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Abstract: We present a new approach to the theory of imperfect competition and apply it to study price competition among differentiated products. The central result provides general conditions under which there exists a pure-strategy price equilibrium for any number of firms producing any set of products. This includes products with multi-dimensional attributes. In addition to the proof of existence, we provide conditions for uniqueness. Our analysis covers location models, the characteristics approach, and probabilistic choice together in a unified framework.

To prove existence, we employ aggregation theorems due to Prékopa (1971) and Borell (1975). Our companion paper [Caplin and Nalebuff (1990)] introduces these theorems and develops the application to super-majority voting rules.

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1. INTRODUCTION

We present a new approach to the theory of imperfect competition and apply it to study price competition among differentiated products. The central result is that there exists a pure-strategy price equilibrium for any number of firms producing any set of products. In addition to the proof of existence, we provide conditions for uniqueness. Our model both unites diverse strands of the earlier literature and opens up uncharted areas for future analysis. In particular, we expand the traditional one-dimensional framework to allow for multi-dimensional product differentiation.

Our approach involves twin restrictions on consumer preferences: one on individuals' preferences, the other on the distribution of preferences across society. These are generalizations of the restrictions supporting 64%-majority rule presented in Caplin and Nalebuff (1988).

To prove existence, we apply a new technique of aggregation. This technique is valuable in a variety of other problems. In the companion paper, we use the aggregation result to generalize our earlier work on 64%-majority rule and to characterize the relationship between the distribution of human capital and the distribution of income [Caplin and Nalebuff (1990)]. There are additional applications in statistics and in search theory.

We begin with a brief review of the early literature on imperfect competition, describing in more detail the existence problem and previous solutions. Section 3 presents our twin assumptions, and shows that they cover many standard cases. In Section 4, we introduce the aggregation theorem and use it in the analysis of demand functions. The proof of existence of equilibrium is in Section 5, and uniqueness results follow in Section 6. Section 7 discusses the importance of dimension in models of imperfect competition. A series of extensions is provided in Section 8. Concluding remarks are in Section 9. All propositions stated without proof are demonstrated in the Appendix.

2. IMPERFECT COMPETITION AFTER HOTELLING

In his celebrated paper, "Stability in Competition," Harold Hotelling (1929) built the pioneering model of product differentiation. Following Bertrand (1883) he believed that price is the relevant strategic variable in oligopoly competition.[1] Unlike Bertrand, he

[1] It may help to set Hotelling's (1929) work in historical context. The theory of imperfect competition begins with Cournot (1838). Cournot solved for equilibrium output of a group of firms producing identical goods at zero cost. Bertrand (1883) objected to Cournot's model on the grounds that price was the natural strategic variable in oligopoly situations. This radically altered the nature of the equilibrium, forcing price to zero in the Cournot model. In his turn Edgeworth (1897) pointed to an instability in the Bertrand model; once capacity constraints are added, there may be no equilibrium in prices – at least not in pure strategies. Beckmann (1965) and Levitan and Shubik (1972) provide an early consideration of mixed strategies; Kreps and Scheinkman (1983) use this mixed strategy approach to show the connection between Bertrand competition and Cournot outcomes in the presence of capacity constraints.

emphasized the importance of imperfect substitutability in mitigating price competition: a firm does not take all its competitors' business by slightly undercutting in price. To capture this, Hotelling modelled the price and location decisions of firms selling goods differentiated by their location along Main Street. He first calculated equilibrium prices for given firm locations and then used this solution to study optimal locations. Hotelling concluded that competitive forces result in too little product diversity – the celebrated "principle of minimal differentiation." Unfortunately, Hotelling's model was a false start. Firms may pick products for which there are no equilibrium prices.[2]

The nature of the existence problem is seen clearly if one tries to base a proof of existence on a standard fixed-point argument. It is straightforward to build models where each firm's best response to other firms' prices is bounded and has standard continuity properties. Specifically, the best response correspondence is typically upper hemi-continuous, suggesting that Kakutani's theorem should be used to demonstrate existence. However the fixed point argument requires that the best-response correspondence be convex valued. This condition may fail in the context of imperfect competition. Without any restrictions on market demand, it may be that two extreme strategies, either charging a high price to a select group of customers (for whom the product is well positioned) or charging a low price to a mass market, both dominate the strategy of setting an intermediate price. It is this issue which has been the major stumbling block in the study of existence.

In response to the existence problem, several different directions have been explored [Gabszewicz and Thisse (1986) provide a valuable survey]. There are a series of papers describing the magnitude of the problem; for example, an entire issue of *Regional Science and Urban Economics* is devoted to negative results [see Macleod (1985) and Shulz and Stahl (1985) for an overview]. A second approach looks for a different type of equilibrium concept; there is the conjectural approach [Eaton (1972)] and price discrimination [Lederer and Hurter (1986)]. Although no pure strategy price equilibrium may exist, Dasgupta and Maskin (1986) show there will be a mixed-strategy solution in Hotelling's model. This solution has been calculated by Osborne and Pitchik (1987); unfortunately, its complexity effectively rules out any comparative static analysis.

On the positive side, several authors have explored special conditions under which a pure strategy price equilibrium exists [Economides (1989), Gabszewicz and Thisse (1979), Hauser (1988), Hauser and Wernerfelt (1988), Lane (1980), and Shaked and Sutton (1982)]. In these models, demand functions are concave in own price (or log-price) so

[2] In Hotelling's model, demand is in fact discontinuous when firms are located close together. Hotelling recognized this, but chose to ignore it as "adventitious". The issue was more serious than he realized. Shubik (1959) and Vickrey (1964) recognized the flaw in Hotelling's reasoning. d'Aspremont *et. al.* (1979) then proved that the discontinuity in demand rules out existence of an equilibrium in the Hotelling model once firms are sufficiently close together. While it is simple to alter the model in ways that restore the continuity of demand, guaranteeing existence of equilibrium is far more challenging.

that there is a unique best-response, and a fixed point argument applies to prove existence. This approach requires a judicious selection of the utility function and strong restrictions on the distribution of consumer types. In some cases all incomes are equal; in the others, all preferences are the same. All are one-dimensional and have consumers uniformly distributed over the market area.

The restriction to product differentiation along one dimension is a real constraint. It limits the scope of consumer preferences (see Proposition 8). It also places an artificial limit on the extent of competition and interaction between firms that is only gradually relaxed in higher dimensions (see Proposition 9). But primarily, it limits applications. Most products are naturally multi-dimensional. Computer printers vary in terms of speed, noise, and clarity of output. Cars vary in size, comfort, sportiness, fuel economy, reliability, and in many other dimensions. The multi-dimensional nature of products is particularly evident in marketing new products. Frequently, the entrant is differentiated from the existing brands by introducing a new dimension to the characteristics space; caffeine-free soda and low alcohol beer are two recent examples. This suggests that a multi-dimensional setting is fundamental to the study of product differentiation.

There are several existing multi-dimensional models of product differentiation. Dixit and Stiglitz (1977) and Spence (1976) use a multi-dimensional C.E.S. model to consider whether a competitive market will provide an optimal amount of product diversity. Their approach relies on a completely symmetric model.[3] This bypasses any issue of product design and questions where asymmetries play a prominent role. Multi-dimensional probabilistic choice models have been developed by Perloff and Salop (1985), Anderson and de Palma (1988), and Anderson, de Palma, and Thisse (1988, 1989). The logit model of Anderson and de Palma is especially notable for the sophisticated statement of the conditions for existence, which are far weaker than the concavity condition on demand applied in the spatial location literature. But once again, the models requires a completely symmetric specification of competing products; issues of product design cannot be addressed. In contrast, the characteristics approach to product differentiation, pioneered by Gorman (1980) and Lancaster (1966), is well-suited to issues of product design. However, application of multi-dimensional characteristics models to imperfect competition has been limited by the difficulty in establishing existence of equilibrium.

We present a new resolution to the existence problem. Our model covers each of these alternative approaches to the theory of differentiated products. Yet there is no imposition of symmetry nor any restriction to one-dimensional products. Thus our existence result opens up the possibility of studying multi-dimensional product differentiation and product design. Behind these advances lie aggregation theorems due to Prékopa (1971) and

[3] These authors consider the C.E.S. demand that arises in the limit with an infinite number of competing firms. They do not consider the issue of whether an equilibrium exists for the standard C.E.S. model with a finite number of firms: Proposition 14 provides this existence result.

Borell (1975). These results provide ideal mathematical tools for the study of imperfectly competitive markets.

3. THE MODEL

There are m firms. Firm i produces a single product, x_i, at cost c_i. Product characteristics are fixed at the stage when firms engage in price competition. Given the set of products in the market, $\bar{x} = [x_1, \ldots, x_m]$, firms simultaneously choose prices.

All products lie in a w-dimensional Euclidean space, $X \subset R^w$. We use χ to represent a general element from the set X and χ_k is its k^{th} characteristic, $\chi = (\chi_1, \ldots, \chi_k, \ldots, \chi_w)$. The reason for this additional notation is to differentiate the k^{th} product, x_k, from the k^{th} characteristic of an arbitrary product, χ_k.

Initially, we consider a market where consumers purchase a single unit of one of the differentiated products, and where all consumers have identical incomes, $Y > 0$. In the extensions section, we allow for non-exclusive and variable levels of consumption, variable income, as well as the possibility that a consumer purchases none of the differentiated goods.

Consumer preferences are defined over the differentiated commodity vector, χ, and a numeraire commodity, z. Preferences vary across consumers as summarized by an n-dimensional index of consumer characteristics, $\alpha \in R^n$. An individual of type α has preferences represented by a utility function $U(\alpha, \chi, z)$.

We introduce two restrictions on the domain of preferences: one on individual preferences, the other on the distribution of preferences across society. These generalize the restrictions used in our study of voting behavior [Caplin and Nalebuff (1988) and (1990)]. With these assumptions, we will demonstrate the existence of a pure strategy price equilibrium for any given set of products. We emphasize that these conditions are sufficient, but not necessary, for existence of a pure-strategy equilibrium.[4]

THEOREM: *Under A1 and A2, for any m firms and arbitrary products $\bar{x}$, there exists a pure strategy Bertrand-Nash equilibrium.*

ASSUMPTION A1: *Preferences are linear in α:*

$$U(\alpha, \chi, z) = \sum_{k=1}^{n} \alpha_k t_k(\chi) + g(z) t_{n+1}(\chi) + t_{n+2}(\chi), \tag{3.1}$$

where $U : R^n \times X \times R_+ \mapsto R$, $t : X \mapsto R^{n+2}$. Additionally, g is a strictly increasing concave function and $t_{n+1}(\chi) > 0$.

[4] See Champsaur and Rochet (1988), Dierker (1988), Hart (1985), and Salop (1979) for examples of existence results that fall outside our framework.

