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LIQUIDITY AND BANKRUPTCY WITH INCOMPLETE MARKETS: PURE EXCHANGE

by

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ABSTRACT

We enlarge the standard model of general equilibrium with incomplete markets (GEI), to incorporate liquidity constraints as well as the possibility of bankruptcy and default. A new equilibrium results, which we abbreviate GELBI (general equilibrium with liquidity, bankruptcy and incomplete markets). When the supply of bank money and bankruptcy/default penalties are taken sufficiently high (the high regime), GEI occur as GELBI. But outside the high regime many new phenomena appear: money is (almost) never neutral, it has positive value and its optimum quantity is often finite; bankruptcy and default not only occur in equilibrium but can have welfare-improving consequences for everyone; there is no real indeterminacy even with financial assets.

We show that GELBI always exists. Indeed when assets pay off in the same commodity in each state, GELBI coincide with GEI in the high regime. The nonexistence of GEI (when assets deliver in multiple commodities) is interpreted in GELBI as a "liquidity trap."

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1. INTRODUCTION.

Liquidity and bankruptcy play a central role in economic activity. But they have not found a place in the standard microeconomic models of general equilibrium, such as the Arrow-Debreu model (GE) ([4]), or its extension to incomplete markets (GEI) ([28], [7], [12], [21]). Were we to imagine money in these models, there would have to be unlimited credit, with any form of bankruptcy a priori ruled out ([6], [29]).

We build a model of an economy with uncertainty about the future and asset markets that are incomplete. Money is introduced as the medium of exchange. It is therefore needed for all transactions.[1] It can be fiat, with no direct utility of consumption, or a commodity, or a mixture. It is durable and so serves as a store of value. It is in finite supply but agents can borrow from a central bank, and have the option of going bankrupt. They can also default on their promises to deliver on assets. (In both cases a penalty is levied which fits the size of the crime.) The model thus breaks out of the confines of GE and GEI, and many new issues can now be raised. Furthermore, unlike GEI, equilibria always exist in our model (Theorems 1 and 2).

Our model is fully in the spirit of general equilibrium. Indeed, if GEI exist, GEI occur as equilibria of our model in the special case when both the supply of bank money and the level of penalties are sufficiently

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[1] Earlier models of GE with cash-in-advance constraints, but without many of the features of our model, include [15], [22], [23], [24], [25], [30]. See [26] for an overview.

high[2] (Theorem 3). We will refer to this situation from now on as the "high regime." In the high regime all interest rates are zero in equilibrium and there is no bankruptcy. But outside of this regime, liquidity constraints and bankruptcy come to the fore, and we get a new kind of equilibrium.

Our wider economic outlook enables the survey of phenomena that were bypassed in GEI analysis. For instance, beyond the high regime, bankruptcy occurs robustly at equilibrium in our model and is thus seen to fit into the orthodox paradigm of market-clearing. This is in contrast to the normal view that bankruptcy is a sure sign of disequilibrium and economy-wide failure. Indeed we find that when markets are incomplete, encouraging bankruptcy or default can be a boon for society, with significant welfare-improving consequences for everyone (Section 10 and [5]).

The neutrality of money is also a phenomenon that holds in the high regime alone. Outside of this regime, money is not neutral in our model: changes in its supply at the bank nearly always alter the real equilibrium outcomes. Nor is the "optimum quantity" of bank money necessarily infinite. For robust values of the parameter space of our model, we find that there are finite stocks of bank money, such that any increase or decrease in their level will cause Pareto-disimprovement of equilibria. (Section 8; and [2], [8], [17], [27] for contrasting views.)

Another striking difference of our model with GEI has to do with the real indeterminacy of equilibria. If financial assets are present in the GEI model then, as is well-known, there is a vast multiplicity of equilibrium allocations ([1], [13]). Our model also allows for such assets; in

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[2] If assets deliver in the same commodity in each state, then in fact GEI and GELBI coincide in the high regime (Theorem 3).

fact, the mysterious "units of account," in which they are imagined to pay off in the GEI model, can be concretely represented in ours by fiat money. It is clear that in the high regime, where GEI occur as equilibria indeterminacy will hold in our model as well. But elsewhere, interest rates are positive, and our equilibrium set shrinks dramatically to a finite set of points (Section 9).

The value of money is borne out as positive by our model. If money is a commodity (i.e. agents have utility of consumption for it), this is hardly surprising. But we show in Section 7 that even when money is fiat (with no direct utility of consumption), its price is always positive in equilibrium. This is so in spite of the fact that (i) we have a finite horizon model (so that money has no future value in the last period) and (ii) agents may hold arbitrary positive amounts of the fiat money in their private endowments. Indeed in the circumstance of (ii), the money interest rate (which, aside from its price, is another measure of the value of money) is also positive in equilibrium. Thus, within our model, we overcome the long-standing puzzle in monetary theory (sometimes called the "Hahn problem," [9], [14], [16], [18], [31]) that abstract proofs of the existence of monetary equilibria may not demonstrate anything more than the existence of trivial equilibria in which the price of money is zero.

One can identify various determinants of the value of money at equilibrium in our model. There is clearly a transactions demand for money, since it is needed to buy each commodity. There is also a speculative demand for money, since inventorying money until after chance moves is the same as holding an asset that promises delivery of one dollar in every state, so money holdings must compete with other assets for a place in each agent's

portfolio. There is a precautionary demand for money to guard against high interest rates in some future state. There is a bankruptcy demand for money from agents who borrow more than they intend to pay back in order to take advantage of lenient penalties. Finally there is the effect of inflation; the demand for money (in period zero) will be enhanced in our model if future prices (in period one) are low, even if all interest rates remain the same.

Let us consider the existence of equilibrium in our model. We permit default in equilibrium, hence to each asset is attached a fraction (between 0 and 1), which indicates how much buyers expect the asset to deliver in the future. If there is positive trade in the asset then this fraction is determined by agents' actions, perfect foresight, and the perfect competition hypothesis. But with an inactive asset market, a peculiar difficulty arises. There is nothing a priori to prohibit the expected fractions from being absurdly pessimistic (close to 0), with the upshot that trade is not induced, and the inactivity in the asset market is sustained. To rule out such spurious instances of market failure, we develop the notion of "rationalizable" expectations. It entails that, if an asset market is inactive, then agents who are "on the verge" of selling must be included in the "potential" market; and, furthermore, those among them who would surely deliver all that they promised must (rationally) be expected to do so (Section 4).

