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ABSTRACT

Let $p = (p_1,\ldots,p_n)$ be a vector of positive integers whose greatest common divisor is unity. The Frobenius problem is to find the largest integer f* which cannot be written as a non-negative integral combination of the p_i. In this note we relate the Frobenius problem to the topic of maximal lattice free bodies and describe an algorithm for $n = 3$.

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I. Introduction

Let $p = (p_1, \ldots, p_n)$ be a vector of positive integers whose greatest common divisor is unity. The Frobenius problem is to find the largest integer f^* which cannot be written as a non-negative integral combination of the p_i. For $n = 2$, it is well known that $f^* = p_1p_2 - p_1 - p_2$. For $n = 3$, there is an algorithm of Rödseth (1977) which finds the Frobenius number f^* in polynomial time. Recently, Kannan (1989) has produced an algorithm for the Frobenius problem which runs in polynomial time for all fixed n, but which is doubly exponential in n.

The question of whether a single linear equation $\Sigma p_i h_i = f$ is solvable in non-negative integers is NP complete, and we cannot expect to resolve its solvability by means of an algorithm which is polynomial in the number of variables as well as the bit size of the data. For fixed n, Lenstra's algorithm (1983) will execute in polynomial time for any particular linear equation. The significance of the Frobenius problem is that it is concerned with a family of linear equations, $\Sigma p_i h_i = f$, as f varies over all positive integers, rather than with a single equation itself. For any instance of the Frobenius problem, the Frobenius number f^* will typically be sufficiently large so that its determination by an exhaustive search over all f less than some established upper bound will not yield a polynomial algorithm.

In this note we shall relate the Frobenius problem to a different area under recent investigation, that of maximal closed convex sets containing no interior lattice points. Given a matrix A, the body $\{x:Ax \le b\}$ is a <u>maximal</u>

lattice free body if it contains no lattice points in its interior and if any strictly larger body obtained by relaxing some of the inequalities does contain an interior lattice point. We demonstrate that if we can maximize a linear function over the set of b's yielding maximal lattice free bodies for a matrix with n rows and n-1 columns, then we can solve the Frobenius problem with n variables. One consequence is an algorithm for the three variable problem - somewhat similar to Rödseth's algorithm - which runs in linear time in the bit size of the integers p_i. We also relate the Frobenius number to the covering radius of a simplex in R^{n-1}, in a somewhat different fashion than that established by Kannan.

Lovász (1988) has conjectured that if n is fixed and A is integral, the set of b yielding maximal lattice free bodies is the union of the set of lattice points in a polynomial number of polyhedra - with a particular lattice for each polyhedron. Maximizing a linear function over the lattice points in each such polyhedron is a standard integer program which can be solved in polynomial time for a fixed number of variables. If the Lovász conjecture were correct, this would yield an alternative polynomial algorithm for the Frobenius problem.

II. The Relationship to Maximal Lattice Free Bodies

Let A be a matrix of size n by n-1, whose columns generate the n-1 dimensional lattice of integers h satisfying $p.h = 0$. In this case the bodies $\{x:Ax \le b\}$ will be simplices which are non-empty if $p.b \ge 0$. Our main result is

Theorem 1. $f^* = \max\{p.b \mid b \text{ is integral and } Ax \le b \text{ contains no lattice points}\}$.

Proof: To demonstrate Theorem 1, we observe that if b is an integer vector such that $Ax \le b$ contains no lattice points, then $f = p.b$ cannot be written as $p.h$ with h non-negative integers. For if this were possible then $0 = p.(b-h)$ so that $b-h$ is in the n-1 dimensional lattice generated by the columns of A. It follows that $b-h = A\xi$ for some integral ξ and therefore the set $Ax \le b$ contains a lattice point.

