alpha'$ and $\alpha''$ is equal to the transformation indexed by $\alpha' + \alpha''$. This is simply an "exponential version" of the affine transformations introduced by Mas-Colell and Neuefind (1977), and by Dierker *et alii* (1984), who parametrized these transformations instead by $\beta_h = e^{\alpha_h} > 0$ . A distribution on the agents' characteristics can then be described, along the lines presented in Dierker *et alii* (1984), by specifying first a marginal distribution over the space of (preferences, incomes) – symbolically, over the horizontal axis in Fig. 1.*b*. Second, we define for each such preference $\succsim$ and income w, a conditional distribution over all transforms ( $\succsim_\alpha$ , w) or equivalently, over the indexing vectors of parameters $\alpha$ – i.e. along any "vertical section" in Fig. 1.*b*. The approach is in fact quite flexible. One can for instance consider directly demand functions instead of preferences on the horizontal axis in Fig. 1.*b*, and accommodate various degrees of "rationality". One can assume that individual demands are generated through utility maximization as we just did in this discussion, or only that they satisfy WARP. Or go to the other extreme and assume only homogeneity and Walras's law. In all cases, the demand functions in any vertical section in Fig. 1.*b* form an *equivalence class* generated by performing affine transformations of the commodity space (and of prices) that are indexed by the $\ell$-dimensional vector $\alpha$ . Any conditional distribution on each vertical section can be therefore viewed as a distribution on the Euclidean space $\mathbb{R}^\ell$. The question we wish to address is whether there are plausible qualitative or quantitative restrictions on the *shapes of these conditional distributions* that lead to interesting properties of aggregate demand.

In this framework, it becomes in principle possible to assess the validity of a conjecture often made in this area, from Wald, Hicks, to Arrow, Hahn and quite a few others, claiming that enough "heterogeneity" should yield a nicely behaved aggregate demand. A common heuristic argument starts with the assumption that some form of "rationality", i.e. WARP, holds at the individual level. Then from the Slutsky equation, any "pathology" that may occur at the microeconomic level must come from the income terms. It is argued, however, that "pathological" market demand features are unlikely to arise at the aggregate level when there is enough behavioral "heterogeneity" in the society, as income terms should then more or less cancel each other. Another example, due to Becker (1962), does not use individual "rationality" but assumes that individual choices are uniformly distributed on each budget plane. Market demand is then quite nice, although special, since it is of the Cobb-Douglas type with expenditures equally divided among commodities. The present framework should enable us to discuss these issues in a meaningful way since we can now talk qualitatively and quantitatively about the "degree of behavioral heterogeneity" in the system by assessing the dispersion, or the variance, of the conditional distributions over the vectors of parameters $\alpha$ within each vertical section in Fig. 1.*b*.

We demonstrate in this paper that if every commodity is desired in the aggregate, then total market demand has indeed good properties if there is enough behavioral heterogeneity in the system, i.e. if the conditional distributions over the vectors of parameters $\alpha$ in each vertical section in Fig. 1.*b* have densities that are "flat enough". What happens is that flat conditional distributions have a regularizing impact on aggregate budget shares, by making them more independent of prices.[1] This will be proved, however, in the spirit of Becker's example, *without assuming any form of individual rationality other than homogeneity and Walras's law* (an important difference with Becker's example is that in our framework, "flat" conditional distributions over the vectors of parameters $\alpha$ do not necessarily imply a large dispersion of choices). Such a result is in a sense comforting since it shows that the Walrasian paradigm is more robust than some might have thought, but it also creates some "embarrassment of riches", as it suggests that individual "rationality" may not be as necessary as some economists would like to believe to the construction of a "sound" macroeconomic theory.

This report is organized as follows. We describe precisely in Section 1 the affine transformations of the commodity space and the linear structures they induce on preferences on one hand, on demand functions on the other. We consider the problem of aggregating market demands when income is independent of prices in Section 2. We show there that when every commodity is desired in the aggregate, total market demand has a negative dominant diagonal Jacobian matrix on a set of prices that fills eventually the whole price space as the degree of behavioral heterogeneity in the system grows. Ultimately, total market demand becomes strictly monotone everywhere, which implies that the weak axiom of revealed preference is satisfied in the aggregate. We apply these methods in Section 3 to the analysis of a private ownership pure exchange economy, in which individual incomes are dependent on prices. We show there that a combination of aggregate desirability and behavioral heterogeneity yields an aggregate excess demand that has the gross substitutability property on a set of prices, the size of which goes up with the degree of behavioral heterogeneity in the system. When the degree of heterogeneity grows, other things being equal, equilibrium becomes unique and stable in any standard *tatonnement* process, with a basin of attraction that fills eventually the whole price space. As a matter of fact, the weak axiom of revealed preference is ultimately satisfied in the aggregate, as between the equilibrium price and any other non-equilibrium price vector, when the degree of heterogeneity goes up. The implications of these results are briefly discussed in Section 4.

## 1. THE $\alpha$-TRANSFORM

The aim of this section is to describe a few facts about specific affine transformations of the commodity space, and the linear structures they induce on preferences and demand functions.

We abstract from the complications arising from production and consider a pure exchange economy. The number of available commodities is $\ell \geqq 2$ . For any vector $x = (x_1,\dots,x_\ell)$ in the commodity space $\mathbb{R}^\ell$ , and any collection of real parameters $\alpha = (\alpha_1,\dots,\alpha_\ell)$, one can generate a new vector $x_\alpha$ by blowing up the amount of each commodity $h$ by the factor $e^{\alpha_h}$

(1.1) $$x_\alpha = e^\alpha \otimes x = (e^{\alpha_1} x_1,\dots,e^{\alpha_\ell} x_\ell) .$$

This condition defines a space of *affine transformations* of the commodity space onto itself, that is indexed by $\alpha$ and is therefore isomorphic to $\mathbb{R}^{\ell}$ : the composition of two transformations indexed by $\alpha'$ and $\alpha''$ is equal to the transformation indexed by $\alpha' + \alpha''$ . These transformations generate in turn linear structures on preference relations and on demand functions.

We look first at the linear structure induced on preferences, as a most convenient intermediary step to justify the corresponding linear structure on demand functions.

To fix ideas, we focus on *preference relations* that are defined by a complete, transitive relation over the nonnegative orthant of the commodity space, i.e. $\mathbb{R}^{\ell}_{+}$ , with the understanding that $x \succsim y$ means "the commodity bundle $x \geqq 0$ is preferred or equivalent to the commodity bundle $y \geqq 0$". Strict preference is noted $x \succ y$ . To simplify matters, we consider exclusively preference relations that are continuous, strictly convex and locally nonsatiated [2] . For any collection of real parameters $\alpha = (\alpha_1, \dots, \alpha_{\ell})$ , the *$\alpha$-transform* of a preference relation $\succsim$ is defined as the relation $\succsim_{\alpha}$ that would be identical to $\succsim$ if the units of measurement of each commodity h had been multiplied by $e^{\alpha_h}$ , that is

$$(1.2) \qquad x \succsim_{\alpha} y \quad \text{if and only if} \quad (e^{-\alpha} \otimes x) \succsim (e^{-\alpha} \otimes y) .$$

The preference relation $\succsim$ is continuous if and only if it has a continuous utility representation u(x), see Debreu (1964). To economize on notation, we shall keep throughout the same symbol or function (here u) to denote an entity and its counterpart generated through an $\alpha$-transformation. With this convention, a continuous utility function representing $\succsim_{\alpha}$ is

$$(1.3) \qquad u(\alpha,x) = u(e^{-\alpha} \otimes x) .$$

It follows immediately that the transformed relation $\succsim_{\alpha}$ inherits from the original preference the properties of continuity, of strict convexity and of local nonsatiation. It is clear that the set of all $\alpha$-transforms of a given preference relation $\succsim$ , when $\alpha$ is arbitrary, is isomorphic to $\mathbb{R}^{\ell}$ .

We look next at the linear structure induced on demand functions by the space of affine transformations under consideration. In order to convey the central idea behind the formal definition, it is convenient to start with demand functions that are generated by utility maximization. Given any preference relation $\succsim$ that is continuous and strictly convex, Walrasian demand $\xi(p,w)$ is well defined for every vector of positive prices p and every nonnegative income $w \geqq 0$ . Specifically, if u(x) is a continuous utility representation of $\succsim$ , then $\xi(p,w)$ is obtained by maximizing u(x) with respect to $x \geqq 0$ subject to the budget constraint $p \cdot x \leqq w$ , in which $p \cdot x$ is the scalar product $\Sigma_h p_h x_h$ . Walrasian demand is clearly homogenous of degree 0 in (p,w) and local nonsatiation implies Walras's Law, i.e. $p \cdot \xi(p,w) \equiv w$ . Next, for every vector of parameters $\alpha$ in $\mathbb{R}^\ell$ , the Walrasian demand function associated to the transformed preference relation $\succsim_\alpha$ is noted $\xi(\alpha,p,w)$ – here again, we keep the same symbol $\xi$ to denote the original demand function and its $\alpha$-transform. Then $\xi(\alpha,p,w)$ is obtained by maximizing $u(\alpha,x) = u(e^{-\alpha} \otimes x)$ with respect to $x \geqq 0$ subject to

$$p \cdot x = (e^\alpha \otimes p) \cdot (e^{-\alpha} \otimes x) \leqq w \ .$$

One gets accordingly

$$\xi(\alpha,p,w) = e^\alpha \otimes \xi(e^\alpha \otimes p,w) \ . \tag{1.4}$$

It is important to note that, although we derived this formula for Walrasian demands, (1.4) defines a transformation indexed by the vector $\alpha$ even for demand functions $\xi$ that may not be generated through utility maximization. We shall adopt throughout this more general viewpoint.

DEFINITION 1.1. *A demand function* $\xi(p,w)$ *is defined for all vectors of positive prices* $p \in \mathrm{Int}\mathbb{R}^\ell_+$ *and all positive incomes* $w > 0$, *takes values in* $\mathbb{R}^\ell_+$ , *is homogenous of degree* 0 *in* (p,w) *and satisfies Walras's Law, i.e.* $p \cdot \xi(p,w) \equiv w$ .

Then condition (1.4) directly defines the *$\alpha$-transform* of such a demand function : it is the demand that would be equal to $\xi(p,w)$ if the units of measurement of each commodity h had been multiplied by $e^{\alpha_h}$ . It follows immediately that the $\alpha$-transform inherits homogeneity, Walras's Law and any regularity property of $\xi$. The $\alpha$-transform would satisfy the weak axiom of revealed preference (WARP) if $\xi$ did, but again, we shall *not* need such rationality conditions in the sequel. Note that, although Walras's Law implies that some commodities are indeed purchased when $w > 0$ , we do not require that all commodities be always bought in positive amounts, i.e. we allow some components $\xi_h(p,w)$ to be equal to 0.