Each individual evaluates a product by a weighted sum of its benefits. These benefits are determined by a function t which maps the w-dimensions of the product characteristics into an $(n+2)$-dimensional vector of utility benefits. For example, the benefits from a car include comfort and speed, which may be complicated functions of the physical attributes. In our framework, all consumers have a common assessment of product benefits but may differ over how they value these benefits. The n-vector of preference parameters α reflects the weights an individual assigns to each of the first n product benefits. The benefit t_{n+1} affects the marginal utility of income; the benefit t_{n+2} is commonly valued across the population. Note that in most applications, preferences take the simpler form $U(\alpha, \chi, z) = \sum_{k=1}^{n} \alpha_k h(\chi_k) + g(z)$, where $h : R \to R$; the k^{th} benefit depends only on the k^{th} characteristic and $w = n$.

Under A1, the function describing the distribution of types can be represented by a density function, $f(\alpha)$, on utility parameters $\alpha \in R^n$. This paper focuses on settings for which $f(\alpha)$ satisfies a weak form of concavity. We first present the definition which provides a general measure of concavity and continue with our specific assumption.

DEFINITION: *Consider $\rho \in [-\infty, \infty]$. For $\rho > 0$, a non-negative function, f, with convex support $B \subset R^n$ is called ρ-concave if $\forall\ \alpha_0, \alpha_1 \in B$,*

$$f(\alpha_\lambda) \geq [\lambda f(\alpha_0)^\rho + (1-\lambda) f(\alpha_1)^\rho]^{1/\rho}, \quad 0 \leq \lambda \leq 1,$$

where $\alpha_\lambda = \lambda\alpha_0 + (1-\lambda)\alpha_1$. For $\rho < 0$, the condition is exactly as above except when $f(\alpha_0)f(\alpha_1) = 0$, in which case there is no restriction other than $f(\alpha_\lambda)$ is non-negative. Finally, the definition is extended to include $\rho = \infty, 0, -\infty$ through continuity arguments as discussed below.

This definition is discussed in detail in our companion paper. Briefly stated, for ρ positive, f^ρ is concave while for ρ negative, $-f^\rho$ is concave.[5] The index ρ is a measure of the degree of concavity of the density; a ρ-concave function is also ρ'-concave for all $\rho' < \rho$. The limiting case of $\rho = \infty$ requires f uniform over its support. The standard definition of concavity corresponds to $\rho = 1$. The case of $\rho = 0$ is another central case and corresponds to log-concavity of f, $\ln[f(\alpha_\lambda)] \geq \lambda \ln[f(\alpha_0)] + (1-\lambda)\ln[f(\alpha_1)]$. Finally, the limiting case of $\rho = -\infty$ corresponds to f quasi-concave.

ASSUMPTION A2: *The probability density of consumers' utility parameters satisfies: $f(\alpha)$ is ρ-concave, $\rho = \frac{-1}{n+1}$, with convex support $B \subset R^n$ with positive volume.*

Before proving the existence theorem, we provide a discussion of the two conditions and show that they cover many standard cases. We begin with examples of probability

[5] This assumes that f^ρ defines a function. If there are multiple solutions (such as when $f(\alpha) = \alpha^2$, $f^{\frac{1}{2}} = \pm\alpha$) then the statement applies to the unique positive root of f^ρ.

distributions that satisfy A2 and utility functions that satisfy A1. We then provide additional examples which exploit the joint nature of A1 and A2. Although written separately, the two assumptions should be read as one joint restriction: all that is required is that both A1 and A2 are satisfied for *some* parameterization of preferences.

Assumption A2 is new to the economics literature. It plays a central role in demonstrating that the aggregate demand functions are well behaved. To understand this assumption, note first that $\frac{-1}{n+1}$-concavity is a weaker condition than log-concavity of f. Hence the entire class of log-concave densities is covered. This includes the multivariate beta, Dirichlet, exponential, gamma, Laplace, normal, uniform, Weibull and Wishart distributions. In some of these cases, log-concavity requires restrictions on the parameter values as provided in Prékopa (1971).

In going beyond log-concavity A2 covers multivariate Cauchy, Pareto, F-distributions and t-distributions. These are all $\rho = \frac{-1}{n}$-concave. Our requirement of $\rho = \frac{-1}{n+1}$ imposes certain conditions on the parameter values. These conditions are provided in Borell (1975) and reproduced in Caplin and Nalebuff (1990). As an example, we illustrates the application of A2 to the multivariate Student's t-distribution.

EXAMPLE 3.1: The density of the n-dimensional t distribution with a degrees of freedom is:

$$f(\alpha) \propto [1 + \frac{1}{a}(\alpha' - \eta')M^{-1}(\alpha - \eta)]^{-(a+n)/2}, \quad M^{-1} \text{ is positive definite.}$$

Note that $f^{-1/(n+a)}$ is the square root of a quadratic form and hence convex. In turn, this implies all functions f^k are convex for $k \leq -1/(n+a)$. Thus the t-distribution satisfies A2 provided $a \geq 1$. The borderline case $a = 1$ is the multivariate Cauchy distribution which therefore satisfies A2 without any additional restrictions.

It is interesting to note that in this example, the restriction on the parameter a is independent of the dimension n. A2 is neither more nor less restrictive in higher dimensions. The higher exponent in $\rho = \frac{-1}{n+1}$ is just offset by the addition of another dimension to the density. In a similar fashion, the parameter restrictions for the Pareto and F-distributions to satisfy A2 are also dimension-free.

In many cases, economic reasoning requires that α is positive. A truncation of the density causes no additional difficulty. A2 includes all truncations of the above distributions provided only that the support set is convex. Other types of transformations must be excluded; for example the lognormal distribution does *not* satisfy A2.

Assumption A1 is also new to the product differentiation literature.[6] To demonstrate the applicability of this condition we first provide a partial listing of utility functions

[6] It is related to the restriction of intermediate preferences introduced by Grandmont (1978) in the theory of social choice.

which satisfy A1. As is evident from the list, an attractive feature of the assumption is that it does not place any *a priori* restriction on the form of product differentiation. It can encompass the characteristics approach of Gorman (1980) and Lancaster (1966), the transport cost approach of Hotelling (1929), the logistic approach of Perloff and Salop (1985), and the vertical approach of Gabszewicz and Thisse (1979) and Shaked and Sutton (1982).[7]

- C.E.S. preferences: Each consumer evaluates a product by a weighted sum of the utility from the characteristics in each dimension.

$$U(\alpha, \chi, z) = \sum_k \alpha_k \chi_k^\rho + z^\rho.$$

There are two special cases of C.E.S. which are of particular interest. When $\rho = 1$, preferences are linear in characteristics. This linear specification leads naturally to a probabilistic choice model if some of the product's characteristics are unobservable. The specific use of logit as a model of product differentiation was pioneered by Perloff and Salop (1985) and de Palma et al. (1985), and further analyzed by Anderson and de Palma (1988), and Anderson et al. (1988, 1989). We provide a discussion showing the general applicability of discrete choice models in Example 3.2.

The other important special case arises as $\rho \to 0$: with the appropriate normalizations, the C.E.S. approaches the Cobb-Douglas utility function,

$$U(\alpha, \chi, z) = \sum_{k=1}^{n} \alpha_k \ln \chi_k + \ln z.$$

A consumer with preferences α seeks to consume characteristics in proportion to α. In this sense, the ray from the origin through α can be thought of as the set of a consumer's most-preferred bundles as the level of consumption varies. In a similar spirit, preferences may be Cobb-Douglas in characteristics and linear in income

$$U(\alpha, \chi, z) = \alpha \cdot \ln(\chi) + z.$$

In a one-dimensional model, this specification is used by Lane (1980).

- Quadratic Transport Costs: Each consumer evaluates a product by its price and its location. When location represents where the product is sold, then transportation costs should be interpreted literally. When the product is located in a characteristic space, the transport costs represent the consumer's loss from purchasing a less than ideal product.

$$U(\alpha, \chi, z) = -||\chi - \alpha||^2 + z.$$

[7] Our model is extended in Section 8 to cover other cases including the Dixit-Stiglitz/Spence model of product differentiation. In the extensions section, we amend A1 and A2 to allow for variable levels of consumption.

By expanding the quadratic term, we find that the interaction between χ and α is linear and thus covered under A1. The one-dimensional quadratic transport cost model has been widely studied, see especially Economides (1984, 1989).

- Vertical differentiation: The vertical characteristic, χ, may be thought of as quality: all consumers prefer more to less. The difference in consumers' preferences, α, is like an income effect and reflects their valuation of quality:[8]

$$U(\alpha, \chi, z) = \chi(z + \alpha), \ \alpha \in R^1.$$

Valuable insights from this specification have been developed by Gabszewicz and Thisse (1979) and Shaked and Sutton (1982).

- Direct Translog utility function [Christensen et. al. (1975)]: In the translog utility model, the α_i terms are the coefficients for the linear approximation to a general utility function, and the α_{ij} represent the second-order terms in the approximation.

$$U(\alpha, \chi, z) = g(z) + \sum_{j=1}^{w} \alpha_j \ln(\chi_j) + (1/2) \sum_{j=1}^{w} \sum_{k=1}^{w} \alpha_{jk} \ln(\chi_j) \ln(\chi_k).$$

Note that in this case the parameter space is of dimension $n = w[w + 3]/2$ since $\alpha_{jk} = \alpha_{kj}$.

The interpretation of α and the meaning of our distributional assumption depend on the model. In the quadratic transport cost model A2 places a restriction on the distribution of most preferred locations, while in the translog model, A2 places a restriction on the distribution of marginal utilities. Our next example shows the applicability A2 in the context of the linear utility models.

EXAMPLE 3.2: Consider the standard qualitative response or discrete choice models used in the econometric literature, such as multinomial logit, multinomial probit and the random coefficients models [see McFadden (1981) for discussion of discrete choice models]. In all three cases the set of product characteristics are divided into two classes consisting of observables and unobservables respectively. The utility function for a consumer of type α is

$$U(\alpha, \chi, z) = \beta \cdot \chi^o + \alpha \cdot \chi^u + z,$$

where χ^o are the observable characteristics of the good and χ^u are the unobservable characteristics. The logit and probit models further specialize to the case where β is common across all individuals and there are as many goods as unobservable characteristics.

[8] This is exactly how we model the case of variable income in the extensions section.

Each of the goods is then taken to be a unit vector $x_i^u = e_i$. Hence the utility function for good i simplifies to

$$U(\alpha, x_i, z) = \beta \cdot x_i^o + \alpha_i + z, \tag{3.2}$$

Logit and probit involve distributional assumptions concerning $f(\alpha)$. Logit uses the Weibull while probit uses the normal; both are covered by A2. The random coefficients model extends probit to allow for the possibility that β is normally distributed across the population. Given that β enters the utility function linearly, this too is covered.

Sometimes a direct application of A1 fails, but with an appropriate change of variables, linearity is restored. Of course these transformations change the restriction implied by A2. The example below shows how we may exploit the joint nature of A1 and A2 to include another interesting case.