Conversely, if b is an integral vector such that $Ax \le b$ contains a lattice point ξ, then $f = p.b = p.(b - A\xi)$, with $b - A\xi$ a non-negative integer vector. It follows from these two observations that f^* is the largest value of $p.b$ for those integral b such that $Ax \le b$ is free of lattice points. ⊗

Theorem 1 permits us to calculate the Frobenius number f^* from a description of the set of vectors b such that $K_b = \{x{:}Ax \le b\}$ is a maximal lattice free body, according to our previous definition. We simply remark that for integral b, the simplex $\{x{:}Ax \le b\}$ contains no lattice points in its interior if, and only if, $\{x{:}Ax \le b-e\}$ contains no lattice points at all, where e is the vector all of whose components are unity. It follows that

$$f^* = \max\{p.b \mid Ax \le b \text{ is a maximal lattice free body}\} - \Sigma p_i.$$

Aside from lattice translates of $\{x{:}Ax \le b\}$, which do not change the value of $p.b$, there are a finite number of maximal lattice free bodies associated with the matrix A.

Kannan shows that the calculation of the Frŏbenius number is equivalent to finding the covering radius of a particular n-1 dimensional simplex. The <u>covering radius</u> of a body K in R^{n-1} is the smallest ρ such that the lattice translates of ρK cover R^{n-1}. Our discussion yields the following relation between the Frobenius number and the covering radius of $\{x : Ax \le b\}$.

Theorem 2. Let the covering radius of $\{x : Ax \le b\}$ be ρ_b, for any particular b with $p.b > 0$. Then $f^* = (p.b)\rho_b - \Sigma p_i$.

Proof: If K_{b*} is that maximal lattice free simplex $\{x : Ax \le b^*\}$ which maximizes p.b, then its covering radius is unity. For if x is not covered by any lattice translate of K_{b*}, then $K_{b*} - x$ contains no lattice points and it can be expanded to a maximal lattice free body strictly larger than K_{b*}. On the other hand, a slight contraction of K_{b*} contains no lattice points, and, therefore, its lattice translates do not cover the origin. The covering radius of K_{b*} is therefore equal to unity. For any other b with $p.b > 0$, the simplex K_b is similar to K_{b*}; it can be brought to K_{b*} by a suitable translation and expansion by a factor $p.b^*/p.b$. It follows that the covering radius of K_b is $p.b^*/p.b = (f^* + \Sigma p_i)/(p.b)$. ⊗

III. Maximal Lattice Free Bodies for n = 3

Relatively little is known about the set of maximal lattice free bodies associated with a general matrix A with n rows and n-1 columns. It is not clear to us how to use the analysis given by Scarf (1985) for the case $n = 4$ to solve the corresponding Frobenius problem. When $n = 3$, Scarf (1981) has demonstrated - under the assumptions that the entries in each row of A have an irrational ratio, that $\pi A = 0$ for a strictly positive vector π and that no two rows are proportional - that there are two maximal lattice free bodies of the form $\{x : A \le b\}$, up to a lattice translation, and that these bodies are easy to find. Specifically, Scarf shows that there is a unimodular coordinate transformation so that the matrix A has the sign pattern

$$\begin{bmatrix} - & - \\ + & - \\ - & + \end{bmatrix},$$

with the sum of the second and third rows strictly positive, and that the two maximal lattice free bodies are given by

$$b^1 = (0,\ a_{2,1},\ a_{3,1} + a_{3,2}) \text{ and}$$
$$b^2 = (0,\ a_{2,1} + a_{2,2},\ a_{3,2}).$$

But the assumption that the entries in each row of A have an irrational ratio is, of course, not satisfied in our case, and the analysis to be presented becomes somewhat more complex; in particular some of the strict inequalities given above may become weak inequalities and there may be more than two maximal lattice free regions.

We shall describe an algorithm which yields a unimodular transformation of coordinates such that the matrix A has the sign pattern

$$\begin{bmatrix} - & \leq \\ + & - \\ \leq & + \end{bmatrix},$$

with the sum of the entries in the second row greater than or equal to zero, and the sum in the third row strictly positive; and then demonstrate that this pattern is sufficient to characterize the maximal lattice free triangles (the symbol $\leq$ appearing in the matrix signifies that the corresponding entry is less than or equal to zero).