Condition (1.4) defines an *equivalence class* among demand functions when $\xi$ is fixed and $\alpha$ varies (which is identical to the equivalence class arising from the concept of *household equivalence scales* developed in the empirical literature). For a given $\xi$, the equivalence class is indexed by $\alpha$ and is thus isomorphic to the finite dimensional vector space $\mathbb{R}^{\ell}$ .

We now describe an elementary but very useful identity that relates the changes of a demand function when some price, or income, or some transformation parameter varies.

LEMMA 1.2. *The $\alpha$-transform $\xi(\alpha,p,w)$ is (continuously) differentiable with respect to* p *if and only if it is (continuously) differentiable with respect to $\alpha$. In that case, for every* $h = 1,\dots,\ell$ ,

$$(1.5) \qquad p_h \frac{\partial \xi_h}{\partial p_h}(\alpha,p,w) + \xi_h(\alpha,p,w) = \frac{\partial \xi_h}{\partial \alpha_h}(\alpha,p,w) \ ,$$

*while for every* $k \neq h$ ,

$$(1.6) \qquad p_k \frac{\partial \xi_h}{\partial p_k}(\alpha,p,w) = \frac{\partial \xi_h}{\partial \alpha_k}(\alpha,p,w) \ .$$

*In addition,*

$$(1.7) \qquad w \frac{\partial \xi_h}{\partial w}(\alpha,p,w) = \xi_h(\alpha,p,w) - \Sigma_k \frac{\partial \xi_h}{\partial \alpha_k}(\alpha,p,w)$$

The proof of (1.5) and (1.6) is immediate by differentiating (1.4) with respect to $\alpha$ and p . Then (1.7) follows from the homogeneity of degree 0 of demand with respect to (p,w) and Euler's identity. Note that if we introduce the *expenditure functions* for the original demand function $w_h(p,w) = p_h \xi_h(p,w)$ and their counterparts for the $\alpha$-transform $w_h(\alpha,p,w) = p_h \xi_h(\alpha,p,w)$ , then (1.5) and (1.6) can be given the more concise form

$$(1.8) \qquad \frac{\partial w_h}{\partial \mathrm{Log} p_k}(\alpha,p,w) = \frac{\partial w_h}{\partial \alpha_k}(\alpha,p,w) \qquad \text{for all } h,k \ .$$

This relation is not surprising if one thinks about the *global* effect on expenditures of a variation of some price, or of some transformation parameter, since one has $w_h(\alpha,p,w) = w_h(e^\alpha \otimes p,w)$. Variations of $\alpha$ and p that leave unaltered $e^\alpha \otimes p$ do not change expenditures. Formula (1.8) shows why the "exponential" parameterization that we chose here to index the space of affine transformations, is in a sense the most "natural" one.

The foregoing argument shows that a demand function is *invariant*, i.e. satisfies $\xi(\alpha,p,w) = \xi(p,w)$ for all $\alpha,p,w$, if and only if $w_h(e^\alpha \otimes p,w) = w_h(p,w)$ for all $\alpha,p,w$ and h. Expenditures on each commodity h are independent of prices and (in view of the homogeneity of a demand function) proportional to income. The demand function is thus of the Cobb-Douglas type. The "if" part of this statement, which turns out to be important for the analysis below, was pointed out in the working paper version of Grandmont (1987) – for converses, see Trockel (1989) and H. Dierker (1989). To sum up

LEMMA 1.3. *A demand function is invariant if and only if it is of the Cobb-Douglas type, i.e. there are real numbers* $r_h \geqq 0$, *with* $\Sigma_h r_h = 1$, *such that* $\xi_h(p,w) = r_h\, w/p_h$ *for all* h.

To conclude this section, we present two diagrams that may help to visualize, in the case of two commodities, what is done when performing an $\alpha$-transformation. The case $\alpha_1 > 0$, $\alpha_2 = 0$ is pictured in Fig. 2.*a*. One stretches then the units of measurement of the first commodity only. The curve DD' is an *offer curve*, i.e. the locus of all demands $\xi(p,w)$ when w and $p_2$ are fixed while $p_1$ varies. The $\alpha$-transform is obtained by applying (1.4) directly. One looks first at the point B of the offer curve corresponding to $\xi(e^{\alpha_1} p_1,p_2,w)$ and one scales it back to get C on the original budget line. In the case of a Cobb-Douglas demand, the offer curve would of course be horizontal, and the $\alpha$-transform C would coincide with the original demand A. Fig. 2.*b* is the counterpart of Fig. 2.*a* when

$\alpha_1 = \alpha_2 = \alpha > 0$. The transformation is then an homothecy of center 0 and ratio $e^{\alpha}$. The curve DD' is now an *Engel curve*, i.e. the locus of all demands $\xi(p,w)$ when income varies, the price system being fixed. If the demand function is homogenous of degree 1 in income, as in the case of a Cobb-Douglas demand, the Engel curve is a ray through the origin. The $\alpha$-transform, represented by C, then coincides with the original demand A.

Fig. 2.*a*

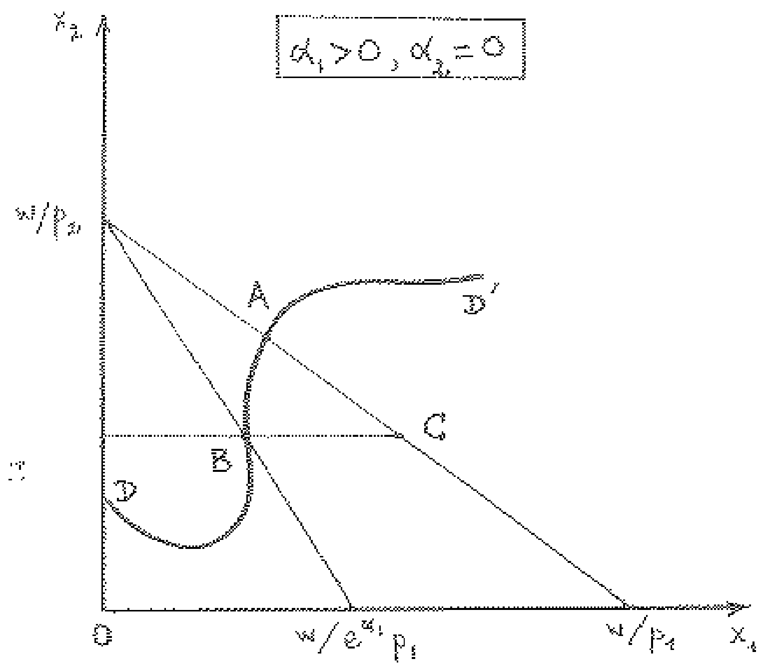
Fig. 2.*b*

*Remark 1.4.* As mentioned above, *homothetic transformations* appear as a particular case of the foregoing affine transformations when all parameters $\alpha_h$ are equal. Homothetic transforms have been analyzed in Grandmont (1987) for the purpose of aggregating demand functions that satisfy the weak axiom of revealed preference.

## 2. MARKET DEMAND

The equivalence class of all $\alpha$-transforms of a given demand function is isomorphic to $\mathbb{R}^{\ell}$. The purpose of this section is to analyze how the shape of the distribution of demand functions in each equivalence class can influence the manner in which price and/or income changes affect aggregate demand.

The characteristics of an individual agent (or of a bunch of individual agents) are assumed to be described by a demand function (in the sense of Definition 1.1) and an income level. We suppose here income to be independent of prices, but this assumption will be relaxed in the next section. We shall generate a distribution over these characteristics in the following way. There is first a set A of "types" which, to simplify matters, we take to be a *separable metric space* [3]. A distribution over types (symbolically, along the horizontal axis in Fig. 3) is then given by a probability measure $\mu$ on A. To each type in A corresponds a demand function $\xi_a(p,w)$ and an income level $w_a > 0$. Second, we specify for each type, a conditional distribution over the space of all $\alpha$-transforms $\xi_a(\alpha,p,w)$ of the demand function $\xi_a$, or equivalently on the space $\mathbb{R}^{\ell}$ of indexing parameters $\alpha$ (along vertical lines in Fig. 3). We assume that this

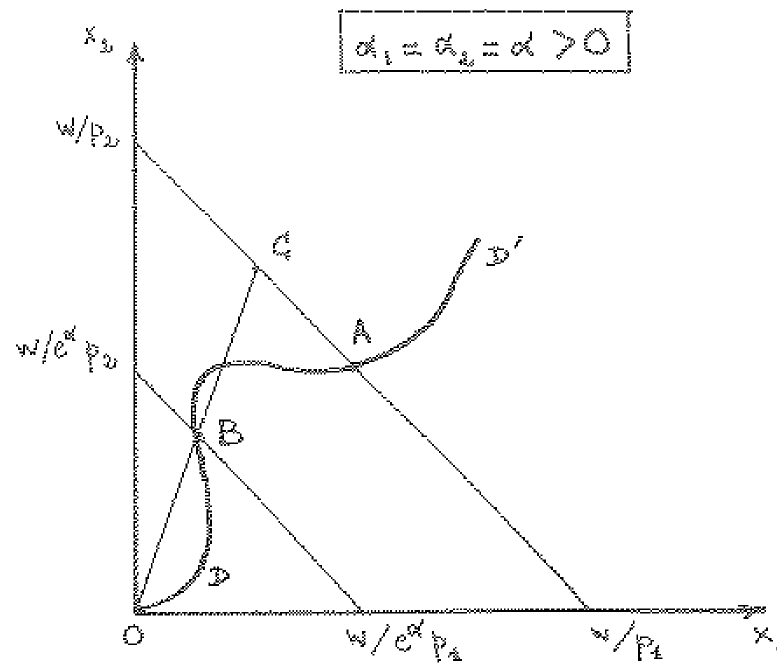



$A = \xi(p_1, p_2, w)$

$B = \xi(e^{\alpha_1} p_1, p_2, w)$

$C = \xi(\alpha, p_1, p_2, w)$

Fig. 2. a




$A = \xi(p_1, p_2, w)$

$B = \xi(e^{\alpha} p_1, e^{\alpha} p_2, w)$

$C = \xi(\alpha, p_1, p_2, w)$

Fig. 2. b

conditional distribution has a density noted $f(\alpha|a)$. The overall *distribution of characteristics* in the society is thus described by the marginal probability measure $\mu$ on A and the conditional densities $f(\alpha|a)$ on $\mathbb{R}^\ell$.

