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THE CAPITAL ASSET PRICING MODEL AS A
GENERAL EQUILIBRIUM WITH INCOMPLETE MARKETS

by

John Geanakoplos and Martin Shubik

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ABSTRACT

We recast the capital asset pricing model (CAPM) in the broader context of general equilibrium with incomplete markets (GEI). In this setting we give proofs of three properties of CAPM equilibria: they are efficient, asset prices lie on a "security market line," and all agents hold the same two mutual funds. The first property requires a riskless asset, the latter two do not.

We show that across all GEI only one of these three properties of equilibrium is generally valid: asset prices depend on covariances, not variances.

We extend CAPM to many consumption goods in such a way that all three properties hold. But now the definition of a riskless asset depends on preferences and endowments, and so cannot be specified a priori.

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1. INTRODUCTION

This paper is devoted to clarifying the relationships between the CAPM stock market trading model and general equilibrium with incomplete markets. CAPM yields three important views of financial markets. First, that they are efficient. Second that asset prices depend not on the variance, but rather on the covariance of the underlying payoffs with a particular, privileged portfolio. Third, that all portfolio holders may be perfectly happy to hold only a few specially designated mutual funds.

By placing CAPM in the broader context of general equilibrium with incomplete markets (GEI) we find that only the second property, that covariances (and not variances) matter to asset pricing, retains validity. Risk averse agents diversify, to be sure, but in general they will not be satisfied with the same mutual funds. In GEI equilibrium there is no arbitrage, but the final allocations are almost never Pareto optimal. In fact, they are almost never constrained Pareto optimal.

Although there is no variant of either the efficiency principle or the mutual fund principle that is precisely true throughout GEI, one may still wonder if perhaps efficiency is approximately true if CAPM is a "reasonable"

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description of reality. CAPM assumes a single consumption good in each state of nature. To regard CAPM as a descriptive model, one must suppose that in reality relative commodity prices are not much affected by redistributions of income. In that case, GEI equilibria do become constrained Pareto efficient, but not Pareto efficient.

When there are many goods, what is the meaning of the riskless asset which is so central to CAPM? Should it promise the same quantity of money in each state, irrespective of the (different) rates of inflation? Should it guarantee equivalent purchasing power measured in terms of some specified basket of goods? How should the basket be chosen? Is there an analogue to the riskless asset in a multi-commodity world?

We shall show that there is a generalization of CAPM to a multi-dimensional world (m-CAPM). To each m-CAPM economy, there corresponds a collection of "riskless" assets. If any one of these is marketed, then all the equilibria will be Pareto optimal. But the rub is that this set of riskless assets depends on the underlying m-CAPM economy. It cannot be calculated without knowing the preferences and aggregate endowments of the economy for each possible state of nature. Without a riskless asset the equilibria will be far from Pareto optimal.

In the classical one good CAPM model the presence of a riskless asset is crucial to Pareto optimality. Perhaps the single most obvious policy recommendation that can be derived from the model is that the government should always engineer the creation of the riskless asset if it is not there already. Yet (at least in the U.S.) for a very long time there has really been no asset which purported to be riskless, when inflation is taken into account. For the one good CAPM, that is a puzzle, since there is no ambigu-

ity about what the riskless asset should be. In the multi-dimensional CAPM, a riskless asset exists, but knowing how to calculate it might be impossible.

In Part I we place the one good CAPM inside GEI. In the presence of a riskless asset (and quadratic preferences, and other CAPM hypotheses) we derive the Pareto efficiency of equilibrium, the security market line, and the mutual fund theorem. We note that efficiency fails when there is no riskless asset, but that the other two theorems remain valid. In Part II we describe a full-blown multi-commodity GEI model. We note that efficiency and the mutual fund theorem fail, but that an analogue to the security market line does hold. We also describe a special multi-good CAPM in which efficiency holds if there is a riskless asset. But the riskless asset cannot be described a priori, independent of the data of the economy.

2. ECONOMIC PRELIMINARIES

In the past few years there has been a great revival of interest by economists in the theory of general equilibrium with incomplete markets (GEI) anticipated by Arrow (1953) and Diamond (1967), and expressed first in general form by Radner (1972) and Hart (1975). The GEI theory focuses on the primitive characteristics of the agents and commodities, treating financial assets as limited devices for transferring wealth across states of nature, rather than starting with reduced form preferences defined on the assets themselves. The extra structure preserved by this approach, which is just beginning to be fully exploited, accounts for the renewed interest in the subject. In a series of papers, for example, by Cass (1984), Werner (1985), Duffie (1987), and Geanakoplos-Polemarchakis (1986) existence of

equilibrium was proved by using the boundedness from below of the consumption sets for models with assets that all deliver in the same numeraire commodity in each state of nature, such as the quadratic-utility capital asset pricing model (CAPM) we shall consider below. In reduced form representations of CAPM, the existence of equilibrium is not always guaranteed (see Nielson (1985)).

Perhaps the most interesting general question that can be posed in the GEI (but not in any reduced form) theory is: how close will equilibrium allocations, constrained as they are by the limited assets available to the market, come to achieving Pareto optimality? Although no satisfactory definition of distance from optimality has been found for this problem, the work of Hart (1975), Greenwald and Stiglitz (1984), and Geanakoplos-Polemarchakis (1986) shows that except for extremely rare choices of utility-endowment characteristics for the agents, any absence of potential insurance contracts will prevent equilibria from being fully Pareto optimal; in fact the missing asset markets will (with rare exceptions) cause equilibrium allocations to fail to be even "constrained" Pareto optimal.¹ One purpose of this paper is to show that the quadratic utility CAPM model, with a riskless asset, is precisely one of those rare economies for which equilibrium allocations always attain full optimality (if every agent's consumption is strictly positive in every state).

¹The reader can consult those papers for definitions of constrained Pareto optimal; Geanakoplos-Polemarchakis (1986) proves (except for rare exceptions) that when some of the asset markets are missing, the equilibrium will not even efficiently use the existing assets, when there are at least two commodities in every state of nature.

The CAPM name has been used to refer to any model in which it is possible to write for each agent a reduced form utility for asset portfolios depending only on the means and variances of their payoffs. It is a famous property of such models that there is a mutual fund theorem and a security market line theorem. The mutual fund theorem states that all portfolios held in equilibrium can be expressed as a combination of two portfolios, the so-called market portfolio and a "zero-beta" portfolio (see Lintner (1965), Sharpe (1964), and for the case where there is no riskless asset see Black (1972)). When there is a riskless asset, the zero-beta portfolio can be taken to be the riskless asset alone. The security market line theorem asserts that in equilibrium there is a linear relationship between the expected payoffs of assets with price equal 1, and the covariances of their payoffs with the market portfolio payoffs. Of course it is impossible to pose the Pareto optimality question in the reduced form version of the model.

There are two well-known GEI models which give rise to reduced form mean-variance utilities for assets. In one, the state space is taken to be infinite, and all assets are assumed to have normally distributed payoffs, and all commodity endowments are kept at zero. The utilities are arbitrary von Neumann-Morgenstern, with common probabilities. In the second version, the utilities must be quadratic von Neumann-Morgenstern with common probabilities, and the endowments of commodities must be zero; but the asset payoffs can be arbitrary, and the state space finite. It is well-known that the normal distribution is specified by its mean and variance, and that the expectation of a quadratic function depends only on the mean and variance of the underlying distribution, so both of these versions clearly give rise to reduced form preferences on assets of the mean-variance kind. Either

version may be considered with or without a riskless asset. The point of Theorem 1 of this paper is that in the quadratic version with a riskless asset, all interior equilibria are fully Pareto optimal, no matter how few assets there are relative to the number of states of nature. By contrast, in the other 3 GEI variations, equilibria are almost never Pareto optimal when the asset markets are incomplete. This shows that Pareto optimality is a property distinct from the mutual fund and security market line properties, though closely related. (For prior discussions of Pareto optimality, see Rubinstein (1974), Mossin (1977), Rothschild (1986).)