We begin with a particular form for the matrix A. Let γ be the greatest common divisor of p_2 and p_3, and write $\gamma = m_3p_2 - m_2p_3$, with m_2 and m_3 integers satisfying $0 \leq m_2 < p_2/\gamma$ and $0 < m_3 \leq p_3/\gamma$. Then the columns of

$$A = \begin{bmatrix} -\gamma & 0 \\ m_3p_1 & -p_3/\gamma \\ -m_2p_1 & p_2/\gamma \end{bmatrix}$$

generate the lattice of integers satisfying $p.h = 0$. The matrix has the sign pattern described above, but without any specific signs for the sums of the second and third rows. We shall systematically add integral multiples of one of the columns of A to the other column, retaining the signs of the entries in A and ultimately achieving the desired signs for these row sums.

The algorithm alternates between two steps:

1. adding the largest integral multiple of column 2 to column 1 so as to preserve the sign pattern $(-, +, \le)$ in column 1, and

2. adding the largest integral multiple of column 1 to column 2 so as to preserve the sign pattern $(\le, -, +)$ in column 2.

After a step of type 1, the sum of the two columns of A will have the sign pattern $(-, \ge, +)$, in which case we terminate, or $(-, -, +)$ and we move to a step of type 2. After a step of type 2, the sum of the two columns of A will have the sign pattern $(-, \ge, +)$, in which case we terminate, or the pattern $(-, +, \le)$ and we continue with a step of type 1. In both of these observations we use the fact that entries in the column sum cannot be all less than or equal to zero, since the pair of columns in a matrix arising from repeated applications of steps 1 and 2 will generate the lattice of integers satisfying $p.h = 0$. The algorithm clearly terminates in a number of steps bounded above by the bit size of p.

Now let us argue that the sign pattern which has just been established is sufficient to identify the maximal lattice free bodies associated with A. Consider the triangle $\{x:Ax \le b^1\}$, with $b^1 = (0, a_{2,1}, a_{3,1} + a_{3,2})$. The inequalities

$$a_{1,1} < 0, \qquad 0 < a_{2,1}, \qquad 0 < a_{3,1} + a_{3,2}$$
$$a_{1,1} + a_{1,2} < 0, \qquad a_{2,2} < 0, \qquad 0 < a_{3,2}$$

state that (0,0), (1,0), (1,1) are each in the relative interior of one of the three faces of the triangle. It is elementary to argue that if the triangle contains any lattice points in its interior, then one of the three lattice points (0,1), (0,-1), or (2,1) will also be interior to the triangle. But this cannot be so since $a_1.(0,-1) \geq 0$, $a_2.(2,1) \geq a_{2,1}$ and $a_3.(1,0) \geq a_{3,1} + a_{3,2}$. Moreover, this triangle is a maximal lattice free triangle since an interior point is obtained if any of the three inequalities is relaxed.

