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A REFINED BARGAINING SET OF AN N–PERSON GAME AND ENDOGENOUS COALITION FORMATION

by

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Abstract. The two most fundamental questions in cooperative game theory are: When a game is played, what coalitions will be formed and what payoff vectors will be chosen? No previous solution concepts or theories in the literature provide satisfactory answers to both questions; answers are especially lacking for the first one.

In this paper we introduce the refined bargaining set, which is the first solution concept in cooperative game theory that simultaneously provides answers to both of the fundamental questions.

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1. INTRODUCTION

MANY PROBLEMS IN ECONOMICS and other social sciences in which people interact in making their decisions can be analyzed by using the model of a cooperative game. A cooperative game, or a game in coalitional form, specifies for every coalition of players a set of payoff vectors that are feasible for players within the coalition if they agree to cooperate. Each player is free to decide with whom to cooperate and how to cooperate. A coalition is formed and a feasible payoff vector is chosen only when the coalition and the payoff vector are agreed to by all players involved. For such a model to contribute to our understanding of the problems it is used to analyze, it is important to know what coalitions will be formed when a game is played and what payoff vectors will be chosen in each of these coalitions. These are the two most fundamental questions in cooperative game theory.

Game theorists usually provide their answers in the form of a solution concept, which identifies for each game some coalition structures and some payoff vectors that are consistent with the coalition structures. Almost all existing solution concepts in the literature are defined, however, for exogenously given coalition structures (see Aumann and Dreze [1] for a collection of them).[1] Hence, these concepts cannot answer the first question directly. The argument for using these concepts is that they provide an answer to the second question given that a particular coalition structure is already formed. Behind this argument is an implicit assumption that each game is played in two stages: In the first stage players decide what coalitions to form, and in the second stage players within each coalition formed in the first stage decide what payoff vectors to choose. This is made explicit in some theories that try to answer the first question using the existing solution concepts. In my view, such an assumption is overly simplistic. When a game

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[1] Ichiishi brought to my attention the paper by Boehm [4] which discussed the core of an economy with production, which was independent of any particular coalition structure.

is played, players' decisions about with whom to cooperate and how to cooperate are not separately made. Therefore, satisfactory answers to the two basic questions must be simultaneously given in the fashion of general equilibrium analysis. [2]

What properties should an ideal solution concept satisfy so that it can provide proper answers to both questions ? First of all, it should have its own intuitive content. But besides that, it also should satisfy the following three properties: (1) It is not a priori confined to payoff vectors of a particular coalition structure; (2) it always includes some payoff vectors of some coalition structure; and (3) it does not always contain payoff vectors of every coalition structure. Such a solution concept would yield for each game simultaneously and endogenously a selected set of coalition structures and payoff vectors consistent with the selected coalition structures. Notice, however, that there is a tension between the second and the third conditions. If the requirements imposed on a solution concept are too strong, as in the case of the core, then the set of solutions can be empty for many games. On the other hand, if the requirements are too weak, as in the case of the Aumann-Maschler bargaining set, then the set of solutions can contain payoff vectors of every coalition structure for all games. Since every solution concept currently in vogue in cooperative game theory violates either one or another of the above properties, one has to wonder whether there exists a solution concept that satisfies all properties.

In this paper we consider a solution concept called the refined bargaining set. This concept is based on the idea of Aumann and Maschler [2], yet it is formalized in a different way. The addition of an essential nonempty intersection condition (among others) remedies a basic problem common to all previous versions of bargaining sets. It makes the outcomes contained in the refined bargaining set far more reasonable than those in other bargaining sets. To our greatest satisfaction, we are also able to demonstrate that the refined bargaining set indeed satisfies all three properties listed above. Hence, we

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[2] This view is certainly not new. It can be found in the paper by Aumann and Dreze [1].

have introduced into the literature for the first time a solution concept that provides proper answers to both of the two most fundamental questions in cooperative game theory.

In Section 2 we give the definition of the refined bargaining set for transferable utility (TU) games, which is independent of any particular coalition structure. We then prove the main result that the refined bargaining set is nonempty for any TU game, and we show by a simple example of a game how some unreasonable coalition structures are eliminated since they never emerge at the refined bargaining set.

We include in Section 3 comparisons between the refined bargaining set and related works in the literature. First, we compare it to the bargaining sets defined by Aumann and Maschler, and by Mas-Colell [6], each of which has influenced our work in its own way. Examples will illustrate the differences between the concepts, which are largely due to a nonempty intersection condition in the refined bargaining set, and we shall discuss the significance of the differences. Second, we compare the refined bargaining set to a few other theories of endogenous coalition formation that go beyond a simple standard solution concept (Shenoy [10], Hart and Kurz [5], Bennett [3]). These two sets of comparisons strongly show that why the refined bargaining set should be favored.