EXAMPLE 3.3: A general additive utility specification is proposed by Johansen (1969),

$$U(\beta, \chi, z) = \sum_k \frac{\beta_k}{\rho_k} \left(\frac{(\chi_k - \gamma_k)}{\beta_k} \right)^{\rho_k} + g(z).$$

Different restrictions on ρ_i and γ_i correspond to important special cases [see Barten (1977)]. To present this in a linear form, we use the transformation $\alpha_i = \beta_i^{1-\rho_i}$. But now, the restriction A2 places on the distribution of β is different from the restriction on α. For example, assume that the underlying density of β is $h(\beta) = k\beta^n$ over some bounded range. This translates to $f(\alpha) = k\alpha^{n+\rho}/(1-\rho)$ which satisfies A2 as it is log-concave. Hence, the original model is covered despite the apparent failure of A1. Note that while the variables ρ_i and γ_i may differ across goods, only the α parameters vary across the population.

Of course, not everything can satisfy A1 and A2. In particular, these assumptions must rule out Hotelling's original specification of product differentiation. In Hotelling's model, preferences are based on Euclidean distance, $U(\alpha, \chi, z) = z - ||\alpha - \chi||$ and the α parameter has a uniform density over the unit interval. Although the preferences as written are not linear, how do we know whether or not there is an equivalent transformed representation that satisfies A1 and A2? There is a qualitative feature of the demand function that tells us no such transformation is possible. In Hotelling's model, demand is discontinuous in price, even when commodities are distinct. Proposition 1 in the next section demonstrates that this is inconsistent with any specification satisfying A1 and A2.

4. AGGREGATION AND DEMAND

In this section, we establish a concavity property for the market demand function faced by an individual firm. The ability to characterize the shape of the market demand

function is central to the proof of existence of equilibrium in Section 5. To establish the concavity property, we exploit a remarkable parallel between our assumptions on consumer preferences and a mathematical literature initiated by Brunn and Minkowski. Specifically, we use recent extensions of the Brunn-Minkowski Theorem due to Prékopa (1971) and Borell (1975); these results are described in detail in our companion paper.[9]

The analysis of this section begins with two elementary though useful observations: (i) individual demand for a given firm is of the reservation price variety; (ii) market demand is continuous. To go further, we introduce the Prékopa-Borell Theorem and show how it applies to our model. The fundamental result is contained in Theorem 1 which combines the preference restriction A1 with the distributional requirement A2 to characterize the concavity of demand functions. We provide two examples that illuminate the theorem.

The reservation price property of individual demand results from exclusive consumption combined with the strict monotonicity of preferences in z. With preferences strictly increasing in z, each consumer has a reservation price for good i: type α maximizes utility with good i if and only if $p_i \leq R_i(\alpha, \bar{x}, p_{-i})$. Formally, to define the reservation price function, we first consider type α's best alternative utility level,

$$A_i(\alpha, \bar{x}, p_{-i})) \equiv \max_{j \neq i} U(\alpha, x_j, Y - p_j).$$

The reservation price is

$$R_i(\alpha, \bar{x}, p_{-i})) = \begin{cases} Y, & \text{if } U(\alpha, x_i, 0) \geq A_i; \\ Y - z_i, & \text{when there exists a solution } U(\alpha, x_i, z_i) = A_i; \\ -\infty, & \text{if } U(\alpha, x_i, z) < A_i \text{ for all } z \in R. \end{cases}$$

The advantage of the reservation price approach is that it allows us to view a complicated oligopoly problem from a monopoly perspective. It will enable us to prove existence of an oligopoly equilibrium without discussing any specific interactions between firms.

The reservation price property also implies that market demand functions are downward sloping. In addition, Proposition 1 shows that unless products are identical, demand is a continuous function of all prices. While an individual's demand at a given firm drops discontinuously to zero at his reservation price, the non-atomic population density (A2) smooths out these discontinuities.

PROPOSITION 1: *Under A1 and any non-atomic density $f(\alpha)$, demand is a continuous function of the price vector p whenever the m commodities are distinct, $t(x_i) \neq t(x_j)\ \forall i \neq j$.*

According to Proposition 1, demand functions are continuous unless some of the products are equivalent, $t(x_i) = t(x_j)$. With equivalent products all consumers prefer the

[9] See Bonnesen and Fenchel (1987) for a presentation of the Brunn-Minkowski Theorem. Das Gupta (1980) provides more accessible proofs of the recent results and clarifies their relation to the work of Brunn and Minkowski.

cheaper good, as in the standard Bertrand model. This complication has no effect on any of the results that follow. It is covered as a special case in all the proofs, and is assumed away in the text for ease of exposition.

With continuous demand functions, the reservation price function provides a compact representation of demand for a given firm's product. Once the other firms' prices are fixed, firm i's demand may be viewed from the perspective of a monopolist facing consumers with the given reservation price function $R_i(\alpha, \bar{x}, p_{-i})$,

$$D_i(p_i) = \int_{\{\alpha: R_i(\alpha,\bar{x},p_{-i}) \geq p_i\}} f(\alpha)d\alpha. \tag{4.1}$$

The simplicity of this expression is in part due to the assumption of unit demand. With variable demand we must consider both the density of consumer types and the demand from each type; this is addressed in the extensions section.

The reservation price expression for demand makes the mathematical issue transparent. How do Assumptions A1 and A2 combine to make the integral in equation (4.1) well-behaved? To answer this we present a statement of the Prékopa-Borell Theorem. This result shows how concavity properties of a population density are transformed under aggregation. For a proof of this theorem and a historical discussion see Das Gupta (1980). Further interpretation and application to social choice and income distribution are provided in Caplin and Nalebuff (1990).

THEOREM (PRÉKOPA-BORELL): *Let f be a probability density function on R^n with convex support B. Take any measurable sets A_0 and A_1 in R^n with $A_0 \cap B \neq \emptyset$ and $A_1 \cap B \neq \emptyset$. For $0 \leq \lambda \leq 1$, define $A_\lambda = \lambda A_0 + (1-\lambda)A_1$, the Minkowski average of the two sets.*[10]

If $f(\alpha)$ is a ρ-concave function, $\rho \geq -1/n$, then

$$\int_{A_\lambda} f(\alpha)d\alpha \geq \left[\lambda \left(\int_{A_0} f(\alpha)d\alpha\right)^{\frac{\rho}{1+n\rho}} + (1-\lambda)\left(\int_{A_1} f(\alpha)d\alpha\right)^{\frac{\rho}{1+n\rho}}\right]^{\frac{1+n\rho}{\rho}}.$$

To interpret the theorem, it is helpful to parameterize the region of integration by λ and define the parameterized cumulative integral,

$$F(\lambda) \equiv \int_{A_\lambda} f(\alpha)d\alpha.$$

The theorem states that ρ-concavity of f translates into $\frac{\rho}{1+n\rho}$-concavity of the cumulative integral.

[10] The Minkowski average A_λ is defined as all points of the form $x_\lambda = \lambda x_0 + (1-\lambda)x_1$, with $x_0 \in A_0, x_1 \in A_1$ and $0 \leq \lambda \leq 1$.

To relate the Prékopa-Borell theorem to our model, recall that demand is given by the integral in equation (4.1). When firm i is considering its optimal response to other firms' prices, p_{-i}, the region of integration in (4.1) depends only on its own price. Thus it is own price which plays the role of parameterizing the region of integration. In order to apply the aggregation result, we need only show that the set of consumers who purchase good i at p_λ contains the Minkowski average of those who purchase the good at price p_0 and those who purchase at price p_1. This is equivalent to the condition that the reservation price rule is a concave function of (α, p_i), which is an implication of A1.

THEOREM 1: *Consider preferences satisfying A1 and $f(\alpha)$ a ρ-concave probability density function on R^n with convex support B. For $\rho \geq -1/n$, all demand functions are $\frac{\rho}{1+n\rho}$-concave over the price interval where demand is strictly positive.*

Theorem 1 shows how a combination of A1 and distributional restrictions on $f(\alpha)$ affects the shape of the demand function. The examples below use the reservation price function to illuminate the result. The proof of Theorem 1 shows the reservation price function is concave in α: the examples use the borderline case of linear reservation prices. For simplicity, we suppress firm subscripts, recognizing that price and demand variables refer to firm i.

Example 4.1: The population is uniformly distributed over the n-dimensional unit simplex, $0 \leq \sum \alpha_k \leq 1$. Reservation prices are linear, $R_i(\alpha) = 1 - \sum \alpha_k$. This corresponds to $\rho = \infty$, so that $\frac{\rho}{1+n\rho} = 1/n$. Theorem 1 shows that the n-th root of demand is concave. As illustrated in Figure 1 below, a direct geometric argument verifies this result; market area shrinks linearly with price in each of n-dimensions, so that $D(p) = (1-p)^n$. In addition, we see the Minkowski average property. The market area at intermediate price p_λ is exactly the Minkowski average of the market areas at p_0 and p_1.

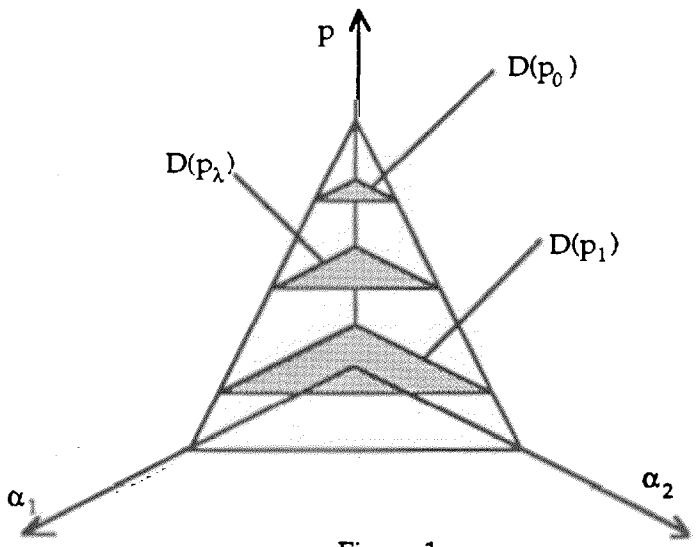

Figure 1

Example 4.2. The geometric argument for the case $\rho = \infty$ can be extended to cover all cases with $\rho > 0$. We illustrate this extension for $\rho = 1/m$, m a positive integer. In this case, Theorem 1 proves that demand is $[\frac{1}{m+n}]$-concave.

To provide a geometric insight into the result, we add m extra dimensions above the support set B, each representing the concave function $f^{1/m}$. This device produces a convex set with uniform density that represents the firm's market area in $n+m$ dimensions, where the Minkowski-average property of the relation between price and market area is retained. This indicates why the bound for the case with $f^{1/m}$ concave in n-dimensions is $1/(n+m)$, the same as the bound for f uniform in $n+m$ dimensions.