Even with this refinement, equilibrium always exists in our model (Theorems 1 and 2). The key mathematical complication in GEI analysis is that desired trades can become arbitrarily large when the asset span approaches a drop in dimension. It turns out that this is not so in our

model. The liquidity constraints, in conjunction with finite penalties, induce an upper bound on desired trades. The matter is more subtle than might appear. Indeed, if the penalties were infinite, we would be thrown back into GEI analysis: even with liquidity constraints, the bound on trades would disappear leading to possible nonexistence of equilibrium ([19]). Note also that our liquidity constraints only require that agents buy, sell and deliver out of the money or goods they have on hand at the time. There is no ad hoc bound imposed on the size of trades, nor is there any transactions cost added to the model. (These are two standard means by which existence has previously been guaranteed in GEI models, see e.g. [28].)

Our equilibrium gives a new perspective on the nonexistence of GEI: we can interpret it as a "liquidity trap" in our model. Take an economy for which GEI fails to exist. No matter how large the penalties and the supply of bank money are made, the demand for money will remain excessive, forcing interest rates to be positive in our equilibrium. (See Section 10.)

By extending the domain of analysis beyond the high regime, we also gain an insight into bankruptcy and default in the economy (see [10], [20] for other points of view). Both entail a deadweight loss of utility to society on account of the penalties that are imposed. Yet, in the presence of incomplete asset markets, they make for a superior allocation of risk-bearing which can outweigh the utility loss [5]. One reason is that, in effect, the dimension of available assets is increased. To see this, first observe that when defaults or bankruptcies are ruled out, any agent will either buy or sell an asset, but not both, since by holding positions on both sides (of the asset market) the agent undoes with one hand what he did

with the other. But the moment default is permitted, then an order of buy is not the exact negative of sell. Thus, a combination of buying and selling (or for that matter, lending and borrowing money at the bank when bankruptcy is permitted) can make for an extra dimension of new possibilities. The advice of old Polonious to "neither borrower nor lender be," needs to be stood on its head: our motto is "both borrower and lender be," if there are incomplete markets.

An added advantage holds for the seller of an asset (or borrower of money). The fact that he can default (go bankrupt) permits him the luxury, if the penalty is light enough, of tailoring the asset he delivers on (the money he repays) to his needs.

The paper is organized as follows. Section 2 recalls the standard underlying economy with incomplete markets. In Section 3, we embed the economy in a larger model in which liquidity constraints, as well as the options of bankruptcy and default, are introduced. Since this is the first presentation of the model, we spend considerable time on all its details, even though they are routine and often tedious. Section 4 defines equilibrium and shows its existence (Theorem 1). Section 5 considers a variant of the model, to facilitate comparison with GEI, and again verifies the existence of equilibrium (Theorem 2). In Section 6 we see that GEI is a special case of our analysis (Theorem 3). The value of money is analyzed in Section 7, its neutrality and optimum quantity in Section 8, the multiplicity of equilibria in Section 9, and finally Section 10 prepares for the discussion of the liquidity trap in Section 11.

For the reader's convenience all proofs are relegated to an Appendix.

2. THE UNDERLYING ECONOMY WITH INCOMPLETE MARKETS

The set of states of nature is $\{0, 1, \ldots, S\}$. State 0 occurs in period 0, and then nature moves and selects one of the states in $S = \{1, \ldots, S\}$ which occur in period 1.

The set[3] of commodities is $L = \{1, \ldots, L\}$, where L will play the role of money, in each state. Thus the commodity space may be viewed as $R_+^L \times R_+^{S \times L}$ whose axes are indexed by $\{0, 1, \ldots, S\} \times \{1, \ldots, L\}$. The pair $s\ell$ denotes commodity ℓ in state s .

The set of agents is the set $H = \{1, \ldots, H\}$. Agent h has initial endowment $e^h \in R_+^L \times R_+^{S \times L}$ and utility function $u^h : R_+^L \times R_+^{S \times L} \to R$. We assume that no agent has the null endowment of commodities in any state, i.e., for $0 \leq s \leq S$ and $1 \leq h \leq H$:

$$e_s^h = (e_{s1}^h, \ldots, e_{s(L-1)}^h) \neq 0$$

and $$\Sigma_h (e_{s1}^h, \ldots, e_{s(L-1)}^h) \gg 0 .$$

Further assume that each u^h is concave and smooth, and[4]

$$\frac{\partial u^h}{\partial x_{s\ell}} > 0 \quad \text{for} \quad h = 1, \ldots, H ; \quad s = 0, \ldots, S ; \quad \ell = 1, \ldots, L-1 .$$

The reason why we do not insist on the above requirement for $\ell = L$ is that we wish to allow for money to be fiat in some states. With this in mind, we add the conditions below. In any state s and for any agent h exactly

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[3] It will be clear from the context whether L (or S) is the set of commodities (or states) or the name of the last commodity (or state).

[4] $\partial u^h / \partial x_{s\ell} > 0$ means $\partial u^h / \partial x_{s\ell}(y) > 0$ for all $y \in R_+^L \times R_+^{S \times L}$, etc.

one of the following holds:

$$\text{(i)} \quad \frac{\partial u^h}{\partial x_{sL}} = 0$$

$$\text{(ii)} \quad \frac{\partial u^h}{\partial x_{sL}} > 0$$

We say, in case (i), that <u>money is fiat</u> for h in state s; and, in case (ii), that <u>money is a commodity</u> for h in state s. We also say that money is fiat (commodity) in state s if it is fiat (commodity) for all h in state s. Finally money is fiat (commodity) if it is fiat (commodity) in all states $s = 0, 1, \ldots, S$.

In every case we take money to be <u>perfectly durable</u>, i.e., it can be carried from period 0 into 1, freely and without depreciation, while all of the commodities $1, \ldots, L-1$ last for just one period. Worthy of note is the fact that, unlike the commodities, the total endowment of money with the agents may be either zero or positive.

The most natural situation, in our opinion, is when money is fiat in state 0 (i.e. in period 0) and is a commodity in each of the states $1, \ldots, S$ (i.e. in period 1). This occurs when we think of money as paper which is made legal tender. Then money will have no utility of direct consumption in period 0, but will have utility at the end of period 1, on account of its reuse in future (unmodeled) periods as a trading chip.

