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THE TRANSACTIONS COST OF MONEY

(A STRATEGIC MARKET GAME ANALYSIS)

by

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(with an example by D. P. Tsomocos)

1. INTRODUCTION

The payments system of a modern economy is a peculiar mix of technological and institutional factors. Trade takes time and involves some form of money or credit. Going to the bank or arranging credits is expensive.

Baumol (1952) and Tobin (1956) address the costs of transactions. However both the Baumol and the Tobin analysis was carried out in a partial equilibrium context. Here we address the task of considering the costs of banking in a closed strategic market game.

In this section a heuristic sketch of the model and results are given. In Sections 2 and 3 the formal model is specified and the proof of the existence of a noncooperative equilibrium is given. The Appendix presents an example calculated by D. P. Tsomocos.

Suppose that there are n different types of trader with a continuum of each type. There are m goods and a fiat money. There are transactions costs which are measured in the consumption of real resources. The transactions costs of physical goods are assumed to be in proportion to the size of the transaction, however the resources used in a single trip to the bank are regarded as independent of the amount of money transacted.

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An individual i has the initial endowment of $(a_1^1, a_2^i, \ldots, a_m^i)$. He submits his bids and offers at the start of the period. The payments pattern consists of when during the time interval he must make his payments for his purchases and when he is paid for his sales.

We assume that by the time T his expenditures amount to:

$$F_i(T) = \int_{t=0}^{T} f_i(t)dt$$

percent of the amount he bids, and his receipts are

$$G_i(T) = \int_{t=0}^{T} g_i(t)dt \ , \quad \text{of his sales,}$$

where as both are given in percentages we have

$$\int_{t=0}^{T} f_i(t)dt = \int_{t=0}^{T} g_t(t)dt = 1 \ .$$

At any point t an individual i's cash flow requirements are:

$$c_i(t) = S_i g_i(t) - B_i f_i(t)$$

where S_i is his total sales and B_i his total bid.

On the assumption that in equilibrium final income will equal expenditures (including transactions costs) Figure 1 shows the cash flow needs of i over the period. The vertical shading indicates shortage and the horizontal surplus. At any point in time t, $Bf(t)$ and $Sg(t)$ represent the total income and expenditures of all individuals.

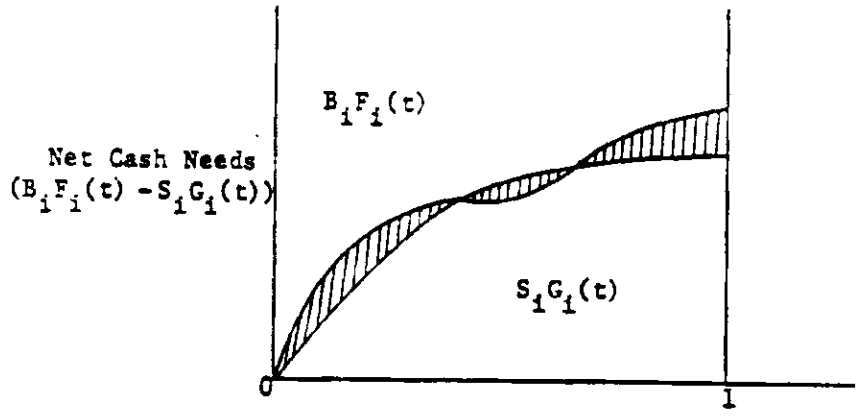

FIGURE 1

At any point of time the overall income of individuals can be less than or equal to their expenditure. The discrepancy is accounted for by money in transit or held by the bank but not earning interest for its owner. At the end of trade we expect that overall banks will balance in equilibrium.

We assume that there is a single outside bank which will lend any amount of fiat at the rate of interest $\rho \geq 0$. It stands ready to lend or to accept deposits at the rate ρ at any time.

There is a bankruptcy penalty μ leveled against any individual who has a negative cash balance at the end. The utility function for an individual can be written in the form

$$\varphi_i(x_1^i, x_2^i, \ldots, x_m^i) + \mu_i \min[0, \text{cash balance}] .$$

The introduction of a sufficiently harsh penalty serves to bound the borrowing of the traders; it eliminates strategic bankruptcy. If the exogenous rate of interest $\rho > 0$, the bank will earn a profit hence in order to balance the books we must consider that bank is required to spend its profits buying goods. We treat the actions of the bank as though it were a

strategic dummy hence it is required to announce in advance how it will allocate its income to various markets.

