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NEIGHBORS OF THE ORIGIN FOR FOUR BY THREE MATRICES

by

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# Neighbors of the Origin for Four by Three Matrices

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## Abstract

Scarf has defined a neighborhood system for families of integer programs where the right-hand side is allowed to vary. This system depends on a matrix  $A$  of constraint and objective function coefficients of the integer programs.

This paper characterizes the set of neighbors of the origin when  $A$  is four by three; showing that it may be described as the set of integer vectors in a union of two-dimensional polyhedra, where the number of polyhedra is quadratic in the bit size of  $A$ .

## 1 Introduction

An important question in optimization is how to recognize an optimal solution. If we are minimizing a convex function over a convex region, we know that any feasible solution that minimizes over a neighborhood of that solution in fact minimizes the function over the entire region. In other words, a local optimum in this case is a global optimum. Here any open set containing the tentative solution will do as a neighborhood. One result is that if we change the problem slightly, we can check whether a solution remains optimal by looking only in this neighborhood and without performing an entire optimization algorithm.

In discrete optimization, a local search is often used as a heuristic to find an optimum. Here a neighborhood is not an arbitrary open set, but

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rather some finite set  $N(h)$  of solutions close in some sense to the tentative solution  $h$ . We may compare a solution with its neighbors; stop if it is locally optimal, otherwise replace it by a superior neighbor; and iterate until we either reach local optimality or tire of the whole affair. To parallel the situation in the continuous convex case, it would be desirable if we could define a neighborhood system  $N()$  so that local optima are global and so that for some set of perturbed problems, the local optima of a perturbed problem are global optima to that perturbed problem.

For a family of integer programming problems  $\min cx$  s.t.  $\bar{A}x \leq b$  with fixed  $m - 1$  by  $n$  constraint matrix  $\bar{A}$  and objective function  $c$ , but variable  $b$  Scarf [1981i] defines a system of neighbors as follows. An  $m$  by  $n$  matrix  $A$  is formed from  $\bar{A}$  by adjoining the vector  $c$  as a new row. An integer point  $x$  is a neighbor of an integer point  $y$  if there is a  $b$  such that  $Ax \leq b$  and  $Ay \leq b$ , but no other integer point lies in the interior of the body  $\{z \in \mathbb{R}^n : Az \leq b\}$ . So that these bodies may be bounded, it is required that the rank of  $A$  be  $n$  and that there exist a positive  $m$  vector  $\pi$  such that  $\pi A = 0$ . For this system,  $N(h) = h + N(0)$ . Also,  $N(0) = -N(0)$ . Scarf shows, under a general positioning assumption, that this neighborhood system is the minimal system with these two properties for which local optima are global for all values of  $b$ . In [Scarf 1981ii], he characterizes the neighbors when  $A$  is a three by two matrix and when  $A$  is four by two. In [Scarf 1985], he shows, again under a general positioning assumption, that when  $A$  is a four by three matrix, there exists an integer vector  $d$ , such that every neighbor  $h$  of the origin satisfies  $d^T h \in \{0, +1, -1\}$ . This paper shall consider details of the structure of these neighbors when  $A$  is a four by three matrix. No general positioning assumption is made.

A straightforward way to use this neighborhood system to prove optimality would be to compute a complete list of the neighbors of the tentative optimum and to check each neighbor for feasibility and objective function value. This list may be very long, however. Even if  $A$  is restricted to being a four by two matrix, there are examples [Scarf 1981ii] where the number of neighbors is linear in the actual components of  $A$ , rather than polynomial in their logarithms as we would desire.

Lovász conjectures that there exists a description of these neighbors as the polynomial union of intersections of lattices and polyhedra of dimension less than that of the full space. Here "polynomial" means polynomial in the number of bits needed to specify  $A$ , for fixed dimension  $n$ . The lattices and polyhedra should also have a description of size polynomial in this specification of  $A$ . Recent results of Kannan suggest that this is so, although

the dependence on dimension in his argument is extreme. Given such a description, we could prove optimality for a problem in  $n$ -space by solving a polynomial number of integer programming feasibility problems of lower dimension of the form: does there exist a point with a better objective function that is feasible in one of these intersections of lattices and lower dimensional polyhedra? It should be noted that there already exists an algorithm for the integer programming feasibility problem which is polynomial in fixed dimension, due to Lenstra [1983]. The characterization of neighbors in the four by three case described below easily gives a description as a polynomial union of the integer points in two-dimensional polyhedra.