In Part I we consider in detail a model in which all consumers have quadratic von Neumann-Morgenstern utilities with common expectations. The state space is finite and the asset returns are arbitrary. The consumers are allowed to have endowments, however, provided that these lie in the span of the assets. In Theorem 1 we prove that if there is a riskless asset, all interior equilibria are Pareto optimal.² In Theorems 2-3' we derive the mutual fund and security market line theorems with (2, 3) and without (2', 3') a riskless asset from a linear algebraic argument that does not appeal to the mean-variance reduced form representation.

In Part II we give a sufficient condition on preferences for all multi-commodity GEI to be Pareto optimal (Theorem 4). One consequence (see Corollary) is that even in the one commodity model it is possible to obtain Pareto optimality with preferences other than quadratic. (For a discussion of these classes of utilities and their connection to mutual funds, see Cass-Stiglitz (1970). For an application to efficiency in the one good

²It is interesting that in CAPM Pareto optimality holds only for interior equilibria, while in the Arrow-Debreu model interiority has no connection with Pareto optimality.

model, see Rubinstein (1974).) We derive a multiple commodity CAPM model and observe that interior equilibria are Pareto optimal in the presence of a riskless asset, and we note the difficulty of finding a riskless asset in practice (Theorem 6). We note that even without the riskless asset, equilibria in the multiple commodity CAPM are constrained Pareto optimal (Theorem 5). Finally, we show that in all GEI equilibria there is a security market line giving prices in terms of covariances with some privileged asset, but not necessarily with the market (Theorem 7).

3. THE MODEL

Let the set of states of nature be denoted by $S = \{0, 1, \dots, S\}$. Let there be only one good in each state of nature. The consumption set is taken to be R_+^{S+1} . Each asset $h \in H = \{1, \dots, H\}$ is characterized by a quasi-concave, monotonic function $V^h : R_+^{S+1} \rightarrow R$, and an endowment vector of commodities $e^h \in R_{++}^{S+1}$. In addition agents hold assets, described below.

The set of financial assets is denoted $A = \{1, \dots, A\}$. Each asset $a \in A$ is represented by a vector $r_a \in R^{S+1}$. Sometimes we shall take the first asset, r_1 , to be the so-called riskless asset $\hat{1}$, which pays one unit of the commodity in every state of nature. $r_1 = \hat{1} = (1, 1, \dots, 1)$. Let us denote by the $(S+1) \times A$ matrix R the entirety of assets. A portfolio $\theta = (\theta_1, \dots, \theta_A)$ is a holding of each asset, and yields a return $R\theta$ across the $S+1$ states of nature. Each individual h begins with an endowment $\bar{\theta}^h = (\bar{\theta}_1^h, \dots, \bar{\theta}_A^h) \geq 0$ of assets.

Equilibrium is defined by a price vector $q \in R^A$, and asset holdings θ_a^h satisfying:

$$(1) \quad \sum_{h \in H} \theta_a^h = \sum_{h \in H} \bar{\theta}_a^h$$

$$\theta^h = \text{Arg}_\theta \text{Max}(V^h(e^h + R\theta) | q \cdot \theta \leq q \cdot \bar{\theta}^h, \text{ and } (e^h + R\theta) \geq 0) .$$

Notice that first trade takes place in the market for assets, then the state is realized, the assets pay-off, and finally consumption occurs.³ Since there is only one commodity, there is no need for the markets to re-open once the state of nature is realized. The definition of equilibrium requires that all promises are honored (consumption is $e^h + R\theta$). In particular, agents are allowed to go arbitrarily short, $\theta_a < 0$, in any asset, provided they ultimately keep their promise (by reducing their consumption by $-\theta_a r_a$).

Allowing short sales violates the standard boundedness from below condition used to guarantee the existence of equilibrium in Arrow-Debreu (1954). Nevertheless, as shown for example in Geanakoplos-Polemarchakis (1986), the possibility of short sales of assets does not interfere with existence of equilibrium, provided that the consumption set is bounded from below. We call an equilibrium $(q, \theta^h; h \in H)$ interior iff $e^h + R\theta^h \gg 0$ for all $h \in H$.

The assets can be variously interpreted. Some may be thought of as shares in a firm, whose production decision has already been made (perhaps $r_a(0) < 0$ and $r_a(s) \geq 0$ for all $s \in \{1, \dots, S\}$). On the other hand

³We have included the state $s = 0$ to make it easy to reinterpret the commodity in state 0 as consumption that occurs simultaneously with the purchase of assets at time 0 provided there is also an asset, say r_2 , which satisfies $r_2 = (1, 0, \dots, 0)$. Under this interpretation the uncertainty $s \in \{1, \dots, S\}$ only affects consumption at time 1.

some may be thought of as permissible contracts. For example, if the initial shares $\bar{\theta}_1^h$ of the riskless asset $r_1 = \hat{1}$ are zero for every household $h \in H$, and if there is another asset $r_2 = (1, 0, \dots, 0)$ with $\bar{\theta}_2^h = 0$ for all h , then trade between assets 1 and 2 can be considered saving and borrowing between consumption at $s = 0$ and consumption at all other s .

Notice that the distribution of consumption across the states of nature, and across individuals, is constrained by the span of the assets. If $A < S$, there are imaginable insurance contracts, i.e. trades of contingent commodities, that are not feasible with the limited asset markets. It can easily be shown (see for example Geanakoplos-Polemarchakis (1986)) that for "almost all" choices of utilities v^h and endowments e^h , all the resulting equilibria are Pareto suboptimal (if $A < S$). The point of our first theorem is to show that for the special case of the quadratic utility capital asset pricing model, full optimality is nevertheless attained even when $A < S$, if there is a riskless asset.

4. THE ASSUMPTIONS

(A1) There is one commodity in every state of nature.

(A2) The endowments $e^h \in \mathbb{R}_{++}^{S+1}$ satisfy:

$$e^h \in \text{sp}[r_1, r_2, \dots, r_A] = \text{sp}[R] \text{ for all } h \in H, \text{ i.e.}$$

$$e^h = R\psi^h \text{ for some } \psi^h \in \mathbb{R}^A, \text{ and } e^h + R\bar{\theta}^h \gg 0 \text{ } \forall h \in H.$$

Consumers are able to trade any fraction of their own initial endowments.

(A3) Quadratic von Neumann-Morgenstern Utilities: For each $h \in H$ there is some number $\alpha_h > 0$ and probabilities $\pi_s^h > 0$, $\sum_{s \in S} \pi_s^h = 1$, such that

$$v^h(w) = v^h(w_0, w_1, \dots, w_S) = \sum_{s \in S} \pi_s^h \left(w_s - \frac{1}{2} \alpha_h w_s^2 \right).$$

Thus the von Neumann-Morgenstern utility of consumption for agent h is $u^h(c) = c - \frac{1}{2} \alpha_h c^2$.

(A4) Common Expectations: $\pi_s^h = \pi_s^{h'} = \pi_s$ for all $h, h' \in H$, $s \in S$.

(A5) Monotonicity: Let $\bar{\theta} = \sum_{h \in H} \bar{\theta}^h$, and let $e = \sum_{h \in H} e^h$. Let $M = R\bar{\theta} + e$. Then $1 - \alpha_h M \gg 0$ for all $h \in H$.

Note that it is infeasible for consumption to exceed M_s in any state s , and $1 - \alpha_h W_s$ is the marginal utility of consumption in state s for agent h , if he is consuming W_s .

We have already noted that under hypotheses A1-A5, existence of equilibrium is guaranteed. We shall shortly show that for any economy satisfying A1-A5, all its interior equilibria are Pareto optimal if there is a riskless asset.

An equilibrium $(q, \theta^h; h \in H)$ gives rise to a consumption allocation $x^h = e^h + R\theta^h$, $h \in H$, with $\sum_{h \in H} x^h = M = R\bar{\theta} + e = R(\sum_{h \in H} \bar{\theta}^h) + \sum_{h \in H} e^h$. We call the equilibrium Pareto optimal if and only if there is no allocation $(y^h; h \in H)$ with $\sum_{h \in H} y^h \leq M$ and $V^h(y^h) > V^h(x^h)$ for all $h \in H$.

5. CAPM WITH A RISKLESS ASSET

In this section we shall assume in addition the presence of a riskless asset:

(A6) Riskless Asset: $r_1 = \hat{1} = (1, 1, \dots, 1)$.