Our analysis is extended to nontransferable utility games in Section 4. All definitions and comments made in previous sections apply to non-transferable utility (NTU) games. Nevertheless, whether the refined bargaining set exists for any NTU game remains an open problem. We are only able to give several partial answers. First, we provide a sufficient condition for the existence of the refined bargaining set, which is inspired by Scarf's work [9] on the existence of the core and, in particular, by Vohra's work [12] on the existence of the Mas-Colell bargaining set. This condition is used to show that a large class of NTU games do have nonempty refined bargaining sets. Second, we consider a concept of the quasi bargaining set which is weaker yet closely related to the refined bargaining set, and we prove that each NTU game has a nonempty quasi bargaining set. We hope that these results shed some light on the open problem of existence of the refined bargaining set for an NTU game.

2. THE REFIND BARGAINING SET OF A TRANSFERABLE UTILITY GAME

Let $N = \{ 1, 2, ..., n \}$ be the set of players and 2^N the set of all nonempty coalitions $S \subset N$. An n-person transferable utility game V in coalitional form is a function from 2^N to subsets of R^n. For each coalition S, $V(S)\mid_{R^S}$, the projection of $V(S)$ on R^S, is the set of payoff vectors that players in S can guarantee themselves if they cooperate. It is assumed that V satisfies the following conditions: [3]

(TU1) $V(N)$ is nonempty, closed, comprehensive ($x \in V(N) \Rightarrow x - R^n_+ \subset V(N)$), and bounded from above (there is an $M > 0$, $u \in V(N) \cap R^n_+ \Rightarrow u_i \leq M$ for all i);

(TU2) For every $S \subsetneq N$, there exists a number $v(S)$ such that $V(S) = \{ x \in R^n \mid \sum_{i \in S} x_i \leq v(S) \}$;

(TU3) For every player i, $b_i = v(i) > 0$.

We assume that (TU1)-(TU3) have specified all aspects of a game that are relevant to our analysis. For example, the fact that a coalition S is not allowed to form (by law or whatever the reasons) is reflected by assigning $v(S)$ some large negative number. Any form of superadditivity thus is not thought as natural and not assumed.[4]

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[3] A TU game in our terminology is more general than a TU game in ordinary terminology in that it does not have the transferable utility property for the grand coalition N. As we shall see, it has a great advantage from a technical point of view. Other notations and definitions are also made as general as possible for later discussion on NTU games.

[4] For a more detailed discussion on non-superadditivity, readers are referred to the paper by Aumann and Dreze [1].

A payoff configuration of a game V is a pair $\{ x; \mathcal{Q} \} = \{ x_1, ..., x_n ; S_1, ..., S_Q \}$, in which x is a vector in R^n, $\mathcal{Q}$ a partition of N, and $x \in V(S_q)$ for all S_q in $\mathcal{Q}$. We often denote a payoff configuration $\{ x; \mathcal{Q} \}$ simply by x when there is no confusion. A payoff configuration $\{x; \mathcal{Q}\}$ is dominated by another one $\{y; \mathcal{R}\}$ if $y_i > x_i$ for all i. Notice that the coalition structures of two payoff configurations in comparison can be different. A payoff configuration $\{x; \mathcal{Q}\}$ is efficient if it is not dominated by others.

DEFINITION 2.1. An objection from S against a payoff configuration x is a pair (S, y), in which $y \in V(S)$ and $y_i > x_i$ for all $i \in S$.

A very intuitive solution concept for a game is the core that consists of payoff configurations against which no coalitions have objections. But except in some classes of games of special features (pure exchange games, simple matching games, etc.), the core of a game is usually empty. Hence the core, as a solution concept, is inadequate for the investigation of general games in coalitional form.

Aumann and Maschler [2] initiated the study of bargaining sets. The key idea is to distinguish objections that are justified from the unjustified. Suppose that at some payoff configuration x a coalition S has an objection to x. It then demands more for its members from the rest of players by threatening to withdraw from x and make themselves better off with payoff y. But some players not in S can counter this objection if they can lure some players in S into a new coalition T to achieve a payoff z that pays themselves no less than x and players lured away from S no less than y. An objection is justified only when it cannot be countered. A payoff configuration is then stable if it has no justified objection. Based upon different formalizations of this idea, various bargaining sets have been proposed since Aumann and Maschler. In this paper we formalize it as follows.

DEFINITION 2.2. Let (S, y) be an objection against a payoff configuration x. A counterobjection from a coalition T against (S, v) is a pair (T, z) in which $z \in V(T)$ and

(RB1) $T \setminus S \neq \emptyset$, $S \setminus T \neq \emptyset$, and $S \cap T \neq \emptyset$;

(RB2) $z_k \geq x_k$ for all $k \in T \setminus S$, and $z_l \geq y_l$ for all $l \in S \cap T$.

An objection (S, y) against x is justified if there exists no counterobjection from any other coalition T to (S, y).

DEFINITION 2.3. The refined bargaining set of a game V, denoted by $RB(V)$, is the set of payoff configurations against which no coalitions have justified objections.

What constitutes the novelty and the essence of the refined bargaining set is (RB1) in Definition 2.2 which has three conditions on the initial objecting coalition S and the subsequent counterobjecting coalition T. First, a counterobjection must be launched by some player who is not in the initial objecting coalition, hence $T \setminus S \neq \emptyset$; second, a counterobjection should not reinforce the initial objection, hence $S \setminus T \neq \emptyset$; and third, a counterobjection should nullify the initial objection, hence $S \cap T \neq \emptyset$. We believe that all three conditions are important for a sensible theory of bargaining sets and that it is the absence of one or another of these conditions (especially the third) that accounts for the inability of the current theory on bargaining sets to provide any direct answer to the question of coalition formation.