## 2 Generalities

We first shall introduce notation that is useful for our work with neighbors, but not specific to the four by three case. Let  $A$  be an  $m$  by  $n$  real matrix of rank  $n$  such that there is a positive  $m$  vector  $\pi$ , with  $\pi A = 0$ . Denote by  $a^i$  the  $i$ th row of  $A$ . We will define a neighbor of the origin in terms of the points that prevent a point from being a neighbor.

**Define:** A point  $x$  dominates  $y$  if  $x$  lies in the interior of the smallest body of the form  $Ax \leq b$  that contains both  $y$  and  $0$ . In other words,

$$\begin{aligned} a^i y \geq 0 &\Rightarrow a^i y > a^i x \\ a^i y \leq 0 &\Rightarrow 0 > a^i x. \end{aligned} \tag{1}$$

**Proposition 1** *If  $x$  dominates  $y$ , then  $x - y$  dominates  $-y$ .*

**Proof:** If  $a^i(-y) \geq 0$  then  $a^i y \leq 0$ . Then  $0 > a^i x$ , so that  $a^i(-y) > a^i(x-y)$ . If, on the other hand,  $a^i(-y) \leq 0$  then  $a^i y \geq 0$ , so that  $a^i y > a^i x$  and  $0 > a^i(x-y)$ .  $\square$

**Define:** An integer point  $x$  is a neighbor (of the origin) if and only if no integer point dominates  $x$ .

We immediately have symmetry in the set of neighbors. We now restrict our attention to the case where  $m = n + 1$ , so that all bodies of the form  $Ax \leq b$  are simplices. To characterize the neighbors we divide  $\mathbf{R}^n$  into cones according to the signs of the products  $a^i x$ . Each cone is of the form  $\{x \in \mathbf{R}^n : a^i x \geq 0 \text{ for } i \in I, a^i x \leq 0 \text{ for } i \notin I\}$ , for some subset  $I$  of the constraints. The constraints in  $I$  will be called positive constraints, the remaining constraints being called negative. The neighbors differ according

to the number of constraints of each sign of the cones in which they lie. We will use caution when considering points on the boundaries of the cones, since these boundaries overlap. For the  $m = n + 1$  case we can determine which of these cones are nontrivial.

**Proposition 2** *The two cones with all of the constraints of the same sign contain only the origin; the  $2^n - 2$  other cones all contain interior points.*

**Proof:** We shall use linear programming duality. Let  $D$  be a diagonal matrix, with each entry of the diagonal being either +1 or -1. The question of nontriviality of the corresponding cone is whether the system

$$DAx > 0 \tag{2}$$

has a solution. This is equivalent to the question of whether the linear program

$$\begin{aligned} \min \quad & 0x \\ \text{s.t.} \quad & DAx \geq e \end{aligned} \tag{3}$$

is feasible, where  $e$  is a vector of ones. This, in turn, by linear programming duality, is equivalent to whether the dual linear program

$$\begin{aligned} \max \quad & ye \\ \text{s.t.} \quad & yDA = 0 \\ & y \geq 0 \end{aligned} \tag{4}$$

is bounded. By our assumption of full rank for  $A$ , if  $yDA = 0$ , the vector  $yD$  must be a multiple of our vector  $\pi$ . If  $D$  is the identity matrix  $I$ , then  $y = \lambda\pi$  is feasible for any nonnegative  $\lambda$ , and (4) is unbounded. The system (2) therefore has no solution. Similarly, if  $D = -I$ , the same values of  $y$  make this l.p. unbounded. For any other sign pattern of  $D$ , 0 is the only multiple  $yD$  of  $\pi$  for which  $y \geq 0$ , so that (4) is bounded, and therefore the system (2) does have a solution.

□

The neighbors in the cones with one constraint of one sign and all the other constraints of the other sign are easy to describe. We describe half of these; the remainder are their negatives.