To illustrate the procedure, take $\alpha \in [0,1]$, $R(\alpha) = 1 - \alpha$ and $f(\alpha) = 2\alpha$. For $0 \leq p \leq 1$,

$$D(p) = \int_0^{1-p} 2\alpha d\alpha = (1-p)^2.$$

This is *identical* to the demand in a two-dimensional market with $R(\alpha) = 1 - (\alpha_1 + \alpha_2)$ and $f(\alpha)$ uniform over the solid simplex. The representation of a concave density as an additional dimension makes this case analogous to Example 4.1; the only difference is that the α_2 axis in Figure 1 is replaced by an $f(\alpha)$ axis.

The example also provides insight into the case $\rho = 0$. In terms of derivatives, $\ln[f(\alpha)]$ strictly concave corresponds to $f''f - f'^2 < 0$, while $f^{1/m}$ concave corresponds to $f''f - f'^2 \leq -f'^2/m$. Thus with f twice continuously differentiable over a bounded support, strict log-concavity implies that $f^{1/m}$ is concave for some positive m, so that $D(p)^{1/(n+m)}$ and hence $\ln(D)$ are also concave.

In each of the examples, note that the bound of Theorem 1 is tight. More generally, the bounds in Theorem 1 are the best available.

PROPOSITION 2: *There are n-dimensional models satisfying A1 with $f(\alpha)$ a ρ-concave probability density function with convex support, $\rho \geq -1/n$, and with $D(p) = (1-p)^{\frac{1+n\rho}{\rho}}$.*

5. EXISTENCE OF EQUILIBRIUM

The heart of the existence problem lies in establishing that each firm's profit function is quasi-concave in own price. A sufficient condition is that $1/D(p)$ is convex for all firms. We show that A1 and A2 imply this property for aggregate demand. Theorem 2 then combines our knowledge of the shape and continuity properties of demand functions to provide the general proof of existence. We continue to suppress firm subscripts where possible.

PROPOSITION 3: *A firm's profit function is quasi-concave in own price provided* $D(p)^{-1}$ *is convex and diminishing in* p *where* $D(p) > 0$.

When the demand function is diminishing, $D(p)^{-1}$ convex is equivalent to $qP(q)$ concave in q, where $P(q)$ is the inverse demand function.[11] But concavity of the revenue function is equivalent to quasi-concavity of the profit function if the firm is allowed an arbitrary convex cost function. Hence the condition on $D(p)$ in Proposition 3 is as weak as possible.

In comparison, much of the earlier literature assumes concavity of $D(p)$. To generate a concave demand function, our model requires that $\rho/(1+n\rho) = 1$. This is possible if and only if $\rho = \infty$ and $n = 1$– a uniform distribution of consumers in a one-dimensional market. This suggests that an approach relying on concavity of demand can neither go beyond one dimension, nor allow a non-uniform density of consumers.

To demonstrate the quasi-concavity of the profit function we combine Theorem 1 and Proposition 3.

PROPOSITION 4: *Under A1 and A2, each firm's profit function is quasi-concave in own price.*

PROOF: By Proposition 3, quasi-concavity follows once $D(p)^{-1}$ is convex where positive. This is equivalent to [-1]-concavity. By Theorem 1, this will follow from A1 and a ρ-concavity condition on $f(\alpha)$ such that $\frac{\rho}{1+n\rho} = -1$. Solving this equation for ρ reveals $\rho = -1/(n+1)$ which is A2, thus completing the proof. *Q.E.D.*

THEOREM 2: *Under A1 and A2, for any m firms and arbitrary products* $\bar{x}$, *there exists a pure strategy Bertrand-Nash equilibrium.*

PROOF: First we assume that no two products are equivalent and $c_i \leq Y$. We then use the best-response correspondence $P = [p_1(p_{-1}), \ldots, p_{\bar{m}}(p_{--m})]$ on the set $Z = \prod_i [c_i, Y]$ with firms restricted to charge between c_i and Y. This correspondence is upper-hemi-continuous, since the range of the best-response correspondence is compact and profit functions are continuous in all prices. By Proposition 4, it is also convex-valued, since the set of maxima of a quasi-concave function is a convex set. Application of Kakutani's fixed point theorem then establishes existence of a fixed point. Note that the restricted range of the correspondence always leaves firms with at least one global best response, so that any such fixed point is a Bertrand-Nash equilibrium.

[11] For twice differentiable functions, this equivalence follows from comparison of the second derivatives. The equivalence holds more generally, as one may confirm by substituting the implied inequalities. Jim Mirrlees has suggested the following simpler approach which helps to provide intuition. Define $\psi(x,y) = P(x/y)x$. By construction, ψ is homogeneous of degree 1 in (x,y). Such a function is concave if and only if $\psi(x,1)$ is concave in x or equivalently, $\psi(1,y)$ is concave in y. Concavity of $\psi(x,1)$ corresponds to concavity of the revenue function in q while concavity of $\psi(1,y)$ is equivalent to convexity of $1/P^{-1}(y)$ since P is a decreasing function.

We now allow for the presence of equivalent goods. For each group of equivalent products, $t(x_i) = t(x_j)$, we select out a lowest cost producer. These selected firms define an oligopoly problem with distinct goods. We use the best response correspondence above with the additional condition that a firm which is the lowest cost producer from a group of equivalent products sets its price no higher than the cost of the next lowest cost producer. This correspondence has a fixed point which we use to construct an equilibrium as follows. For all the selected firms use the prices corresponding to the fixed point. For all other firms, set price equal to cost. At these prices, the higher cost producers of equivalent products are undercut and receive zero demand. Hence all firms have selected a global best response.

Finally, if there are firms with $c_i > Y$ we use the arguments above to establish existence for a market without these firms, and recognize that this remains an equilibrium when these firms charge $p_i = c_i$. *Q.E.D.*

At this point it is appropriate to comment on the relationship between the approach to existence in this paper and the independent work of Dierker (1989). Both papers explore how distributional assumptions on consumer preferences can lead to quasi-concavity of the demand function and thus existence of equilibrium. It is primarily in the description of consumer preferences that the two approaches differ. Dierker takes the consumer valuations for the goods as the primitive; he assumes that the joint distribution of these valuations is log-concave. Application of the log-concave (or $\rho = 0$) version of the Prékopa-Borell theorem establishes log-concavity of the demand function. Our approach begins with an explicit model of the commodity space and the distribution of a multi-dimensional utility parameter for consumers. The consumer valuations are then derived from the utility function. In the context of Dierker's work, our approach provides a large class of utility models for which the joint distribution of product valuations is log-concave for any set of products.

6. UNIQUENESS OF EQUILIBRIUM

In this section, we provide sufficient conditions for uniqueness of equilibrium.[12] Three cases are covered: duopoly competition in any number of dimensions, multi-firm competition in one dimension, and the logit model. These three cases span the horizontal characteristics model of Lane (1980), the horizontal location model of Economides (1989), and the logit model of Anderson et. al. (1989). In Section 8, the uniqueness results are extended to include the vertical differentiation models of Gabszewicz and Thisse (1979) and Shaked and Sutton (1982)

[12] The results in this section have been greatly simplified and improved by the comments of Paul Milgrom.

In addition to uniqueness, the cases covered also have a "log-supermodularity" property [Milgrom and Roberts (1990a)]. The results from Milgrom and Roberts (1990a,b) show that the combination of log-supermodularity and uniqueness gives rise to several desirable properties:

- There are no other equilibria either in mixed or correlated strategies.
- The equilibrium strategy vector is globally stable under many learning and adjustment processes including best-response dynamics, Bayesian learning, and fictitious play.
- The game is dominance solvable so that the equilibrium strategy is the unique rationalizable strategy for each player.
- Any parameter change that increases the marginal return to price for some firm results in an equilibrium in which all firms charge higher prices; for example, an increase in one firm's cost c_i results in all firms charging higher prices.

These results are especially valuable when there is a stage of product design before price competition takes place. In these multi-stage games, the uniqueness and robustness of the price equilibrium allows the product designer to predict profits for any hypothetical product he might produce.

The proof of uniqueness for all three cases is based on a dominant diagonal argument. If among any given set of firms, $|\frac{\partial^2 \pi_i}{\partial p_i^2}| > \sum_{j \neq i} |\frac{\partial^2 \pi_i}{\partial p_i \partial p_j}|$ then there is at most one equilibrium in which these firms are all active in the market [see, for example, Friedman (1977)]. In combination with our earlier existence result, this allows us to prove uniqueness.

Dominant diagonal arguments are predicated on the condition that all firms have a unique best response, and that profit functions are twice differentiable. The first of these conditions is not an issue when considering active firms: Proposition 4 shows that provided ρ is strictly above $-1/(n+1)$, each firm's profit function is strictly quasi-concave where demand is strictly positive, and best responses are unique. But the differentiability condition is restrictive; it requires that $g(z)$ from A1 is twice-differentiable and that the density $f(\alpha)$ from A2 is differentiable.[13] We refer to these additional restrictions as the differentiable case.

Before turning to the proofs, there is a question about the interpretation of uniqueness. There is a potential indeterminacy in the price of a firm that gets zero demand in equilibrium. Such a firm will be content with any price provided it gets zero demand. This raises the issue of whether there may be multiple equilibria driven only by the alternative

[13] Our reason for studying the differentiable case is that it allows us to use the standard arguments for uniqueness and so simplifies the proofs. The concerned reader can confirm that the differentiability conditions in Propositions 6 and 7 can be dispensed with by replacing the infinitesimal dominant diagonal argument with its discrete alternative.

pricing strategies of inactive firms. Proposition 5 shows, to the contrary, that any indeterminacy of equilibrium prices set by inactive firms has no effect on the prices chosen by the active firms.[14] Thus in the proofs that follow, we can adopt the convention that inactive firms charge prices equal to their marginal costs without influencing the decision of any consumer or the profits of any firm.[15] Our claim of uniqueness refers only to uniqueness of market outcomes.

PROPOSITION 5: *Consider an equilibrium in which a given set I of firms has zero demand. Let p_A denote a vector of equilibrium prices for the active firms. There exists an equilibrium which leads to exactly the same demand and profits in which all active firms continue to charge prices p_A while all firms in I charge their marginal cost.*

We now turn to duopoly competition. Our sufficient condition for uniqueness strengthens A2 to log-concavity.

PROPOSITION 6: *In the differentiable case with A1 and with $\ln[f(\alpha)]$ concave where $f(\alpha) > 0$, the duopoly equilibrium is unique. In addition, the pricing game is log-supermodular.*

Proposition 7 applies to one-dimensional markets with a log-concave density of consumers. In this case, uniqueness applies to a given set of active firms.