Fig. 3

Per capita aggregate demand is also defined in two steps. *Conditional market demand* is obtained by integration over the space of all $\alpha$-transforms of the demand function corresponding to a given type

$$(2.1) \qquad X(a,p,w) = \int_{\mathbb{R}^\ell} \xi_a(\alpha,p,w)\, f(\alpha|a)\, d\alpha\ .$$

The integral is well defined if $\xi_a$ is continuous in $(p,w)$, since for each commodity $h$, one has $0 \leqq p_h\, \xi_{ah}(\alpha,p,w) \leqq w$. Conditional market demand $X(a,p,w) \geqq 0$ is in fact continuous, with $p \cdot X(a,p,w) \equiv w$, if the demand function $\xi_a(p,w)$ is continuous in $(a,p,w)$ and if the conditional density $f(\alpha|a)$ is continuous in $(\alpha,a)$. Thus if we assume in addition

(2.a) *The income level* $w_a > 0$ *depends continuously on the type* $a$. *Per capita income is finite, i.e.*

$$\bar{w} = \int_A w_a\, \mu(da) < +\infty$$

then *total market demand* given by

$$(2.2) \qquad X(p) = \int_A X(a,p,w_a)\, \mu(da)$$

is well defined, nonnegative, continuous in p and satisfies $p \cdot X(p) \equiv \bar{w}$. [4]

The main thrust of our approach will be to study how the qualitative or quantitative properties of the conditional densities $f(\alpha|a)$ affect aggregate demand, and more precisely its partial derivatives with respect to prices. Clearly, we have to assume that the demand functions $\xi_a$ and the

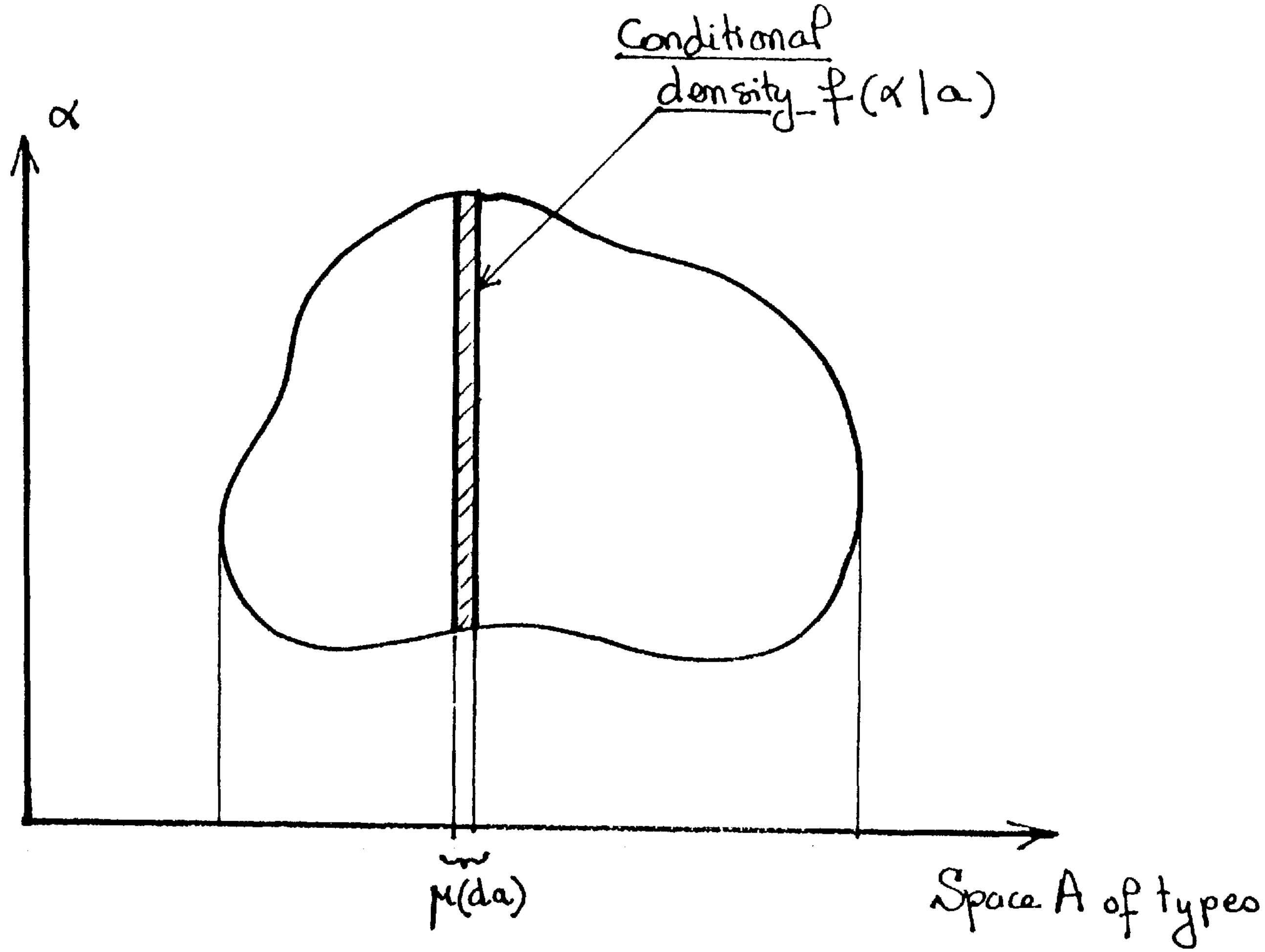



Fig. 3

conditional densities are regular enough to ensure differentiability in the aggregate. This is done through the following assumptions.

(2.*b*) 1) *The demand function* $\xi_a(p,w)$ *is continuous in* $(a,p,w)$ .

2) *The conditional density* $f(\alpha|a)$ *is continuous in* $(\alpha,a)$. *It has partial derivatives* $\frac{\partial f}{\partial \alpha_k}(\alpha|a)$ , *and they are continuous in* $(\alpha,a)$. *Moreover, for each type* a , *partial derivatives are uniformly integrable, i.e. for every* $k = 1,\dots,\ell$

$$m_k(a) = \int_{\mathbb{R}^\ell} \left|\frac{\partial f}{\partial \alpha_k}(\alpha|a)\right| \, d\alpha < +\infty$$

(2.*c*) *For every commodity* k, $m_k(a)$ *is bounded above by* $m_k$ *for* $\mu$*-almost every type* a.

As an incidental remark, individual demands $\xi_a$ need not be differentiable in the present context : smooth conditional densities are enough to yield a nice smoothing of aggregate demand. Also, since $m_k(a)$ is continuous, assumption (2.*c*) would be automatically satisfied if A, or the support of the distribution $\mu$, were compact.

*Asymptotic invariance*

Our approach exploits the fact that with enough behavioral heterogeneity in each equivalence class of $\alpha$-transforms, i.e. if the conditional densities $f(\alpha|a)$ are "flat enough", then aggregate demand is nicely behaved, in the sense that aggregate expenditures on each commodity become more or less independent of prices. In the limit, each conditional market demand would be of the Cobb-Douglas type.

To see heuristically the intuition underlying this point, consider the $\alpha_o$-transform of conditional market demand (2.1), with $\alpha_o$ being an arbitrary $\ell$-dimensional vector of parameters, which we note $X(\alpha_o,a,p,w)$. We have clearly

$$X(\alpha_0,a,p,w) = \int_{\mathbb{R}^\ell} \xi_a(\alpha+\alpha_0,p,w) \, f(\alpha|a) \, d\alpha ,$$

which yields after the change of variables $\beta = \alpha + \alpha_0$ ,

$$(2.3) \qquad X(\alpha_0, a, p, w) = \int_{\mathbb{R}^\ell} \xi_a(\beta, p, w)\ f(\beta - \alpha_0 | a)\ d\beta$$

This expression shows that *the $\alpha_0$-transform of conditional market demand is obtained by translating the conditional distribution in the equivalence class of $\alpha$-transforms by $\alpha_0$.* Now, if conditional densities $f(\alpha|a)$ are "flat", they are "close" to being invariant by an arbitrary translation. But then (2.3) implies that conditional market demands are "close" to being invariant by an arbitrary $\alpha_0$-transformation and thus, in view of Lemma 1.3, "close" to being of the Cobb-Douglas type. Total market demand (2.2), being not "too far" from an average of Cobb-Douglas demands, should then be rather "well behaved".

*Aggregate price effects*

The foregoing heuristic argument suggests that we should try to assess the extent to which aggregate demand is "well behaved" as a function of the degree of behavioral heterogeneity in the system. To implement this program, we shall design *approximate* quantitative evaluations of the partial derivatives of aggregate demand, that depend on some measure of the degree of "flatness" of the conditional densities $f(\alpha|a)$ in each equivalence class of $\alpha$-transforms, and that are valid well before the "Cobb-Douglas limit".

We first have to show that conditional market demand (2.1) has, for each *a*, partial derivatives with respect to prices, and that these partial derivatives are jointly continuous in (a,p,w). To demonstrate this point, we proceed indirectly by using Lemma 1.2 : we consider the $\alpha_0$-transform of conditional market demand and show differentiability with respect to $\alpha_0$ . Under assumption (2.*b*), the $\alpha_0$-transform (2.3) has indeed partial derivatives with respect to $\alpha_0$ . These partial derivatives can be computed by differentiation under the integral sign in (2.3), and they are jointly continuous in $(\alpha_0, a, p, w)$ [5] . If we perform these operations and evaluate the outcome at $\alpha_0 = 0$ , we get

$$\frac{\partial X_h}{\partial \alpha_k}(0,a,p,w) = -\int_{\mathbb{R}^\ell} \xi_{ah}(\alpha,p,w)\,\frac{\partial f}{\partial \alpha_k}(\alpha|a)\,d\alpha\ .$$

We can now conclude from Lemma 1.2 that conditional market demand has indeed partial derivatives with respect to prices. With the convention that $\delta_{hk}$ is the Kronecker symbol, i.e. $\delta_{hk}$ is equal to 1 if $h = k$ and to 0 otherwise, we get then for every h , k, in view of (1.5) and (1.6)

$$(2.4)\qquad p_k\,\frac{\partial X_h}{\partial p_k}(a,p,w) + \delta_{hk}\,X_h(a,p,w) = -\int_{\mathbb{R}^\ell} \xi_{ah}(\alpha,p,w)\,\frac{\partial f}{\partial \alpha_k}(\alpha|a)\,d\alpha\ .$$

Our strategy is to estimate bounds for the partial derivatives of conditional market demand that depend only on some characteristic features of the conditional density $f(\alpha|a)$. Now the nonnegative demand $\xi_{ah}(\alpha,p,w)$ is bounded above by $w/p_h$ . So, from (2.4), we get immediately the bounds

$$(2.5)\qquad \left|p_k\,\frac{\partial X_h}{\partial p_k}(a,p,w) + \delta_{hk}\,X_h(a,p,w)\right| \leqq w\ m_k(a)/p_h\ .$$