Unlike the Aumann-Maschler bargaining set that is defined for an exogenously given coalition structure and is nonempty for each coalition structure, the refined bargaining set is defined free of any particular coalition structure and contains payoff configurations of a selection of coalition structures only. Therefore, it determines endogenously coalition structures as well as payoff vectors. Coalition structures that do not emerge at any payoff configurations in the refined bargaining set are thus regarded as incompatible with the game and eliminated. It can be illustrated by a simple example.

EXAMPLE 2.5. Let V be a 4-person TU game with $v(12) = v(34) = 4$, and $v(S) = |S|$ otherwise. Its refined bargaining set contains the following payoff configurations: { 2+t, 2-t , 2+s, 2-s ; <12>, <34> }, $0 \leq t \leq 1$, and $0 \leq s \leq 1$. Hence the refined bargaining set selects { <12>, <34> } as the only coalition structure that is compatible with V.

Of course, it is important to know whether the refined bargaining set exists for every game. A potential problem with nonexistence could be that for some game we might eliminate all coalition structures. The most important result in this paper is the following theorem that shows that the refined bargaining set does exist for every TU game.

THEOREM 2.4. *The refined bargaining set is nonempty for every TU game V.*

Before giving a proof of Theorem 2.4, we consider other interesting properties of the refined bargaining set. Some of them follow directly from the definitions. First, an objection from an individual i has no counterobjections since no coalition T satisfies both $\{i\} \setminus T \neq \emptyset$ and $\{i\} \cap T \neq \emptyset$. Hence payoff configurations in the refined bargaining set are individually rational. Second, an objection from the grand coalition N has no counterobjections since $T \setminus N = \emptyset$ for all coalitions T. Hence payoff configurations in the refined bargaining set satisfy collective rationality at the grand coalition. But this is short of overall efficiency. The efficiency property of the refined bargaining set deserves a more careful consideration.

We say that a game V is superadditive at N if $\bigcap_{S_k \in Q} V(S_k) \subset V(N)$ for every partition Q of N. Obviously payoff configurations in the refined bargaining set of a superadditive V must be efficient. But generally the refined bargaining set of a game can contain payoff configurations that are not efficient.

EXAMPLE 2.6. V is a 4-person TU game with $v(T) = 4$ for any coalition T of two players and $v(S) = |S|$ otherwise. The payoff configuration { (1, 1, 1, 1) ; <1234> } is in the refined bargaining set, and it is obviously not efficient.

However, the refined bargaining set in this example does have payoff configurations that are efficient such as { (2, 2, 2, 2) ; <12>, <34> }. In fact, we will see from the proof of Theorem 2.4 that the refined bargaining set of any game contains at least one efficient payoff configuration. So even had we imposed overall efficiency in the definition of the refined bargaining set, we still could have the universal existence result. Some may prefer doing so since it further restricts payoff configurations in the final solution set. We refrain from doing so because we are not quite sure about players' capabilities of achieving overall Pareto improvements, which usually require changes from one coalition structure to another entirely new one. They seem too complicated to take for granted.

The properties of the refined bargaining set are summarized in the following theorem.

THEOREM 2.7. (i) *All payoff configurations in the refined bargaining set are individually rational, and collectively rational for the grand coalition*;

(ii) *all payoff configurations in the refined bargaining set are efficient for a game that is superadditive at N ; and at least one of them is efficient for a general game .*

The rest of the section is now devoted to a proof of Theorem 2.4.

A Proof of Theorem 2.4.

We begin with games that are superadditive at N. Define for each coalition $S \subset N$ a subset O_S of $Bd(V(N))$ as follows:

$$O_S = \{ x \in Bd(V(N)) \mid S \text{ has a justified objection against } x \}.$$

If we can show that there is an $x \in Bd(V(N)) \setminus (\bigcup_{S \in 2^N} O_S)$, then $\{x; N\} \in RB(V)$. In order to do this, we investigate some properties of $\{O_S\}_{S \in 2^N}$ that are stated in the form of three lemmas.

LEMMA 2.8. *If* $x_i = 0$, *then* $x \in O_{\{i\}}$. Obvious from the definition.

LEMMA 2.9. O_S *is open (relative in* $Bd(V(N))$ *) for each coalition* S.

Proof. Suppose that O_S is not open for a coalition S of two players or more. Then there would be some x, and a sequence $\{x^m\}$ such that $x \in O_S$, $x^m \in Bd(V(N)) \setminus O_S$, and $x^m \to x$. Since $x \in O_S$, there is a justified objection (S, y) against x. Because $x^m \to x$, (S, y) is also an objection against x^m for large m. But $x^m \in Bd(V(N)) \setminus O_S$ means that (S, y) is not a justified objection against x^m. Thus for each m there is a counterobjection $z^m \in V(T^m)$ from some coalition T^m against y (viewed as an objection against x^m) with

(1) $T^m \setminus S \neq \emptyset$, $S \setminus T^m \neq \emptyset$, and $S \cap T^m \neq \emptyset$;

(2) $z^m_k \geq x^m_k$ for all $k \in T^m \setminus S$, and $z^m_l \geq y_l$ for all $l \in S \cap T^m$.