Even if all transactions costs did not consume real resources the presence of a positive rate of interest makes the system nonconservative with the bank making a profit and being able to remove real resources. We may regard this as the link in this model to a growth model where the real resources not consumed may be considered as the capital stock of the economy.

2. DESCRIPTION OF THE MODEL

The specific model is now described formally. We somewhat simplify the cash flow conditions shown in Figure 1.

There are n types of traders, each type consisting of a continuum of individuals.[1] At the beginning, trader i has an endowment $a^i = (a^i_1, \ldots, a^i_m)$ $(a^i_j \geq 0)$. Trader i can borrow an amount u^i of money from an outside bank for trading, the interest rate ρ for the whole period is given exogenously. Every one must pay linearly w.r.t. time, but will not receive payment until the end. A trader can go to the bank as many times as he wants, but there is a fixed cost $(\alpha^i_{m+1,1}, \ldots, \alpha^i_{m+1,m})$ for each time he goes to the bank. Similarly, to sell one unit of good j, he must spend α^i_{jk} unit of good k $(k \neq j)$. That is, there is a transaction matrix

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[1] For convenience we assume that the set of each type of traders has Lebesgue measure 1.

$$\begin{bmatrix} -1 & \cdots & \alpha^i_{1m} & 0 \\ \alpha^i_{21} \quad 0 & \cdots & \alpha^i_{2m} & 0 \\ \cdots\cdots\cdots\cdots & & & 0 \\ \alpha^i_{m1} & \cdots & \alpha^i_{m,m-1} - 1 & 0 \\ \alpha^i_{m+1,1} & \cdots & \alpha^i_{m+1,m} & -1 \end{bmatrix}$$

At the end of the trade, the bank buys commodities according to a set of prices given in advance: $(\bar{p}_1, \ldots, \bar{p}_m)$, so traders can return the debt (with interest) by selling suitable amounts of goods to the bank.

A strategy of a trader i is

$$(1) \qquad s^i = (u^i, k^i; b^i_1, \ldots, b^i_m, q^i_1, \ldots, q^i_m; r^i_1, \ldots, r^i_m))$$

where u^i = the total amount of money he borrows,

k^i = number of times he goes to the bank,

b^i_j = bid for good j by i,

q^i_j = offer of good j by i,

r^i_j = percentage of good, sold to the bank by i.

The quantities appear in s^i subject to the following constraints:

$$(2) \qquad \begin{cases} 0 \le u^i \le U \text{ [2]} \\ 0 \le k^i \le K, \text{ [2] } k^i \text{ integer} \\ q^i_j + \sum_{k \ne j} q^i_k \alpha^i_{kj} + k^i \alpha^i_{m+1j} \le a^i_j, \quad q^i_j \ge 0 \\ \sum_j b^i_j \le u^i, \quad b^i_j \ge 0 \\ 0 \le r^i_j \le 1 \end{cases}$$

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[2] Because of the positive transaction costs, k^i has a natural upper bound; on the other hand, $\rho > 0$ and $\bar{p}_j$ fixed imply u^i must have an upper bound.

Let $\Sigma^i = \{s^i \text{ in (1) subject to (2)}\}$. Then Σ^i is the strategy set for i. Obviously Σ^i is compact. But Σ^i is not convex due to k^i assuming discrete values.

The market prices are given by

$$(3) \qquad p_j = \begin{cases} \dfrac{\int b_j^i}{\int q_j^i} & \text{provided } \int b_j^i > 0 \text{ and } \int q_j^i > 0 \\ 0 & \text{otherwise} \end{cases}$$

After trading, the holding of goods for i is given by

$$(4) \qquad x_j^i = \begin{cases} a_j^i - q_j^i - \sum\limits_{k \neq j} q_k^i \alpha_{kj}^i - k^i \alpha_{m+1j}^i + b_j^i/p_j & (p_j \neq 0) \\ a_j^i - \sum\limits_{k \neq j} q_k^i \alpha_{kj}^i - k^i \alpha_{m+1j}^i & (p_j = 0) \end{cases}$$

The final holding of goods and money are given by

$$(5) \qquad z_j^i = (1 - \bar{r}_j) x_j^i - \sum_{k \neq j} \alpha_{kj}^i \bar{r}_k x_k^i$$

$$(6) \qquad z_{m+1}^i = \frac{-\rho(1 + k^i)}{2k^i} u^i - \sum_j b_j^i + \sum_j q_j^i p_j + \sum_j \bar{p}_j \bar{r}_j^i x_j^i \,.$$

Hence the payoff to trader i is

$$(7) \qquad \pi^i = \varphi^i(z_1^i, \ldots, z_m^i) + \lambda^i \min\{0, z_{m+1}^i\}$$

where φ^i is assumed to be continuous differentiable in R_{++}^m, strictly concave and increasing, and $\lambda^i > 0$ is assumed to be sufficiently large.