At any equilibrium we know from monotonicity that the price of the riskless asset cannot be zero. Moreover it is evident that the equilibrium conditions are homogeneous of degree zero in q , hence without loss of generality we shall suppose that at equilibrium if r_1 is the riskless asset, then $q_1 = 1$.

Pareto Optimality

Theorem 1: Let $(q, \theta^h; h \in H)$ be an interior equilibrium ($e^h + R\theta^h \gg 0$ for all $h \in H$) for an economy satisfying assumptions A1-A6. Then the equilibrium allocation $x^h = e^h + R\theta^h$, $h \in H$, is Pareto optimal.

Proof: Let $\mu^h \in \mathbb{R}^{S+1}$ be agent h 's marginal utility vector at the equilibrium allocation: $\mu_s^h = 1 - \alpha_h x_s^h$ for all $s \in S$, $h \in H$, i.e. $\mu^h = \hat{1} - \alpha_h x^h$ for all $h \in H$. Since the utilities are concave and differentiable, and since there is agreement on the probabilities π_s , it is enough to show that all the μ^h are colinear.

Observe that on account of monotonicity, w.l.o.g. $q_1 = 1$. Let \hat{A} be the set of assets a for which $q_a \neq 0$. Then $\hat{1} \in \hat{A}$. For each $a \in \hat{A}$, let $\hat{r}_a = (1/q_a)r_a$ be the per dollar return on asset a . Now for any agent $h \in H$, observe that equilibrium requires that $\sum_{s \in S} \pi_s \mu_s^h (\hat{r}_a(s) - 1) = 0$, for otherwise the agent could buy (sell) one unit of the riskless asset $\hat{1}$ and sell (buy) $1/q_a$ units of asset a . Similarly if $a \in A/\hat{A}$,

so that $q_a = 0$, then for all h , $\sum_{s \in S} \pi_s \mu_s^h r_a(s) = 0$.

Let us now define an inner product for two vectors x and y in R^{S+1} by $x \square y = \sum_{s \in S} \pi_s x_s y_s$. We write $x \perp y$ iff $\sum_{s \in S} \pi_s x_s y_s = 0$. Using this notation, we have from above that $\mu^h \perp (\hat{r}_a - \hat{1})$ for all $h \in H$. Let $K = \text{sp}\{(\hat{r}_a - \hat{1}); a \in \hat{A}\} \cup \{r_a; a \in A \setminus \hat{A}\}$, where for any set T of vectors in R^{S+1} , $\text{sp}[T]$ denotes the span of T , i.e. the smallest subset of R^{S+1} containing T . Then $\mu^h \perp K$, for all $h \in H$.

Observe that since $e^h \in \text{sp}[R]$ and since consumption $x^h = e^h + \theta^h$, it follows that $x^h \in \text{sp}[R] = \text{sp}[1, r_2, \dots, r_A]$ for all $h \in H$. But recall that $\mu^h = \hat{1} - \alpha^h x^h$, hence $\mu^h \in \text{sp}[R]$ for all $h \in H$. But clearly $\text{sp}[R] = \text{sp}[K; \hat{1}]$. Hence all the μ^h must lie in the same one-dimensional subspace in $\text{sp}[R]$ perpendicular to K ; i.e., they are all colinear. Q.E.D.

Mutual Fund Theorem 2: Let $(q, \theta^h; h \in H)$ be an interior equilibrium for a CAPM economy with a riskless asset satisfying assumptions A1-A6. Let $x^h = e^h + R\theta^h$ be the equilibrium consumption of agent h , and let $M = \sum_{h \in H} e^h + R\bar{\theta}$ be the market consumption. Then there are scalars s_h and t_h for each agent h such that $x^h = s_h \hat{1} + t_h M$. If R has full column rank and if $e^h = 0$ for all h , then $\theta^h = s_h(1, 0, \dots, 0) + t_h \bar{\theta}$, where $\bar{\theta}$ is the market portfolio of assets. $\bar{\theta} = \sum_{h \in H} \bar{\theta}^h = \sum_{h \in H} \theta^h$.

Proof: We have already seen that if $\mu^h = \hat{1} - \alpha_h x^h$, then the μ^h are all colinear. Hence the consumption vectors $x^h = (1/\alpha_h)[\hat{1} - \mu^h]$ all lie in some two-dimensional space, spanned by $\hat{1}$ and some vector μ , colinear with all the μ^h . Since $\sum x^h = M$, we have that $M \in \text{sp}[\hat{1}, \mu]$. Since $M = \sum_{h \in H} x^h = \sum_{h \in H} (1/\alpha_h)[\hat{1} - \mu^h] = s\hat{1} - t\mu$ it follows that if M and $\hat{1}$ are

colinear, then so are \hat{l} and μ , and hence $x^h = s_h \hat{l}$ for some scalar s_h , for all $h \in H$. If \hat{l} and μ are not colinear, then $sp[\hat{l}, \mu] = sp[\hat{l}, M]$ and we can write $x^h = s_h \hat{l} + t_h M$, for all $h \in H$.

If $e^h = 0$ for all h , then $x^h = R\theta^h$ and $M = R\bar{\theta}$. If R has full column rank, then θ^h is the unique portfolio with $x^h = R\theta^h$. Hence we must have that in portfolio space, $\theta^h = s_h(1, 0, \dots, 0) + t_h \bar{\theta}$. Q.E.D.

Before proving our next theorem, let us recall the definition of expectation and covariance between two vectors x and y in R^{S+1} . Ex denotes the expectation of x with respect to the common priors π , $Ex = x \square \hat{l} = \sum_{s \in S} \pi_s x_s$. $cov(x, y) = \sum_{s \in S} \pi_s (x_s - Ex)(y_s - Ey)$, and $var x = cov(x, x)$. Notice that $cov(x, y) = \sum_{s \in S} \pi_s [x_s y_s - x_s Ey - y_s Ex + ExEy] = \sum_{s \in S} \pi_s x_s y_s - ExEy = x \square y - ExEy$.

Security Market Line Theorem 3: Let $(q, \theta^h; h \in H)$ be an interior equilibrium for a CAPM economy with a riskless asset satisfying assumptions A1-A6, taken with normalization $q_1 = 1$. Let $r : R^{S+1} \rightarrow R$ represent the payoffs of any potential asset, marketed ($r = R\psi$ for some ψ) or not ($r \neq R\psi$ for all $\psi \in R^A$). Then there is a unique price $q(r)$ at which every consumer h will be satisfied to continue to hold exactly θ^h , despite the new opportunity of buying or selling short the new asset at price $q(r)$. (Of course if $r = R\psi$, then $q(r) = q \cdot \psi$.) Moreover, if $q(r) \neq 0$, so that $\hat{r} = r/q(r)$ is well-defined, then again denoting $M = e + R\bar{\theta}$ and $\hat{M} = M/q(M)$ we must have:

$$(*) \quad E\hat{r} - 1 = \frac{cov(\hat{r}, \hat{M})}{var M} (EM - 1) .$$

Proof: The existence of the price $q(r)$ follows immediately from the Pareto optimality theorem. Indeed, letting μ^h be the marginal utility vector at equilibrium of any agent h , $q(r) = (\mu^h \square r) / (\mu^h \square \hat{1})$. Since in any equilibrium satisfying our hypothesis, $\mu^h \gg 0$, $q(r)$ is well defined and independent of h , since all the μ^h are colinear.

Observe that the security market equation (*) defines a linear relation between $E\hat{r}$ and $\text{cov}(\hat{r}, \hat{M})$ for all securities \hat{r} with $q(\hat{r}) = 1$. Any line is determined by any two distinct points on the line. It can trivially be checked that $(\text{cov}(\hat{1}, \hat{M}), E\hat{1}) = (0, 1)$ and $(\text{var} \hat{M}, E\hat{M})$ satisfy (*), for the portfolio payoffs $\hat{r} = \hat{1}$ and $\hat{r} = \hat{M}$, respectively. Thus if there is a linear relationship between $\text{cov}(\hat{r}, \hat{M})$ and $E\hat{r}$ for all portfolio payoffs \hat{r} with $q(\hat{r}) = 1$, then (*) is the right formula. (If $\text{var} \hat{M} = 0$, then from Pareto optimality we know that for any h , μ_s^h is a constant independent of s , and so $(\mu^h \square \hat{r}) / (\mu^h \square \hat{1}) = E\hat{r} / E\hat{1} = E\hat{r}$, so * trivially holds.)