Without loss of generality, we can assume $T^m = T$ for some fixed coalition T since there are only finitely many coalitions. We also assume that z^m converges to some z because $\{z^m\}$ is bounded from above through $V(T)$ and bounded from below through x^m. That $V(T)$ is closed implies $z \in V(T)$. Hence when we take the limit, we have

(2') $z_k \geq x_k$ for all $k \in T \setminus S$, and $z_l \geq y_l$ for all $l \in S \cap T$.

(1) and (2') means that (T, z) is a counterobjection against y (viewed as an objection against x). But this contradicts that (S, y) is a justified objection against x. Q.E.D.

The next lemma involves the notion of a balanced collection of coalitions. A collection $\mathcal{F}$ of coalitions is balanced if there exist positive numbers λ_S for all $S \in \mathcal{F}$ such that

$$\sum_{\substack{i \in S \\ S \in \mathcal{F}}} \lambda_S = 1, \text{ for all } i \in N.$$

LEMMA 2.10. *$\bigcap_{S \in \mathcal{F}} O_S = \emptyset$ for any balanced collection $\mathcal{F}$ of coalitions.*

Proof. First notice that if a balanced collection $\mathcal{F}$ of coalitions contains no partitions, then there are two coalitions S and T in $\mathcal{F}$ such that

$$T \setminus S \neq \emptyset, \; S \setminus T \neq \emptyset, \text{ and } S \cap T \neq \emptyset.$$

Here is a proof.[5] It is obvious when $n = 2$. Assume that it is true for all $n < k$. Consider the case $n = k$. Let $\mathcal{F}^* = \{S \setminus \{k\} \mid S \in \mathcal{F}\}$. Then $\mathcal{F}^*$ is balanced. If $\mathcal{F}^*$ contains a partition, then by the definition of $\mathcal{F}^*$ and the condition that $\mathcal{F}$ contains no partitions we can find the desired coalitions. If $\mathcal{F}^*$ does not contain a partition, then we can find the desired coalitions by the induction hypothesis.

Now suppose that there is an $x \in Bd(V(N))$ and a balanced collection of coalitions $\mathcal{F}$ such that every coalition in $\mathcal{F}$ has a justified objection against x. V is superadditive at N implies that $\mathcal{F}$ does not contain a partition. Find coalitions S and T in $\mathcal{F}$ such that $T \setminus S \neq \emptyset$, $S \setminus T \neq \emptyset$, and $S \cap T \neq \emptyset$. Given that S has a justified objection y against x, the excess of T at x must be less than that of S, i.e., $v(T) - \sum_{i \in T} x_i < v(S) - \sum_{i \in S} x_i$. For if it is not, we define z by

$$z_i = x_i \text{ for } i \in T \setminus S, \text{ and } z_i = y_i \text{ for } l \in S \cap T,$$

then (T, z) would be a counterobjection against (S, y) since

$$\begin{aligned}\sum_{i \in T} z_i &= \sum_{i \in T \setminus S} x_i + \sum_{i \in T \cap S} y_i \\ &= \sum_{i \in T \setminus S} x_i + \sum_{i \in T \cap S} y_i + \sum_{i \in S \setminus T} y_i - \sum_{i \in S \setminus T} y_i + \sum_{i \in T \cap S} x_i - \sum_{i \in T \cap S} x_i\end{aligned}$$

[5] I thank Peleg for allowing me to use his simple proof here.

$$= \sum_{i\in T} x_i + \sum_{i\in S} y_i - \sum_{i\in S\setminus T} y_i - \sum_{i\in T\cap S} x_i$$

$$< \sum_{i\in T} x_i + \sum_{i\in S} y_i - \sum_{i\in S\setminus T} x_i - \sum_{i\in T\cap S} x_i$$

$$\leq \sum_{i\in T} x_i + v(S) - \sum_{i\in S} x_i$$

$$\leq v(T) .$$

But the same argument also leads to the conclusion that the excess of S at x must be less than that of T. We thus have a contradiction. Q.E.D.

We now map $Bd(V(N)) \cap R^n_+$ homeomorphically onto the standard (n-1)-simplex $\Delta^N = \{ p \in R^n \mid p_i \geq 0, \text{ and } \sum_{i=1}^{n} p_i = 1\}$ by g,

$$g : x \rightarrow \frac{x}{\Sigma x_i} .$$

Given that $V(N)$ is comprehensive and bounded from above and that V is superadditive at N, it is easy to check that g is indeed a homeomorphism. Denote by $g(O_S)$ the image of O_S under g. So far, Lemmas 2.8 - 2.10 prove three properties of $\{g(O_S)\}_{S\in 2^N}$:

(a) $\Delta^{N\setminus\{i\}} \subset g(O_{\{i\}})$ for all $i \in N$; where $\Delta^{N\setminus\{i\}} = \{ p \in \Delta^N \mid p_i = 0 \}$;

(b) $g(O_S)$ is open (relative in Δ^N) for each coalition $S \subset N$; and

(c) $\bigcap_{S\in \mathcal{F}} g(O_S) = \emptyset$ for any balanced collection $\mathcal{F}$ of coalitions.