There are assets $j \in J = \{1, \ldots, J\}$. The seller of one unit of asset j is obliged to deliver a state contingent vector of commodities $A^j \in R_+^{S \times L}$, where we assume $A^j \neq 0$. In particular, promised deliveries could take the form of money (fiat or commodity), or commodities, or both.

3. LIQUIDITY AND BANKRUPTCY WITH INCOMPLETE MARKETS

3.1. Outline of the Model

The model is built, in brief, as follows. There are two time periods. In period zero agents borrow a fixed stock of money M_0 from the bank at an interest rate θ_0, and use it to trade commodities and assets. Then nature moves and we enter one of the states $s = 1, \ldots, S$ in period one. In any state s the first thing that happens is a fresh disbursal of bank money M_s at interest rate θ_s. Next the loan on M_0 comes due. After this, asset deliveries take place, followed by another round of trade in the commodities.[5] Then, at the end, there is the settlement of the loan on M_s.

Agents who borrow money have the option of not fully honoring their debt. But then they must incur penalties in proportion to the amount by which they go bankrupt at the rate $\tilde{\lambda}$. Similarly sellers of assets are free to default on their promises to deliver, but penalty rates λ apply. (Note that $\tilde{\lambda}$ and λ are vectors.)

What we need is a penalty which increases without limit as the bankruptcy or default increases, and which can be varied in intensity. The precise form the bankruptcy penalties takes is not critical. In particular it need not be separable or linear, though we restrict to that form for ease of exposition (see footnote 6 for more details).

In our model every transaction that an agent undertakes requires the physical transfer of commodities or money out of what the agent has on hand

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[5] To fix ideas we have chosen a particular sequence of events, but this can be permuted quite freely without affecting any of our qualitative results. For instance, the loan on M_0 could come due before the move of chance, or after the delivery of assets, or still later, after the conclusion of trade in commodities in period one, etc.

at the time. For money this amounts to what we have called the liquidity constraint. These constraints apply however not only to the payment of money but also to the delivery of commodities promised by assets. The upshot is that we have a well defined physical process in which effect follows cause in a time sequence. By contrast, general equilibrium analysis steers clear of all liquidity constraints because all transactions are imagined to occur simultaneously. Of course we can recover the general equilibrium outcome in our sequential model by introducing enough credit for money and for commodities, accompanied by sufficiently harsh penalties for nonrepayment (see below). The point of our paper is to go beyond this and to analyze the effects of liquidity constraints and bankruptcies.

3.2 The Formal Model $E(M,\tilde{\lambda},\lambda)$

We now describe the sequence of events in detail. This is best visualized as a tree (see FIG I). Let $\eta = (\theta_0, p_0, p_{0L}, \pi, (\theta_s, p_s, p_{sL}, K_s)_{s\in S}) \in R_+ \times R_+^{L-1} \times R_+ \times R_+^J \times (R_+ \times R_+^{L-1} \times R_+ \times R_+^{J\times L})^S = R_+^N$ be a list of macrovariables, with the interpretation:

$0 \le \theta_0$ = interest rate of money in state 0

$0 \le \theta_s$ = interest rate of money in state $s = 1, \ldots, S$

$0 < p_{s\ell}$ = price of commodity ℓ in state $s = 0, 1, \ldots, S$;
$\ell = 1, \ldots, L-1$

$0 \le p_{0L}$ = price of money in state 0

$0 \le p_{sL}$ = price of money in state $s = 1, \ldots, S$

$0 \le \pi_j$ = price of asset $j = 1, \ldots, J$

$0 \le K_{sj\ell} \le 1$ = percentage that agents expect to receive of the promised delivery of commodity ℓ out of asset j in state s ;
$s = 1, \ldots, S$; $j = 1, \ldots, J$; $\ell = 1, \ldots, L$.