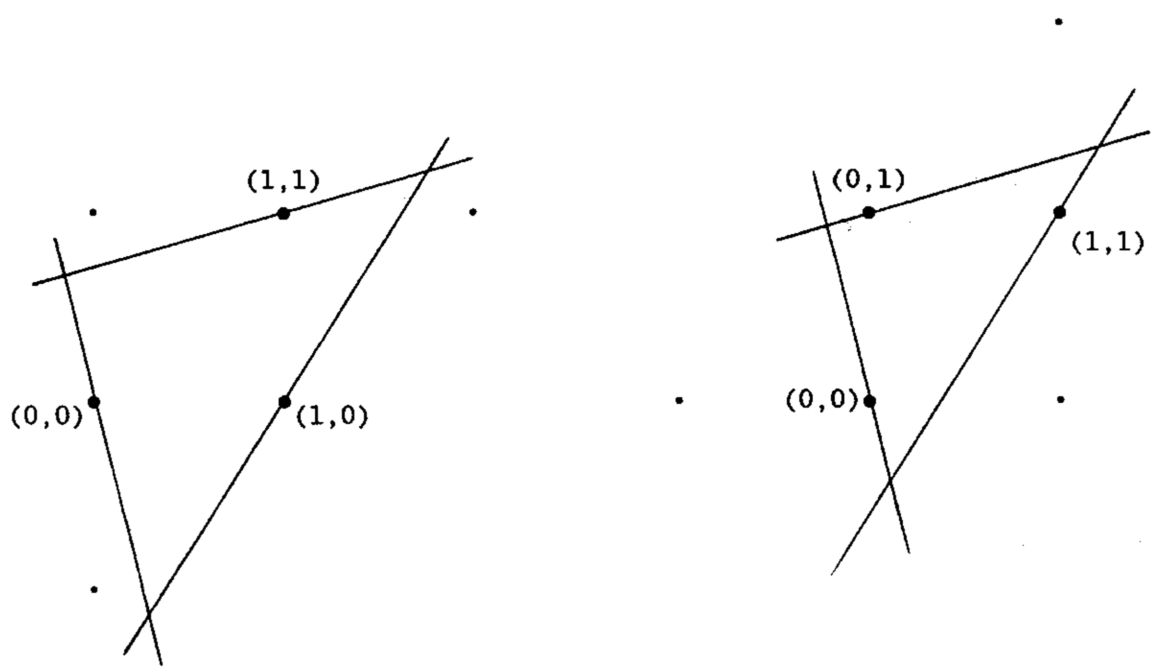

FIGURE 1

If the three inequalities $a_{1,2} \leq 0$, $a_{2,1} + a_{2,2} \geq 0$ and $a_{3,1} \leq 0$ are all strict, then an inspection of Fig.1 reveals that the triangle with

$b^2 = (0, a_{2,1} + a_{2,2}, a_{3,2})$ is also a maximal lattice free body. In this case it is easy to see that there are no other maximal lattice free triangles. Such a body must have a lattice point in the relative interior of each side; we may translate the body so that (0,0) is in the relative interior of the side $a_1 x = b_1$. Since (0,1) is not interior to the triangle we must have $a_3.(0,1) \geq b_3$ and similarly $a_2.(1,0) \geq b_2$. Moreover, the lattice point (1,1) must satisfy $a_i x \geq b_i$ for $i = 2$ or 3; if this inequality holds for $i = 3$, then the body is contained in $\{x:Ax \leq b^1\}$. If the inequality is satisfied for $i = 2$, then the body is contained in $\{x:Ax \leq b^2\}$. We see that $f^* = \max\{p.b^1, p.b^2\} - p_1 - p_2 - p_3$.

If $a_{1,2} = 0$, the second lattice free body is not maximal, since b_3 may be relaxed slightly without introducing interior lattice points. In this case, the bodies with $a_2 x = b_2$ passing through $(1,t)$ and $a_3 x = b_3$ passing through $(1,t+1)$, so that $b = b^1 + t(0, a_{2,2}, a_{3,2})$, will also be maximal lattice free bodies for all non-negative integral t with $a_{2,1} + ta_{2,2} \geq 0$. But all of these bodies will be lattice translates of each other and yield the same value of p.b. More specifically, if $a_{1,2} = 0$, then if γ is the greatest common divisor of p_2 and p_3, we have $a_{2,2} = -p_3/\gamma$, $a_{3,2} = p_2/\gamma$, $a_{1,1} = -\gamma$ and $p.b = p.b^1 = \gamma p_1 + p_2 p_3/\gamma$. The other cases, $a_{2,1} + a_{2,2} = 0$ or $a_{3,1} = 0$, yield the same conclusion, i.e., that the Frobenius number is $p.b^1 - \Sigma p_i$.