2.1. The ε Modified Game

In order to get rid of the singularity at $p_j = 0$, consider a modified game Γ_ε, where the prices are calculated by

$$(8) \qquad p_j(\varepsilon) = \frac{\varepsilon + \int_i b_j^i}{\varepsilon + \int_i q_j^i} .$$

On the other hand, to overcome the difficulty from nonconvexity, instead of Σ^i we sometimes consider Σ_C^i, the convex hull of Σ^i. Let $\Sigma_C = \times_i \Sigma_C^i$, and Σ_{CS} the type-symmetric subset of Σ_C. For any $s \in \Sigma_{CS}$, there is always an $s' \in \Sigma = \times_i \Sigma^i$ such that the aggregate effect of s' on the market is the same as s. In fact, assume that i is of type α, and

$$(9) \qquad s^i = \sum_{k^i} \lambda_{k^i} s^i(k^i) , \quad s^i(k^i) \in \Sigma^i , \quad \lambda_{k^i} \geq 0 , \quad \sum_{k^i} \lambda_{k^i} = 1 .$$

Then s' can be achieved by letting λ_{k^i} portion of the individuals of type α play $s^i(k^i)$. ($\forall i \in \alpha$; α runs over all different types).

Note that a mapping from Σ_{CS} into $2^{\Sigma_{CS}}$ induces a mapping from $\times_\alpha \Sigma_C^\alpha$ into $2^{\times_\alpha \Sigma_C^\alpha}$ < α runs over all n different types. Here Σ_C^α can be interpreted as the quotient set of $\times_{i \in \text{type}\alpha} \Sigma_C^i$, where s^i and s^j should be regarded equivalent if i and j are of the same type α and the corresponding components of s^i and s^j have the same magnitude.

Now for any $s \in \times_\alpha \Sigma_C^\alpha$, denote also by s the corresponding element in Σ_{CS}. Then assume that $s' \in \times_i \Sigma^i$ is the one as described in (9). Consider the best responses of i to s' in Σ^i. Due to the nonconvexness of Σ^i, i may have more than one best response. But for a given k^i

as the number of times for i to go to the bank, i can have no more than one best response. This follows directly from (5), (6) and that φ^i is strictly concave. Let S^i be the set of i's best responses, and S^i_C the convex hull of S^i. Then $S^i_C \subset \Sigma^i_C$.

Let $s^i(k^i)$ be the best strategy i can play with k^i fixed in advance. $s^i(k^i)$ can be a best response of i to s' in Σ^i or not. It is easy to see that

$$(10)\quad S^i_C = \left\{ s^i = \sum_{k^i=1}^{K} \lambda_{k^i} s^i(k^i) \;:\; \lambda_{k^i} \begin{array}{l} \lambda_{k^i} = 0 \text{ if } s^i(k^i) \text{ is not a best response} \\ \text{runs from 0 to 1 with } \sum_{k^i} \lambda_{k^i} = 1 \\ \text{if } s^i(k^i) \text{ is a best response} \end{array} \right\}$$

The mapping $\psi^i : \psi^i(s) = S^i_C$ has the property: $\psi^i = \psi^j$ if i and j belong to a same type.

Lemma 1. The mapping $\psi^i : \times_\alpha \Sigma^\alpha_C \to 2^{\Sigma^i_C}$ is upper-semi continuous.

Proof. Consider a convergent sequence $\{ s^{(v)} \}$ in $\times_\alpha \Sigma^\alpha_C$ with limit s. Assume that $\psi^i(s^{(v)}) = S^{(v)_i}_C$ and $\psi^i(s) = S^i_C$. Let $s^{(v)_i} \in S^{(v)_i}_C$ and $\lim_v s^{(v)_i} = s^i$. It suffices to show that $s^i \in S^i_C$.