This expression sets upper and lower limits for the partial derivatives of conditional market demand, that depend only on the behaviour of the conditional density $f(\alpha|a)$, as characterized by the coefficients $m_k(a)$ . An important feature of the result is that it is "additive", in the sense that if one puts together different types, then we may set the bound for the mixture as the supremum of the elementary bounds. More precisely, it is not difficult to verify that, under assumptions (2.*a*), (2.*b*), (2.*c*), total market demand is continuously differentiable, that partial derivatives can be computed by differentiating (2.2) under the integral sign, and that bounds for these partial derivatives can be obtained as in (2.5) by replacing the income level w by per capita income $\bar{w}$ and $m_k(a)$ by $m_k$ . Formally [6]

PROPOSITION 2.1. *Assume* (2.*a*), (2.*b*), (2.*c*). *Then total market demand* X(p) *is continuously differentiable, with*

$$\frac{\partial X_h}{\partial p_k}(p) = \int_A \frac{\partial X_h}{\partial p_k}(a,p,w_a)\ \mu(da)\ ,$$

*and these partial derivatives satisfy*

(2.6) $$\left|p_k \frac{\partial X_h}{\partial p_k}(p) + \delta_{hk}\ X_h(p)\right| \leqq \bar{w}\ m_k/p_h\ .$$

*Remark 2.2. (Expenditure functions)* If one uses the expenditure functions $w_{ah}(\alpha,p,w) = p_h\ \xi_{ah}(\alpha,p,w)$ , and their aggregate counterparts $W_h(a,p,w) = p_h\ X_h(a,p,w)$ or $W_h(p) = p_h\ X_h(p)$ , then (2.4) and (2.6) become respectively, for every h,k

(2.7) $$\frac{\partial W_h}{\partial \mathrm{Log} p_k}(a,p,w) = -\int_{\mathbb{R}^\ell} w_{ah}(\alpha,p,w)\ \frac{\partial f}{\partial \alpha_k}(\alpha|a)\ d\alpha$$

(2.8) $$\left|\frac{\partial W_h}{\partial \mathrm{Log} p_k}(p)\right| \leqq \bar{w}\ m_k$$

This more concise formulation shows immediately how to proceed if one wishes to get the expression of, and estimate some bounds for, higher order derivatives of aggregate demand. Let us strengthen (2.*b*) by requiring the conditional densities $f(\alpha|a)$ to have continuous, uniformly integrable, partial derivatives of higher orders. By the same kind of arguments as in the text, one gets the following general formula, for all commodities h and k, and all relevant orders $r \geqq 1$

(2.9) $$\frac{\partial^r W_h}{\partial \mathrm{Log} p_{k_1} \ldots \partial \mathrm{Log} p_{k_r}}(a,p,w) = (-1)^r \int_{\mathbb{R}^\ell} w_{ah}(\alpha,p,w)\ \frac{\partial^r f}{\partial \alpha_{k_1} \ldots \partial \alpha_{k_r}}(\alpha|a)\ d\alpha.$$

Then if $m^{(r)}_{k_1\ldots k_r}$ is an upper bound of the integrals

$$\int_{\mathbb{R}^\ell} \left|\frac{\partial^r f}{\partial\alpha_{k_1}\ldots\partial\alpha_{k_r}}(\alpha|a)\right| d\alpha$$

over the support of $\mu$ , the general counterpart of (2.8) is

$$\left|\frac{\partial^r W_h}{\partial \mathrm{Log} p_{k_1}\ldots\partial \mathrm{Log} p_{k_r}}(p)\right| \leqq \bar{w}\, m^{(r)}_{k_1\ldots k_r} \quad . \tag{2.10}$$

*Diagonal dominance and the Law of Demand*

The coefficients $m_k$ appearing in (2.c) measure in some sense the heterogeneity of the distribution of the agents' demand functions within each equivalence class generated by $\alpha$-transforms, across all types $a$ of traders that are present in the society : the smaller these coefficients, the "flatter" the conditional densities $f(\alpha|a)$ [7] , and thus the larger the degree of heterogeneity of (conditional) market behaviour. We already argued that in such a case, market demand should mimic eventually what we would get from a single aggregate consumer who would maximize a Cobb-Douglas utility. Proposition 2.1 tells us how to make this statement precise. Indeed if all $m_k$ tend to 0, other things being equal, all cross partial derivatives $\frac{\partial X_h}{\partial p_k}(p)$, $h \neq k$ go to 0 uniformly on any compact set of positive prices, while the elasticity of aggregate demand for commodity h with respect to its own price, i.e. $p_h \frac{\partial X_h}{\partial p_h}(p)/X_h(p)$ , gets close to -1 at any vector of prices such that market demand $X_h(p)$ is positive. Alternatively, (2.8) tells us that aggregate budget shares become asymptotically independent of prices, uniformly on compact sets of prices, as the coefficients $m_k$ converge to 0. One should expect accordingly aggregate demand to be "well behaved" when there is enough heterogeneity of the agents' demand functions, i.e. when all coefficients $m_k$ are small, provided that aggregate demand for any commodity does not vanish.

In order to ensure a nonvanishing aggregate demand, we shall employ two simplifying assumptions. The first one is that all conditional densities within each $\alpha$-equivalence class are actually independent of the agents' type

(2.*d*) *For $\mu$-almost every* a, *the conditional density* $f(\alpha|a)$ *is independent of* a .

The second assumption is essentially an *aggregate desirability* condition. We assumed that an *individual* trader (or type) may choose not to purchase some commodities, in particular if their prices are too high. We want now to require that if we consider all types of traders present in the system, the *aggregate* budget share devoted to the consumption of any commodity is uniformly bounded away from 0.

(2.*e*) *For every commodity* h , *there exists* $\varepsilon_h > 0$, *with* $\Sigma_h \varepsilon_h \leqq 1$, *such that for all vectors of positive prices* p

$$p_h \int_A \xi_{ah}(p, w_a)\ \mu(da) \geqq \varepsilon_h \bar{w} .$$

The aggregate desirability condition is a condition accross types that is, in view of (2.*d*), independent of any assumption we make about distributions in each equivalence class of $\alpha$-transforms. It does not appear to be too unrealistic on first approximation if we think of each lower bound $\varepsilon_h$ on aggregate budget shares as relatively small. [8]

We are now ready to demonstrate that total market demand X(p) is nicely behaved if it does not vanish and if, other things being equal, the conditional density in each equivalence class of $\alpha$-transforms is flat enough.

THEOREM 2.3. *Assume* (2.*a*), (2.*b*), (2.*c*), (2.*d*), (2.*e*). *Then* $p_h\, X_h(p) \geqq \varepsilon_h\, \bar{w}$ *for every* h *and every* p. *The price elasticities of aggregate demand satisfy*

$$(2.11) \qquad \left| \frac{\partial \mathrm{Log} X_h}{\partial \mathrm{Log} p_k}(p) + \delta_{hk} \right| \leqq m_k / \varepsilon_h \ .$$

*This implies in particular*

1) *Total market demand for commodity* h *is a decreasing function of its own price, i.e.* $\frac{\partial X_h}{\partial p_h}(p) < 0$ , *if* $m_h < \varepsilon_h$ .

2) *Assume* $m_h < \varepsilon_h$ *for every commodity* h *and let* DD(m,ε) *be the set of prices* p *in* $\mathrm{Int}\mathbb{R}^{\ell}_{+}$ *such that* $\Sigma_k (m_k/p_k) < \varepsilon_h/p_h$ *for every* $h = 1,\ldots,\ell$ . *Then the Jacobian matrix of total market demand is such that*

$$\frac{\partial X_h}{\partial p_h}(p) < 0 \ , \qquad \left|\frac{\partial X_h}{\partial p_h}(p)\right| > \sum_{k \neq h} \left|\frac{\partial X_h}{\partial p_k}(p)\right|$$

*for every* p *in* DD(m,ε) *and has therefore a dominant diagonal on that set.*

3) *Assume* $m_k \ell < \varepsilon_h$ *for all commodities* h, k. *Then total demand has a negative quasi-definite Jacobian matrix, i.e.* $\Sigma_{h,k}\, v_h \frac{\partial X_h}{\partial p_k}(p)\, v_k < 0$ *for every* $v = (v_1,\ldots,v_\ell) \neq 0$ *and every price system* p *in* $\mathrm{Int}\mathbb{R}^{\ell}_{+}$ , *and is thus strictly monotone, i.e.* $(p-q)\cdot[X(p)-X(q)] < 0$ *whenever* $p \neq q$. *In particular, the weak axiom of revealed preference is satisfied in the aggregate, i.e.* $p\cdot X(q) \leqq \bar{w}$ , $X(q) \neq X(p)$ *implies* $q\cdot X(p) > \bar{w}$ .

*Proof*. We have from the independence assumption (2.*d*)

$$p_h\, X_h(p) = \int_{\mathbb{R}^\ell} p_h\, e^{\alpha_h} \left[\int_A \xi_{ah}(e^{\alpha} \otimes p, w_a)\, \mu(da)\right] f(\alpha)\, d\alpha$$

$$\geqq \int_{\mathbb{R}^\ell} \varepsilon_h\, \bar{w}\, f(\alpha)\, d\alpha = \varepsilon_h\, \bar{w} \ .$$

Then (2.11) follows immediately from (2.6), and 1) is clear. To prove 2), remark that from 1), $\frac{\partial X_h}{\partial p_h}(p) < 0$ in $DD(m,\varepsilon)$. In that case, (2.11) implies

$$\frac{1}{X_h(p)} \left|\frac{\partial X_h}{\partial p_h}(p)\right| \geqq (\varepsilon_h - m_h)/(p_h\, \varepsilon_h) \ ,$$

and for $k \neq h$

$$\frac{1}{X_h(p)} \left|\frac{\partial X_h}{\partial p_k}(p)\right| \leqq m_k/(p_k\, \varepsilon_h)$$

and the result follows. To prove 3), remark that $m_k \ell < \varepsilon_h$ for all h, k implies $\frac{\partial X_h}{\partial p_h}(p) < 0$ . Then if $v = (v_1,\ldots,v_\ell) \neq 0$, we are sure that $\Sigma_{h,k}\, v_h \frac{\partial X_h}{\partial p_k}(p)\, v_k < 0$ if

$$|\Sigma_h\, v_h^2 \frac{\partial X_h}{\partial p_h}(p)| > \Sigma_{h\neq k}\, |v_h||v_k|\left|\frac{\partial X_h}{\partial p_k}(p)\right| \ .$$

From (2.6), the left hand side of the foregoing inequality is bounded below by $\bar{w}\ \Sigma_h\ v_h^2(\varepsilon_h - m_h)/p_h^2$ , whereas the right hand side does not exceed $\bar{w}\ \Sigma_{h\neq k}\ |v_h||v_k|m_k/p_h\ p_k$ . If we let $v_h = p_h\ u_h$ , the Jacobian matrix of total market demand is thus negative quasi-definite if

$$\Sigma_h\, u_h^2\, \varepsilon_h > \Sigma_{h,k}\, |u_h||u_k|m_k \ .$$

The result follows under the assumption $m_k \ell < \varepsilon_h$ for all h, k, as one has always $\ell\ \Sigma_h\ u_h^2 \geqq \Sigma_{h,k}\ |u_h||u_k|$. Strict monotonicity of total market X(p) and the weak axiom of revealed preference in the aggregate follow from standard arguments, see Hildenbrand (1983). Q.E.D.