Consider now that $1 = q(\hat{r}) = (\mu^h \square \hat{r}) / (\mu^h \square \hat{1})$. Recalling that for any two vectors x and y , $x \square y = \text{cov}(x, y) + ExEy$, we have $\mu^h \square \hat{r} = \text{cov}(\hat{r}, \mu^h) + E\mu^h E\hat{r}$. Finally, recall that for any h we could find s_h and t_h with $\mu^h = s_h \hat{1} + t_h \hat{M}$. Hence we have

$$\mu^h \square \hat{r} = \text{cov}(\hat{r}, s_h \hat{1} + t_h \hat{M}) + E\mu^h E\hat{r}, \text{ or}$$

$$\mu^h \square \hat{r} = t_h \text{cov}(\hat{r}, \hat{M}) + E\mu^h E\hat{r}.$$

Taking $\mu^h \square \hat{1}$, t_h and $E\mu^h$ as constants gives us a linear relationship between $\text{cov}(\hat{r}, \hat{M})$ and $E\hat{r}$. Q.E.D.

Note the necessity of the hypothesis that $q(\hat{r}) = 1$.

Portfolios that are free (for example buying 1 unit of \hat{M} and selling short one unit of the riskless asset 1) have "betas" and returns lying on the line through the origin parallel to the security market line.

6. CAPM WITHOUT A RISKLESS ASSET

There are other versions of mean-risk behavior in which the mutual fund theorem and the security market line for marketed securities still holds, but Pareto optimality fails. This shows that Pareto optimality is a separate, stronger property attaching to the quadratic utility, riskless asset version of the mean variance model. To make our point we shall prove the mutual fund theorem and a security market line theorem for marketed assets in a quadratic-utility CAPM model without a riskless asset, in which optimality need not obtain.

Definition: Let $M = e + R\bar{\theta} - R\theta_M$ be the market payoff. Let $z = R\theta_z$. We call θ_z a zero-beta portfolio iff $\text{cov}(z, M) = 0$.

Lemma: Let $r = R\theta_r$ and $M = R\theta_M$. Then there is a zero-beta portfolio $\theta_z = \theta_r + \lambda\theta_M$, for some λ , such that $\text{sp}[r, M] = \text{sp}[z, M]$.

Proof: If $\text{var}(M) = 0$, there is nothing to prove. So suppose $\text{var}(M) > 0$. Let $\theta_z = \theta_r + \lambda\theta_M$. Then $\text{cov}(R\theta_z, M) = \text{cov}(R\theta_r, M) + \lambda\text{cov}(M, M)$, so choose $\lambda = -[\text{cov}(R\theta_r, M)/\text{var}(M)]$. Then $\text{cov}(z, M) = 0$ and z and M are linearly independent if r and M are noncolinear. Hence $\text{sp}[r, M] = \text{sp}[z, M]$.

Q.E.D.

Mutual Fund Theorem 2': Let $(q, \theta^h; h \in H)$ be an interior equilibrium for an economy satisfying assumptions A1-A5, that is, possibly without a riskless asset, and let $x^h = e^h + R\theta^h$ be the final consumption of agent $h \in H$, and let $M = e + R\bar{\theta}$ be the market portfolio payoff. Then there is a zero beta portfolio payoff z , $z = R\theta_z$, satisfying $\text{cov}(z, M) = 0$, such that for all $h \in H$, there are scalars s_h and t_h for which $x^h = s_h z + t_h M$. Moreover, if R has full column rank and $e^h = 0$ for all $h \in H$, then $\theta^h = s_h q + t_h \bar{\theta}$ for all $h \in H$.

Proof: Let $\mu^h = 1 - \alpha_h x^h$ be the marginal utility vector of consumer h . Let \hat{A} be the set of assets a with $q_a \neq 0$, and let $A \setminus \hat{A}$ be the remaining assets a' with $q_{a'} = 0$. Noting that $M = e + R\bar{\theta} \gg 0$, we must have that $q \cdot \theta_M \neq 0$ for θ_M solving $M = R\theta_M$. Let $\hat{M} = M / (q \cdot \theta_M)$, and similarly for all $a \in \hat{A}$ let $\hat{r}_a = r_a / q_a$. Then as we saw in the Pareto optimality proof, at an interior equilibrium it must be that $\mu^h \perp (\hat{r}_a - \hat{M})$ for all $a \in \hat{A}$ and $\mu^h \perp r_a$ for all $a \in A \setminus \hat{A}$. Letting $K = \text{sp}\{(\hat{r}_a - \hat{M}) | a \in \hat{A}\} \cup \{r_a | a \in A \setminus \hat{A}\}$, $\mu^h \perp K$. But $\mu^h \in \text{sp}\{K, \hat{1}, M\}$ for all $h \in H$. Thus the μ^h all lie in a two-dimensional set. Note the fact that $\hat{1}$ might not be in $\text{sp}\{R\}$ costs a dimension in this argument, and allows for the possibility of Pareto suboptimality.

If $\hat{1} \in \text{sp}\{R\}$, then we have already proved the result (letting $z = \hat{1}$). If $\hat{1} \notin \text{sp}\{R\}$, then we can still deduce that there is a two-dimensional space V containing all the x^h , $h \in H$. For if there were 3 linearly independent consumption vectors x^h , then the vectors $\mu^h = \hat{1} - \alpha_h x^h$ would vary over at least 3 dimensions, since $x^h \in \text{sp}\{R\}$ for all h .

Finally, let V be the (at most) two-dimensional subspace containing

all the x^h . Since $M = \sum_{h \in H} x^h$, $M \in V$. From the lemma, V is spanned by M and a zero-beta portfolio payoff $z = R\theta_z$. Hence $x^h = s_h z + t_h M$ for all $h \in H$. If $e^h = 0$ and R has full column rank, then $x^h = R\theta^h$, and $\theta^h = s_h \theta_z + t_h \bar{\theta}$. Q.E.D.

Security Market Line 3': Let $(q, \theta^h; h \in H)$ be an interior equilibrium for a CAPM economy satisfying A1-A5, with prices q normalized so that $q \cdot \theta_M = 1$. Then there is a linear relationship between the covariance of returns with the market return, and expected return, for assets with unit price. Precisely, there are scalars $\lambda_1, \lambda_2, \lambda_3$ such that for any portfolio $\hat{\psi}$ with $q \cdot \hat{\psi} = 1$ and $\hat{r} = R\hat{\psi}$, $\lambda_1 \text{cov}(\hat{r}, M) + \lambda_2 E\hat{r} + \lambda_3 = 0$. Furthermore, if there exists a zero-beta portfolio $\hat{\theta}_z$ with $q \cdot \hat{\theta}_z = 1$, then letting $\hat{z} = R\hat{\theta}_z$ this linear relationship takes the form:

$$(**) \quad E\hat{r} - E\hat{z} = \frac{\text{cov}(\hat{r}, M)}{\text{var}(M)} (EM - E\hat{z}) .$$

Proof: For any portfolio ψ and any agent $h \in H$, letting $r = R\psi$,

$$q \cdot \psi = \frac{\mu^h \square r}{\mu^h \square M} = \frac{\text{cov}(\mu^h, r)}{\mu^h \square M} + \frac{E\mu^h E r}{\mu^h \square M}$$

Restricting attention to $\hat{\psi}$ with $q \cdot \hat{\psi} = 1$, $\hat{r} = R\hat{\psi}$,

$$\begin{aligned} \mu^h \square M &= \text{cov}(\mu^h, \hat{r}) + E\mu^h E\hat{r} \\ &= \text{cov}(\mathbb{1} - \alpha_h x^h, \hat{r}) + E\mu^h E\hat{r} \\ &= -\alpha_h \text{cov}(x^h, \hat{r}) + E\mu^h E\hat{r} . \end{aligned}$$

So $(\mu^h \square M) / \alpha_h = -\text{cov}(x^h, \hat{r}) + (E\mu^h / \alpha_h) E\hat{r}$.