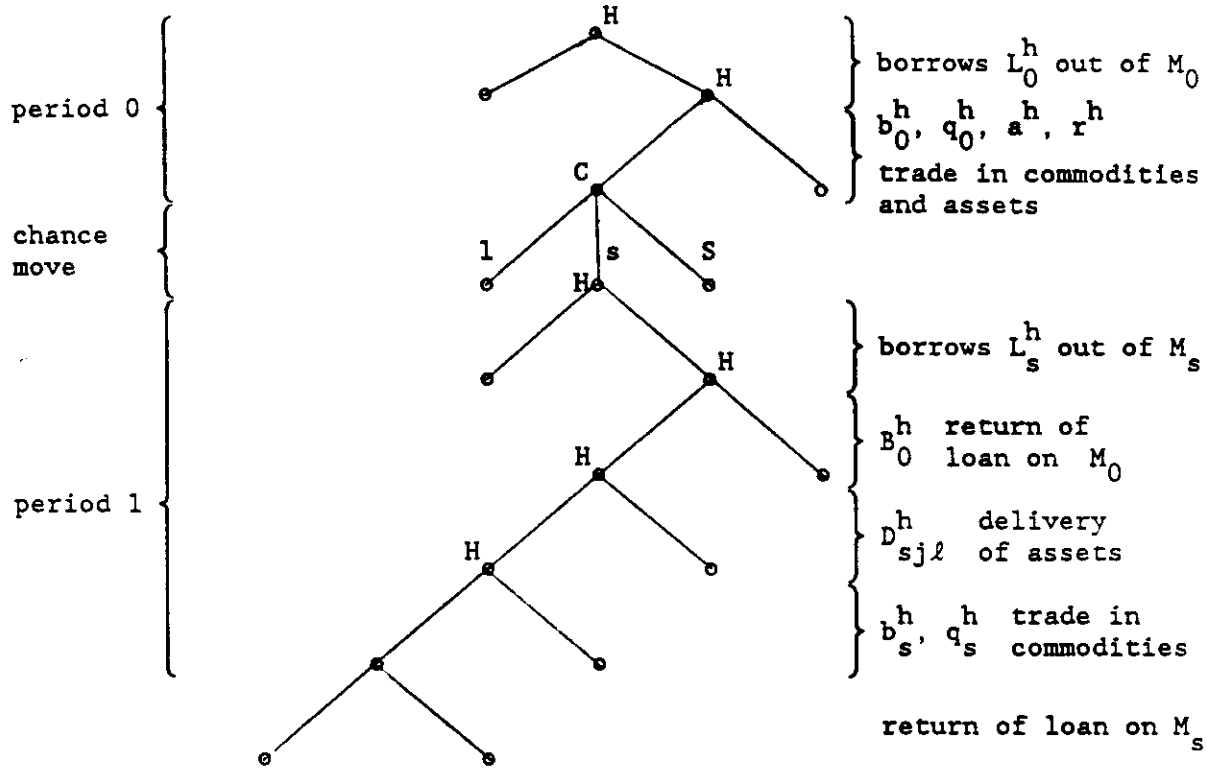

FIGURE I

(H denotes the choices of the agents, and C the move of chance.) To avoid division by zero, we first describe[6] trade opportunities for each agent h at η when $0 < p_{sL}$ for $s = 0, 1, \ldots, S$, and $0 < \pi_j$ for $j = 1, \ldots, J$. At the start, each agent h borrows money L_0^h at the interest rate θ_0, from a bank which has a total supply $M_0 > 0$ of money. Thus he owes the bank $\mu_0^h = (1 + \theta_0)L_0^h$. Next the agents trade in the L commodities and J assets, using money for purchases. Thus the choices available to agent h at this point are given by

[6]This description is already clear from the tree and our verbal description in Section 3.1. But we go through all the tedious details of the accounting to make the model completely formal.

$$\left\{(b_0^h,\ q_0^h,\ a^h,\ r^h) \in R_+^{L-1} \times R_+^{L-1} \times R_+^J \times R_+^J\ :\ 0 \le q_{0\ell}^h \le e_{0\ell}^h,\right.$$

$$\left.\sum_{\ell=1}^{L-1} b_{0\ell}^h + \sum_{j=1}^{J} a_j^h \le L_0^h + e_{0L}^h\right\} .$$

Here $b_{0\ell}^h$ (a_j^h) is the money spent by h for purchase of commodity ℓ (asset j); and $q_{0\ell}^h$, r_j^h the quantities of these that he puts up for sale. Thus agent h winds up with the commodity bundle $x_0^h \in R_+^{L-1}$, where

$$x_{0\ell}^h = e_{0\ell}^h - q_{0\ell}^h + \frac{b_{0\ell}^h p_{0L}}{p_{0\ell}} , \quad \text{for } \ell = 1, \ldots, L-1$$

and the amount of money

$$\hat{L}_0^h = e_{0L}^h + L_0^h + \sum_{\ell=1}^{L-1}\left(\frac{p_{0\ell}}{p_{0L}} q_{0\ell}^h - b_{0\ell}^h\right) + \sum_{j=1}^{J}\left(\frac{\pi_j r_j^h}{p_{0L}} - a_j^h\right) \ge 0 .$$

Also he holds asset j in the amount

$$y_j^h = \frac{a_j^h p_{0L}}{\pi_j} .$$

Now agent h chooses $x_{0L}^h \le \hat{L}_0^h$ that he will consume in period 0, and is left with

$$\bar{L}_0^h = \hat{L}_0^h - x_{0L}^h \ge 0$$

of money.

Then chance picks one of S states and we leave period 0 and enter period 1. In state s, the first thing that happens is a fresh disbursal

of $M_s > 0$ additional bank money. Denote by L_s^h the amount borrowed by h in state s, at the interest rate θ_s. Thus he owes $\mu_s^h = (1 + \theta_s)L_s^h$ to the bank in state s. The money at hand for h now is:

$$\hat{L}_s^h = \bar{L}_0^h + e_{sL}^h + L_s^h .$$

Agent h now decides how much of $\hat{L}_s^h$ to repay on his bank loan made in period 0. Denote this by B_{0s}^h. So h is left with $\hat{L}_s^h - B_{0s}^h \geq 0$ of money, and his debt outstanding on M_0 is $c_{0s}^h = L_0^h(1 + \theta_0) - B_{0s}^h$ $= \mu_0^h - B_{0s}^h$.

The next step is the delivery of assets. Agents are free to default on their promises to deliver. Let $D_{sj\ell}^h$ be the amount of commodity ℓ that agent h chooses to deliver on asset j in state s, and D_{sjL}^h the amount of money h delivers on asset j in state s. Clearly (without further borrowing) we must require

$$\sum_{j=1}^{J} D_{sjL}^h \leq \hat{L}_s^h - B_{0s}^h$$

$$\sum_{j=1}^{J} D_{sj\ell}^h \leq e_{s\ell}^h \quad (\ell = 1, \ldots, L-1) .$$

Then, recalling the meaning of $K_{sj\ell}$, agent h obtains commodities and money from his purchase of assets to end up with

$$\hat{e}_{s\ell}^h = e_{s\ell}^h - \sum_{j=1}^{J} D_{sj\ell}^h + \sum_{j=1}^{J} y_j^h K_{sj\ell} A_{sj\ell} , \quad \ell = 1, \ldots, L-1$$

of the $L-1$ commodities and

$$\bar{L}_s^h = \hat{L}_s^h - B_{0s}^h - \sum_{j=1}^{J} D_{sjL}^h + \sum_{j=1}^{J} y_j^h K_{sjL} A_{sjL}$$

of money. He uses these stocks of money and commodities to trade in the L-1 markets as before (via the choices b_s^h and q_s^h); and finally chooses the amount x_{sL}^h that he will consume of money (the balance is returned to defray the loan on M_s).