From (10) one can write

$$(11)\qquad s^{(v)_i} = \sum_{k^i=1}^{K} \lambda^{(v)}_{k^i} s^{(v)_i}(k^i) .$$

Choose a subsequence of $\{ s^{(v)_i} \}$, say $\{ s^{(v')_i} \}$ such that

$$(12)\qquad \lim_{v'} \lambda^{(v')}_{k^i} = \lambda_{k^i} \;;\; \lim_{v'} s^{(v')_i}(k^i) = s^i(k^i) .$$

Claim. $s^i(k^i)$ is a best response of i in Σ^i to s provided that $\lambda_{k^i} > 0$. In fact $s^i(k^i) \in \Sigma^i$ is a direct consequence of the compactness of Σ^i. Assume, by contradiction, that $s^i(k^i)$ is not a best response. Then $\exists \bar{s}^i \in \Sigma^i$ such that

$$\pi^i(s^i(k^i), s') < \pi^i(\bar{s}^i, s') \tag{13}$$

where $s' \in \times_i \Sigma^i$ is the strategy selection in Γ_ε corresponding to s as mentioned in (9).

But then by the continuity of π^i on $(s^i(k^i), s')$ in Γ_ε we would have

$$\pi^i(s^{(\nu')}(k^i), s'^{(\nu')}) < \pi^i(\bar{s}^i, s'^{(\nu')}) \text{ for } \nu' \text{ large enough} \tag{14}$$

which contradicts with the fact that $s^{(\nu')}(k^i)$ is a best response to $s^{(\nu')}$, since $\lambda_{k^i}{}^{(\nu')} > 0$ for large ν'.

Our claim is proved. Now $s^i = \sum_{k^i=1}^{K} \lambda_{k^i} s^i(k^i) \in S_C^i$. Therefore Lemma 1 is true.

Recall that $\psi^i(s) = \psi^j(s)$ when i and j are of the same type. Therefore the mappings ψ^i (all i of type α) induce a mapping

$\psi^\alpha : \times_\alpha \Sigma_C^\alpha \to 2^{\Sigma_C^\alpha}$. ψ^α is upper-semi continuous.

Define $\psi : \times_\alpha \Sigma_C^\alpha \to 2^{\times_\alpha \Sigma_C^\alpha}$ by

$$\psi(s) = \psi^{\alpha_1}(s) \times \ldots \times \psi^{\alpha_n}(s) \tag{15}$$

($\alpha_1, \ldots, \alpha_n$ are n different types of traders.) Then ψ is also upper-semi continuous.

Proposition 1. Γ_ε has at least one NE.

Proof. Let $\psi : \underset{\alpha}{\times} \Sigma_C^\alpha \to 2^{\underset{\alpha}{\times} \Sigma_C^\alpha}$ be defined as in (15). Since $\underset{\alpha}{\times} \Sigma_C^\alpha$ is compact and ψ is upper-semi continuous, by the Kakutani theorem, $\exists \overset{*}{s} \in \underset{\alpha}{\times} \Sigma_C^\alpha$ such that

$$\psi(\overset{*}{s}) \ni \overset{*}{s} . \tag{16}$$

Let $\overset{*}{s}'$ be the corresponding strategy selection of $\overset{*}{s}$ as mentioned in (9). (16) implies that $\overset{*}{s}'$ is an NE of Γ_ε.

Remark. If the initial endowment is not Pareto optimal and if the α_{jk}^i are all very small, the allocation corresponding $\overset{*}{s}'$ is closed to a CE allocation, and hence $\overset{*}{s}'$ is nontrivial.

Now we look at the boundedness of the prices. For simplicity, assume that the utility functions of type α_1 have the following property: for any j, regardless of the values of $x_{j'}^{\alpha_1}$ $(j' \neq j)$, we have

$$\lim_{x_j^{\alpha_1} \to 0} \varphi^{\alpha_1}(x^{\alpha_1}) = \infty . \tag{17}$$

Moreover, we assume that $a^{\alpha_1} > 0$ and hence $u^\alpha(a^{\alpha_1}) > 0$.

Lemma 2. Assume that $\alpha_{jk}^i \leq 1/2m$, $\rho \leq 1/2$. Then there exist $R > 0$ such that, for any NE obtained as in Lemma 1, the associated prices $p_1(\varepsilon), \ldots, p_m(\varepsilon)$ satisfy

$$(18) \qquad \frac{p_j(\varepsilon)}{p_k(\varepsilon)} \leq R$$

where R is independent of ε.