Summing over $h \in H$ gives

$$\sum_{h \in H} \frac{\mu^h \square M}{\alpha_h} = - \sum_{h \in H} \text{cov}(x^h, \hat{r}) + \sum_{h \in H} \frac{E\mu^h}{\alpha_h} E\hat{r}$$

or $\lambda_1 = -\text{cov}(M, \hat{r}) + \lambda_2 E\hat{r}$.

Finally, observe that if there is a zero-beta portfolio $\hat{\theta}_z$ with $q \cdot \hat{\theta}_z = 1$, then the above linear relationship must take the form (**) since any linear relation is determined by two points, namely $(\text{var } M, EM)$, $(0, E\hat{r})$.

Q.E.D.

II. GENERALIZATIONS TO MULTIPLE COMMODITIES AND NON-QUADRATIC UTILITIES

In this section we take a broader perspective, allowing for multiple consumer commodities and therefore trade on spot markets and assets, and also for nonquadratic utilities. The essential point that emerges here is that when asset markets are incomplete, there is typically a lack of coordination between the desires of consumers and the desires of shareholders. It is thus not correct to assert that under conditions of information symmetry, perfect competition, etc., that asset markets are efficient, or nearly so.

We argue our case in two ways. We quote an argument in Geanakoplos-Polemarchakis (1986), that shows that generically, when the asset market is incomplete, the equilibrium trade in assets is inefficient in the strong sense that all traders could be made better off if they made different asset trades, even if the subsequent spot markets were allowed to clear at competitive prices, on account of the effect on relative commodity prices when the spot markets subsequently clear. When there is only one physical commodity

in each state, the desires of shareholders and consumers are necessarily the same, and indeed it is easy to see that no reallocation of existing assets can Pareto dominate the competitive allocation.

Second, we consider the case where spot market relative prices are unaffected by asset trades (the case with identical income effects on consumption). This is the case presumably that is represented by the parable of the single commodity, assumed by the models of Sections 2 and 3. Under this knife-edge hypothesis on income effects, asset reallocations alone cannot Pareto improve on interior equilibria. If, in addition, attitudes toward risk are quadratic, one would expect full Pareto optimality to obtain, as in Theorem 1. But this turns out to be false, unless the assumption of a riskless asset is augmented by a far stronger hypothesis. This abstraction further clarifies the role of the riskless asset in the one-commodity world.

We shall begin by describing an equilibrium with multiple commodities and proving the analogue to Theorem 1 for Pareto optimality in the general case.

From now on the commodity consumption space $R_+^{(S+1)(L+1)}$ consists of $L+1$ physical commodities, $(0, 1, \dots, L)$, in each of $(S+1)$ states of nature. Consumers $(W^h, e^h, \bar{\theta}^h)$ are characterized by utility functions W^h that are smooth and strictly concave, and by endowments $e^h \in R_{++}^{(S+1)(L+1)}$, and by their holdings $\bar{\theta}^h$ of initial assets. It is often the case that a utility function W^h can be extended to a convex set X^h containing $R_+^{(S+1)(L+1)}$, while retaining smoothness and strict concavity; the quad-

ratic utility is an obvious example.⁴ This is information which can be useful, so we shall take $W^h : X^h \rightarrow \mathbb{R}$, where X^h is any closed, convex set, that is bounded from below, containing $\mathbb{R}_+^{(S+1)(L+1)}$. Of course we shall always restrict consumption to $\mathbb{R}_+^{(S+1)(L+1)}$.

Assets \bar{r}_a , $a \in A : \{1, \dots, A\}$ now yield $(L+1)$ dimensional vectors of commodities in each state, so \bar{r}_a is given by an $(S+1) \times (L+1)$ matrix. The totality of assets is represented by the collection \bar{R} . The notation $(\bar{R}\theta)_s$, for $\theta \in \mathbb{R}^A$, will mean the vector commodity payoff that occurs in state s , given the portfolio $\theta = (\theta_1, \dots, \theta_A)$. We assume that asset 1 is not identically 0, and that it pays off a nonnegative amount in every commodity, and a positive amount of some commodity in each state. A consequence of this mild assumption is that in equilibrium the price of asset 1 can be taken to be strictly positive.

Asset prices are denoted, as before, by $q \in \mathbb{R}^A$. Commodity prices are $p \in \mathbb{R}_{++}^{(S+1)(L+1)}$. The product $p_s (\bar{R}\theta)_s$ means the payoff in units of account in state s that is obtained by selling the commodity vector payoff $(\bar{R}\theta)_s$ at prices $p_s = (p_{s0}, \dots, p_{sL})$.

As before, we let $\bar{\theta} = \sum_{h \in H} \bar{\theta}^h$, and $e = \sum_{h \in H} e^h$, and $M_s = e_s + (R\bar{\theta})_s$.

An interior equilibrium for the above economy is a tuple $(q, p, (\theta^h, x^h); h \in H)$ satisfying $(\theta^h, x^h) \in \mathbb{R}^A \times \mathbb{R}_{++}^{(S+1)(L+1)}$ for all $h \in H$, $\sum_{h \in H} \theta^h = \bar{\theta}$, $\sum_{h \in H} x^h = \sum_{h \in H} e^h + \bar{R} \sum_{h \in H} \bar{\theta}^h = M$, and

⁴We also require that $DW^h(x^h) \gg 0$ for all $x^h \in \mathbb{R}_{++}^{(S+1)(L+1)}$ that satisfy $x \leq M$, where M is the aggregate social endowment. It is not necessary to the theory that W^h be monotonic outside the feasible set; indeed, the quadratic utility is not monotonic for large x^h .

$(\theta^h, x^h) \in D^h(q, p) = \text{ArgMax}\{W^h(x) \mid (\theta, x) \in \mathbb{R}^A \times X^h, q \cdot \theta - q \cdot \bar{\theta}^h \text{ and } p_s(x_s - e_s^h) \leq p_s(\bar{R}\theta)_s \text{ for all } s \in S\}$, for all $h \in H$.⁵

An extremely useful construction in the following is the asset constrained demand: $D^h(p \mid \theta^h) = \text{Arg Max}\{W^h(x) \mid x \in X^h, p_s \cdot (x_s - e_s^h) \leq p_s(\bar{R}\theta^h)_s \text{ for all } s \in S\}$. The choice $x \in D^h(p \mid \theta^h)$ iff agent h would choose x if he was forced to hold the portfolio θ^h , but could trade freely on the spot commodity markets at prices $p \gg 0$. An asset constrained interior equilibrium is a tuple $(p, (\theta^h, x^h); h \in H)$ satisfying all the requirements of an equilibrium except that the optimality condition $(\theta^h, x^h) \in D^h(q, p)$ is replaced by the weaker requirement $x^h \in D^h(p \mid \theta^h)$. We shall shortly say more about these.

Finally, let $D^h(p) = \bigcup_{\theta^h} D^h(p \mid \theta^h)$ be the set of all conceivable commodity bundles h might demand, if he did not have to worry about his budget constraint for assets, but if he could only spread his limitless wealth across states of nature through assets. Consider for example the case where there is only one commodity per state of nature, and where X^h is all of \mathbb{R}^{S+1} . Then the prices p are irrelevant, since there is no spot trade. If the asset span, $\text{sp}[R]$, has full rank $S+1$, so that the asset market is complete, then $D^h(p)$ is all of \mathbb{R}^{S+1} . If the number of assets $A < S+1$, then $D^h(p)$ is an A -dimensional affine subspace of \mathbb{R}^{S+1} (a subspace translated by the endowment e^h).

⁵Notice that we have allowed the choice set for commodities in calculating demand $D^h(q, p)$ to include all of X^h . Since we are restricting attention to demands x^h interior to $\mathbb{R}^{(S+1)(L+1)}$, this makes no difference on account of the concavity of W^h . We have already mentioned the great importance for Pareto optimality of considering only interior equilibria.

1. A SUFFICIENT CONDITION FOR PARETO OPTIMALITY WITH MULTIPLE COMMODITIES,
NON-QUADRATIC UTILITIES, AND INCOMPLETE MARKETS

For general $X^h \subset \mathbb{R}^{(S+1)(L+1)}$, let us give:

Definition: For $x \in X^h$, let $\mu^h(x) = DW^h(x)$ be the marginal utility of agent h .⁶ Given commodity prices $p \gg 0$, we say that the vector $\mu \gg 0$ is ray p-reachable for agent h iff there is some $\lambda > 0$ and $x \in (D^h(p) \cap \text{interior } X^h)$ with $\mu^h(x) = \lambda\mu$.