PROPOSITION 7: *Consider the differentiable case with A1 and with $\ln[f(\alpha)]$ concave over an interval $B \subset R$. If there is an equilibrium in which a given set of firms are active, then there is no other equilibrium with the same set of active firms. In addition, the pricing game among these active firms is log-supermodular.*

Our final result is based on the logit model of Example 3.2. Here the utility of good i for a consumer of type α is

$$U(\alpha, x_i, z) = \beta \cdot x_i^0 + \alpha_i + z,$$

where $f(\alpha)$ represents the product of n identical Weibull distributions. Uniqueness and log-supermodularity in the logit model follow directly from differentiation of the profit function. The uniqueness of equilibrium in the logit model was first noted by Anderson and de Palma (1988) for the case of duopoly.

[14] Note that this does not mean that the presence of inactive firms has no effect on the equilibrium prices charged by the active firms. However, it follows from Proposition 5 that if adding a new firm changes the equilibrium prices among the pre-existing firms but leaves the new firm inactive, then the new firm must be charging marginal cost in the equilibrium.

[15] Proposition 5 requires that inactive firms never charge a price *below* their marginal cost. Hence in characterizing the unique equilibrium, we rule out strategies that yield losses in the event that demand is positive.

Our characterization of uniqueness is still incomplete. The results in this section either place restrictions on the number of firms or the dimensionality of consumer preferences, or impose strong symmetry requirements. Assumption A2 on the distribution of preferences has been strengthened to log-concavity. While these conditions are sufficient, there may be other cases of interest which satisfy the dominant diagonal criterion for uniqueness. In addition, there may be more general examples of uniqueness which do not fall under the dominant diagonal argument or log-supermodularity.

7. THE IMPORTANCE OF DIMENSION

A feature which distinguishes our approach is the consideration of multi-dimensional models. How can we assess an appropriate dimension for modelling a particular market? At one level, we could list all potentially relevant attributes of the product. But consumers may evaluate products according to only a few attributes, or according to a real valued function of all the attributes. It is the dimension of the space of utility parameters rather than the number of product characteristics which is of greater economic interest. For this reason we refer to n as the dimension of the market.

There are two economic features that help distinguish an appropriate dimension for a given market: variety of preferences and the scope of competition. Proposition 8 shows that a low n imposes strong *a priori* restrictions on preferences. Proposition 9 shows that the number of a firm's competitors is greatly restricted in low dimensions.

PROPOSITION 8: *Under A1 with $\alpha \in R^n$, consider m firms selling goods at prices p:*

(a) For $m \leq n+1$ no value restriction is implied; all $m!$ preference orderings of the m products can coexist;[16]

(b) For $m > n+1$, value restrictions are implied; some of the $m!$ orderings are ruled out.

In a one-dimensional model, Proposition 8 implies that preferences are restricted as soon as there are three products. Consumers in a one-dimensional market act exactly like voters who see an election solely in terms of the traditional left-right spectrum.

A second limitation of the one-dimensional model is that no firm faces more than two competitors regardless of the number of firms in the market. In higher dimensions the corresponding limitation is more subtle. Even in two dimensions, it is possible for one firm to be competitive with all other firms. Nevertheless, Proposition 9 shows that the *average* number of a firm's competitors is bounded above by 6.[17]

Given prices p, consider all active firms, and define firm j as a competitor to firm i if a marginal reduction in the price of firm i reduces demand for good j. Let C_i denote the number of firm i's competitors at the given prices.

[16] We require that a linear independence condition is satisfied, as seen in the proof.

[17] Archibald and Rosenbluth (1975) demonstrate a similar result for the Lancaster model: when goods are represented in a three-dimensional space of characteristics, the upper bound on the

PROPOSITION 9: *Under A1 and A2 with $\alpha \in R^2$, the average number of competitors per active firm is bounded above by 6: $[\sum_i C_i]/m' \leq 6$, where m' represents the number of firms with $D_i(p) > 0$.*

An example where each firm has exactly six competitors is when market areas tile the plane into regular hexagons as in Lösch (1954).[18] Our proof also implies that as the number of firms rises, for a generic distribution of product characteristics and prices, the average number of competitors converges exactly to six.

Proposition 9 suggests that the dimension of the product space will affect the competitive structure of the market. This issue was first addressed in the pioneering analysis of Stiglitz (1986). In his model market areas are n-dimensional cubes. The first-order conditions imply that the markup of price over marginal cost is inversely proportional to the ratio of market surface area to volume, which falls with dimension. Markets become more competitive as the dimensionality of the product increases. Of course these conclusions rely on the fact that the first-order conditions characterize an equilibrium; this is established in Section 8.3.

8. EXTENSIONS

The results of the earlier sections are robust to various changes in the underlying assumptions. We allow for income differences, variable demand, and non-exclusive consumption. The consumer is also given an option to purchase none of the differentiated goods. Finally, the model extends to cases with an infinite number of firms and unbounded markets.

8.1 Equal Incomes

We relax the assumption that all consumers have equal income. Income is treated as one of the parameters of preference orderings over differentiated commodities. Thus it is required to enter the utility function linearly, and the restriction on the joint distribution of parameters extends to the income term.

average number of competitors per firm is six. While our result relates to cases of exclusive unit consumption, theirs relates to goods which are freely divisible and combinable. This similarity in results breaks down in higher dimensions. Archibald and Rosenbluth show that there are no bounds on the number of neighbors per firm in four dimensions, a result which is not valid in the three-dimensional version of our model. There is a direct geometric explanation for this: the number sought by Archibald and Rosenbluth in their n-dimensional model turns out to be the average number of sides in any partition of an $(n-1)$-dimensional surface into polygons, regardless of whether or not the polygons are convex. Hence their n-dimensional bound must be at least as big as our $(n-1)$-dimensional bound. The numbers agree in two dimensions only because the restriction to convex polygons makes no difference for two-dimensional partitions.

[18] Proposition 9 is related to the fact that no regular polygon with more than six sides can tile the plane [Grunbaum and Shepherd (1987)]. In three dimensions, space can be tiled with regular 14-sided figures although this is not necessarily the maximum. In higher dimensions, the general bound is not known.

ASSUMPTION A1′: *Preferences satisfy A1 with the additional restriction that* $g(z) = z$.

A special case of A1′ is $U = \chi z$ considered by Gabszewicz and Thisse (1979, 1980) and Shaked and Sutton (1982) in their models of vertical product differentiation.

ASSUMPTION A2′: *The joint probability density of consumers' utility parameters and income,* $f(\alpha, Y)$, *satisfies* $f(\alpha, Y)^{-\frac{1}{n+2}}$ *is a convex function over its support,* B, *which is a convex subset of* R^{n+1} *with positive volume. In addition,* $Y' = \sup_{\{(\alpha,Y)\in B\}} Y$ *is finite.*

With these assumptions we have placed the problem of existence in the framework of A1 and A2. It is readily seen that the reservation price rule is now concave in (α, Y),

$$R_i(\alpha_\lambda, Y_\lambda) \geq \lambda R_i(\alpha_0, Y_0) + (1-\lambda) R_i(\alpha_1, Y_1).$$

To apply the existence result in Theorem 2, we replace the upper bound Y on prices with the new bound Y'.

PROPOSITION 10: *Under A1′ and A2′, there exists a price equilibrium.*

The linearity of preferences in income allows us to extend the uniqueness arguments of Section 6. The one-dimensional result of Proposition 7 corresponds to the case of pure vertical differentiation; all preferences are identical and only incomes differ. Thus the duopoly uniqueness result holds even when consumers differ both in income and in preferences. With A1′ and $\ln[f(\alpha, Y)]$ concave and differentiable where $f(\alpha, Y) > 0$, the duopoly equilibrium is unique and the pricing game is log-supermodular.

8.2 Unit Demand

The simplest extension is to permit each consumer the option of purchasing none of the differentiated goods. In many instances, purchasing the good $\chi = 0$ at price zero corresponds to dropping out of the market. Thus if we create a fictitious firm 0 which sells $x_0 = 0$ at $p_0 = 0$ the consumers then have the option to purchase none of the differentiated goods. Note that this has no effect on the arguments for existence of an equilibrium; all the real firms still have convex-valued best response correspondences and the 0 firm has its price fixed at 0.

There are some subtleties involved in introducing the fictitious firm. Equilibrium prices may be affected since the market is more competitive. Additionally, the uniqueness argument for duopoly does not apply; the reason is that the presence of the fictitious firm takes us from a duopoly to a three firm oligopoly. Finally, there are models where the $x_0 = 0$ approach is not an appropriate way to model non-consumption. For example, in location models, the 0 good does not correspond to zero consumption of the differentiated

commodity, but rather a differentiated commodity located at the origin. [Even so, it is still possible to allow the consumer to purchase none of the differentiated goods if we specify a utility loss, concave in α, for choosing not to purchase any of the differentiated goods.]

The extension to variable demand is more difficult since we lose the ability to apply the reservation price method. However the Prékopa-Borell Theorem continues to play a central role in extending the results to cases with variable and/or non-exclusive consumption. Here we require a version of the theorem which applies when the integrand is a function of both α and p.

In the case of general demands,

$$D_i(p_i) = \int_{R^{n+1}} f(\alpha, Y) D_i(\alpha, Y, p_i)\, d\alpha dY, \tag{8.1}$$

where $D_i(\alpha, Y, p_i)$ measures the demand for good i at price p_i from an (α, Y) type. Since we are integrating a product of two terms, we require properties on this product. If each term is log-concave, then the product of two log-concave functions is itself log-concave. Prékopa's theorem show that this log-concavity property is inherited through the integral sign.

THEOREM [PRÉKOPA (1973)]: *Let $h(\alpha, p)$ be a log-concave non-negative measurable function on $R^n \times R$ with non-empty support B:*

$$h(\alpha_\lambda, p_\lambda) \geq h(\alpha_1, p_1)^\lambda h(\alpha_2, p_2)^{1-\lambda},$$

where $\alpha_\lambda = \lambda\alpha_1 + (1-\lambda)\alpha_2$ and $p_\lambda = \lambda p_1 + (1-\lambda)p_2$, $0 \leq \lambda \leq 1$. Then

$$D(p) \equiv \int_{\{\alpha \in R^n:\ (\alpha, p) \in B\}} h(\alpha, p) d\alpha$$

is log-concave in p.

The critical stage in proving existence of equilibrium is showing that the profit function is quasi-concave. From (8.1), log-concavity of aggregate demand follows from the generalized Prékopa theorem when both $\ln[D(\alpha, Y, p_i)]$ and $\ln[f(\alpha, Y)]$ are concave in their arguments. Quasi-concavity of profits then follows from Proposition 3.

However, the condition that $\ln[D(\alpha, Y, p_i)]$ is concave is not satisfied for many standard demand specifications including C.E.S.. We must employ a significantly weaker condition on demand and prove quasi-concavity of profits through a sufficient condition other than convexity of $1/D(p)$. Proposition 13 employs a change of variables to establish a new sufficient condition for quasi-concavity of profits. For simplicity of notation, we drop the i subscripts where it is understood that price and demand both refer to firm i.