Proof. Consider type α_1 traders. In the equilibrium, they are divided into no more than K subsets each of which the individuals all play the same strategy and hence have the same allocation. The largest subsets must have positive measure greater than $1/K$ of the measure of the set of Type α_1 traders. Due to the property (17) of its utility function we must have

$$d \leq x_j^{\alpha_1} \leq D$$

where d and D are two constants. Hence we should have

$$(19) \qquad \frac{\partial u^{\alpha_1}}{\partial x_j} \Big/ \frac{\partial u^{\alpha_1}}{\partial x_{j'}} \leq R' .$$

WLOG, assume that $p_1(\varepsilon) \leq p_2(\varepsilon) \leq \ldots \leq p_m(\varepsilon)$. We want to show that

$$(20) \qquad \frac{p_m(\varepsilon)}{p_1(\varepsilon)} \leq 2mR' .$$

In fact, if $p_m(\varepsilon) > 2mR'p_1(\varepsilon)$, any trader i of type α_1 can get an improvement by selling a small amount Δq_m^i more and at the same time buying more $\alpha_{mj}^i \Delta q_j^i$ and more Δq_1^i. For buying $\alpha_{mj}^i \Delta q_m^i$, he needs money

$$M_1 = \sum_j p_j(\varepsilon)\alpha_{mj}^i \Delta q_m^i$$

including interest the money he must borrow at the beginning is

$$(1+\rho)M_1 \leq \frac{1.5}{2m} \sum_j p_j(\varepsilon)\Delta q_m^i \leq 0.75 p_m(\varepsilon)\Delta q_m^i \ .$$

But he can spend $0.25p_m(\varepsilon)\Delta q_m^i$ on good 1 again

$$\Delta q_1^i \geq + \frac{0.25p_m(\varepsilon)\Delta q_m^i}{p_1(\varepsilon)} > 0.5mR'\Delta q_m^i \geq R'\Delta q_m^i \ .$$

Therefore (20) must be true and (18) follows.

Now we can also show that the ratios $p_j(\varepsilon)/\bar{p}_j$ are bounded. In fact, we need only show $\gamma_1 > 0$, $R_1 > 0$ and j_0 , such that

$$(21) \qquad r_1 \leq \frac{p_{j_0}(\varepsilon)}{\bar{p}_{j_0}} \leq R_1 \ .$$

Choose j_0 such that at the second stage there are some traders selling good j_0 to the bank. Consider two different possibilities:

(a) $\dfrac{p_{j_0}(\varepsilon)}{\bar{p}_{j_0}}$ is very small,

(b) $\dfrac{p_{j_0}(\varepsilon)}{\bar{p}_{j_i}}$ is very large,

In case (a) i can buy a little bit more $\Delta q_{j_0}^i$ of good j_0 at stage 1 and sell part of $\Delta q_{j_0}^i$ more at second stage to make an improvement. In case (b), i can sell a little good j_0 at first stage and reserve more of good j_0 at the second stage and make an improvement. So in any case (21) holds.

Finally (18) and (21) implies that $\exists\ p > 0$ and $P > 0$ such that

(22) $\quad p \le p_j(\varepsilon) \le P\ ,\quad j = 1, \ldots, m\ ;\quad \text{all}\quad \varepsilon > 0$

Proposition 2. There is a $p > 0$ and a $P > 0$ such that for any NE of Γ_ε, the associated prices $(p_1(\varepsilon), \ldots, p_m(\varepsilon))$ satisfy (22). p and P are independent on ε.

3. THE EXISTENCE OF NE FOR Γ

Let s'^*_ε be an NE as in Proposition 1 of the game Γ_ε ($\varepsilon = 1/2, \ldots, 1/n, \ldots$). From the discussion in Section 2, by a limit process, it is easy to see that there is a s'^*, which is a limit point of (s'^*_ε), is an NE of Γ. Moreover, s'^* is nontrivial if the initial endowment is not Pareto optimal and the α^i_{jk} are all sufficiently small. So we have the following

Theorem 1. Assume that the utility functions φ^i are C^1, strictly concave and increasing in R^m_{++}. Assume that the initial endowment is not Pareto optimal. Then the game Γ has at least one nontrivial equilibrium.

3.1. Comment on Convexification by Type Nonsymmetric Behavior

We obtain convexity by considering the possibility that behavior of individuals of the same type at equilibrium is not necessarily the same. A simple example illustrates this possibility. Consider two types of traders trading in two commodities, gin and tonic water. All traders have the same preferences. Each likes either a strong drink or a weak drink. A strong drink has proportions of gin to tonic of 2:1 and a weak drink has proportions of 1:2. Their preferences can be illustrated by the nonconvex indifference curves shown in Figure 2. Suppose that the two types are dif-

ferentiated by their endowments. The first type has (3,0) and the second type has (0,3). At the prices 1 for gin and for tonic the market can clear in many different ways with an arbitrarily sized set of traders of type 1 drinking strong drinks provided that they are offset buy a set of type 2 drinking weak drinks. Unfortunately the price system does not determine the specifics of the distribution. In terms of actual banking this seems to indicate that after competition has equalized the rate of interest and loan availability competition to attract different customers must be carried on at a different level of micro-variables. Thus there appears to be room for the giving away of toasters and television sets. But in terms of the theory the loss of finiteness of the set of equilibria weakens the value of prices as a guide.