Definition: We say that the map μ^h has the ray property at prices p for the economy $(\bar{R}, (W^h, X^h, e^h, \bar{\theta}^h); h \in H)$ iff whenever the image $\mu^h(D^h(p))$ of $D^h(p)$ under μ^h contains $\mu \gg 0$, $\{\lambda > 0 \mid \exists x \in D^h(p), \mu^h(x) = \lambda\mu\}$ is an interval and $\inf\{W^h(x) \mid \mu^h(x) = \lambda\mu, \lambda > 0, x \in D^h(p)\} < W^h(e^h)$ and $\sup\{W^h(x) \mid \mu^h(x) = \lambda\mu, \lambda > 0, x \in D^h(p)\} \geq W^h(M)$.

Definition (co-reachable hypothesis): We say that the asset structure \bar{R} allows for co-reachable agents at prices p iff (1) every map μ^h has the ray property at p , and (2) a vector μ is ray p-reachable by any agent h iff μ is ray p-reachable by all agents $h' \in H$.

The co-reachable hypothesis means that the images of the $D^h(p)$ under the maps μ^h contain the same rays for all agents $h \in H$, and furthermore that these ray images are large intervals. No matter what the utilities W^h , if $\mu^h(X^h)$ has range including all of $\mathbb{R}_{++}^{(S+1)(L+1)}$ (as may be assured by the proper boundary conditions), and if the asset markets are complete, then the co-reachable hypothesis is trivially satisfied. There are

⁶This definition is slightly different from the one given in Section 2, since here it includes the probability weights π_s if they are used in the definition of W^h .

cases where the hypothesis is satisfied even when the asset market is incomplete. For example, under the one good, quadratic von Neumann-Morgenstern hypothesis of Section 2, $\mu^h(x) = \hat{1} - \alpha_h x$. If X^h is taken to be \mathbb{R}^{S+1} and all the $e^h \in \text{sp}[R]$, then the sets $D^h(p)$ are identical vector spaces, and the sets $\mu^h(D^h(p))$ are each identical translations of $D^h(p)$ by the vector $\hat{1}$. If $\hat{1} \in \text{sp}[R] - D^h(p)$, then $\mu^h(D^h(p))$ is a vector space, hence it includes all multiples $\lambda\mu$ of any vector μ that it contains. In particular, suppose that $0 << \mu = \mu^h(x) = \hat{1} - \alpha_h x$. Then letting $x_\lambda = \frac{1}{\alpha_h}(\hat{1} - \lambda\mu) \in D^h(p) - \text{sp}[R]$, we find that for $\lambda \rightarrow 0$, $W^h(x_\lambda) \rightarrow W^h(\frac{1}{\alpha_h} \hat{1})$, which is the bliss point or maximum achievable utility, and for $\lambda \rightarrow \infty$, $W^h(x_\lambda) \rightarrow -\infty < W^h(e^h)$. Hence the ray property holds, and co-reachability is confirmed.

Note that if for all $h \in H$ we choose $X^h = \{x \in \mathbb{R}^{S+1} \mid x_i \geq -A\}$, for $A > \max_h 1/\alpha_h$, co-reachability could similarly be confirmed, and at the same time we would have X^h bounded from below. The images $\mu^h(D^h(p))$ would be ray identical (although perhaps not pointwise identical). If $A > 1/\alpha_h$, then it can easily be checked that for any $\mu \gg 0$, there is $x \in X^h$, with $\mu^h(x) = \lambda\mu$, for some $\lambda > 0$, and $W^h(x) < W^h(0) < W^h(e^h)$. The significance of the co-reachability hypothesis, when X^h is bounded from below, is illustrated by the following generalization of Theorem 1 to many commodities and possibly nonquadratic utilities.

Theorem 4: Let $(q, p, (\theta^h, x^h); h \in H)$ be an interior equilibrium for the economy $(\bar{R}, (W^h, X^h, e^h, \bar{\theta}^h); h \in H)$. Suppose that the co-reachability hypothesis holds at p . Then the allocation $(x^h; h \in H)$ is fully Pareto optimal.

Proof: Let $\mu = \mu^{h'}(x^{h'})$ for some $h' \in H$. We must show that $\mu^h(x^h) = \lambda^h \mu$ for all $h \in H$. For this it suffices to find $(\hat{\theta}^h, \hat{x}^h)$ with $\hat{x}^h \in D^h(p|\hat{\theta}^h)$ and $\mu^h(\hat{x}^h) = \lambda \mu$, for some $\lambda > 0$, with $W^h(\hat{x}^h) = W^h(x^h)$, since it follows at once that then $\hat{x}^h = x^h$. To see this, note that if h had asset income $q \cdot \hat{\theta}^h$, his demand at prices (q, p) would be $(\hat{\theta}^h, \hat{x}^h)$, since there the first order conditions are satisfied. But $q \cdot \hat{\theta}^h$ is either greater, less, or equal to $q \cdot \bar{\theta}^h = q \cdot \theta^h$. If less, then from the fact that W^h is monotonic near x^h , and from the availability of the nonnegative asset \bar{r}_1 , it follows that it would be possible to make a better choice than (θ^h, x^h) at prices (q, p) , and asset income $q \cdot \bar{\theta}^h$.⁷ A similar argument shows that $q \cdot \hat{\theta}^h > q \cdot \bar{\theta}^h$ would contradict the optimality of $(\hat{\theta}^h, \hat{x}^h)$. Hence the conclusion that $q \cdot \bar{\theta}^h = q \cdot \hat{\theta}^h$, and $x^h = \hat{x}^h$, follows from the strict concavity of W^h .

From the ray property there exists (θ_1^h, x_1^h) and (θ_2^h, x_2^h) with $W^h(x_1^h) < W^h(e^h) < W^h(x^h) < W^h(x_2^h)$ and $\mu^h(x_1^h) = \lambda_1 \mu$ and $\mu^h(x_2^h) = \lambda_2 \mu$. Note that $\lambda_2 < \lambda_1$, for by concavity of W^h , we must have that $(\lambda_1 \mu) \cdot (x_2^h - x_1^h) > 0$. From the diminishing marginal returns that are a

⁷The choice $(\alpha \theta^h + (1-\alpha) \hat{\theta}^h, \alpha x^h + (1-\alpha) \hat{x}^h)$ is strictly preferred to (θ^h, x^h) , for $0 < \alpha < 1$, and satisfies the state by state budget sets exactly, while leaving asset income to spare. For α near 1, W^h is monotonic at $\alpha x^h + (1-\alpha) \hat{x}^h$, and so spending the extra asset income on \bar{r}_1 makes W^h higher still.

consequence of strict concavity, we must have $0 < (\lambda_2 \mu) \cdot (x_2^h - x_1^h) < (\lambda_1 \mu) \cdot (x_2^h - x_1^h)$, hence $\lambda_2 < \lambda_1$.

Again from the ray property, to each $\lambda_2 \leq \hat{\lambda} \leq \lambda_1$, there is $(\hat{\theta}, \hat{x})$ with $\mu^h(\hat{x}) = \hat{\lambda} \mu$, and $\hat{x} \in D^h(p|\hat{\theta})$. From the last paragraph, we know that all such \hat{x} satisfy $W^h(\hat{x}) \leq W^h(x_2^h)$, and hence $q \cdot \hat{\theta} \leq q \cdot \theta_2^h$. But from the fact that $p \gg 0$, and (q, p) is an equilibrium, we know that q does not permit arbitrage, i.e. there is no θ with $\bar{R}\theta > 0$ and $q \cdot \theta \leq 0$. Since X^h is closed and bounded from below, the set of x that satisfy the budget constraints with asset income less or equal to $q \cdot \theta_2$, at asset prices q , is compact. A standard argument now shows that there must be some $(\hat{\theta}, \hat{x})$ as above with $W^h(\hat{x}) = W^h(x^h)$.