To complete the proof, we need yet one more lemma on open coverings of a simplex.

LEMMA 2.11. *If $\{C_S\}_{S\in 2^N}$ is a family of open sets of Δ^N that satisfy $\Delta^{N\setminus\{i\}} \subset C_{\{i\}}$ for all $i \in N$, and $\bigcup_{S\in 2^N} C_S = \Delta^N$, then there is a balanced collection of coalitions $\mathcal{F}$ such that $\bigcap_{S\in \mathcal{F}} C_S \neq \emptyset$.*

Proof. Scarf [9] proved the lemma under the identical assumptions except that all sets involved are closed. In the case of an open covering $\{C_S\}_{S\in 2^N}$, we take an $\varepsilon > 0$ and consider for each coalition S a closed set defined by

$$F_S = \{ x \in \Delta^N \mid dist(x, \Delta^N \setminus C_S) \geq \varepsilon \}.$$

It is obvious that F_S is a closed subset of C_S for each S. Since Δ^N and $\Delta^{N\setminus\{i\}}$ are compact, so if we choose an ε small enough, then we still have $\Delta^{N\setminus\{i\}} \subset F_{\{i\}}$ for each i and $\bigcup_{S\in 2^N} C_S = \Delta^N$. Thus Scarf's result implies that there exists a balanced collection of coalitions $\mathcal{F}$ such that $\bigcap_{S\in\mathcal{F}} F_S \neq \emptyset$, which in turn leads to $\bigcap_{S\in\mathcal{F}} C_S \neq \emptyset$. Q.E.D.

Now given that $\{g(O_S)\}_{S\in 2^N}$ satisfy (a)-(c), Lemma 2.11 immediately leads to the conclusion $\bigcup_{S\in 2^N} g(O_S) \neq \Delta^N$. Since g is a homeomorphism between $Bd(V(N)) \cap R^n_+$ and Δ^N, we have $\bigcup_{S\in 2^N} O_S \neq Bd(V(N))$. Thus $RB(V) \neq \emptyset$ because $\{ x; N\} \in RB(V)$ for any $x \in Bd(V(N)) \setminus (\bigcup_{S\in 2^N} O_S)$.

The proof of the existence of the refined bargaining set of a general game V is now straightforward. For any game V, we construct an auxiliary game $\tilde{V}$:

$\tilde{V}(S) = V(S)$, for all $S \subset N$ but $S \neq N$, and

$\tilde{V}(N) = \bigcup_{Q\in\mathcal{P}} \bigcap_{S_k\in Q} V(S_k)$, in which $\mathcal{P}$ is the family of all partitions of N.

Since the only change made is to $\tilde{V}(N)$, $\tilde{V}$ is still a TU game in our terminology. It is obvious that $\tilde{V}$ is superadditive at N. Thus there is an $x \in Bd(\tilde{V}(N))$ such that $\{ x; N\} \in RB(\tilde{V})$. But $Bd(\tilde{V}(N))$ is precisely the set of all efficient payoff configurations. Hence, if Q is a coalition structure of N that generates x, then $\{ x; Q \}$ belongs to the refined bargaining set of V. Q.E.D.

3. Comparisons with Other Works in the Literature

The discussion in this section is largely expository. Since the refined bargaining set uses the basic idea of Aumann and Maschler that is behind all variants of bargaining sets, it is important to see what the distinct features of the refined bargaining set are and how significant they are. We choose to illustrate them in comparisons with the bargaining sets by Aumann and Maschler [2], and by Mas-Colell [6]. Also, since one main motivation behind the introduction of the refined bargaining set is to provide a solution concept that endogenizes coalition formations, we will compare it to few theories by other authors on the same issue.

3.A. Comparisons with the Other Bargaining Sets

Although we use in the refined bargaining set the same idea Aumann and Maschler used in their original bargaining set $M_1^{(i)}$, our formalization of this idea has several major differences from theirs.

In the refined bargaining set objections and counterobjections are defined through coalitions of players, while in $M_1^{(i)}$ they are defined through individual players. A pair (S, v) that is an objection against some u is considered in $M_1^{(i)}$ as an objection of any player in S against any player not in S. It thus has many identities. It is justified in $M_1^{(i)}$ as long as there are some $i \in S$ and $j \notin S$ such that j has no counterobjection against it when viewed as an objection of i against j. Hence it allows a coalition to justify its objection by choosing cunningly a particular representative and a particular target. But such a manipulation should not be successful if the players outside the coalition are sophisticated enough. We believe that it is more reasonable to assume that an objection (S, v) is justified only if for all $i \in S$ and $j \notin S$, j has no counterobjection towards i, which is equivalent to the first two conditions in (RB1) of Definition 2.2. The following example is an illustration of this difference.