**Lemma 3** *If we define*

$$\alpha_i = \min\{a^i x : x \in \mathbb{Z}^n, a^i x > 0, a^j x < 0 \forall j \neq i\}, \tag{5}$$

*then the neighbors in the cone  $C_i = \{x : a^i x \geq 0, a^j x \leq 0 \forall j \neq i\}$  are precisely the integer points in  $C_i$  with  $a^i x \leq \alpha_i$ .*

**Proof:** Suppose  $x$  is in  $C_i$  but  $a^i x > \alpha_i$ . Then the point  $y$  that achieves the minimum which defines  $\alpha_i$  dominates  $x$  as follows.

$$\begin{aligned} a^j x \leq 0 &\Rightarrow j \neq i \Rightarrow 0 > a^j y. \\ a^j x > 0 &\Rightarrow j = i \Rightarrow a^i x > \alpha_i = a^i y. \end{aligned} \quad (6)$$

Conversely, if  $x$  is in  $C_i$  but not a neighbor, and so is dominated by some  $z$ , it violates this constraint as follows. For  $j \neq i$ ,  $a^j x \leq 0$ , so that  $a^j z < 0$ . By our last proposition,  $a^i z > 0$ . Then by definition of  $\alpha_i$ ,  $a^i z \geq \alpha_i$ . Since  $z$  dominates  $x$ , and  $a^i x \geq 0$ , we must have  $a^i x > a^i z \geq \alpha_i$ .  $\square$

In fact, domination by an integer point in  $C_i$  is almost equivalent to being beyond the hyperplane  $a^i x = \alpha_i$ . There are, however, some problems associated with integer points  $x$  such that  $a^j x = 0$  for some  $j$ . Therefore we shall have to define a few more values:

$$\alpha_{i,j} = \min\{a^i x : x \in \mathbf{Z}^n, a^i x > 0, a^j x \leq 0, a^k x < 0 \forall k \notin \{i,j\}\}. \quad (7)$$

### 3 The four by three case

The remainder of this paper considers only the case in which  $A$  is a four by three matrix. In this case there is only one type of sign pattern on the constraints not dealt with in the previous section: two constraints positive and two constraints negative. The main result of this paper is the characterization of neighbors in the cones with these sign patterns.

Given two distinct constraints  $i$  and  $j$ , let the remaining constraints be  $k$  and  $l$ . Then let

$$K_{i,j} = \{x \in \mathbf{R}^3 : a^i x > 0, a^j x > 0, a^k x < 0, a^l x < 0\}. \quad (8)$$

Let  $KI_{i,j}$  denote the convex hull of integer points in  $K_{i,j}$ .

**Theorem 4** *If we define*

$$S_{i,j} = (\partial KI_{i,j}) \cap \{x \in \mathbf{Z}^3 : a^i x \leq \alpha_{i,j}, a^j x \leq \alpha_{j,i}, a^k x \geq -\alpha_{k,l}, a^l x \geq -\alpha_{l,k}\} \quad (9)$$

(where  $\partial B$  denotes the boundary of  $B$ ), then  $S_{i,j}$  consists of the neighbors in  $K_{i,j}$ .

The proof consists of three lemmas.

**Lemma 5** *If  $x$  is in  $S_{i,j}$ , then  $x$  is a neighbor.*

**Proof:** Suppose  $x$  lies in  $K_{i,j}$  but is not a neighbor. Let  $y$  be an integer vector that dominates  $x$ , preventing it from being a neighbor. We will show for all sign patterns of  $Ay$  and  $A(x-y)$  that  $x$  does not lie in  $S_{i,j}$ . We know that

$$\begin{aligned} a^i(x-y) &> 0 & a^k y &< 0 \\ a^j(x-y) &> 0 & a^l y &< 0. \end{aligned} \quad (10)$$

Suppose  $a^i y \leq 0$ . By Proposition 2 we must have  $a^j y > 0$ , and so  $a^j x \geq \alpha_{j,i}$ . Then  $a^j x > \alpha_{j,i}$ , so that  $x$  is not in  $S_{i,j}$ . Symmetrically, if  $a^j y \leq 0$  then  $x$  is not in  $S_{i,j}$ .

If  $a^k(x-y) \geq 0$ , then  $a^l(x-y) < 0$ , and so  $a^l(x-y) \leq -\alpha_{l,k}$ . We can then conclude  $a^l x < -\alpha_{l,k}$ , and so  $x \notin S_{i,j}$ . Symmetrically, if  $a^l(x-y) \geq 0$  then  $x \notin S_{i,j}$ .