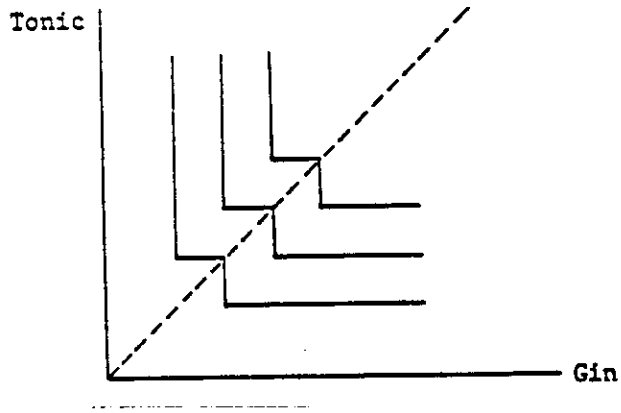

FIGURE 2

APPENDIX

A Simple Example

In order to demonstrate the model, we will consider an example with two commodities, a fiat money and log separable utility functions. Such an exchange economy is the smallest possible which can be modeled as a strategic market game with a single fiat money. This is true since in an economy with one commodity and one money both goods can be considered as moneys. In addition, we will exclude the possibility of wash sales by setting the following restriction on the player's strategy set,

$$b_j^i q_j^i = 0$$

i.e. a trader i can not both bid an offer in the same market. Even though the existence of an equilibrium point with active wash sales and the fact that the market with heavy wash sales is best for all traders has been proved by Dubey-Shubik, we conjecture that the margin for such an improvement under the presence of transaction costs will be severely attenuated. Thus, the simplification we employ thereafter, without altering the substance of our arguments, fundamentally improves the tractability of mathematics in our example.

Our example consists of three steps:

(a) Solution for the competitive equilibrium

(b) Solution for the non-cooperative equilibrium with transaction costs and $r = 0$

(c) Solution for the non-cooperative equilibrium with transaction costs and $r > 0$

Solution for the Competitive Equilibrium without Transaction Costs and $r = 0$

Consider two types of traders i and j with initial endowments (10,30) and (30,10) respectively and utility functions of the form

$$u_i = \log(x_1^i) + \log(x_2^i) \ .$$

We expect that trader of type i will try to maximize by bidding for the first good and offer for sale good 2 whereas trader of type j will exemplify the opposite behavior. Thus, trader i tries to maximize his payoff G_i

$$G_i = \log\left[10 + b_1^1 \frac{a_1}{b_1}\right] + \log(30 - q_2^i)$$

subject to his cash-flow constraint

$$u^1 - b_1^i + q_2^i \frac{b_2}{\xi_2} - u^i \geq 0 \Rightarrow$$

$$\Rightarrow b_1^{\iota} \leq q_2^{\iota} \frac{\beta_2}{q_2} \ .$$

And trader j will try to maximize his payoff G_j

$$G_j = \log(30 - q_1^j) + \log\left[10 + b_2^j \frac{q_2}{b_2}\right]$$

$$\text{s.t.} \quad b_2^j \leq q_1^j \frac{b_1}{q_1}$$

where $q_1 = nq_1^j$, $q_2 = nq_2^i$, $b_2 = nb_2^j$, $b_1 = nb_1^i$. The Lagrangians of the system are:

$$\mathcal{L}_i = \log\left[10 + b_1^i \frac{a_1}{b_1}\right] + \log(30 - q_2^i) - \lambda_1\left[b_1^i - q_2^i \frac{b_2}{q_2}\right]$$

$$\mathcal{L}_j = \log(30 - q_1^j)\log\left[10 + b_1^i \frac{a_1}{b_1}\right] - \gamma_1\left[b_1^i - q_2^i \frac{b_2}{q_2}\right]$$

After a certain amount of unedifying calculation we emerge with:

$$\Rightarrow \boxed{q_1^j - q_2^i - \frac{10(2n^2 - 6n + 1)}{(2n^2 - 2n + 1)}} \tag{A1}$$

We see that no trade occurs for $n = 1$ and $n = 2$.