EXAMPLE 3.1. Let V be a 4-person TU game with $v(23) = v(34) = v(42) = 4.1$, $v(1234) = 6$, and $v(S) = 1$ for others. Take any payoff configuration $\{ x; \langle 1234 \rangle \}$ in the Aumann-Maschler bargaining set $M_1^{(i)}(1234)$. Since $x_i \geq 1$ for each i and $x_1 + x_2 + x_3 + x_4 = 6$, $x_2 + x_3 \leq 4$. Now if $x_1 > 1$, player 2 can raise an objection against 1 through coalition <23> to which 1 has no counterobjection. Hence, $x_1 = 1$ for any $\{ x; \langle 1234 \rangle \}$ in $M_1^{(i)}(1234)$. But the refined bargaining set contains many other payoff configurations. For instance, it contains $\{ (1.5, 1.5, 1.5, 1.5); \langle 1234 \rangle \}$. This is because when 2 has an objection against 1 through coalition <23>, 1 can count on 4 to counterobject even though he has no counterobjection of his own. We believe that this is a very plausible outcome. Player 1, on the one hand, is the weakest among all judging from marginal contributions to various coalitions; but the rest of the players, on the other hand, are severely hurt by their conflicts (which are reflected by $v(23) = v(34) = v(42) = 4.1$, and $v(123) = 1$). Hence 1 can take advantage of the latter factor and gain considerably!

Some readers might not share our view on this point. Also some might think that if this were the only difference, then the refined bargaining set would simply enlarge the Aumann-Maschler bargaining set. What really makes the refined bargaining set different from the Aumann-Maschler bargaining in an essential way is the following point.

It is required in the refined bargaining set that the objecting and the counterobjecting coalitions have a nonempty intersection. But it is not so in the definition of the Aumann-Maschler bargaining set. As we have argued in the last section, such a requirement is conceptually indispensable. Here we further discuss some undesirable features of the Aumann-Maschler bargaining set that are consequences of the lack of this nonempty intersection requirement. First, the Aumann-Maschler bargaining set has no intrinsic linkage to coalition structures. The key result by Peleg [7] states that for any nonnegative TU game the Aumann-Maschler bargaining set $M_1^{(i)}(Q)$ is nonempty for every coalition structure Q. It exists even for obviously unreasonable coalition structures. Hence it does not select coalition structures endogenously as the refined bargaining set does. Second, even for a reasonable coalition structure it includes many unreasonable payoff configurations .

EXAMPLE 3.2. V is a 5-person TU game with $v(12) = v(34) = v(45) = v(35) = 4.1$, $v(12345) = 10$, and $v(S) = 1$ for all others. The payoff configuration { (2, 2, 2, 2, 2); (12345) } belongs to the Aumann-Maschler Bargaining set $M_1^{(i)}(12345)$. But intuitively this is not a reasonable outcome. If players 1 and 2 leave the grand coalition and get 2.05 to each, players 3, 4, and 5 can do nothing to undermine it. Hence a reasonable payoff configuration should give players 1 and 2 a total of at least 4.1. Many configurations in $M_1^{(i)}(12345)$ do not satisfy this property. In contrast, all configurations in the refined bargaining set do satisfy this property. [6]

Of the two differences that the refined bargaining set has with the Aumann-Maschler bargaining set, the first makes counterobjections easier in the refined bargaining set but the second makes them more difficult. Hence, there is no set inclusion relation between the refined bargaining and the Aumann-Maschler bargaining set. At the conceptual level, although it is arguable which has a better view concerning the first difference, it is no doubt that the refined bargaining set represents a much sounder approach concerning the second and essential difference. Another important point is that in the Aumann-Maschler

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[6] It may be worth mentioning that { (2, 2, 2, 2, 2); (12345) } is the nucleolus for this game and (1.85, 1.85, 2.10, 2.10, 2.10) the Shapley value, both of which are not in the refined bargaining set. Hence, it is in general impossible to have an algbraic proof for the nonemptiness of the refined bargaining set. This shows that the refined bargaining set and Theorem 2.4 are technically deeper than the Aumann-Maschler bargaining set and the corresponding existence results.

bargaining set of any coalition structure, players across the coalitions cannot bargain with each other; therefore, it is a concept for exogenously given coalition structures. While the refined bargaining set places no restrictions on bargaining players. Hence, in our judgement, the refined bargaining set is an overall more satisfactory solution concept than the Aumann-Maschler bargaining set.

Another recent version of a bargaining set was proposed by Mas-Colell [6], in which objections and counterobjections were entirely defined through coalitions (Shapley and Shubik [11] earlier considered a similar bargaining set but to a lesser extent). However, a very serious weakness of the Mas-Colell bargaining set is that it imposes none of the three conditions in (RB1) on a counterobjection. Thus it is very difficult to make sense of a counterobjection defined there. Our criticisms on the Aumann-Maschler bargaining set clearly applies to the Mas-Colell bargaining set as well. Even Mas-Colell himself pointed out that "the really serious problem is that the set is too large." [7] In fact, it is almost a superset of both the refined bargaining set and the Aumann-Maschler bargaining set. It is not exactly so only because it requires that at least one inequality in (RB2) must be strict. Of course, we could have required the same in the refined bargaining set. But we did not do it for the following reasons.