In the remaining case, we know both  $y$  and  $x-y$  lie in  $K_{i,j}$ . Since these are integer points, they lie in  $KI_{i,j}$ . We will show that  $x$  must then lie in the interior of  $KI_{i,j}$ . Since  $K_{i,j}$  is an open set, there exists an  $\epsilon$ -ball about  $x$  in  $K_{i,j}$ . For any integer  $h$  greater than  $1/\epsilon$ , there is a unit sphere about  $hx$  in  $K_{i,j}$ . This sphere contains  $hx$  plus or minus each of the three unit vectors, giving six integer points all in  $K_{i,j}$ . The point  $hx$  lies in the interior of the convex hull of these six points, which are all in  $KI_{i,j}$ , so  $hx$  lies in the interior of  $KI_{i,j}$ . Now  $x$  is a strictly convex combination of  $y$ ,  $x-y$ , and  $hx$ , and so lies in the interior of  $KI_{i,j}$ , and not in  $S_{i,j}$ :

$$x = \frac{1}{2h-1}hx + \frac{h-1}{2h-1}y + \frac{h-1}{2h-1}(x-y). \quad (11)$$

□

The converse is established in the next two lemmas.

**Lemma 6** *If  $x$  is a neighbor in  $K_{i,j}$ , then  $x$  is in  $\{x \in \mathbf{Z}^3 : a^i x \leq \alpha_{i,j}, a^j x \leq \alpha_{j,i}, a^k x \geq -\alpha_{k,i}, a^l x \geq -\alpha_{l,k}\}$ .*

**Proof:** Each constraint corresponds to domination by a particular point. If  $a^i x > \alpha_{i,j}$ , then for the  $y$  that achieves the minimum in the definition of  $\alpha_{i,j}$ ,  $y$  dominates  $x$  as follows:

$$\begin{aligned} a^i x &> 0, & a^i x &> \alpha_{i,j} = a^i y \\ a^j x &> 0, & a^j x &> 0 \geq a^j y \\ a^k x &< 0, & 0 &> a^k y \\ a^l x &< 0, & 0 &> a^l y. \end{aligned} \quad (12)$$

Similarly if  $a^j x > \alpha_{j,i}$ ,  $x$  is dominated. If  $a^k x < -\alpha_{k,l}$ , then for the  $y$  that achieves the minimum in the definition of  $\alpha_{k,l}$ ,  $y + x$  dominates  $x$  as follows:

$$\begin{array}{lll}
a^i x > 0, & a^i y < 0 & \Rightarrow a^i x > a^i(x+y) \\
a^j x > 0, & a^j y < 0 & \Rightarrow a^j x > a^j(x+y) \\
a^k x < 0, & a^k x < -\alpha_{k,l} = -a^k y & \Rightarrow 0 > a^k(x+y) \\
a^l x < 0, & a^l y \leq 0 & \Rightarrow 0 > a^l(x+y).
\end{array} \tag{13}$$

Similarly if  $a^l x < -\alpha_{l,k}$ ,  $x$  is dominated. Thus if  $x$  is a neighbor, all of these  $\alpha$  constraints must hold.  $\square$

**Lemma 7** *No neighbor can lie in the interior of  $KI_{i,j}$ .*

**Proof:** We will prove this by contradiction, using the following claim: For any  $x$  interior to  $KI_{i,j}$ , such that neither  $x$  nor  $-x$  is dominated by any integer point, there are two integer points  $y$  and  $z$  on the same facet of  $KI_{i,j}$ , such that for each constraint  $h$ , either  $|a^h y| \geq |a^h x|$  or  $|a^h z| \geq |a^h x|$ . Here the absolute values are to avoid the distinction between the positive constraints  $i$  and  $j$  and the negative constraints  $k$  and  $l$ .

To find these two points, consider the line segment from  $x$  to 0. Because  $x$  lies in the interior of  $KI_{i,j}$ , and 0 outside of  $KI_{i,j}$ , this segment must intersect the boundary of  $KI_{i,j}$  at some point  $v$ . Because  $v$  lies strictly between 0 and  $x$ , and  $x$  lies in  $KI_{i,j}$ , we know that

$$0 < |a^h v| < |a^h x| \quad \forall h \in \{1, 2, 3, 4\}. \tag{14}$$

As  $v$  lies on the boundary of  $KI_{i,j}$ , it lies on some facet of this boundary. Let  $r$ ,  $s$ , and  $t$  be three integer points in this facet in whose convex hull  $v$  lies. We will pick  $y$  and  $z$  from among these three points.

For each constraint  $h$ , at least one of these integer points, say  $r$ , must satisfy  $|a^h r| < |a^h x|$ . Let us say, when this happens, that  $r$  covers  $h$ . Since three points must cover four inequalities, some point must cover at least two constraints.