The competitive equilibrium can be found as $n \to \infty$.

$$\boxed{q_1^j - q_2^i = 10} \text{ as } n \to \infty 1$$

The competitive price vector is

$$\boxed{p_1 - p_2 = \frac{6}{9} = 5}$$

Finally, the final allocations will be:

for trader i (20,20)

and for trader j (20,20)

Solution for the Non-Cooperative Equilibrium with Transaction Costs and $r = 0$

In this variant we have to explicitly introduce transaction costs via the transaction costs matrix. This matrix in our example is

$$M^i = \begin{bmatrix} -1 & \mu^i_{12} & 0 \\ \mu^i_{21} & -1 & 0 \\ \mu^i_{31} & \mu^i_{32} & -1 \end{bmatrix}$$

and the vector X^i which represents the amount of goods purchased, except the last entry which is the number of bank visits.[1]

$$X^i = \begin{bmatrix} -\frac{b^i_1}{p_1} - \frac{b^i_2}{p_2} - 1 \end{bmatrix}$$

where X^i gives final allocations and the last entry indicates the number of trips to the bank. In this case $\underline{k = 1}$ because in this variant $r = 0$. So,

$$X^i \cdot M^i = \mathcal{L}_i = \left[+ \frac{b^i_1}{p_1} - \mu^i_{21}\left(\frac{b^i_2}{p_2}\right) - \mu^i_{31},\ \mu^i_{12}\left(\frac{b^i_1}{p_1}\right) + \frac{b^i_1}{p_2} - \mu^i_{31},\ 1 \right] .$$

In order to make the solution mathematically tractable we assume

$$M^i = M^j = M .$$

So, after modifying the final allocations and the cash-flow constraint, we have trader i trying to maximize his payoff G_i

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[1] The negative signs are for computational reasons.

$$G_i = \log\left[10 + b_1^i \frac{a_1}{p_1} - \mu_{21} \frac{b_2^i}{p_2} - \mu_{31}\right] + \log\left[30 - q_2^i - \mu_{12}\left[b_1^i \frac{q_1}{b_1}\right] - \mu_{32}\right]$$

$$\text{s.t.} \quad b_1^i \leq q_2^i \frac{b_2}{p_2} \ .$$

Similarly, trader j tries to maximize his payoff G_j .

For simplicity in calculation we add the symmetric condition:

$$\mu_{31} = \mu_{32} \quad \text{and} \quad \mu_{21} = \mu_{12} \tag{A2}$$

This symmetrization does <u>not</u> alter the substance of our model. Using a completely analogous process with the one used in the solution for the competitive equilibrium we end up with

$$q_1^j = q_2^i \tag{A3}$$

$$\lambda_1 = \gamma_1 \ . \tag{A4}$$

We obtain:

$$q_1^j = q_2^i = \frac{30 - \mu_{32} - \left[\mu_{12} + \left(\frac{n}{n-1}\right)^2\right][10 - \mu_{32}]}{1 + 2\mu_{12} + \left(\frac{n}{n-1}\right)^2} \ . \tag{A5}$$

Taking the limit as $n \to \infty$

$$q_2^i = q_1^j = \frac{10 - \frac{1}{2}\cdot\mu_{12}(10 - \mu_{32})}{1 + \mu_{12}} \tag{A6}$$

Solution for the Non-Cooperative Equilibrium with Transaction Costs and $r > 0$[2]

In this variant of the example k (i.e. the number of times that the individual goes to the bank) varies so that it becomes a decision variable of the trader's strategy. Moreover the cash-flow constraint becomes complicated and it takes the form developed in the description of the basic model. Thus, u becomes a decision variable of the traders as well. Finally, the price formation mechanism takes the form described in the basic model.

For the sake of mathematically tractability we will still maintain our simplification that $M^i = M^j = M$.

Therefore, after modifying the final allocations and the cash-flow constraint, we have trader i trying to maximize his payoff G_i

$$G_i = \log\left[10 + b_1^i \frac{1}{p_1} - \mu_{21} \frac{b_2^i}{p_2} - k^i \mu_{31}\right] + \log\left[30 - q_2^i - \mu_{12}\left[b_1^i \frac{1}{b_1}\right] - k^i \mu_{32}\right]$$

$$\text{s.t.} \quad u^i - b_1^i \leq q_2^i p_2 - \left[\frac{k^i + 1}{2k^i} r\right] u^i \geq 0$$

$$\Rightarrow - b_1^i \leq q_2^i p_2 - \left[\frac{k^i + 1}{2k^i} r\right] u^i \geq 0$$

$$\Rightarrow b_1^i + \left[\frac{k^i + 1}{2k^i} r\right] u^i \leq q_2^i p_2 .$$

Similarly, trader j will try to maximize his payoff G_j. In this example

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[2] In this example, the interest earned by the bank is divided into two parts (percentages) for buying two different commodities at the first stage --the mechanism is different from what we described before. In this case, the boundedness of prices may be no longer true. On the other hand, since k assumes integer values, the maximization problem may have no type-symmetric solutions.