First, the refined bargaining set, as a correspondence, is upper-hemi-continuous for suitable topologies on the set of all games, such as the closed-convergence topology. If one wants this property, considered by many as a necessity for a reasonable solution concept, then one has to be content with weak inequalities. The Mas-Colell bargaining set, on the other hand, is not upper-hemi-continuous as shown by Example 2.2 in Vohra [12]. Second, if one wants to prove for a general game the existence of the refined

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[7] But keep in mind that the emphasis of Mas-Colell's paper was not on the bargaining set pre se. The primary result there was the equivalence of his bargaining set and Walrasian equilibrium allocations in an atomless exchange economy. So an enlarged bargaining set only makes the equivalence result even stronger.

bargaining set with (RB1), then again one has to be content with weak inequalities. The existence of the Mas-Colell bargaining set was proved only for games with a weak superadditivity property (Vohra [12]). Third, in most games the requirement of at least one strict inequality shrinks the refined bargaining set only marginally anyway. [8]

3.B. Other Theories of Endogenous Coalition Formation

There have been few theories in the literature that address the issue of endogenous coalition formation. Here we briefly discuss those theories by Shenoy [10], Hart and Kurz [5], and Bennett [3], and compare them with our theory based on the refined bargaining set. Noticeably, all these theories were constructed beyond a simple standard solution concept. It seems that their authors never believed that any standard solution concept could have a chance at endogenizing coalition formation.

In Shenoy's theory, one first chooses a solution concept with coalition structures and calculates the solution sets for all coalition structures. One then uses these solution sets to induce a dominance relation on coalition structures and select the undominated ones as the only likely coalition structures for a game. In his paper [10], Shenoy also illustrated the theory using the Aumann-Maschler bargaining set as the solution concept. We find two major defects with Shenoy's theory. First, by choosing a solution concept with coalition structures, the theory implicitly accepts the faulty two-stage assumption mentioned in the introduction. Second, the theory was actually more of a suggestion in that it was never carried out for any chosen solution concept. Choosing the Aumann-Maschler bargaining set as the solution concept, Shenoy was only able to show that for any three-player game there exists an undominated coalition structure. No one has ever reported more successes

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[8] A precise formulation of this statement was contained in Zhou [13].

since. It is believed that there are games for which every coalition structure is dominated according to Shenoy' theory. In this case, all coalition structures are eliminated!

Hart and Kurz's [5] theory of endogenous coalition formation is quite different from other approaches including ours. Their view was that "the reason coalition forms in not in order to get their worth, but to be in a better position when bargaining with the others on how to divide the maximal amount available (i.e., the worth of the grand coalition, which for superadditive games is no less than $\Sigma v(B_k)$)." Although this view has a point for superadditive games, it does not apply to games that are not superadditive. But it is precisely the lack of superadditivity that to a large extent leads to the issue of endogenous coalition formation. This was missed in Hart and Kurz's theory. Finally, even taken as a theory of endogenous coalition formation for superadditive games, it still possesses the same two shortcomings that Shenoy's theory has.

The aspiration approach by Bennett [3] offers yet another theory of endogenous coalition formation. Unlike standard solution concepts that select feasible payoff vectors for a game, various aspiration solutions specify various prices demanded by the players. These prices are realizable in the sense that each player belongs to a coalition which can pay all of its players their demanded prices. The theory concludes that "the coalitions which can afford to pay these prices are the coalitions which are predicted to form in the game." But we have strong reservations towards this theory. It is known that generally these demanded prices are not feasible for any coalition structure; therefore, they cannot be realized simultaneously. When some coalitions which can afford to pay their players form, there must be other players (possibly many) not in these coalitions who cannot get their demanded prices. The theory says nothing about what they will do and how much they will get. Hence, the theory is clearly incomplete. Furthermore, it is no doubt that what these players will do and how much they get should have an effect on players in those supposedly formed coalitions. Hence, even the theory's prediction on those coalitions and players is also highly questionable.

To summarize, all theories of endogenous coalition formation reviewed have various shortcomings. On the other hand, the refined bargaining set offers a much simpler yet far more satisfactory theory. Of course, we do not mean that the problem of endogenous coalition formation is actually solved by our approach. Any complicated problem like this cannot be solved by any single solution concept, and as a matter of fact, not even by any single theory. However, we have shown that it can be approached by the refined bargaining set, and we do believe that the refined bargaining set represents a good starting point for a more satisfactory theory.

4. REFIND BARGAINING SETS AND NON-TRANSFERABLE UTILITY GAMES

In this section we extend the analysis to nontransferable utility games. An n-person nontransferable utility game V in coalitional form is a function from 2^N to subsets of R^n. It is assumed that V satisfies the following conditions:

(NTU1) For each coalition S, $V(S)$ is nonempty, closed, comprehensive, bounded from above, and it is a cylinder ($x \in V(S)$, $x_i = y_i$ for all $i \in S \Rightarrow y \in V(S)$);

(NTU2) For every i, there is a $b_i > 0$, such that $V(\{i\}) = \{ x \in R^n \mid x_i \leq b_i \}$.

The definitions of objections, counterobjections, and the refined bargaining set are the same as in Section 2 since they all were defined through $V(S)$ without any reference to the transferable utility property. However, the existence of the refined bargaining set of a NTU game remains an open problem. We do not have a proof of it yet, nor do we have any example of a game with an empty refined bargaining set. Given the arguments we have had in favor of the refined bargaining set, this is certainly an important problem to be answered. In this section we provide some partial answers that may shed some lights on solving this open problem.