Some combinations are impossible. Because neither  $x$  nor  $-x$  are dominated by integer points, we cannot have both  $a^i r < a^i x$ , and  $a^j r < a^j x$ , nor can we have both  $a^k(-r) < a^k(-x)$  and  $a^l(-r) < a^l(-x)$ , similarly for  $s$  and  $t$ . Thus no point can cover both  $i$  and  $j$ , nor both  $k$  and  $l$ . Some point must cover, therefore, exactly two constraints. Without loss of generality,  $r$  covers  $i$  and  $k$ , and therefore does not cover  $j$  or  $l$ . Let a point covering  $j$



be  $s$ . If  $s$  does not cover  $k$ , then, since it cannot cover  $i$ , we may use  $r$  and  $s$  for  $y$  and  $z$ .

$$\begin{array}{l} |a^i s| \geq |a^i x| \quad |a^j r| \geq |a^j x| \\ |a^k s| \geq |a^k x| \quad |a^l r| \geq |a^l x| \end{array} \quad (15)$$

If  $s$  does cover  $k$ , then it cannot cover  $l$ . The point  $t$  must cover  $l$ , and hence not cover  $k$ . If  $t$  also covers  $j$ , we may use it as  $s$  in the preceding case. If  $t$  does not cover  $j$ , then we have  $s$  and  $t$  for  $y$  and  $z$ .

$$\begin{array}{l} |a^i s| \geq |a^i x| \quad |a^j t| \geq |a^j x| \\ |a^k t| \geq |a^k x| \quad |a^l s| \geq |a^l x| \end{array} \quad (16)$$

Having established our claim, let  $x$  be a neighbor which lies in the interior of  $KI_{i,j}$ . Determine  $y$  and  $z$  as in the claim, and let  $w = y + z - x$ . Using the fact that  $y$  and  $z$  are in  $KI_{i,j}$ , and our claim, we have

$$a^i w = a^i y + a^i z - a^i x \geq \min\{a^i y, a^i z\} > 0. \quad (17)$$

Similarly  $a^j w > 0$ ,  $a^k w < 0$ , and  $a^l w < 0$ . Thus  $w$  is in  $KI_{i,j}$ . Now let  $u = (y + z)/2 = (w + x)/2$ . Because  $y$  and  $z$  lie on the same facet of the boundary of  $KI_{i,j}$ , so does  $u$ . On the other hand, since  $x$  is a neighbor and so integer,  $w$  is also integer, and hence in  $KI_{i,j}$ . But  $x$  is by assumption in the interior of  $KI_{i,j}$ , so that  $u$  is also in the interior of  $KI_{i,j}$ , giving a contradiction.  $\square$

Thus we have established the theorem of this section, that everything in  $S_{i,j}$  is a neighbor in  $KI_{i,j}$ , and that all neighbors in  $KI_{i,j}$  are in  $S_{i,j}$ .

If  $A$  is rational, the conjecture of Lovász mentioned in the introduction is also satisfied. This depends upon being able to bound the number of facets of  $KI_{i,j}$  in terms of the matrix  $A$ . For this purpose, following Schrijver, we define the size of a rational number  $r = p/q$  to be  $1 + \log_2(|p| + 1) + \log_2(|q| + 1)$ . To determine the number of these facets, we use the following result of Cook, Hartman, Kannan, and McDiarmid.

**Theorem 8** (*Cook, Hartman, Kannan, and McDiarmid*) *If  $A$  is in  $\mathbf{Q}^{m \times n}$ , and  $b$  is in  $\mathbf{Q}^n$ , and  $\phi = \max_i(n + 2 + \sum_j \text{size}(a_{i,j}) + \text{size}(b_i))$ , then the convex hull of integer points  $x$  satisfying  $Ax \leq b$  has at most  $2m^n(12n^2\phi)^{n-1}$  vertices.*

Using this theorem, and the well-known Euler relation on the numbers of vertices, edges, and facets of a polytope, we can give a polynomial bound on the number of facets of  $KI_{i,j}$ . This is not quite trivial, because  $KI_{i,j}$  is unbounded.

**Theorem 9**  $KI_{i,j}$  has at most  $14,929,920\phi^2$  facets, where  $\phi$  is defined as in the previous theorem, using  $b = 0$ .