PROPOSITION 11: *A firm's profit function is quasi-concave in own price where profits are strictly positive provided that $\ln D(p)$ is concave in $\ln p$ where demand is strictly positive.*

PROOF: Since the logarithmic and exponential functions are both monotonic, it suffices to show that $\ln \pi(p)$ is concave in $\ln p$ where profits are strictly positive. We rewrite profits as

$$\ln \pi(p) = \ln[p - c] + \ln D(p).$$

The first term is directly concave in $\ln p$, and the proposition follows. *Q.E.D.*

Although this may appear unfamiliar, the condition that $\ln[D(p)]$ is concave in $\ln p$ is equivalent to an increasing elasticity of demand. It is satisfied for all C.E.S. demand functions and other cases discussed below. Although the condition in Proposition 11 is more restrictive than the condition in Proposition 3, Cobb-Douglas demand, $D(p) = 1/p$, stretches both to their limit: $1/D(p) = p$ and $\ln D(p) = -\ln p$, the two linear cases.

Prékopa's theorem suggests a way to generalize Assumptions A1′ and A2′ to generate a market demand function with the increasing elasticity property required in Proposition 11.

ASSUMPTION A1″: *For $1 \leq i \leq m$ and any given p_{-i}, $\ln[D_i(\alpha, Y, p_i)]$ is concave in $(\alpha, Y, \ln p_i)$ over the convex region where demand is strictly positive.*

ASSUMPTION A2″: *The joint density of consumers' utility parameters and income, $\ln[f(\alpha, Y)]$, is a concave function in (α, Y) over its support, B, which is a non-empty subset of R^{n+1}.*

PROPOSITION 12: *Under A1″ and A2″, the profit function is quasi-concave where profits are strictly positive.*

PROOF: By A1″ and A2″, the product $D(\alpha, Y, p)f(\alpha, Y)$ is log-concave in $(\alpha, Y, \ln p)$. Thus application of Prékopa's Theorem shows that

$$D(p) \equiv \int_{\{(\alpha,Y)\in R^{n+1}:(\alpha,Y,p)\in B\}} D_i(\alpha, Y, p) f(\alpha, Y)\, d\alpha dY$$

is log-concave in $\ln p$. Note that concavity is in $\ln p$, not p because the integrand is log-concave in $\ln p$. The result follows directly from Proposition 11. *Q.E.D.*

To show how to A1″ applies to economic problems, we consider Cobb-Douglas preferences, relaxing the assumption of unit demand. Consumers purchase only one of the differentiated commodities, but may do so on any scale, μ, consistent with their budget.

$$U(\alpha, \mu\chi, z) = \prod_{k=1}^{n} (\mu\chi_k)^{\alpha_k} z^{\beta} \text{ with } \alpha \in R^{n-1}, \alpha_n = 1 - \sum_{k=1}^{n-1} \alpha_k, \text{ and } \beta > 0. \tag{8.2}$$

The preference parameters now reflect the consumer's ideal mix of characteristics: if all characteristics were available separably and at equal prices, the consumer would

purchase a good proportional to his α vector. The actual choices are more complicated since characteristics come in bundles. Even with bundled characteristics, consumers with Cobb-Douglas preferences spend a fixed fraction $1/(1+\beta)$ of their income on the differentiated commodity. In addition, the optimal differentiated commodity is independent of income. These observations allow us to verify A1'' directly.

PROPOSITION 13: *Cobb-Douglas preferences (8.2) satisfy A1''.*

Thus when the distribution of Cobb-Douglas parameters satisfies A2'', Proposition 12 applies and each firm's profit function is quasi-concave where profits are strictly positive. The remaining barriers to existence of an equilibrium are that prices may not be bounded and that continuity may be violated. Given the non-atomic density, continuity is not a problem. On the other hand, while boundedness frequently holds, it is hard to establish. We return to the Cobb-Douglas example as this forces us to confront the unboundedness problem head on. Without unit demand, bounded income need not lead to bounded reservation prices A monopolist facing Cobb-Douglas demand would like to charge an infinite price. Thus in an oligopoly model, the bounding of the best response function must depend on competitive forces. In the appendix, we show that once two or more firms compete, it is possible to find a compact set on which to apply Kakutani's fixed point theorem.

THEOREM 3: *Under Cobb-Douglas preferences (8.2) and A2'', there exists a pure strategy Bertrand-Nash equilibrium for $n \geq 2$ firms, each with strictly positive costs, $c_i > 0$; for the duopoly, this equilibrium is unique.*

This extends the Caplin-Nalebuff (1986) existence result for 2-firms competing in 1-dimension. Existence and uniqueness results apply to the Cobb-Douglas model with variable but exclusive demand for any number of firms competing over any number of dimensions. This captures an important feature of many markets; for example, most PC-consumers choose between IBM or Macintosh compatibility, but their level of consumption is widely variable.

Our approach is specifically designed for models without a representative agent. Recently, Anderson et. al. (1988, 1989) have illuminated the connection between representative agent models of imperfect competition, such as the C.E.S. model of Dixit-Stiglitz/Spence, and models with heterogeneous consumers. Here we follow up on their approach, and apply Theorem 3 to prove existence of equilibrium in the C.E.S model.

In the C.E.S. model there exists a representative agent with income Y and preferences

$$U = \left(\sum_{k=1}^{n} \mu_k^{\gamma} \right)^{1/\gamma} z^{\beta},$$

where μ_i represents the aggregate amount of good i consumed. As before, each good is produced by a separate firm. In this model, utility is maximized subject only to the standard budget constraint

$$\sum_{k=1}^{n} p_k \mu_k + z = Y.$$

In particular, there is no restriction either to unit demand or to exclusive consumption.

There is a reformulation which places the C.E.S. model in our framework. Anderson et al. (1988) demonstrate that C.E.S. demand functions are obtained in a logit model where consumers choose variable quantities of a single commodity. Specifically, C.E.S. demands arise when the indirect utility of consuming good i at p_i for a consumer of type α is

$$v_i(\alpha, p_i) = -\ln p_i + (1+\beta)\ln[Y/(1+\beta)] + \beta \ln \beta + \alpha_i, \tag{8.3}$$

and the α_i coordinates are independently and identically Weibull distributed. We use this equivalence to prove existence of equilibrium for the C.E.S. model.

PROPOSITION 14: *In the representative agent C.E.S. model, there exists a pure strategy Bertrand-Nash equilibrium for $n \geq 2$ firms, each with strictly positive costs, $c_i > 0$; for the duopoly, this equilibrium is unique.*

PROOF: The indirect utility function in (8.3) is exactly the indirect utility arising in our variable consumption Cobb-Douglas preference model above, specialized to the case with n goods, where goods are of the form,

$$x_1 = (e, 1, ..., 1), \ldots, x_n = (1, 1, ..., e).$$

In addition, the Weibull distribution is log-concave in α. Hence the C.E.S. demand functions arise in a model which satisfies the requirements of Theorem 3, so that the existence and uniqueness results apply also to the C.E.S. model. *Q.E.D.*

The issue of existence is side-stepped in prior versions of the Dixit-Stiglitz/Spence model. The earlier work makes the approximation that an index of prices including the firm's own price is unaffected when the firm increases its own price.[19]

Our final example shows how we can combine both variable and non-exclusive demand when there is no representative consumer. We specify homothetic preferences according

[19] In an early version of their paper, Dixit and Stiglitz (1974) explicitly work with a continuum of firms, so that own price does not affect the price index. With a large but finite number of firms in the market, the effect of own price on the index will be "close" to zero. Ignoring one's own effect is surely inappropriate when there are a small numbers of firms. In all cases, to compare the approximate equilibrium to the true equilibrium requires that the true model has an equilibrium.

to an Indirect Translog Model [Christensen et. al. (1975)]. This gives rise to the demand functions

$$D_i(\alpha, Y, p_i) = (Y/p_i)(\alpha_i + \sum_{j=1}^{w} \alpha_{ij} \ln[p_j]),$$

where the number of firms equals the number of characteristics, $w = m$. Direct calculations show that $\ln[D_i(\alpha, Y, p_i)]$ is concave in all arguments $\{\alpha, Y, \ln p_i\}$ provided α_{ii} is constant across the population. With this restriction, quasi-concavity of the profit function follows from Proposition 12 provided that $f(\alpha, Y)$ is log-concave over its support.

8.3 Unbounded Markets

Our approach also applies to models with an infinite number of firms distributed over an unbounded space. There is a possibly infinite set, I, of products, located in R^w. In this setting, we return to the case of unit demand.

We no longer prove existence of equilibrium. We show that any joint solution to the first-order conditions is an equilibrium. In this approach, we need only consider preferences at the candidate price vector. Here, all the relevant information is provided by the reservation price function.

A1(p): *All consumers have an optimal choice at p. The reservation price function for firm i's product is concave across α at the given p_{-i}, for all i.*

ASSUMPTION A2(p): *The measure of consumers' utility parameters has a density that satisfies: $f(\alpha)$ is ρ-concave, $\rho = \frac{-1}{n+1}$, with convex support $B \subset R^n$ with positive volume. In addition, given p_{-i} and $p_i \geq c_i$, demand for firm i's product is finite.*

With these assumptions, quasi-concavity of the profit function follows by Theorem 1. This justifies using the first-order approach.

PROPOSITION 15: *Consider a price vector p^* where all first-order conditions are satisfied. With A1(p^*) and A2(p^*), p^* is an equilibrium.*

A direct application is to the symmetric case of products distributed over a uniform grid in R^n.[20] When consumers have a quadratic transportation cost, and using a rectangular grid, Stiglitz (1986) demonstrated that the equal price solution to the first-order

[20] Unbounded markets offers one route to model complete symmetry; the circle model provides an alternative for the case with a finite number of firm. The applicability of our framework to the circle model [as in Economides (1989)] is outlined in our working paper, Caplin and Nalebuff (1989).

condition was dependent on dimension in an interesting way.[21] Integrating this with a model of entry, he argued that in low dimensions there will be excess product diversity, while in high dimensional markets, too little diversity. These conclusions require that the solution to the first-order conditions is in fact a price-equilibrium. For quadratic transportation costs, this is a correct hypothesis by Proposition 15.

9. CONCLUSION

This paper presents a new approach to imperfect competition. We provide conditions under which there exists a pure strategy price equilibrium in a multi-dimensional model of product differentiation. The primary innovation is the move away from one-dimensional products and a uniform distribution of consumer types. The existence result extends to any number of products with any number of characteristics; the permissible multi-dimensional distributions of consumer types include the Normal, Pareto, Weibull, and Beta distributions. Our analysis covers location models, the characteristics approach, and probabilistic choice together in a unified framework.

Our existence results are based on competition between a fixed (but arbitrary) set of products with a given set of characteristics. The natural next question is: how will firms choose the characteristics of their products, rationally anticipating the ensuing price competition? This has been studied by Prescott and Visscher (1977) in a model with sequential entry and fixed prices. Having established the existence of a price equilibrium, we may begin to examine the pattern of sequential locations in the presence of price competition.