Note first that μ^h is a one-to-one function. Take $y \neq z$; by strictly diminishing returns, $\mu(y) \cdot (y-z) < \mu(z) \cdot (y-z)$, so $\mu(y) \neq \mu(z)$. Hence we can write $\hat{x}(\hat{\lambda})$, for $\lambda_2 \leq \hat{\lambda} \leq \lambda_1$. Moreover, the function \hat{x} is continuous in $\hat{\lambda}$. If $\hat{\lambda}_t \rightarrow \lambda^*$, then since the $\hat{x}(\hat{\lambda}_t)$ lie in a compact set, by passing to subsequences we have $\hat{x}(\hat{\lambda}_t) \rightarrow x^*$ and $\mu^h(x^*) = \lambda^* \mu$. So the function $\varphi : \{\hat{\lambda} | \lambda_2 \leq \hat{\lambda} \leq \lambda_1\} \rightarrow \mathbb{R}$ given by $\varphi(\hat{\lambda}) = W^h(\hat{x}(\hat{\lambda}))$ is continuous. The continuous image of a connected set must be connected, so the image of φ must contain $W^h(x^h)$. Q.E.D.

As a first consequence of Theorem 4, let us consider the models of Section 2, but now extended to allow for a class of nonquadratic utilities:

Corollary: Under assumptions (1), (2), (4), (5), (6), let

$W^h(x) = \sum_{s \in S} \pi_s u^h(x_s)$, where all the u^h belong to one of the following exclusive classes: (a) $u^h(z) = \frac{1}{b}(1 + \alpha_h z)^b$, $\alpha_h > 0$, $b < 1$, $b \neq 0$, (b) $u^h(z) = \log(1 + \alpha_h z)$, $\alpha_h > 0$, (c) $u^h(z) = -e^{-(1+\alpha_h z)}$, $\alpha_h > 0$.

Any interior equilibrium for such an economy must be fully Pareto optimal.

Proof: We need only apply Theorem 4, taking advantage of the fact that each u^h can be extended to $X^h = \{x \in R^{S+1} | x_s \geq -1/\alpha_h + \epsilon\}$, where $0 < \epsilon$. Consider case (a), where $\mu_s^h(x) = \alpha_h(1 + \alpha_h x_s)^{b-1} \pi_s$. Note that for any $x \in X^h$, $\mu^h = \mu^h(x)$, $\mu_s^h = \alpha_h \pi_s m_s^{b-1}$, where $m \in \text{sp}[R]$. So for any $h' \in H$, take $x_\lambda^{h'} = (1/\alpha_{h'})[\lambda m - 1] \in \text{sp}[R]$. For λ big enough, $x_\lambda^{h'} \in X^{h'}$ and $\mu^{h'}(x_\lambda^{h'}) = (\alpha_{h'}/\alpha_h) \lambda^{b-1} \mu^h$. As $\lambda \rightarrow \infty$, $W^{h'}(x_\lambda^{h'}) \rightarrow \infty$. As $\lambda \rightarrow 0$, eventually $-(1/\alpha_{h'})1 \ll x_\lambda^{h'} \ll 0$, so $W^{h'}(x_\lambda^{h'}) < W^{h'}(e^{h'})$, and the co-reachability hypothesis is confirmed. A similar argument works for cases (b) and (c). Q.E.D.

2. CONSTRAINED OPTIMALITY

The most important application for us of Theorem 4 is to economies with many goods, but where the relative prices in each state of nature can be taken to be independent of the distribution of income. Consider now utilities of the form: $W^h(x) = \sum_{s \in S} \pi_s^h u^h(v_s^h(x_{s0}, x_{s1}, \dots, x_{sL}))$, where $v_s^h : R_+^{L+1} \rightarrow R$ represents "in state" utility, and $u^h : R \rightarrow R$ represents the attitude toward risk of agent h . We shall suppose that the concave and strictly quasi-concave, monotonic v_s^h give rise to the same "income effects." To be precise, let $x_s^h(p_s, I) = \text{Arg Max}_{x_s} (v_s^h(x_s) | x_s \in R_+^{L+1}, p_s x_s \leq I)$ be the state s demand of agent h .

Assumption 1.1: $x_s^h(p_s, I)$ is a differentiable function at all (p_s, I) for which $x_s^h(p_s, I) \gg 0$. Furthermore, if at (p_s, I^h) , $x_s^h(p_s, I^h) \gg 0$ for all $h \in H$, then $[\partial x_s^h(p_s, I^h)/\partial I] = \bar{z}_s(p_s) \geq 0$, independent of I^h and h .

Two examples of utilities satisfying 1.1 are (1) all individuals have

the same homogeneous utility $v_s = v_s^h$ or (2) $v_s^h(x_{s0}, x_{s1}, \dots, x_{sL}) = x_{s0} + \bar{v}_s^h(x_{s1}, \dots, x_{sL})$ for all $h \in H$. In the first case, $\bar{z}_s(p_s) = x_s^h(p_s, I)/I$, which is independent of I and h , but not of p_s . In the second case, $\bar{z}_s(p_s) = (1, 0, \dots, 0)$ (if $p_{s0} = 1$), which is independent of h , I , and p_s . It is well-known, after Gorman, that all distributions of income that allow for strictly positive spot market clearing prices p_s give rise to the same price vector $p_s = \bar{p}_s$ (if $\bar{p}_{s0} = 1$).

For our two examples, these spot market clearing prices \bar{p}_s are easy to calculate. In (1), $\bar{p}_s = Dv(M_s)/[\partial v(M_s)/\partial x_{s0}]$. In (2) \bar{p}_s is the marginal utility vector for each consumer at (any) allocation which maximizes the sum of utilities, given the aggregate endowment M_s . Notice that in both cases \bar{p}_s will depend on s , if the aggregate endowment is not constant.

The consequences of assumption 1.1 can be seen immediately for constrained optimality.

Theorem 5: Let $(\bar{R}, (W^h = (u^h, v^h, \pi^h), e^h, \bar{\theta}^h))$ be an economy in which the $(v^h, e^h, \bar{\theta}^h)$ satisfy assumption 1.1, and suppose that $(q, p, (\theta^h, x^h); h \in H)$ is an interior equilibrium. Then the allocation $(x^h; h \in H)$ is constrained Pareto optimal.

Proof: Since all interior asset constrained equilibria $(\hat{p}, (\hat{\theta}^h, \hat{x}^h); h \in H)$ satisfy $p = \hat{p}$, it follows that if $W^h(\hat{x}^h) > W^h(x^h)$, then $q \cdot \hat{\theta}^h > q \cdot \bar{\theta}^h$. But then $q \cdot \sum_{h \in H} \hat{\theta}^h > q \cdot \sum_{h \in H} \bar{\theta}^h$, contradicting $\sum_{h \in H} \hat{\theta}^h = \sum_{h \in H} \bar{\theta}^h$.
Q.E.D.

3. CAPM WITH MULTIPLE COMMODITIES: THE RISKLESS ASSET REVISITED

Assumption 1.1 generalizes the requirement that there is only one physical commodity per state of nature, at least for the purposes of obtaining constrained optimality. A natural conjecture is that if in addition we carry over assumptions 2-6, including the hypothesis that the u^h all belong to one of the same utility classes (quadratic, or exponential, etc.) given by the corollary to the last theorem, then we might be able to prove full Pareto optimality for interior equilibria. A moment's reflection in the one commodity world, where 1.1 and 1.2 hold trivially, shows however that even when all the "state utilities" v^h are identical and homothetic, full Pareto optimality cannot be expected. It is clearly necessary that the "in state" marginal utility of income is constant; otherwise it has implications for risk aversion, and we know that only special risk averse behavior can give rise to full optimality. Note, incidentally, that both of our examples (1) and (2) of acceptable v^h do give rise to constant "in state" marginal utility of income. But there is a more interesting problem, having to do with the meaning of the riskless asset in a many commodity world.

When there is only one commodity, in every state, the riskless asset should provide for delivery of one unit of the good in each state, or 1 unit of account in every state, if we take the price of the good to be one. But when there are multiple commodities, with different relative prices across the states of nature, then what should the riskless asset provide? Should it identify some significant commodity bundle, and pay off in constant purchasing power relative to this bundle? The right answer is that it should distribute purchasing power across the various states so that the increase

in "within state" utility is the same across all states.