The constraints on his choices are:

$$q_{s\ell}^h \leq \hat{e}_{s\ell}^h$$

$$\sum_{\ell=1}^{L-1} b_{s\ell}^h \leq \bar{L}_s^h$$

$$x_{sL}^h \leq \bar{L}_s^h - \sum_{\ell=1}^{L-1} b_{s\ell}^h + \sum_{\ell=1}^{L} \left(\frac{q_{s\ell}^h p_{s\ell}}{p_{sL}} \right) = \tilde{L}_s^h .$$

The consumptions $x_{s\ell}^h$ for $1 \leq \ell \leq L-1$ are given by

$$x_{s\ell}^h = \hat{e}_{s\ell}^h - q_{s\ell}^h + \frac{b_{s\ell}^h p_{sL}}{p_{s\ell}} ;$$

the debt outstanding on M_s is

$$c_s^h = L_s^h(1 + \theta_s) - (\tilde{L}_s^h - x_{sL}^h) = \mu_s^h - (\tilde{L}_s^h - x_{sL}^h) ;$$

and the default on $A_{sj\ell}$ is

$$c_{sj\ell}^h = r_j^h A_{sj\ell} - D_{sj\ell}^h .$$

The final outcome to h from his choices

$$\sigma^h = (\mu_0^h,\ b_0^h,\ q_0^h,\ a^h,\ r^h,\ x_{0L}^h,\ (\mu_s^h,\ B_{0s}^h,\ D_s^h,\ b_s^h,\ q_s^h,\ x_{sL}^h)_{s\in S})$$

$$\in \Sigma_\eta^h \subset \Sigma = R_+ \times R_+^{L-1} \times R_+^{L-1} \times R_+^J \times R_+^J \times R_+ \times (R_+ \times R_+ \times R_+^{J\times L} \times R_+^{L-1} \times R_+^{L-1} \times R_+)^S$$

is a bundle $F_\eta^h(\sigma^h) = w^h = (x^h,\ c^h) \in (R_+^L \times R_+^{S\times L}) \times (R^S \times R^S \times R^{S\times J\times L})$. The map F_η^h has been defined by our previous accounting. It remains to specify the utility to h of the outcome w^h. This is given by:[7]

$$U^h(w^h) = u^h(x_0^h,\ x_1^h,\ \ldots,\ x_S^h) - \sum_{s=1}^{S} \tilde{\lambda}_{0s}^h (c_{0s}^h)^+$$

$$- \sum_{s=1}^{S} \tilde{\lambda}_s^h (c_s^h)^+ - \sum_{s=1}^{S} \sum_{j=1}^{J} \sum_{\ell=1}^{L} \lambda_{sj\ell}^h (c_{sj\ell}^h)^+$$

where, for any real number c, $c^+ = \max\{0,c\}$.

We shall now describe the trading opportunities available to any agent h at an η for which some prices may be zero. Since we are ultimately interested only in equilibrium, we restrict attention to η in which zero prices (among π_j, p_{0L}, p_{sL}) occur only under the following conditions:

(1) $\pi_j = 0 \Rightarrow \sum_{s=1}^{S} \sum_{\ell=1}^{L} p_{s\ell} A_{sj\ell} = 0$

(2) $p_{sL} = 0$ for some $s = 0, 1, \ldots, S \Rightarrow$ money is fiat in state s

(3) $p_{0L} = 0 \Rightarrow p_{sL} = 0$ for all $s = 1, \ldots, S$

In (1) we permit the price of asset j to be zero only if it delivers nothing of value. In (2), we do not permit money to be valueless if it has

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[7] Much more general default and bankruptcy penalties would have served our purpose. What is crucial is that $U^h(x^h,\ c^h)$ is concave; $U^h(x^h,\ c^h) \leq U^h(x^h,\ 0)$; and for any x^h, increasing <u>any</u> coordinate of c^h sufficiently far eventually yields $U^h(x^h,\ c^h) < U^h(e^h,\ 0)$.

utility of consumption to any agent. (This does not rule out a positive price for fiat money. In fact we will show that typically fiat money has positive price.) Finally prices that violate condition (3) could never occur at equilibrium since money is durable. Thus (1), (2) and (3) define the only relevant occurrence of zero prices at equilibrium.

For η with such zeros, the trading opportunities are limited as follows. When $\pi_j = 0$, an agent cannot buy or sell asset j. When $p_{sL} = 0$ for $s = 1, \ldots, S$, he cannot buy or sell commodities in state s, but c_{0s}, c_s and c_{sjL} are taken to be zero for any choices made by the agents. Finally, if $p_{0L} = 0$, no trade is permitted in asset markets or for any commodities. This defines Σ_η^h and F_η^h at η with (π_j, p_{0L}, p_{sL}) possibly zero, but satisfying (1), (2) and (3).

Let us interpret our rules of trade. When $\pi_j = 0$, (1) implies that asset j delivers nothing of value. So nobody will want either to buy or sell it, and to avoid division by 0 in our previous formulae we find it convenient to simply rule out trade. When $p_{sL} = 0$, for some $s = 1, \ldots, S$, money has no value. Since our model postulates money as the medium of exchange, it stands to reason that no purchases can be made in state s. On the other hand, selling merely an infinitesimal amount of any commodity $s\ell$ will raise enough money (since $e_s^h \neq 0$) to defray any loan on M_0 or M_s in state s, so no agent is motivated to sell more than an infinitesimal amount of any commodity (recall that by (2) there is no consumption value of money). This justifies our convention of setting $c_{0s} = c_s = c_{sjL} = 0$, and trades equal to zero when $p_{sL} = 0$. Finally, if $p_{0L} = 0$ and $p_{sL} = 0$ for all $s = 1, \ldots, S$, then money cannot purchase any commodities or assets anywhere; and since money is fiat, and not desired for itself, no

agent has any reason to sell commodities or assets for money.

4. EQUILIBRIUM

Consider an H-tuple of choices $\sigma = (\sigma^1, \ldots, \sigma^H)$ and a list η of macrovariables

$$\eta = (\theta_0, P_0, P_{0L}, \pi, (\theta_s, P_s, P_{sL}, K_s)_{s\in S} .$$

We define (η,σ) to be a pre-General Equilibrium with Liquidity and Bankruptcy and Incomplete Markets (preGELBI) if, for all $s = 0, 1, \ldots, S$,

i(a) $\quad \sum_h L_s^h = M_s ,$

i(b) $\quad \sum_h b_{s\ell}^h = \sum_h p_{s\ell} q_{s\ell}^h , \quad \ell = 1, \ldots, L-1 ,$

i(c) $\quad \sum_h a_j^h = \sum_h \pi_j r_j^h , \quad j = 1, \ldots, J ,$

i(d) $\quad \sum_h D_{sj\ell}^h = K_{sj\ell} A_{sj\ell} \sum_h r_j^h , \quad j = 1, \ldots, J , \quad \ell = 1, \ldots, L ;$

(ii) $\quad \sigma^h$ maximizes $U^h \circ F_\eta^h$ on Σ_η^h for $h = 1, \ldots, H .$

At a preGELBI agents must maximiize utility ((ii)), and demand must equal supply for money, commodities, and assets (i(a), i(b) and i(c)). In addition we require in i(d) that the aggregate delivery on assets is equal to the expected proportion of aggregate promises. Notice that the delivery on assets is to the market, and not on a bilateral basis. In effect we have assumed that all the default risk for each asset is pooled among its buyers, i.e. there is perfect financial intermediation.