Given the $\varepsilon_h$'s , the pointed open cone $DD(m,\varepsilon)$ appearing in the theorem can be made to include any a priori given compact set of positive prices in $\mathrm{Int}\mathbb{R}_+^\ell$ , by making the $m_h$'s small enough. Other things being equal, increasing heterogeneity of the distribution of demand functions in each equivalence class of α-transforms yields negative diagonal dominance on a set of prices that fills eventually the whole region of admissible prices. At some point, total market demand becomes strictly monotone, and the weak axiom of revealed preference is satisfied in the aggregate, for all prices [9].

*Aggregate income effects*

The foregoing arguments dealt with the influence of the distribution of the agents' characteristics on the partial derivatives of aggregate demand with respect to prices. We ask now the same sort of question about aggregate income effects, or more specifically about how dispersion of the conditional densities $f(\alpha|a)$ influences the partial derivatives of aggregate demand with respect to per capita income $\bar{w}$ .

To make the question meaningful, we must of course specify how the distribution of income varies when per capita income $\bar{w}$ changes. We choose here to keep this distribution fixed. This means that $w_a = \theta_a \bar{w}$ ,in which the $\theta_a$'s are fixed with $\int_A \theta_a \mu(da) = 1$. Then conditional market demand $X(a,p,w)$ is defined as before by (2.1), while total market demand is

$$(2.12) \qquad X(p,\bar{w}) = \int_A X(a,p,\theta_a \bar{w}) \mu(da) \ .$$

Under assumption (2.*b*), conditional market demand has continuous partial derivatives. Upper and lower bounds for the partial derivatives with respect to income are then obtained by remarking that $X_h(a,p,w)$ is homogenous of degree 0 in $(p,w)$, using Euler's identity as well as the inequalities (2.5). This yields

$$(2.13) \qquad \left| w \frac{\partial X_h}{\partial w} (a,p,w) - X_h(a,p,w) \right| \leqq w \, \Sigma_k \, m_k(a)/p_h \ .$$

We can follow the same procedure for total market demand by using the inequalities (2.6) in Proposition 2.1 and (2.11) in Theorem 2.3. This leads to

PROPOSITION 2.4. *Assume* (2.*a*), (2.*b*), (2.c). *Then total market demand* $X(p,\bar{w})$ *defined in* (2.12) *is continuously differentiable, with*

$$(2.14) \qquad \left|\bar{w}\frac{\partial X_h}{\partial \bar{w}}(p,\bar{w}) - X_h(p,\bar{w})\right| \leqq \bar{w}\,\Sigma_k\, m_k/p_h\ .$$

*If in addition* (2.*d*), (2.*e*) *hold, the income elasticity of aggregate demand satisfies*

$$(2.15) \qquad \left|\frac{\partial \mathrm{Log}X_h}{\partial \mathrm{Log}\bar{w}}(p,\bar{w}) - 1\right| \leqq \Sigma_k\, m_k/\varepsilon_h\ .$$

*In particular,* $\dfrac{\partial X_h}{\partial \bar{w}}(p,\bar{w}) > 0$ *whenever* $\Sigma_k\, m_k < \varepsilon_h$ .

*Unbounded supports*

The main result of this section is in a sense the identity (2.4) – or its counterpart (2.7). The method followed in the text in order to get a bound for the right hand side of (2.4) was chosen because it apparently lends itself to an easy generalization to the general exchange equilibrium framework that will be considered in the next section, in which income is price dependent. Here is another variant that could also be useful, and that yields sharper bounds without assumptions (2.*d*), (2.*e*), if we are willing to assume that conditional densities have unbounded support.

Let us assume (2.*a*), (2.*b*) and strengthen assumption (2.*c*) by replacing it by

(2.*c*') *For each commodity* k , *there exists* $m_k$ *such that*

$$\left|\frac{\partial f}{\partial \alpha_k}(\alpha|a)\right| \leqq m_k\, f(\alpha|a)$$

*for* $\mu$*-almost every type* a *and for every* $\alpha$.

Remark that this assumption requires that for $\mu$-almost every type, the support of $f(\alpha|a)$ is the whole space $\mathbb{R}^{\ell}$ , i.e. $f(\alpha|a) > 0$ for every $\alpha$ . By using (2.c') to bound the right hand side of (2.4), inequality (2.6) becomes

$$(2.16) \qquad |p_k \frac{\partial X_h}{\partial p_k}(p) + \delta_{hk} X_h(p)| \leqq m_k X_h(p) \leqq \bar{w}\, m_k/p_h .$$

The counterpart of (2.11) is therefore that if $X_h(p) \neq 0$ , the price elasticities of $X_h(p)$ satisfy for $k = 1,\ldots,\ell$

$$(2.17) \qquad |\frac{\partial \mathrm{Log} X_h}{\partial \mathrm{Log} p_k}(p) + \delta_{hk}| \leqq m_k .$$

To get the equivalent of Theorem 2.3 from there, we need a weak form of aggregate desirability ensuring simply that total market demand for every commodity h never vanishes, i.e. $X_h(p) \neq 0$ for every p in $\mathrm{Int}\mathbb{R}^{\ell}_{+}$ and every h . This can be achieved without the independence condition (2.*d*), under rather weak postulates. It suffices for instance to assume that *for every commodity* h *, there exists a type* $a_0$ *in the support of* $\mu$ *, who demands a positive amount of* h *at some price-income configuration* (p,w) . Since $f(\alpha|a_o) > 0$ for all $\alpha$ , this implies that $X_h(a_o,p,w)$ is positive for all price systems p . By continuity, this will be also true for a near $a_o$ , and thus for total market demand $X_h(p)$. If the above desirability condition failed for some h , then we would have $X_h(p) = 0$ for all prices, and we could in fact exclude this particular commodity from the model. Then one can conclude from (2.17) that 1) $\frac{\partial X_h}{\partial p_h} < 0$ if $m_h < 1$ , that 2) if one assumes $m_h < 1$ for all h , the Jacobian matrix of total market demand has a dominant diagonal (in the sense of Theorem 2.3) on the set DD*(m) of prices p in $\mathrm{Int}\mathbb{R}^{\ell}_{+}$ such that $\Sigma_k (m_k/p_k) < 1/p_h$ for every $h = 1,\ldots,\ell$ , and that 3) the Jacobian matrix of total market demand is negative quasi-definite whenever $m_k\, \ell < 1$ for all k . The reader will also easily verify that under assumptions (2.*a*), (2.*b*), (2.*c'*), the counterpart of (2.14) becomes

$$(2.18) \qquad |\bar{w} \frac{\partial X_h}{\partial \bar{w}}(p,\bar{w}) - X_h(p,\bar{w})| \leqq \Sigma_k\, m_k\, X_h(p,\bar{w}) \leqq \bar{w}\, \Sigma_k\, m_k/p_h ,$$

which yields whenever $X_h(p,\bar{w}) \neq 0$

$$(2.19) \qquad |\frac{\partial \mathrm{Log} X_h}{\partial \mathrm{Log} \bar{w}}(p,\bar{w}) - 1| \leqq \Sigma_k\, m_k .$$

## 3. MARKET EXCHANGE EQUILIBRIUM

We showed in the preceding section that under some aggregate desirability condition, total market demand had a negative dominant diagonal Jacobian matrix on a large set of prices if the conditional distribution in each equivalence class of demand functions was flat enough. In fact, the "Law of Demand", and thus the weak axiom of revealed preference, is at some point satisfied in the aggregate. We wish to demonstrate now that the same sort of assumptions implies uniqueness and stability of equilibrium in a private ownership competitive exchange economy by establishing gross substitutability, and the weak axiom of revealed preferences between the equilibrium price vector and other prices, for aggregate excess demand. The novel feature here, as compared to the previous section, is of course that income, being equal to the market value of each trader's commodities endowment, is now price dependent.

In this context, an individual's characteristics are a demand function, in the sense of Definition 1.1, and a commodity endowment. We generate a distribution over individual characteristics in the same manner as in Section 2 (see Fig. 3), by specifying first a marginal probability distribution $\mu$ on the space of types A. To each type *a* corresponds here a demand function $\xi_a(p,w)$ and an initial endowment of commodities $\omega_a$ in $\mathbb{R}^{\ell}_{+}$, $\omega_a \neq 0$ . Given *a* , one defines a conditional distribution over the space of $\alpha$-transforms of $\xi_a$ through a density $f(\alpha|a)$ . We suppose the regularity conditions (2.*b*), (2.*c*) as in the preceding section, and the counterpart of assumption (2.*a*) becomes here

(3.*a*) *The endowment* $\omega_a \in \mathbb{R}^{\ell}_{+}$, $\omega_a \neq 0$ *depends continuously on the type* a. *Per capita endowment of commodities is finite and has all its components positive, i.e.*

$$\bar{\omega} = \int_A \omega_a \, \mu(da) \in \mathrm{Int}\mathbb{R}^{\ell}_{+}$$

Conditional market demand is thus defined for every vector of positive prices p and every positive income w , exactly as in (2.1)

$$(3.1) \qquad X(a,p,w) = \int_{\mathbb{R}^\ell} \xi_a(\alpha,p,w)\ f(\alpha|a)\ d\alpha\ .$$

*Conditional market excess demand* is accordingly

$$(3.2) \qquad Z(a,p) = X(a,p,p\cdot\omega_a) - \omega_a\ .$$

Under assumption (2.*b*), $Z(a,p)$ is well defined, continuous, bounded below by $-\omega_a$, homogenous of degree 0 in prices and satisfies Walras's law, i.e. $p\cdot Z(a,p) \equiv 0$ . *Total market excess demand* is finally obtained by integration over all types

$$(3.3) \qquad Z(p) = \int_A Z(a,p)\ \mu(da)\ .$$

Under assumptions (3.*a*), (2.*b*), $Z(p)$ is well defined, continuous, homogenous of degree 0 , bounded below by $-\bar{\omega}$ and satisfies Walras's Law, i.e. $p\cdot Z(p) \equiv 0$ . A pure exchange *market equilibrium* is described by a price vector $p^*$ in $\mathrm{Int}\mathbb{R}^\ell_+$ such that $Z(p^*) = 0$ .