The logit model of consumer choice provides a framework for addressing this question. We have shown both the existence and *uniqueness* of equilibrium given any number of competing products. This means that firms have a precise prediction of profits given the choices of competing products. A second arena in which we have both existence and uniqueness is duopoly competition. This is the setting in which Hotelling originally espoused the principle of minimum differentiation. Here we can investigate how the dispersion of consumer preferences affects the distribution of product locations, either in physical or in characteristics space.

Our multi-dimensional framework also brings us closer to marketing applications. For example, Feenstra and Levinsohn (1989) use a variant of our model to map out consumer preferences for the characteristics of cars. Given the existence of a price equilibrium,

[21] In the Stiglitz model, each firm has $2n$ neighbors in n-dimensions. Demand elasticity is measured by market surface area to volume, which is proportional to $2n$. Hence demand is more elastic in higher dimensions. This leads to greater competition and endogenously lower profits.With a fixed cost of entry, fewer firms can cover entry costs despite the fact that in higher dimensions consumers have further to go and so the efficient solution would have more firms per unit area.

they proceed to analyze the competitive structure of the automobile industry and make predictions about optimal pricing policies.

To prove existence, we introduce an aggregation technique due to Prékopa and Borell. Our companion paper provides a more detailed discussion of the theorem, showing how it applies to generalize our earlier results on 64%-majority rule. We also use it to characterize the income distribution arising in a Roy model of self-selection in the labor market. These different applications suggest the value of ρ-concavity conditions in diverse branches of economic theory. In the present case, ρ-concavity and the Prékopa-Borell Theorem are ideal mathematical tools to handle the complications inherent in imperfectly competitive markets. The generality of this approach opens up the possibility of studying product design. Given that there exists a price equilibrium, we may begin a systematic study of how firms choose products, along the lines envisioned by Hotelling.

10. APPENDIX

PROPOSITION 1: *Under A1 and any non-atomic density $f(\alpha)$, demand is a continuous function of the price vector p whenever the m commodities are distinct, $t(x_i) \neq t(x_j)\ \forall i \neq j$.*

PROOF: Under A1, the consumers indifferent between i and j are defined by

$$\alpha \cdot \eta_{ij} = b_{ij}, \tag{10.1}$$

where the k^{th}-coordinate of the n-vector η_{ij} equals $t_k(x_j) - t_k(x_i)$, and $b_{ij} = t_{n+2}(x_i) - t_{n+2}(x_j) + t_{n+1}(x_i)g(Y - p_i) - t_{n+1}(x_j)g(Y - p_j)$. By assumption, $\eta_{ij} \neq 0$ so that equation (10.1) defines a hyperplane in R^n. Hence the set of indifferent consumers has zero population. Continuity of demand then follows from the continuity of g. *Q.E.D.*

THEOREM 1: *Consider preferences satisfying A1 and $f(\alpha)$ a ρ-concave probability density function on R^n with convex support B. For $\rho \geq -1/n$ all demand functions are $\frac{\rho}{1+n\rho}$-concave over the price interval where demand is strictly positive.*

PROOF: Assume that a consumer of type α purchases good i when it is priced at p_i and a consumer of type α' purchases the good when its price is $p_i' > p_i$, with other goods priced at the fixed levels p_{-i}. We show that good i at price p_λ is among the most preferred goods for type α^λ. Consider the comparison between good i and any other good j. Substitution in the utility function from A1 gives the following inequalities,

$$\sum_{k=1}^{n} \alpha_k t_k(x_i) + g(Y-p_i)t_{n+1}(x_i) + t_{n+2}(x_i) \geq \sum_{k=1}^{n} \alpha_k t_k(x_j) + g(Y-p_j)t_{n+1}(x_j) + t_{n+2}(x_j), \text{ and}$$

$$\sum_{k=1}^{n} \alpha_k' t_k(x_i) + g(Y-p_i')t_{n+1}(x_i) + t_{n+2}(x_i) \geq \sum_{k=1}^{n} \alpha_k' t_k(x_j) + g(Y-p_j)t_{n+1}(x_j) + t_{n+2}(x_j).$$

As $g(z)$ is concave and $t_{n+1}(x_i) > 0$, these inequalities combine to give,

$$\sum_{k=1}^{n} \alpha_k^\lambda t_k(x_i) + g(Y-p_\lambda)t_{n+1}(x_i) + t_{n+2}(x_i) \geq \sum_{k=1}^{n} \alpha_k^\lambda t_k(x_j) + g(Y-p_j)t_{n+1}(x_j) + t_{n+2}(x_j).$$

Thus good i at price p_λ is among type the most preferred goods for type α^λ.

When goods i and j are distinct, the mass of consumers for which the final inequality is an equality is zero so that the reservation price expression from equation (4.1) is valid. Hence the Prékopa-Borell Theorem applies and the result follows.

There is no problem of breaking ties when goods i and j are equivalent, since good i must be strictly preferred to good j at p_λ. The reason is that $p_\lambda < p_i'$ by definition and $p_i' \leq p_j$ as otherwise α' would have strictly preferred good j. Hence the set of consumers at price p_λ contains the Minkowski sum of the consumers at prices p_i and p_i' and the Prékopa-Borell Theorem applies directly, completing the proof. *Q.E.D.*

PROPOSITION 2: *There are n-dimensional models satisfying A1 with $f(\alpha)$ a ρ-concave probability density function with convex support, $\rho \geq -1/n$ and with $D(p) = (1-p)^{\frac{1+n\rho}{\rho}}$.*

PROOF: We show the bounds are tight for a specific case of duopoly competition. Let a consumer of type α have utility function $U(\alpha, \chi, z) = \alpha \cdot \chi + z$. Good 1 has characteristics $(1, 1, \ldots, 1)$ and good 2 is the zero vector, $(0, 0, \ldots, 0)$. The density of types is $f(\alpha) = c(\sum \alpha_k)^{1/\rho}$ over the unit simplex. With $p_1 = 1$ the reservation price function for good 2 is $1 - \sum \alpha_k$. Thus demand for good 2 is

$$D(p_2) = \int_{\{\alpha: \sum \alpha_k \leq 1 - p_2\}} f(\alpha) d\alpha = \int_0^{1-p_2} k^{n-1} k^{1/\rho} dk = (1 - p_2)^{\frac{1+n\rho}{\rho}}.$$

Q.E.D.

PROPOSITION 3: *A firm's profit function is quasi-concave in own price provided that $D(p)^{-1}$ is convex and diminishing in p over the region where $D(p) > 0$.*

PROOF: The profit function is monotonically increasing for all prices below marginal cost: an increase in price reduces sales and per-unit losses. Hence we can restrict attention to prices above cost. A failure of quasi-concavity requires there exists $c \leq p_0 < p_1$, and $0 < \lambda < 1$ such that

$$(p_0 - c)D(p_0) > (p_\lambda - c)D(p_\lambda);$$

$$(p_1 - c)D(p_1) > (p_\lambda - c)D(p_\lambda).$$

If $D(p_1) > 0$, we may divide both sides of the first inequality by the product $D(p_0)D(p_\lambda)$ and both sides of the second inequality by $D(p_1)D(p_\lambda)$ and then add the weighted average to show:

$$(p_\lambda - c)/D(p_\lambda) > \lambda(p_\lambda - c)/D(p_0) + (1 - \lambda)(p_\lambda - c)/D(p_1).$$

This inequality contradicts the assumption that $D_i(p_i, p_{-i})^{-1}$ is convex in p_i. If $D(p_1) = 0$ then the second inequality requires $p_\lambda < c$, contrary to the assumption. *Q.E.D.*

PROPOSITION 5: *Consider an equilibrium in which a given set I of firms has zero demand. Let p_A denote a vector of equilibrium prices for the active firms. There exists an equilibrium which leads to exactly the same demand and profits which all active firms continue to charging prices p_A while all firms in I charge their marginal cost.*

PROOF: First observe that no firm in I can become active at the new lower price vector. The reason is that even with other inactive firms charging their original higher prices, market shares at marginal cost must have been zero or else the firm could have achieved positive profits. With zero demand, charging marginal cost is still profit-maximizing. For the active firms, their profits cannot have been improved by the price reduction of inactive firms. The fact that they can still achieve their earlier profit levels demonstrates that the old prices must still be optimal. *Q.E.D.*

PROPOSITION 6: *In the differentiable case with A1 and with $\ln[f(\alpha)]$ concave where $f(\alpha) > 0$, the duopoly equilibrium is unique. In addition, the pricing game is log-supermodular.*

PROOF: We use a change of variables to make the dominant diagonal argument transparent. Let each firm choose $u_i \equiv g(Y - p_i)t_{n+1}(x_i)$ instead of p_i. Then

$$\ln[\pi_i(u_1, u_2)] = \ln[h_i(u_i) - c_i] + \ln[D_i(u_1 - u_2)],$$

where $h_i(u_i) = Y - g^{-1}[u_i/t_{n+1}(x_i)]$ is concave. Note that the first term is strictly concave in u_i since it is the composition of an increasing strictly concave function with a concave function. To complete the dominant diagonal argument, it suffices to show that

$$\frac{\partial^2 \ln D_i}{\partial u_i^2} + \frac{\partial^2 \ln D_i}{\partial u_i \partial u_j} \leq 0,$$

with the first term negative and the second term positive. In the duopoly problem, the same conditions yield log-supermodularity [Milgrom and Roberts (1990a)].

Both conclusions follow from the observation that

$$\frac{\partial^2 \ln D_i}{\partial u_i^2} = -\frac{\partial^2 \ln D_i}{\partial u_i \partial u_j} < 0.$$

This is a consequence of the log-concavity of D in u_i (by Prékopa-Borell) and the fact that demand depends only on $u_i - u_j$.

This demonstrates that there is at most one equilibrium in which both firms are active. Additionally, there can be at most one equilibrium with an inactive firm. If there is an equilibrium with firm 1 inactive, then it must receive zero demand at price equal to its marginal cost. But then to become active, it would have to charge a higher price which cetainly leaves firm 2 a positive market share if it charges its marginal cost. In addition, Proposition 5 shows that all equilibria with a given inactive firm are equivalent. Finally, it is impossible for there to be two equilibria, one where both firms are active and the other where one firm is inactive. The reason is that the firm active in both cases must increase its u_i by more than the change in u_j when moving to the solution where both firms are active (as otherwise the other firm would still have no market share). This cannot be optimal since the diagonal dominance argument implies that the reaction curves have slope less than one. *Q.E.D.*

PROPOSITION 7: *Consider the differentiable case with A1 and with* $\ln f$ *concave over an interval* $B \subset R$. *If there is an equilibrium in which a given set of firms are active, then there is no other equilbrium with the same set of active firms. In addition, the pricing game among these active firms is log-supermodular.*

PROOF: Because the consumers lie along a line, it is possible to order the firms according to their market positions. For any given set of active firms, each firm must have the same neighbors even in distinct equilibria. The linear market reduces the number of neighbors to one or two. We look at the case for a representative firm with two neighbors; the argument for a firm with one neighbor is analogous.