4.A. A Sufficient Condition for a Nonempty Refined Bargaining Set

If we examine the proof of Theorem 2.4 closely, we can find a sufficient condition for the existence of the refined bargaining set of a NTU game. In fact, Lemma 2.10 was the only place in the proof where the transferable utility property was used. It states that for an efficient payoff configuration x of a TU game the collection of coalitions that have justified objections against x does not have a balanced subcollection. We can no longer prove this for a general NTU game. However, if we simply assume it, then it implies the existence of the refined bargaining set.

DEFINITION 4.1. A NTU game V is RB-balanced if for any efficient payoff configuration x the collection of the coalitions that have justified objections against x does not have a balanced subcollection.

THEOREM 4.2. *Any RB-balanced game V has a nonempty refined bargaining set.*

The proof of Theorem 4.2 is the same as that of Theorem 2.4 since Lemma 2.10 is now replaced by the assumption of RB-balancedness. This result is analogous to that by Scarf [9] on the existence of the core and that by Vohra [12] on the existence of the Mas-Colell bargaining set. Its main weakness is that the condition of RB-balancedness is not intuitive. But before there is a complete answer to the question of the existence of the refined bargaining set, this approach is still the most powerful one.[9] Here we show that an important class of NTU games are in fact RB-balanced.

DEFINITION 4.3. A NTU game V is called a pairwise NTU game if it satisfies

(PNTU3) for every coalition S with $2 < |S| < n$, there is a number $v(S)$

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[9] To appreciate this point, readers are reminded that even though Theorem 2.4 deals with TU games only, it is still necessary to adopt our approach or at least some form of fixed-point argument (see footnote 6).

such that $V(S) = \{ x \in R^n \mid \sum_{i \in S} x_i \leq v(S) \}$.

A pairwise NTU game best describes a situation in which cooperation between any two players is quite successful and can take place in a nonlinear fashion, but cooperation with more parties involved has to depend on a common medium of exchange, say money. The class of pairwise NTU games is reasonably rich: It contains all TU games, all games with three players, and all games of pairs (Peleg [8]).

THEOREM 4.4. *Any pairwise NTU game V is RB-balanced, thus has a nonempty refined bargaining set.*

Proof. As in the proof of Lemma 2.10, we have to show that for any $x \in Bd(V(N))$ and any two coalitions S and T with $T \setminus S \neq \emptyset$, $S \setminus T \neq \emptyset$, and $S \cap T \neq \emptyset$, it cannot be true that both S and T have a justified objection against x. When both S and T have more than two players, the proof of Lemma 2.10 applies. When either has two players, the conditions on S and T imply that $S \cap T$ contains a single player. It is then obvious from the definition that at most one of S and T can have a justified objection against x.

Q.E.D.

4.B. THE QUASI-BARGAINING SET

Here we consider another weaker version of the refined bargaining set for which the existence is guaranteed. The basic idea is as follows. Once a payoff configuration x is listed, players may try to form coalitions and raise objections against x. The question is what coalitions and objections are most likely to emerge, especially when various coalitions have to compete for some players in common. Obviously, if a coalition has an objection that gives each of its players more than he can ever get by participating in other coalitions, then such an objection is very stable and it definitely defeats x. But if there is no such an objection, then it is not clear what coalitions will be formed and what objection will be raised. In this case, one might believe that no coalition will be able to raise any objection. This is formalized by the next definition.

DEFINITION 4.5. Let (S, y) be an objection against x. A *QB*-counterobjection from another coalition T to (S, y) is a pair (T, z) in which $z \in V(T)$ and

(QB1) $T \setminus S \neq \emptyset$, $S \setminus T \neq \emptyset$, and $S \cap T \neq \emptyset$;

(QB2) $z_k \geq x_k$ for all $k \in T$, and $z_l \geq y_l$ for at least one $l \in S \cap T$.

An objection (S, y) against x is strongly justified if there exists no *QB*-counterobjection from any other coalition T to (S, y). The quasi-bargaining set of a game is the set of all payoff configurations against which no coalitions have strongly justified objections.

The proof of the existence of the quasi-bargaining set is the same as the proofs of Theorems 2.4 and 4.4. We leave it to the readers.

THEOREM 4.6. *Any NTU game V has a nonempty quasi-bargaining set.*

Of course, many payoff configurations in the quasi-bargaining set are unreasonable. Some of them could be dismissed through further refinements. But this does not mean that we should dismiss the quasi-bargaining set as a meaningful concept, at least not now. Since it is obvious that payoff configurations not in the quasi-bargaining set are not plausible, the quasi-bargaining set at least eliminates some coalition structures that are incompatible with a NTU game just as the refined bargaining set does to a TU game. This feature is not held by any other existing solution concepts of NTU games. Before we can prove the existence of the refined bargaining set or any other reasonable solution concepts with the same feature for a general NTU game, the quasi-bargaining set is the only solution concept that endogenously provides some answers to the two fundamental questions in game theory.

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