We know in fact from section 2 that conditional market demand has continuous first order partial derivatives. Thus

$$(3.4) \qquad \frac{\partial Z_h}{\partial p_k}(a,p) = \frac{\partial X_h}{\partial p_k}(a,p,p\cdot\omega_a) + \frac{\partial X_h}{\partial w}(a,p,p\cdot\omega_a)\ \omega_{ak}$$

exists and is continuous in $(a,p)$ for every $h$ , $k$. It follows then from (2.5), (2.13) and assumption (2.*c*) that

$$p_h\ p_k\ \left|\frac{\partial Z_h}{\partial p_k}\ (a,p)\right| \leqq (p\cdot\omega_a)\ (1 + m_k) + p_k\ \omega_{ak}(1 + \Sigma_j\ m_j)$$

and this implies that total market excess demand has indeed continuous partial derivatives which can be obtained by differentiating (3.3) under the integral sign [10].

PROPOSITION 3.1. *Assume (3.a), (2.b), (2.c). Then total market excess demand is continuously differentiable and for every* h , k

$$\frac{\partial Z_h}{\partial p_k}(p) = \int_A \frac{\partial Z_h}{\partial p_k}(a,p)\ \mu(da)$$

As mentioned at the outset of this section, our goal is to demonstrate unicity and stability of equilibrium when, other things being equal, the distribution of demand functions in each equivalence class of $\alpha$-transforms is flat enough. We shall need, here as previously, conditions ensuring that every commodity is desired in the aggregate. These conditions will perform two distinct roles. First, they will ensure the mere *existence* of an equilibrium price system. Second, they will guarantee that market demand does not vanish when the densities in each $\alpha$-equivalence class become flat. We shall accordingly postulate assumption (2.*d*) – the conditional densities $f(\alpha|a)$ are actually independent of *a* – and an aggregate desirability condition like (2.*e*). In order to avoid the complications arising from the double role played by this condition, we shall simplify the exposition by working here with a stronger version, which exploits the fact that a type actually specifies a demand function *and* an endowment. One can thus, without loss of generality, let the space A of types be a subset of $B \times \mathbb{R}^{\ell}_{+}$, with $a = (b,\omega)$ , in which $b \in B$ stands for the type of the demand function $\xi_b$ and $\omega \neq 0$ is the corresponding commodity endowment. The distribution $\mu$ on A is then generated by a marginal distribution $\nu(d\omega)$ on the space of these endowments (along the horizontal axis in Fig. 4) and by regular conditional probabilities $\nu(db|\omega)$ on B (along vertical lines in Fig. 4). The aggregate desirability condition we employ here states that budget shares are uniformly bounded away from 0 after integration along each *vertical section*.

(3.*e*) *For each commodity* h , *there exists* $\varepsilon_h > 0$ , *with* $\Sigma_k\ \varepsilon_k \leqq 1$ , *such that for* $\nu$*-almost every endowment* $\omega$ , *for all vectors of positive prices* p *and for all positive incomes* w

$$p_h \int_B \xi_{bh}(p,w)\ \nu(db|\omega) \geqq \varepsilon_h\ w\ .$$

Fig. 4

It is now routine to verify that this aggregate desirability assumption ensures the existence of a market equilibrium.

PROPOSITION 3.2. *Assume* (3.*a*), (2.*b*), (2.*d*), (3.*e*). *Then* $p_h(Z_h(p) + \bar{\omega}_h) \geqq (p\cdot\bar{\omega})\ \varepsilon_h$ *for every* p *and every* h . *There exists an equilibrium price vector* $p^*$ *in* $\mathrm{Int}\mathbb{R}^{\ell}_{+}$ . *It satisfies* $p^*_h\ \bar{\omega}_h \geqq (p^*\cdot\bar{\omega})\ \varepsilon_h$ *for all* h .

*Proof.* One has clearly

$$\int_A p_h\ \xi_{ah}(p, p\cdot\omega_a)\ \mu(da) = \int_{\mathbb{R}^{\ell}_{+}} [p_h \int_B \xi_{bh}(p, p\cdot\omega)\ \nu(db|\omega)]\ \nu(d\omega)$$

$$\geqq (p\cdot\bar{\omega})\ \varepsilon_h$$

and this implies $p_h(Z_h(p) + \bar{\omega}_h) \geqq (p\cdot\bar{\omega})\ \varepsilon_h$ as in the first lines of the proof of Theorem 2.3. If we let $\Delta$ be the simplex of prices p in $\mathrm{Int}\mathbb{R}^{\ell}_{+}$ such that $p\cdot\bar{\omega} = 1$ , then $||Z(p_n)||$ diverges to $+\infty$ when the sequence $p_n$ in $\Delta$ tends to $\bar{p}$ in $\bar{\Delta}\setminus\Delta$ . As Z(p) is continuous, bounded below and satisfies Walras's law, it is a standard result of equilibrium analysis that there exists $p^*$ in $\Delta$ such that $Z(p^*) = 0$ (Debreu (1982, Theorem 8)). Then

$$p^*_h\ \bar{\omega}_h = p^*_h(Z_h(p^*) + \bar{\omega}_h) \geqq (p^*\cdot\bar{\omega})\ \varepsilon_h \qquad \text{Q.E.D.}$$

We would like to show next that under the aggregate desirability condition (3.*e*), commodities are gross substitutes in a region of prices that is large when the conditional distribution in each equivalence class of $\alpha$-transforms is flat enough. To see why this might be true, let us consider (3.4). For any price vector p , the first term of the right hand side can be made uniformly small by using (2.5) while, in view of (2.13), the second term should be nonvanishing and positive when all coefficients $m_j$ become small, other things being equal. From Proposition 3.1, one should expect gross substitutability to prevail accordingly at p and thus locally. The following result makes this intuition precise and demonstrates that it holds in effect globally, on a set of prices that becomes larger and larger

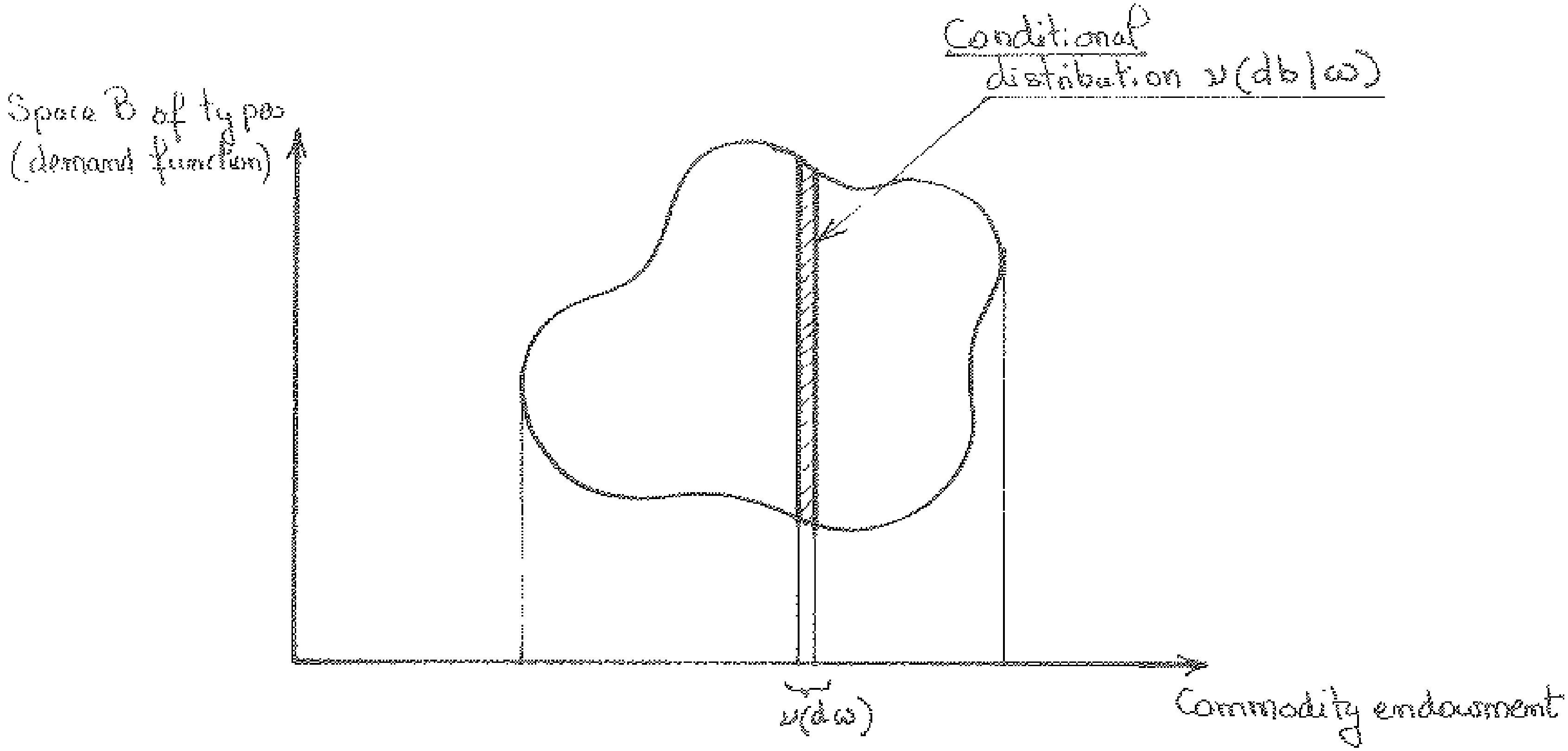

Space B of types
(demand function)
Conditional
distribution ν(db|ω)
ν(dω)
Commodity endowment

Fig. 4

when all $m_j$ tend to 0, *ceteris paribus*. Proposition 3.2 tells us that equilibrium prices are bound to stay in the limited domain defined by the inequalities $p^*_h \bar{\omega}_h \geqq (p^* \cdot \bar{\omega})\, \varepsilon_h$ for all h . One should expect accordingly, when all $m_j$ go to 0 , the region for which one gets gross substitutability to contain eventually the set of equilibrium prices, in which case equilibrium is indeed unique (of course up to a scalar multiplication of all prices). On the top of that, one gets in fact at some stage that the weak axiom of revealed preference holds in the aggregate, as between an equilibrium price vector and any other non-equilibrium price system.

THEOREM 3.3. *Assume* (3.a), (2.b), (2.c), (2.d), (3.e). *Let* GS(m,ε) *be the convex cone of prices in* $\mathrm{Int}\mathbb{R}^{\ell}_{+}$ *such that*

$$p_k \bar{\omega}_k (\varepsilon_h - \Sigma_j m_j) > (p \cdot \bar{\omega})\, m_k \text{ , all } k \neq h . \tag{3.5}$$

1) *Commodities are gross substitutes i.e.* $\dfrac{\partial Z_h}{\partial p_k}(p) > 0$ *for* $k \neq h$ , *when the price vector* p *lies in* GS(m,ε) .