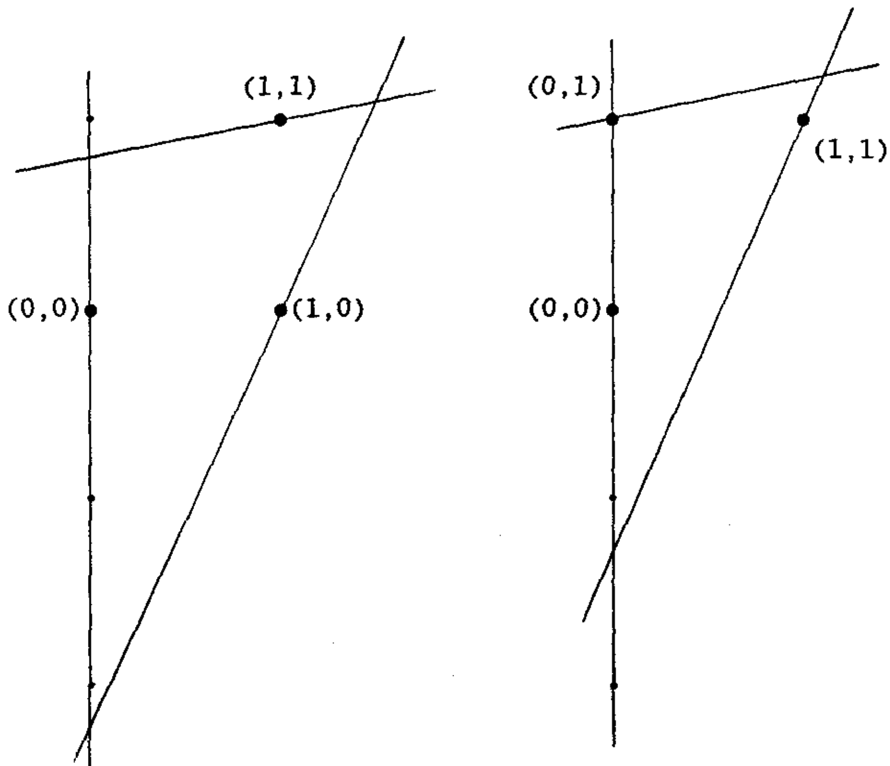

FIGURE 2

As a final topic let us turn to the possibility of finding the Frobenius number f^* by studying a pair of problems with two variables. As before, let γ be the greatest common divisor of p_2 and p_3 so that p_2/γ and p_3/γ are integers. From the analytic solution of the two variable problem we know that every integer

$$h \geq h^* = (p_2/\gamma)(p_3/\gamma) - (p_2/\gamma) - (p_3/\gamma) + 1$$

can be represented as a non-negative integer combination of p_2/γ and p_3/γ and therefore for every integer $h \geq h^*$, $\gamma h = p_2h_2 + p_3h_3$ for non-negative integral h_2 and h_3. By considering the Frobenius problem based on the two integers p_1 and γ we also know that every integer $\geq p_1\gamma - p_1 - \gamma + 1$ can be represented as $p_1h_1 + \gamma h$ for h_1 and $h \geq 0$. From this last remark it follows that every integer $\geq p_1\gamma - p_1 - \gamma + 1 + \gamma h^*$

$$= p_1\gamma + p_2p_3/\gamma - p_1 - p_2 - p_3 + 1$$

can be represented as $a_1h_1 + \gamma h$ for $h_1 \geq 0$ and $h \geq h^*$ and, therefore, as a non-negative integral combination of p_1, p_2, p_3. We conclude that $f^* \leq p_1\gamma + p_2p_3/\gamma - p_1 - p_2 - p_3$, with equality if we can exhibit a maximal lattice free triangle $\{x:Ax \leq b\}$ with $p.b = p_1\gamma + p_2p_3/\gamma$. In the case in which one of the three inequalities $a_{1,2} \leq 0$, $a_{2,1} + a_{2,2} \geq 0$ and $a_{3,1} \leq 0$, obtained by our algorithm, is satisfied with equality, we have constructed precisely such a maximal lattice free body.

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