We turn to two problems that can occur at a preGELBI. First note that

there exist trivial preGELBI in which the price of money $p_{sL} = 0$ for any subset of states s. The moment the value of money drops to zero, it is unable to move any commodities or assets in trade, and so supply and demand are both trivially zero. Of course there are rare instances when this does not matter, because no gains can be made from trade even with perfect liquidity. In our definition of equilibrium, we shall permit the value of money to be zero only in these instances.

To make ideas formal let $(w^h = (x^h, c^h))_{h=1}^H$ be an H-tuple of outcomes and let $s = 0, 1, \ldots, S$ be a state. Consider net trades $(z_s^h)_{h=1}^H$ in commodities in state s, with $\sum_h z_s^h = 0$. Suppose $\hat{x}_s^h = x_s^h + z_s^h \geq 0$, and $\hat{x}_{s'}^h = x_{s'}^h$ if $s' \neq s$. Then if the condition:

$$U^h(\hat{x}^h, c^h) > U^h(x^h, c^h) \quad \text{for all } h$$

never holds for any net trades $(z_s^h)_{h=1}^H$, we say that $(w^h)_{h=1}^H$ is Pareto-optimal in state s.

Consider a preGELBI (η,σ). We define money to be essential at (η,σ) in state s if

(iii) $p_{sL} > 0$ or else the outcome of (η,σ) is Pareto-optimal in state s

We say <u>money is essential</u> at (η,σ) if it is essential in every state $s = 0, 1, \ldots, S$. Condition (iii) requires that money have positive value unless trade in commodities is itself unnecessary. (This condition can be further refined without disturbing the existence of equilibrium—see Section 5).

There is a second serious problem with the definition of a preGELBI. It does not place any restrictions on expected rates of delivery $K_{sj\ell}$ when

there is no trade on asset market j. If these expectations are allowed to be arbitrary, then nothing prevents them from being absurdly pessimistic. In that event, however, we would regard the failure of the market to induce trade as spurious.

The point is easily seen by a simple example. Consider a preGELBI when there are no assets. This can be sustained as a preGELBI even when assets $j \in J$ are introduced. Take prices $\pi_j > 0$ but very small. No agent will want to sell, because by doing so he undertakes a real obligation either to deliver commodities, or to incur default penalties. Set ("virtual") $K_{sj\ell} > 0$, but even smaller. Then in spite of the cheapness of the asset, there will also be no buyers because they do not expect to recover anything from their investment. However, the $K_{sj\ell}$ ought to represent rational conjectures of agents about each other. This leads us to further refine the notion of a preGELBI.

The idea is to prohibit unduly pessimistic expectations about how much the assets will deliver. Consider an asset that is not traded. If there are agents who are "on the verge" of selling the asset at the given price, then they define the potential market. Among these sellers we can identify (in each state for each commodity) those who would surely deliver all they promised for very small sales. Such sellers must be (rationally) expected to do so.

In our next definition we shall require that when an asset market is inactive, its potential market is nevertheless well-defined, i.e. the set of sellers on the verge of selling is nonempty, and the $K_{sj\ell}$ are derived from rational expectations about them.

To make ideas formal, consider a preGELBI (η,σ). Partition the

assets into sets J_σ and $J\backslash J_\sigma$, where J_σ is the set of assets for which positive trade occurs. Let $V^h(\eta, \lambda^h, \bar{\lambda}^h, J_\sigma)$ be the maximum utility agent h can achieve in Σ^h_η under the additional constraint: if $j \in J\backslash J_\sigma$ then h can <u>not</u> buy asset j. Thus V^h is well-defined without knowing $K_{sj\ell}$ for assets j that are not traded.

We say that h <u>is on the verge of selling</u> $j \in J\backslash J_\sigma$ if (1) $V^h(\eta, \lambda^h, \bar{\lambda}^h, J_\sigma)$ is achieved with $r^h_j = 0$, and (2) whenever $\hat{\eta}$ differs from η only in that $\hat{\pi}_j > \pi_j$, then $V^h(\hat{\eta}, \lambda^h, \bar{\lambda}^h, J_\sigma)$ $> V^h(\eta, \lambda^h, \bar{\lambda}^h, J_\sigma)$. (Clearly in order to attain the higher utility, h must choose to sell asset j.)

Furthermore such an agent h will be called <u>completely reliable</u> in commodity $s\ell$ on asset j if

$$V^h(\hat{\eta}, \hat{\lambda}^h, \bar{\lambda}^h, J_\sigma) = V^h(\hat{\eta}, \lambda^h, \bar{\lambda}^h, J_\sigma)$$

for all $\hat{\eta}$ and $\hat{\lambda}^h$ close enough to η and λ, with $\hat{\lambda}^h$ the same as λ^h except for $\hat{\lambda}^h_{sj\ell} < \lambda^h_{sj\ell}$; and $\hat{\eta}$ (as before) different from η only in $\hat{\pi}_j > \pi_j$. (Evidently agent h will not default on the delivery of $sj\ell$ when he is induced to sell a little of asset j at the slightly higher price $\hat{\pi}_j$, to get the utility $V^h(\hat{\eta}, \lambda^h, \bar{\lambda}^h, J_\sigma)$. Otherwise he could further increase his utility $V^h(\hat{\eta}, \hat{\lambda}^h, \bar{\lambda}^h, J_\sigma)$ by continuing to default, when the penalty decreases to $\hat{\lambda}^h_{sj\ell}$.)