As in Proposition 6, we change variables from p_i to u_i and reduce the proof to properties of $\ln D$. Firm i with neighbors j and k has demand function $D(u_i - u_j, u_i - u_k)$ over the range where j and k remain active competitors. Using this change of variable ensures that

$$\frac{\partial^2 \ln D}{\partial u_i^2} + \frac{\partial^2 \ln D}{\partial u_i \partial u_j} + \frac{\partial^2 \ln D}{\partial u_i \partial u_k} = 0.$$

When all firms raise u_i equally, demand is uneffected. Thus diagonal dominance, uniqueness, and log-supermodularity all follow provided that each cross partial is positive. Differentiation yields

$$\frac{\partial^2 \ln D}{\partial u_i \partial u_j} = \frac{-(D_{11} + D_{21})D + D_1(D_1 + D_2)}{D^2},$$

where subscripts denote partial derivatives.

Because there is no interaction between firms j and k (as they are separated by i), the D_{21} term is zero. The sum $-D_{11}D + D_1D_1$ is positive because $\ln D$ it is concave in each of its arguments by the Prékopa-Borell theorem. The remaining term, D_1D_2, is positive because demand for firm i is increasing in both u_j and u_k. *Q.E.D.*

PROPOSITION 8: *Under A1 with $\alpha \in R^n$, consider m firms selling goods at prices p:*

(a) For $m \leq n+1$ no value restriction is implied; all m! preference orderings of the m products can coexist.

(b) For $m > n+1$, value restrictions are implied; some of the m! orderings are ruled out.

PROOF: (a) With A1, those who prefer (x_i, p_i) to (x_j, p_j) lie in a half-space. With $m \leq (n+1)$, any preference ordering is possible since there exists a solution to any n linear inequalities with independent gradient vectors.

(b) For $m > (n+1)$, the social choice results of Greenberg (1979, Theorem 2) translated into this setting imply there exists a product which captures at least $1/(n+1)$ of the market in duopoly competition against each of the other $(m-1)$ products. If there were no value restrictions, consider a population of m individuals, one with each of the cyclic preferences

$$\{(x_1, p_1) \succ (x_2, p_2) \succ \ldots \succ (x_m, p_m)\}; \ldots; \{(x_m, p_m) \succ (x_1, p_1) \succ \ldots \succ (x_{m-1}, p_{m-1})\}.$$

Here, no firm captures at least $1/(n+1)$ against each of its rivals in duopoly competition, a contradiction. *Q.E.D.*

PROPOSITION 9: *Under A1 and A2 with $\alpha \in R^2$, the average number of competitors per active firm is bounded above by 6: $[\sum_i C_i]/m' \leq 6$, where m' represents the number of firms with $D_i(p) > 0$.*

PROOF: Under A1 and A2, at any prices p, firm i's customers lie in a convex subset of B. This collection of convex market areas form a partition of B. If there is a tie for the best commodity, we break this arbitrarily in favor of the good with the lowest index. This is irrelevant to demand as the set of indifferent consumers is generally of measure zero.

Let each market area be a convex polyhedron. This involves no loss of generality since the number of neighbors is unchanged when we replace B by the convex hull of the points on the boundary of B where some consumer is indifferent between two products. The number of a firm's competitors then equals the number of sides of its polyhedral market interior to B. The average number of sides, s, is bounded by counting angles.

At any vertex of the market partition interior to B, the consumer is indifferent between at least three products: hence the average interior angle is 120° or less. At vertices in $Bd(B)$, the average interior angle is 90° or less. It is easy to confirm that for any s-sided polygon, the sum of the interior angles is $(180°)(s-2)$. Thus the average over the m' active firms is $(180°)(s-2)/s$. This is increasing in s and equals 120° when $s = 6$ yielding the result. *Q.E.D.*

PROPOSITION 13: *Cobb-Douglas preferences (8.2) satisfy A1″.*

PROOF: Since revenue is constant over the region where demand is positive, $D(\alpha, Y, p) = [p(1+\beta)]^{-1}$ and thus $\ln[D(\alpha, Y, p)] = k - \ln p$, a linear function of log price. The only remaining issue is to show that demand is positive over a convex set of α types. This follows as the log of indirect utility is linear in α and $\ln p$: if type α_0 purchases good i when $\ln p_i = q_i$ and type α_1 purchases good i when $\ln p_i' = q_i'$, then type α_λ will choose to purchase good i when the log of the price equals q_λ. *Q.E.D.*

THEOREM 3: *Under Cobb-Douglas preferences (8.2) and A2', there always exists a pure strategy Bertrand-Nash equilibrium for $n \geq 2$ firms, each with strictly positive costs, $c_i > 0$; for the duopoly, this equilibrium is unique.*

PROOF: Once again, we use Kakutani's fixed point theorem to prove existence. Note that Proposition 1 extends to this model; demand is continuous unless goods are identical. Hence, the proof will go through exactly as in Theorem 2 once we establish that there exists a best-response mapping which takes a compact set Z into itself. This is demonstrated in Lemma 1 below.

To prove uniqueness, we reconsider the first-order conditions (6.2) and (6.3). Uniqueness for duopoly follows exactly as in the proof of Proposition 6, where $b(p_i)$ is linear in $\ln p_i$. *Q.E.D.*

Given a price vector p, let $p_i(p_{-i})$ be the best response correspondence. Define

$$m_{-i} \equiv \min_{j \neq i} p_j \text{ and } M_{-i} \equiv \max_{j \neq i} p_j.$$

For convenience, we set $Y = (1+\beta)$ so that total market revenue equals 1.

LEMMA 1: *There exists a vector $z \in R^n$ and corresponding set $Z = \prod_i [c_i, z_i]$ such that for all $p \in Z$*

$$p_i(p_{-i}) \cap [c_i, z_i] \neq \emptyset.$$

PROOF: The proof of Lemma 1 itself uses Lemmata 2 and 3. Lemma 2 demonstrates the existence of a scalar k such that for any i and p_{-i},

$$p_i \geq k m_{-i} \Longrightarrow D_i(p_i, p_{-i}) = 0. \tag{10.1}$$

Lemma 3 defines firm specific scalars r_i with $r_i r_j \leq 1, i \neq j$ and a constant p^* such that,

$$m_{-i} > p^* \Longrightarrow p_i(p_{-i}) \cap [c_i, M_{-i} r_i] \neq \emptyset \tag{10.2}$$

We take $r_1 = \max_i r_i$, and set $r = Max[r_i, 1]$.

With (10.1) and (10.2), set $z_1 = kp^* r$ and $z_i = kp^*$, $i \neq 1$. With this vector z, consider a $p \in Z$. We separate cases according to the value of m_{-i} to show that firm i always has a best response below z_i.

(i) If $m_{-i} \leq p^*$, then (10.1) above shows that for $p_i > kp^*$, $D_i^*(p_i, p_{-i}) = 0$. Hence profits are zero for $p_i > z_i$ and it is immediate that $p_i = c_i$ is as profitable as any price outside $[c_i, z_i]$.

(ii) If $m_{-i} > p^*$, firm i has a best response in the set $[c_i, M_{-i} r_i]$ by Lemma 3. Hence we need only confirm that $M_{-i} r_i \leq z_i$. For all firms other than firm 1, $M_{-i} r_i \leq kp^* r r_i \leq kp^*$, where the second inequality follows from the definition of r and the fact that $r_i r_j \leq 1, i \neq j$. For firm 1, $M_{-1} r_1 < kp^* r$, by the definition of r. *Q.E.D.*

To prove Lemmata 2 and 3 we note that with A1', a type (α, Y) chooses x_i over x_j if and only if

$$\alpha \cdot f(x_i) + h(x_i) - \ln[r_i] \geq \alpha \cdot f(x_j) + h(x_j), \tag{10.3}$$

where $r_i = p_i/p_j$.

LEMMA 2: *There exists a scalar k such that for any i and p_{-i},*

(i) $p_i \geq km_{-i} \Longrightarrow D_i(p_i, p_{-i}) = 0$;

(ii) $p_i \leq m_{-i}/k \Longrightarrow p_i D_i(p_i, p_{-i}) = 1$

PROOF: Equation (10.3) shows that for any two firms (i, j) there exists some number k_{ij} such that if $p_i \geq k_{ij}p_j$ then firm i gets zero demand. It is then clear that picking $k \equiv \max_{(i,j)} k_{ij}$ establishes the Lemma. Note that we have normalized income so that a firm that captures the entire market has revenue equal to 1. *Q.E.D.*

Lemma 2(ii) establishes that any firm can capture the entire revenue by undercutting its rivals suficiently. Furthermore the necessary level of undercutting can be made independent of other prices. Lemma 3 establishes that once the minimum price of other firms is high enough, a firm does better undercutting its rivals and capturing the entire market rather than accepting at most (1/2) the revenue at the higher price $M_{-i}r_i$.

LEMMA 3: *There exist scalars r_i with $r_i r_j \leq 1, i \neq j$ and a constant p^* such that,*

$$m_{-i} > p^* \Longrightarrow p_i(p_{-i}) \cap [c_i, M_{-i}r_i] \neq \emptyset$$

PROOF: We define $p^* = \max_i 2kc_i$ and scalars r_i so that,

$$r_i D_i(1, 1, \ldots, r_i, 1, \ldots, 1) = 1/2.$$

By Lemma 2(i), with r_i high enough, firm i's revenue is zero. By Lemma 2(ii), with r_i low enough, firm i has revenue 1. The existence of r_i follows by the continuity of D under A1′.

To establish (10.2) we first note that for $p_i \geq M_{-i}r_i$ revenue and hence profits are below (1/2). In contrast, a price $p_i = m_{-i}/k$ generates revenue of 1 and costs of $c_i k/m_{-i}$, which are below (1/2) as

$$c_i k/m_{-i} \leq c_i k/p^* \leq c_i/(2c_i) = (1/2).$$

Note that the first inequality follows by assumption and the second from the definition of p^*. Finally, observe that only one firm can have strictly more than 50% of the total revenue when all firms charge the same price. If such a firm exists, it is labelled as firm 1. For $i \neq 1$,

$$1/2 = r_1 D_1(r_1, 1, \ldots, 1) = 1D_1(1, 1/r_1, \ldots, 1/r_1, \ldots, 1/r_1) \leq D_1(1, 1, \ldots, 1/r_1, \ldots, 1).$$

This implies that $r_i D_i(1, 1, \ldots, 1/r_1, \ldots, 1) \leq (1/2)$ so that $r_i r_1 \leq 1$ as required. If no firm has more than 50%, then $r_i \leq 1$ for all i and the lemma follows directly. *Q.E.D.*

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