2) *Assume*

$$\varepsilon_k(\varepsilon_h - \Sigma_j m_j) > m_k \text{ , all } k \neq h . \tag{3.6}$$

*Then* GS(m,ε) *contains the set of prices* p *in* $\mathrm{Int}\mathbb{R}^{\ell}_{+}$ *such that* $p_k \bar{\omega}_k \geqq (p \cdot \bar{\omega})\, \varepsilon_k$ *for all* k , *and therefore the set of equilibrium prices. In that case, the equilibrium price vector is unique (up to multiplication by a scalar).*

3) *The condition*

$$\varepsilon_k^2 (\varepsilon_h - \Sigma_j m_j) > m_k \text{ , all } k \neq h , \tag{3.7}$$

*(which is stronger than* (3.6) *since* $0 < \varepsilon_k < 1$ *for every* k) *implies that the weak axiom of revealed preference holds in the aggregate, i.e.* $p^* \cdot Z(p) > 0$ , *as between an equilibrium price vector* $p^*$ *and any other price system* p *in* $\mathrm{Int}\mathbb{R}^{\ell}_{+}$ *that is not colinear to* $p^*$ .

*Proof.* 1) In view of (2.5), we have for every $k \neq h$

$$(3.8) \qquad \left| p_h\, p_k \int_A \frac{\partial X_h}{\partial p_k}\,(a, p, p\cdot\omega_a)\ \mu(da) \right| \leqq (p\cdot\bar{\omega})\ m_k\ .$$

On the other hand, (2.13) implies that the integral

$$(3.9) \qquad \int_A p_k\ \omega_{ak} [p_h\ \frac{\partial X_h}{\partial w}\ (a, p, p\cdot\omega_a)]\ \mu(da)$$

is bounded below by

$$(3.10) \qquad \int_A p_k\ \omega_{ak} [p_h\ X_h(a, p, p\cdot\omega_a)/(p\cdot\omega_a)]\ \mu(da) - (p_k\ \bar{\omega}_k)\ \Sigma_j\ m_j\ .$$

But the integral appearing in (3.10) is equal to, on account of (2.*d*) and (3.*e*)

$$\int_{\mathbb{R}^\ell} [\int_{\mathbb{R}^\ell_+} p_k\ \omega_k\ [\int_B p_h\ e^{\alpha_h}\ \xi_{bh}(e^{\alpha} \otimes p, p\cdot\omega)/(p\cdot\omega)\ \nu(db|\omega)]\ \nu(d\omega)]\ f(\alpha)\ d\alpha$$

$$\geqq \int_{\mathbb{R}^\ell} [\int_{\mathbb{R}^\ell_+} (p_k\ \omega_k)\ \varepsilon_h\ \nu(d\omega)]\ f(\alpha)\ d\alpha$$

$$\geqq (p_k\ \bar{\omega}_k)\ \varepsilon_h\ .$$

Thus a lower bound for (3.9) is $(p_k\ \bar{\omega}_k)\ (\varepsilon_h - \Sigma_j\ m_j)$ while a lower bound for (3.8) is $-(p\cdot\bar{\omega})\ m_k$ . Therefore Proposition 3.1 implies for all $k \neq h$

$$p_h\ p_k\ \frac{\partial Z_h}{\partial p_k}\ (p) \geqq (p_k\ \bar{\omega}_k)\ (\varepsilon_h - \Sigma_j\ m_j) - (p\cdot\bar{\omega})\ m_k\ .$$

Hence market excess demand has the gross substitutability property when p lies in GS $(m, \varepsilon)$.

2) If we have (3.6) then $p_k\ \bar{\omega}_k \geqq (p\cdot\bar{\omega})\ \varepsilon_k$ implies (3.5) for $k \neq h$. In view of Proposition 3.2, GS$(m, \varepsilon)$ contains the set of equilibrium

prices in that case. To conclude that the equilibrium price vector is unique, up to a multiplication by a scalar, one may apply known results – see Arrow and Hahn (1971, Theorem 9.7.7). An elementary direct proof is the following one. Let $p^*$ be an equilibrium price system and let $p$ be an arbitrary price vector in $GS(m,\varepsilon)$ that is not colinear to $p^*$ . Choose $r$ such that $rp^* \leqq p$ , with equality for some component $h$ . When moving from $rp^*$ to $p$ along the segment $[rp^*,p]$, the $h$-th component of the vector remains constant while the others never go down and some are actually increased. Since $Z_h(rp^*) = 0$ , one gets $Z_h(p) > 0$ as the segment lies entirely in $GS(m,\varepsilon)$, and $p$ cannot be an equilibrium (Note that if we had chosen $s$ such that $sp^* \geqq p$ , with equality for some component $k$ , the same argument would have yielded $Z_k(p) < 0$).

3) The main line of the argument is to prove that $p^*\cdot Z(p)$ is uniquely minimized at price vectors $p$ that are colinear to $p^*$. The difference with standard studies of this issue, such as Arrow and Hahn (1971, proof of Theorem 9.7.9) is that here gross substitutability does not necessarily hold everywhere. The first observation is that under (3.7) the set $K_s$ of prices $p$ in $\mathrm{Int}\mathbb{R}^\ell_+$ that satisfy

$$(3.11) \qquad (p\cdot\bar{\omega})\ \Sigma_h(\varepsilon_h^2/p_h\ \bar{\omega}_h) \leqq 1 + s$$

is a proper subset of $GS(m,\varepsilon)$ for $s = 0$ . Indeed we get from (3.11) when $s = 0$

$$(p\cdot\bar{\omega})/p_k\ \bar{\omega}_k < 1/\varepsilon_k^2 < (\varepsilon_h - \Sigma_j\ m_j)/m_k$$

for all $k \neq h$ , which is (3.5). One may also remark that $K_0$ contains the set $C$ of prices such that $p_h\ \bar{\omega}_h \geqq (p\cdot\bar{\omega})\ \varepsilon_h$ for all $h$, and thus any equilibrium price $p^*$. Indeed one gets for every $p$ in $C$, $(p\cdot\bar{\omega})\ \varepsilon_h^2/p_h\ \bar{\omega}_h \leqq \varepsilon_h$ and thus (3.11) for all $s \geqq 0$ by summation over $h$ since $\Sigma_h\ \varepsilon_h \leqq 1$ .

We choose an $s > 0$ small enough so that the inclusion $K_s \subset GS(m,\varepsilon)$ still holds, i.e. such that

$$(1+s)/\varepsilon_h^2 < (\varepsilon_h - \Sigma_j\ m_j)/m_k\ , \text{ all } k \neq h\ .$$

The second observation is that if p in $\mathrm{Int}\mathbb{R}^{\ell}_{+}$ is such that

$$(3.12) \qquad (p\cdot\bar{\omega})\ \Sigma_h(\varepsilon_h^2/p_h\ \bar{\omega}_h) \geqq 1 + s\ ,$$

then $p^*\cdot Z(p) \geqq (p^*\cdot\bar{\omega})\ s > 0$ for any equilibrium price system $p^*$ . Indeed, one has from Proposition 3.2

$$\begin{aligned} p^*\cdot Z(p) &\geqq -\ (p^*\cdot\bar{\omega}) + (p\cdot\bar{\omega})\ \Sigma_h\ p_h^*\ \varepsilon_h/p_h \\ &\geqq -\ (p^*\cdot\bar{\omega}) + (p\cdot\bar{\omega})\ \Sigma_h(p^*\cdot\bar{\omega})\ \varepsilon_h^2/p_h\ \bar{\omega}_h \\ &\geqq (p^*\cdot\bar{\omega})\ [-\ 1 + (p\cdot\bar{\omega})\ \Sigma_h\ \varepsilon_h^2/p_h\ \bar{\omega}_h] \\ &\geqq (p^*\cdot\bar{\omega})\ s > 0\ , \end{aligned}$$

(One can remark that the statement holds in fact if $p^*$ is replaced by any price vector $\bar{p}$ such that $\bar{p}_h\ \bar{\omega}_h \geqq (\bar{p}\cdot\bar{\omega})\ \varepsilon_h$).

From this, one concludes that given any equilibrium price system $p^*$ , since $p^*\cdot Z(p^*) = 0$ and $p^*$ belongs to $K_s$ , the minimum of $p^*\cdot Z(p)$ exists and is reached at price vectors p for which the inequality (3.12) is violated, i.e. in the interior of $K_s$ . Since $K_s$ is contained in GS(m,$\varepsilon$), known arguments show that the minimum is reached at price vectors of the form $rp^*$ (see e.g. the proof of Theorem 9.7.9 in Arrow and Hahn, 1971). Thus $p^*\cdot Z(p) > 0$ for all price vectors p that are not colinear to $p^*$.

Q.E.D.

The foregoing result makes precise how increasing dispersion of the distribution of demand functions in each equivalence class of $\alpha$-transforms leads to gross substitutability and WARP in the aggregate. We should expect the same circumstances to be associated to an "increasing stability" of market equilibrium. Specifically, let us choose the first commodity as *numeraire* (our formulation being symmetric, this is immaterial : we could have chosen any other commodity as well), and let us denote by P a price vector in $\mathrm{Int}\mathbb{R}^{\ell}_{+}$ , the first component of which is equal to 1. A standard *tatonnement* process is then a rule for adjusting the remaining prices

(3.13) $$\dot{P}_h = G_h(Z_h(P)) \ , \qquad h = 2,\ldots,\ell \ ,$$

in which each $G_h$ satisfies as usual $G_h(0) = 0$ , $G'_h > 0$ and is thus sign preserving. Since, according to the theorem, WARP is satisfied in the aggregate when (3.7) holds, it follows from well known results (see e.g. Arrow and Hahn, 1971, Theorem 12.3.2) that the unique equilibrium price vector $P^*$ is then *globally stable* when the tatonnement process is specialized to

(3.14) $$\dot{P}_h = \gamma_h Z_h(P) \ , \text{ with } \gamma_h > 0 \ , \text{ for } h = 2,\ldots,\ell \ .$$

Under assumption (3.6), commodities are gross substitutes near the unique equilibrium price vector $P^*$ and we know then, again from standard results (see e.g. Hahn 1982, Section 2.1), that $P^*$ is locally stable in the tatonnement (3.13). Were the gross substitutability property valid for all prices, the unique equilibrium price vector would be globally stable in the tatonnement (3.13), see Arrow and Hahn (1971, Theorem 12.3.4). In the present context, increasing the dispersion of the distribution in each $\alpha$-equivalence class, i.e. lowering the $m_j$'s , other things being equal, makes the range of prices for which gross substitutability obtains eventually larger and larger while, in view of Proposition 3.2, the unique equilibrium price vector $P^*$ remains in the compact region of prices P in $\text{Int}\mathbb{R}^{\ell}_{+}$ defined by the inequalities

(3.15) $$P_h \bar{\omega}_h \geqq (P \cdot \bar{\omega}) \ \varepsilon_h \qquad \text{for } h = 1,\ldots,\ell.$$

We should accordingly expect here the basin of attraction of the equilibrium price vector to become also large in that case, and to cover eventually the whole region of admissible prices. The following fact makes this intuition precise.