Consider a preGELBI (η,σ). We will say that K (given by η) is <u>rationalizable at</u> (η,σ) if $p_{0L} = 0$ or else, for all $j \in J\backslash J_\sigma$ with $\pi_j > 0$,

(i) The set H(j) of agents who are on the verge of selling j is nonempty

(ii) There exist t^h_j and $\rho^h_{sj\ell}$ such that $K_{sj\ell} \geq \sum_{h\in H(j)} t^h_j \rho^h_{sj\ell}$ if $p_{0L} > 0$

where

$$0 \leq t^h_j \leq 1 \ , \quad \sum_{h\in H(j)} t^h_j = 1$$

$$0 \leq \rho^h_{sj\ell} \leq 1$$

$$\left.\begin{array}{l} t^h_j > 0 \\ h \text{ is completely reliable in} \\ \text{commodity } s\ell \text{ on asset } j \end{array}\right\} \Rightarrow \rho^h_{sj\ell} = 1 \ .$$

Since, on account of liquidity constraints, no trade is permitted when $p_{0L} = 0$, the question of rationalizable K can not arise. However see Remark 2.

A preGELBI (η,σ) is called a GELBI if K (given by η) is rationalizable at (η,σ), and if money is essential at (η,σ).

THEOREM 1. There always exists a GELBI of $\Gamma(M,\lambda,\bar{\lambda})$ for any M, λ, $\bar{\lambda} > 0$.

It is worth remarking that much of the complication in the proof (steps 6, 10) occurs because we allow for fiat money. The proof is much simpler for commodity money. In fact, it is also simple in the natural case when money is fiat in period 0 and commodity in period 1 (states 1, ..., S).

5. CREDIT MARKETS FOR COMMODITIES: THE ECONOMY E*

In our model $E(M,\bar{\lambda},\lambda)$ we have departed from the general equilibrium approach in some essential ways. First there are liquidity constraints on purchases brought about by the finite supply of money. Second, the loans on M_0 come due after borrowing from M_s, thereby enabling agents to refinance. Third, the delivery on assets has to take place out of endowments in

period 1, prior to trade. Fourth, both bankruptcies and defaults are permitted.

We defined E in this manner to incorporate important features of economic life. To bring the analysis more in line with GEI, we shall remove the second and third features and change our model to E^*.

In the GEI model, agents are required to balance budgets in period $0, 1, \ldots, S$ separately. Accordingly, let us resequence in our tree and make the loans on M_0 due prior to the move of chance, leaving the rest of the tree the same. Now, of course, the credit terms $\{c^h_{0s}\}_{s=1}^S$ will be replaced in the model by the single term c^h_0, and $\{\tilde{\lambda}^h_{0s}\}_{s=1}^S$ by $\tilde{\lambda}^h_0$. Also (in the definition of the essentiality of money) when $p_{sL} = 0$ for some $s = 1, \ldots, S$, the credit terms c^h_s are ignored in the payoff to h, but the penalty on c^h_0 is retained, since c^h_0 comes due <u>before</u> we enter state s where money is valueless.

In general equilibrium analysis all transactions are imagined to occur simultaneously. This is impossible if transactions involve the movement of real commodities. The idea behind E^* is that the timeless nature of GEI can be recast in a sequential model, where all actions involve time, provided we introduce sufficient credit for commodities as well as money.

We alter the model by imagining that agents can borrow each commodity $\ell = 1, \ldots, L-1$ in state $s = 1, \ldots, S$ at rate[8] of interest $\rho_{s\ell} \geq 0$, at the same time as they borrow M_s. Let $\gamma^h_{s\ell} > 0$ be the penalty for default on returning commodity $s\ell$ at the end of the tree in state s. Call this model $E^*(M, \tilde{\lambda}, \lambda, \gamma, \rho) = E^*$.

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[8] Fixing bank supply and making interest rate endogenous is equivalent to fixing interest rate and making bank supply endogenous. We take the latter course here, since we wish to focus on the case where $\rho_{s\ell}$ is zero.

Assume in addition that utilities are bounded, i.e.,

$\sup\{u^h(x) - u^h(e^h) : x \in R_+^L \times R_+^{S\times L},\ h \in H\} < u^{**}$ for some finite u^{**} .

GELBI can be defined for E^* exactly as for E .

THEOREM 2. There exists a GELBI of $E^*(M,\tilde{\lambda},\lambda,\gamma,\rho)$ for any M, $\tilde{\lambda}$, λ, $\gamma > 0$ and $\rho \geq 0$.

6. GELBI vs. GEI

Throughout this section, we take money to be fiat, and the model to be E^* . Suppose that in $E^*(M,\tilde{\lambda},\lambda,\gamma,\rho)$ there is no default (against λ) or bankruptcy (against γ or $\tilde{\lambda}$) . The resulting GELBI is nevertheless not a GEI of the underlying economy, unless the interest rates θ_s are all zero, and the (virtual) $K_{sj\ell}$ of all untraded assets are identically 1. (The $K_{sj\ell}$ for traded assets are obviously 1 since there is no default.) When these conditions are met, the result is a general equilibrium with incomplete markets (GEI). We take this as a definition of GEI, since (i), (ii), (iii) of the following theorem is obvious (in the theorem "GEI" means the standard definition given in the literature--see [7], [12], [21]).

THEOREM 3

(i) Suppose that a GELBI $E^*(M,\tilde{\lambda},\lambda,\gamma,0)$ gives rise to no default or bankruptcy, all interest rates are zero, and all $K_{sj\ell} = 1$. Then the GELBI allocation is a GEI allocation of the underlying economy.

(ii) Take any GEI of an economy. Consider $E^*(M,\tilde{\lambda},\lambda,\gamma,0)$, by adding an extra fiat money which is not initially held by any agent. If assets in the GEI economy promise delivery in units of account,

denote these promises in fiat money in E^*. For large enough M, $\tilde{\lambda}$, λ, γ there is a GELBI of $E^*(M,\tilde{\lambda},\lambda,\gamma,0)$ which gives rise to the GEI allocation; furthermore at this GELBI there is no default or bankruptcy, interest rates are zero, and all $K_{sj\ell} = 1$.

(iii) Finally if no asset of the underlying economy pays off in units of account, then for any single large enough choice of M, $\tilde{\lambda}$, λ, γ all the GEI of the economy are GELBI of $E^*(M,\tilde{\lambda},\lambda,\gamma,0)$ in the sense of (ii).

(iv) Suppose, in each state $s = 1, \ldots, S$ there is a single commodity $s\ell$ in which all assets exclusively promise delivery. We have the following equivalence. Consider $E^*(M,\tilde{\lambda},\lambda,\gamma,0)$, by adding an extra fiat money which is not initially held by any agent. Then for a single large enough choice of M, $\tilde{\lambda}$, λ, γ the GELBI allocations of $E^*(M,\tilde{\lambda},\lambda,\gamma)$ coincide with the GEI of the underlying economy.