bank reserves which will be denoted by R are equal to:

$$R = \left[\frac{k^i + 1}{2k^i}\right]u^i + \left[\frac{k^j + 1}{2k^j}\right]u^j \quad . \tag{A7}$$

Using symmetry by setting $\mu_{31} = \mu_{32}$ and $\mu_{21} = \mu_{22}$ and taking the limit as $n \to \infty$ we emerge with

$$q_1^i - q_2^j = \frac{10 - \frac{1}{2}\mu_{12}(10-k)\mu_{32}}{1 + \mu_{12}} \tag{A8}$$

and

$$u^i - u^j = \frac{1}{r}\left[\frac{2\mu_{32}(\mu_{12} + 1)(2 + \mu_{12})}{\mu_{12}(10 - k\mu_{32}) - 90}\right]k^2 \quad . \tag{A9}$$

Discussion of the Solutions Obtained from the Simple Example

We commence our discussion of solutions from the second variant (i.e. non-cooperative equilibrium with transaction costs and $r = 0$). We first see that if we set $\mu_{12} = \mu_{32} = 0$ we arrive at the solution of the competitive equilibrium as $n \to \infty$. Second, as μ_{32} increases then q_2^i increases as well. However, as μ_{12} increases then q_2^i decreases. Analogous observations apply for q_1^j , μ_{31} and μ_{21} . Thus, we have our first proposition.

Proposition 1. The presence of transaction costs influences quantities offered in the market. So, prices are affected by the introduction of transaction costs. This holds true if transaction costs are not too high so that trade is feasible.

Knowing that $\mu_{ij} \geq 0$, the condition for transaction costs not to be prohibitive for trade is:

$$20 - 10\mu_{12} + \mu_{12}\mu_{32} > 0 \ .$$

Finally, we have to note that our proposition contradicts Saving's basic assumption in his paper that the introduction of costs leaves prices unaffected.

We now proceed to the analysis of the third variant (i.e. non-cooperative equilibrium with transaction costs and $r > 0$). We again see that if we set the transaction costs equal to zero, we arrive at the competitive equilibrium solution as $n \to \infty$. Moreover, we still observe the same relationship between transaction costs and quantities offered in the market. Going now to equation (III.3.44) we observe the following:

- as r <u>increases</u> then u <u>decreases</u>
- as r <u>increases</u> then k <u>increases</u>
- as μ_{32} <u>increases</u> then k <u>decreases</u>
- as μ_{32} <u>increases</u> then u <u>increases</u>

Therefore, we are now able to state our second proposition.

<u>Proposition 2</u>. Under the presence of transaction costs and non-negative interest rates in an exchange economy, transactions demand for cash (i.e. u in our case) is <u>inversely proportional</u> to interest rates and <u>proportional</u> to transaction costs associated with monetary exchanges. This holds true if transaction costs are <u>not</u> too high so that trade is feasible.

We have to note that our proposition is in accordance with Baumol-Tobin's analysis of the transactions demand for cash.

Finally, we see that

$$\frac{\partial R}{\partial k^j} < 0$$

since r , u^j , k^j are all positive in the third variant. Therefore, the function of revenues is a strictly-decreasing function. Thus, we have our third proposition.

Proposition 3. The revenues of the bank are inversely proportional to the number of times a trader goes to the bank.

Note on Maximization of Bank's Revenue

We evaluated, when describing the basic model, the bank's revenue being equal to,

$$R = \sum_{i=1}^{n} \left[\left(\frac{k^i + 1}{2k^i} \right) r u^i \right] .$$

Thus, the bank's decision variable is the interest rate. However, we cannot maximize R with respect to r since k and u cannot be treated as constants and they are related with r as stated in Proposition 2. It might be the case that the bank will set interest rates so high that exchange will be blocked altogether since traders will not borrow any amount of money from the bank. Therefore, our model is not well-defined for the case of maximizing bank revenues.

REFERENCES

Baumol, W. J., 1952. "The transactions demand for cash: An inventory theoretic approach," Quarterly Journal of Economics, 56: 545-556.

Tobin, J., 1956, "The interest elasticity of the transactions demand for cash," Review of Economics and Statistics, 38: 241-247.