COROLLARY 3.4. *Assume* (3.6) *and let* $P^*$ *be the unique equilibrium price system. Then* $P^*$ *is asymptotically stable in the tatonnement* (3.13). *Let* K *be an arbitrary compact set of price vectors* P *in* $\text{Int}\mathbb{R}^{\ell}_{+}$ *containing all price vectors* P *satisfying* (3.15), *hence* $P^*$ . The *basin of attraction of* $P^*$ *contains the set* K *if, other things being equal, the* $m_j$*'s are small enough.*

*Proof.* We know that

$$V(P) = \text{Max}_j \left|\frac{P_j}{P^*_j} - 1\right|$$

is a Lyapounov function for (3.13) if gross substitutability holds everywhere, see Arrow and Hahn (1971, proof of Theorem 12.3.4). In the present case, under (3.6), the unique equilibrium price vector $P^*$ lies in the intersection of the open cone $GS(m,\varepsilon)$ and of the hyperplane defined by $P_1 = 1$. Then gross substitutability holds for all price vectors P such that $V(P) < \sigma$ when $\sigma$ is small enough and local asymptotic stability follows by the same argument.

To show the last part of the corollary, we remark first that the set $I(\rho)$ of prices P such that $P_h \geqq \rho_h$ for $h = 2,\ldots,\ell$, in which the $\rho_h$ satisfy

$$0 < \rho_h \leqq \bar{\omega}_1 \varepsilon_h / [\bar{\omega}_h(1 - \varepsilon_h)] \qquad \text{for } h = 2,\ldots,\ell,$$

is invariant in the dynamics induced by (3.13). Indeed, in view of Proposition 3.2, we have for $h \geqq 2$

$$Z_h(P) \geqq [\varepsilon_h \sum_{k \neq h} P_k \bar{\omega}_k / P_h] - \bar{\omega}_h(1 - \varepsilon_h)$$

$$> (\bar{\omega}_1 \varepsilon_h / P_h) - \bar{\omega}_h(1 - \varepsilon_h) \ ,$$

and thus $Z_h(P) > 0$ , hence $\dot{P}_h > 0$ , when $P_h = \rho_h$.

We choose the $\rho_h$'s small enough so that $I(\rho)$ contains the given compact set K. Clearly, when all the coefficients $m_j$ go to 0, other things being equal, the unique equilibrium price vector $P^*$ stays in the bounded region defined by (3.15) while $GS(m,\varepsilon)$ fills eventually the whole price space. Thus if the $m_j$'s are small enough, one can find $\sigma$ such that the set $J(\rho,\sigma)$ of price vectors P in $I(\rho)$ satisfying $V(P) < \sigma$ contains K, and such that gross substitutability holds everywhere in $J(\rho,\sigma)$. The arguments of Arrow and Hahn (1971, proof of Theorem 12.3.4) show that $J(\rho,\sigma)$ is invariant in the dynamics induced by (3.13) and that V(P) is a Lyapounov function for (3.13) restricted to $J(\rho,\sigma)$. Thus any trajectory starting in $J(\rho,\sigma)$ stays there and converges to $P^*$ as t goes to $+ \infty$. Q.E.D.

## 4. CONCLUSION

We showed that increasing the degree of behavioral heterogeneity in the household sector or more specifically, making the conditional distributions in each equivalence class of $\alpha$-transforms of demand functions flat enough, has an important regularizing influence on aggregate budget shares : market demand has a dominant diagonal Jacobian matrix, aggregate excess demand has the gross substitutability property on a large set of prices. These facts have strong consequences for the unicity and stability of equilibrium, for the prevalence of the weak axiom of revealed preferences in the aggregate in a private ownership Walrasian exchange model.

Some of these findings were obtained with the help of strong assumptions (independence of conditional densities in each equivalence class of $\alpha$-transforms, aggregate budget shares uniformly bounded away from 0). It is hoped that further studies will weaken these assumptions, and extend the analysis so as to incorporate production, sequential temporary equilibrium or imperfect competition. Beyond the specifics of the present paper, however, I believe these results already demonstrate that the finite dimensional linear structure, which affine transformations of the commodity space (or household equivalence scales) induce on preferences or more generally, on demand functions, does provide us with a fruitful formal language (that is not exclusive, of course, of others) to talk about and quantify such things as the dispersion, the variance, the shape, of the distribution of behavioral characteristics in a socioeconomic system. We can hope to be able to develop within this formal framework distributional hypotheses leading not only to quantitative testable predictions about aggregate demand as here, but also about the distribution of the agents' observable choices. Progress along a distributional approach of this sort, based upon the *heterogeneity* of actual observed behaviors, would provide a most welcome alternative to the naive, and presumably misleading, representative optimizing agent approach to macroeconomics.

An important feature of the analysis presented in this paper is that it did not rely upon any hypothesis about the "rationality" of individual demand functions other than homogeneity (absence of money illusion) and

Walras's Law (compliance with individual budget constraints). This feature is in a sense comforting since it shows that general Walrasian equilibrium theory may be more robust than some might have thought, but it also creates some kind of "embarrassment of riches". The present analysis raises again some (old) questions about the actual status of the postulates of individual "rationality" in the Walrasian paradigm. It suggests in particular that such postulates might not be as necessary as some would like to believe to the construction of a sound quantitative macroeconomics. An alternative research strategy might be indeed to rely more on particular features of the distribution of behavioral characteristics among the members of the system under consideration. An important issue to investigate would then be how such macroeconomic distributions might arise *endogenously* from specific socioeconomic interactive processes at the micro-level. One could for instance envision a more "adaptive" viewpoint, in the spirit of Hildenbrand (1971), Föllmer (1974), in which the decision rule (here the demand function) or the preferences of an individual are influenced in a stochastic (Markovian) fashion by those of his immediate neighbor(s), and generate endogenously a macroeconomic distribution by looking for invariant distributions. The properties of these invariant distributions might in turn generate enough strong macroeconomic structure to allow us to proceed on secure grounds. These avenues, which might lead eventually to some kind of "*Statistical Economics*" (in the sense we talk of "Statistical Mechanics"), may not sound quite orthodox after so much emphasis put for so long on "individual rationality" as the main structuring language in our profession. Yet they are presumably worth exploring.

FOOTNOTES

* The bulk of the research reported in this paper was essentially done at Yale University during the fall semester of 1990. I wish to thank all my colleagues for their support and comments, especially Martin Shubik, Herbert Scarf and Truman Bewley. I had also very useful conversations with Jorgen Weibull, Mike Jerison, Andreu Mas-Colell, Christian Gouriéroux, Camille Bronsard, Guy Laroque, Bruno Jullien and Bernard Caillaud. I am much indebted to Al Klevorick for his comments on an earlier draft. But my greatest debt is to Werner Hildenbrand, whose pioneering insights and stubborn insistance have kept over the years quite a few of us struggling, beyond the fads of the moment, with the aggregation problem. Our profession is fortunate indeed to have him around.

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[1] Such a statement, like any claim involving the *shape* of a distribution over a vector space, depends of course on the particular parameterization under consideration and is not invariant to a nonlinear change of variables. Had we chosen to work with the parameters $\beta_h = e^{\alpha_h} > 0$, the required property would be that the conditional distributions of the vector $(\mathrm{Log}\beta_1, \ldots, \mathrm{Log}\beta_\ell)$ have "flat enough" densities.

[2] Continuity means that for all sequences $x_n \to x$, $y_n \to y$, $x_n \succsim y_n$ for all $n$ implies $x \succsim y$. Strict convexity means that for every $x_1$, $x_2$, $y$ with $x_i \succsim y$ for $i = 1,2$, $x_1 \neq x_2$, then $\lambda x_1 + (1-\lambda) x_2 \succ y$ for all $\lambda$ in $(0,1)$. Local nonsatiation means that for any $x$ and any neighborhood $V$ of $x$, there exists $y$ in $V$ with $y \succ x$.

[3] We shall rely essentially, therefore, on continuity considerations in order to define the integrals representing aggregate behavior. An alternative viewpoint would have been to rely exclusively on measure-theoretic methods, as developed in Hildenbrand (1974).

[4] Continuity of these integrals with respect to the relevant parameters can be shown by repeated applications of Lebesgue dominated convergence Theorem. See Dieudonné (1969, Theorem 13.8.6). It should be noted that total market demand $X(p)$ is *not* a demand function in the sense of Definition 1.1. It is not even a well defined function of per capita income $\bar{w}$ , unless one fixes for instance its distribution among types.

[5] Again, these facts follow from the dominated convergence theorem, see Dieudonné (1969, Theorem 13.8.6).

[6] Again, these statements follow from Dieudonné (1969, Theorem 13.8.6).

[7] An easy way to generate distributions having small $m_k$'s is the following one (I owe it to C. Gouriéroux). Let us fix *any* distribution defined by a smooth density $g(\beta)$ over vectors $\beta$ in $\mathbb{R}^\ell$, that has uniformly integrable partial derivatives. The density of the variable $\alpha = \sigma\,\beta$ , in which $\sigma > 0$ is an arbitrary real number, is $f_\sigma(\alpha) = g(\alpha/\sigma)/\sigma$. The coefficients

$$m_k = \int_{\mathbb{R}^\ell} \left|\frac{\partial f_\sigma}{\partial \alpha_k}(\alpha)\right| \, d\alpha$$

can be made as small as one wishes by increasing the parameter $\sigma$.

[8] Assumptions (2.*d*), (2.*e*) are there only to guarantee that aggregate budget shares $p_h\, X_h(p)/\bar{w}$ are uniformly bounded away from 0. I believe that the independence assumption (2.*d*) can be relaxed, at the cost of course of strengthening (2.*e*). As an obvious example, one can dispense with (2.*d*) altogether if one is willing to assume (2.*e*) for each type, i.e. $p_h\, \xi_{ah}(p,w) \geqq \varepsilon_h\, w$ for all h, p, w and every *a*. For another example, see the subsection after Proposition 2.4.

[9] The foregoing argument has been stated in terms of demand functions to keep the interpretation in line with the main development of the paper. One may remark however, that we have *not* used homogeneity or Walras's Law. The $\alpha$-transform (1.4) is defined for any function. The reader will easily

verify that the results stated up to now (with the exception of (1.7) and Lemma 1.3) apply to the aggregation of a set of functions $\xi_a(p)$ taking values in $\mathbb{R}_+^\ell$ such that $p_h\ \xi_{ah}(p) \leqq w_a$ . This point may be useful in other contexts.

[10] This follows again from the dominated convergence theorem, see Dieudonné (1969, Theorem 13.8.6).

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