Definition: Let $W^h(x) = \sum_{s \in S} \pi_s^h u^h(v_s^h(x_s))$, and suppose that at equilibrium, consumption is $\bar{x}^h = (\bar{x}_s^h; s \in S)$. A riskless asset r for consumer h satisfies $Dv_s^h(\bar{x}_s^h) \cdot r_s = 1$ for each $s \in S$.

Assumption 6.1: The collection of assets includes a riskless asset for at least one agent $h \in H$.

Notice that the riskless asset for consumer h cannot be defined without knowledge of the spot market clearing consumptions \bar{x}^h (or equivalently, without knowing the spot market clearing prices \bar{p}), unless there is only one commodity per state. On the other hand, observe that under the hypothesis that all the "in state" utilities v^h are identical and homogeneous, the same asset r is a riskless asset for every consumer. Furthermore, under this homogeneity hypothesis, all initial endowment allocations $(\bar{R}, (e^h, \bar{\theta}^h); h \in H)$ that give rise to the same aggregate endowment M also give rise to the same riskless asset, at every interior equilibrium. Thus the following theorem is not without content.

Theorem 6: Let $(\bar{R}, (W^h = (u^h, v^h, \pi^h), e^h, \bar{\theta}^h); h \in H)$ be an economy in which all $v^h = v$ are identical and homogeneous of degree one.

Furthermore, suppose that assumptions 2 (the span of \bar{R} includes e^h for all $h \in H$), 3.1 (that all u^h are drawn from one of the same special classes designated earlier), 4 (the priors π^h are common), 5 (monotonicity), and 6.1 (there is a riskless asset for at least one individual h). Then any interior equilibrium is fully Pareto optimal.

The proof of the above follows immediately from Theorem 4. Let us repeat that the above theorem shows that the capital asset pricing model can be extended to a model with a genuine multiplicity of commodities, if all consumers have the same homogeneous preferences of degree one within each state. To achieve full Pareto optimality, only one asset is required, beyond the initial endowments of all the consumers. By choosing the spot market price normalizations appropriately, one can of course arrange it so that this asset pays off commodities with value equal to one in every state. We have therefore called this the riskless asset, by analogy to the one commodity model of the last section (to which it clearly reduces when there is indeed only one physical commodity). On the other hand, when there are many commodities the correct price normalization cannot be predicted in advance, i.e. it is impossible to know what asset will serve as the riskless asset without knowing the "in state" preferences and the aggregate endowment.

Under the conditions of Theorem 6, let us normalize prices $\bar{p}_s = DV_s(M_s)$. Let the money payoffs of each asset $r \in R^{(S+1)(L+1)}$ be $\bar{r} \in R^{S+1}$ defined by $\bar{r}_s = \bar{p}_s \cdot r_s$, $s = 0, 1, \dots, S$. If in addition the u^h are quadratic for all h , then one can easily derive the exact analogues of Theorems 2, 3, 2' and 3' in this multiple commodity world.

4. SECURITY MARKET LINE

A central relationship in the capital asset pricing model is the security market line, which suggests that the return to an asset is linearly and negatively related to the covariance of the asset return and the market return. The fact that it is the covariance, and not the variance, which is important to the pricing of assets is one of the enduring lessons of CAPM.

But this is a lesson which holds much more generally than for the CAPM model. We shall derive a security market line for any GEI model, provided that we are permitted to substitute an arbitrary asset $\mu \in \mathbb{R}^{(S+1)(L+1)}$ for the market asset. In particular, after making this substitution we can always find a "riskless" asset without making any assumptions about the return matrix \bar{R} or the preferences W^h . Let $(\bar{R}, (W^h, X^h, e^h, \bar{\theta}^h); h \in H)$ be any multi-commodity general equilibrium model with incomplete markets. Let $(q, p, (\theta^h, x^h); h \in H)$ be an interior equilibrium. For each $r \in \mathbb{R}^{(S+1)(L+1)}$ let $\bar{r} \in \mathbb{R}^{S+1}$ be defined by $\bar{r}_s = p_s \cdot r_s$, $s = 0, 1, \dots, S$.

Theorem 7. Let $(\bar{R}, (W^h, X^h, e^h, \bar{\theta}^h); h \in H)$ be a multi-commodity GEI. Let $(q, p, (\theta^h, x^h); h \in H)$ be an interior equilibrium. Let $\pi \in \mathbb{R}_{++}^{S+1}$ be arbitrary, and satisfy $\sum_{s \in S} \pi_s = 1$. Then there is always a renormalization of prices p that maintains equilibrium, such that with respect to these prices there is $z \in \text{sp}[\bar{R}]$ with $\bar{z}_s = K$ for all $s \in S$, and there is $\bar{\mu} \in \mathbb{R}^{(S+1)(L+1)}$ such that for all $r \in \text{sp}[\bar{R}]$ with $q(r) = 1$,

$$[E_{\pi} \bar{r} - E_{\pi} \bar{z}] = \frac{\text{Cov}_{\pi}(\bar{\mu}, \bar{r})}{\text{Var}_{\pi}(\bar{\mu})} [E_{\pi} \bar{\mu} - E_{\pi} \bar{z}].$$

Remark. Note that the security market line holds only for marketed securities (with price equal 1).

Proof. Choose any $h \in H$. From the separating hyperplane theorem there is a vector $\hat{\mu}^h \in \mathbb{R}_{++}^{(S+1)(L+1)}$ such that for all $y \in \mathbb{R}_+^{(S+1)(L+1)}$, if $W^h(y) \geq W^h(x^h)$, then $\hat{\mu}^h \cdot y \geq \hat{\mu}^h \cdot x^h$. Clearly $\hat{\mu}_s^h = \lambda_s^h p_s$, for some $\lambda_s^h > 0$, for each $s = 0, 1, \dots, S$. Moreover, there is some $\lambda > 0$ such that for each $r \in \text{sp}[\bar{R}]$, $\lambda q(r) = \hat{\mu}^h \cdot r$. Let $\mu \in \mathbb{R}_{++}^{(S+1)(L+1)}$ be defined by $\mu_s = (1/\lambda \pi_s) \hat{\mu}_s^h$. Then for all $r \in \text{sp}[\bar{R}]$, $q(r) = \sum_{s \in S} \pi_s \bar{\mu}_s \bar{r}_s = \bar{\mu} \square_{\pi} \bar{r}$.

Let us observe that without loss of generality we can always renormalize p so that there is $r \in \text{sp}[\bar{R}]$ such that $p_s \cdot r_s = k$ for all $s = 0, 1, \dots, S$, and $\bar{\mu}$ and \bar{r} are linearly independent. Asset r_1 satisfies $p_s \cdot r_s(1) > 0$ for all $s = 0, 1, \dots, S$ (recall the definition of GEI in the beginning of Section II) so that r_1 will do unless \bar{r}_1 and $\bar{\mu}$ are colinear. If all assets $r \in \text{sp}[\bar{R}]$ with $q(r) = 1$ yield money payoffs \bar{r} colinear with $\bar{\mu}$, the theorem is vacuously true. If there is some $r \in \text{sp}[\bar{R}]$, with $q(r) > 0$, and \bar{r} not colinear with \bar{r}_1 and $\bar{\mu}$, then let $z = \alpha r_1 + (1-\alpha)r$, with $0 < \alpha < 1$. Then $q(z) > 0$, and if α is close enough to 1, $p_s \cdot z_s > 0$ for all $s = 0, 1, \dots, S$. Moreover, \bar{z} is linearly independent of $\bar{\mu}$ and \bar{r}_1 . So we can renormalize \bar{z} to be the riskless asset. Thus without loss of generality suppose we have $z \in \text{sp}[\bar{R}]$, with $\bar{z}_s = k$ for all s , and $\bar{\mu}$ and \bar{z} linearly independent.

If $q(r) = 1$ for $r \in \text{sp}[\bar{R}]$, then $1 = \bar{\mu} \square_{\pi} \bar{r} = \text{Cov}_{\pi}(\bar{\mu}, \bar{r}) + E_{\pi} \bar{\mu} E_{\pi} \bar{r}$. Hence there is a (negative) linear relation between $\text{Cov}_{\pi}(\bar{\mu}, \bar{r})$ and $E_{\pi} \bar{r}$. As long as $\text{Var}_{\pi} \bar{\mu} > 0$, the claimed equation must describe this relationship. Q.E.D.

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