**Proof:** Let  $KI_{i,j}$  have  $v$  vertices,  $e$  edges, of which  $e_u$  are unbounded, and  $f$  facets. Let  $c = a^i + a^j - a^k - a^l$ , where  $l$  and  $k$  are the remaining constraints as in the previous proofs. Let  $c_0$  be large enough that all vertices  $x$  of  $KI_{i,j}$  satisfy  $cx < c_0$ . Then the intersection of the half space  $\{x : cx \leq c_0\}$  is bounded, and has  $v' = v + e_u$  vertices,  $e' = e + e_u$  edges, and  $f' = f + 1$  facets. Because this is bounded, we can apply the Euler relation

$$f' - e' + v' = 2. \quad (18)$$

Since each facet is incident on at least three edges, and each edge is incident on exactly two facets, we know  $3f' \leq 2e'$ , giving us

$$f' \leq 2v' - 4. \quad (19)$$

Substituting back to the original problem, we have

$$f \leq 2v + 2e_u - 5. \quad (20)$$

To determine the number of unbounded edges of  $KI_{i,j}$ , we will look at the rays of  $KI_{i,j}$ . A direction  $d$  will be called a *ray* of a convex set  $X$  if  $x + \mu d$  is in  $X$  for all  $x$  in  $X$ , and all nonnegative  $\mu$ . The set of rays of a polyhedron is necessarily a cone. Every ray of  $KI_{i,j}$  must lie in the cone  $\bar{K}_{i,j} = \{x \in \mathbb{R}^3 : a^i x \geq 0, a^j x \geq 0, a^k x \leq 0, a^l x \leq 0\}$ . For any direction  $d$  that does not lie in this cone, and any  $x$  whatever, there is sufficiently large  $\mu$  so that  $x + \mu d$  is not in  $K_{i,j}$ , and so not in  $KI_{i,j}$ .

The four extreme rays of  $\bar{K}_{i,j}$  are solutions of linear equations with rational coefficients, and so may be scaled to be integer vectors. To show that these are rays of  $KI_{i,j}$ , let  $d$  be one of these integer vectors, and let  $x$  be any point in  $KI_{i,j}$ . The point  $x$  is some convex combination of integer vectors  $z^i$  in  $K_{i,j}$ . For any nonnegative  $\mu$ ,  $x + \mu d$  is the same convex combination of the vectors  $z^i + \mu d$ . Now any of these is a convex combination of the vectors  $z^i$  and  $z^i + [\mu]d$ , both of which must be in  $KI_{i,j}$ . Since  $KI_{i,j}$  is convex, we have  $x + \mu d \in KI_{i,j}$ . Thus the four extreme rays of  $\bar{K}_{i,j}$  are rays of  $KI_{i,j}$ . Since we have shown that the rays of  $KI_{i,j}$  all lie in  $\bar{K}_{i,j}$  and so are nonnegative combinations of these four rays, these four rays are the extreme rays of  $KI_{i,j}$ .

Every unbounded edge of a polyhedron must run in the direction of one of its extreme rays, so  $KI_{i,j}$  has at most four times as many unbounded edges as vertices, that is  $u_e \leq 4v$ . Thus we can conclude that

$$f \leq 10v - 5 \quad (21)$$

and, by Theorem 8, substituting for  $m$  and  $n$ , we establish the current theorem.  $\square$

## 4 Conclusion

We may now describe the set of neighbors in the four by three case as the set of integer points in a polynomial union of the following two dimensional polyhedra. Neighbors in the cones  $C_i$ ,  $i = 1, \dots, 4$  lie in the polyhedra

$$\{x : a^i = \alpha_i, a^j < 0 \forall j \neq i\} \quad (22)$$

and, for all  $j \neq i$ ,

$$\{x : a^i x \leq \alpha_i, a^j x = 0, a^k x < 0 \forall k \neq i, j\}. \quad (23)$$

Neighbors in the negatives of these cones lie in the negatives of these polyhedra. Neighbors in the cones  $KI_{i,j}$  lie in the polyhedra of the form

$$\{x : cx = c_0, \alpha_{i,j} \geq a^i x > 0, \alpha_{j,i} \geq a^j x > 0, -\alpha_{k,l} \leq a^k x < 0, -\alpha_{l,k} \leq a^l x < 0\}, \quad (24)$$

where  $\{i, j, k, l\} = \{1, 2, 3, 4\}$ , and  $cx = c_0$  ranges over the set of equalities determining facets of  $KI_{i,j}$ .

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