We see from Theorem 3 that our GELBI model includes the GEI as a special case. It also gives a concrete representation for the units of account that appear in the GEI model.

The following remark also needs no proof.

Remark 3. Reread Theorem 3 as follows. Fix λ and assume that the columns of A are linearly independent. Delete "no default" and "$K_{sj\ell} = 1$." Replace GEI by GEI_λ. Let "large enough" refer now only to the triple M, $\tilde{\lambda}$, γ. The statement so obtained is still true.

We see from Remark 3, that our GELBI model also includes the GEI_λ (as defined in [5]) as a special case.

7. THE VALUE OF MONEY

The discussion of this section, and all its propositions, refer to both the models $E = E(M,\tilde{\lambda},\lambda)$ and $E^* = E^*(M,\tilde{\lambda},\lambda,\gamma,0)$.

Prposition 1. Suppose money is a commodity for some agent h in at least one state $s = 0, 1, \ldots, S$. Then in any GELBI, $p_{0L} > 0$ and $p_{sL} > 0$.

This is immediate from (ii) of Step 10 and (iii) of Step 6 in the proof of Theorem 1.

Proposition 2. Suppose money is fiat. Fix $(M,\tilde{\lambda},\lambda,\gamma)$. Fix assets A in the underlying economy. Then, for fixed utilities u and generic endowments e (or for fixed endowments e and generic utilities u), $p_{0L} > 0$ at every GELBI of the economy (A,e,u) .

This is also clear since, from the essentiality of money, $p_{0L} > 0$ unless the initial endowments are statewise Pareto optimal in every state $s = 0, 1, \ldots, S$. Such pairs (e,u) are clearly degenerate. The rest are called nondegenerate.

Propositions 1 and 2 pertain to a longstanding puzzle in monetary theory (sometimes called the Hahn problem [18]), that abstract proofs of the existence of monetary equilibria may not demonstrate anything more than the existence of trivial equilibria in which the price of money is zero. Even when money is fiat, we overcome the Hahn problem in our framework in two ways. First, by Proposition 2, the price of money is generically positive

at equilibrium. Second, even when it drops to zero, there is a sense in which the equilibrium is not trivial. Since money is essential, we could clear all commodity markets by announcing prices at which no agent would lwant to trade.

In fact the equilibrium with $p_{0L} = 0$ is nontrivial in a more refined sense. If we removed the fiat money and the cash-in-advance constraints from our model, we could think of an equilibrium with default, but obviously without bankruptcy. Call this a GEI_λ equilibrium, following [5]. Refine the definition of the essentiality of money by requiring $p_{0L} > 0$ unless no trade is a GEI_λ equilibrium. In Remark 1 in the Appendix, we show that this refinement leaves the existence proof of equilibrium intact.

To sum up, when $p_{0L} = 0$, not only commodity, but also asset markets, can be cleared via prices at which no agent would want to trade.

Aside from $p_{0L} > 0$ and $p_{sL} > 0$, there is another indicator of the value of money, namely θ. We will exhibit two situations with fiat money where $\theta \neq 0$ at any GELBI.

Proposition 3. Suppose (e,u) of the underlying economy is nondegenerate, money is fiat, and $\sum_h e^h_{0L} > 0$. Then at any GELBI of E or E^*, $\theta \neq 0$.

There is a surprising aspect to this simple proposition. Consider a nondegenerate economy with fiat money which has a GELBI with $\theta = 0$. Now give the agents arbitrary positive endowments of the fiat money. A new GELBI with $\theta \neq 0$ will emerge. So an increase in the private supply of fiat money increases, and not decreases, interest rates. The reason is that price levels will inflate so far as to make even the larger supply of money become scarce and trade at positive interest rates.

Proposition 4. Consider a nondegenerate economy with fiat money. Fix M, λ. Then, for sufficiently small $\tilde{\lambda}$, all GELBI of $E(M,\tilde{\lambda},\lambda)$ have $\max\{\theta_s : s = 0, 1, \ldots, S\}$ uniformly bounded away from 0. (The same proposition holds for $E^*(M,\tilde{\lambda},\lambda,\gamma,0)$ if we also choose γ large enough.)

Note: Propositions 3 and 4 exhibit robust regions in the parameter space of $\tilde{\lambda}$ in which GELBI are disjoint from GEI or GEI_λ.

8. THE NEUTRALITY OF MONEY AND THE OPTIMUM QUANTITY OF MONEY

Let us consider a nondegenerate underlying economy with fiat or commodity money for which GEI exists. Choose $(M,\tilde{\lambda},\lambda,\gamma)$ large enough, in accordance with (i) and (ii) of Theorem 3, so that the GEI is a GELBI (η,σ) of $E^*(M,\tilde{\lambda},\lambda,\gamma,0)$, with no default or bankruptcies and $\theta = 0$. It is clear that if we raise some M_s and alter σ simply by having the agents borrow, hoard, and return without use the additional amount of M_s, then we continue at a GELBI with the same macrovariables η, and the same real trades.

There is a second sense in which money can be neutral. Suppose that in the above situation money is fiat and each asset delivers exclusively in commodities or exclusively in fiat money (i.e. assets are real or financial, not mixed). Scale up the whole vector M. Alter σ by scaling up money borrowings, money bids, etc. in the same proportion as M. Alter η by scaling up prices in the same proportion. It is again clear that we remain at a GELBI with no real changes.

Note that the hoarding strategy sustains the neutrality of money for any increase of the vector M, but it does so only if $\theta = 0$. The more traditional scaling strategy for neutrality works only in the case when every M_s increases by the same proportion, but on the other hand it con-

tinues to apply if $\theta > 0$, provided there is no bankruptcy.

But for the cases described above, money will typically not be neutral in our model. Changes in its stock will affect the real outcome at equilibrium. Indeed we might be often able to determine "optimal" levels of the vector M.

For instance let $(M,\bar{\lambda},\lambda,\gamma)$ be as before and consider this time lowering M while $(\bar{\lambda},\lambda,\gamma)$ are held fixed. By the same argument as in the proof of Proposition 4, bankruptcies and positive interest rates must appear as M falls low enough. What is striking is that Pareto-improvements may occur throughout the descent of M until optimum levels of M are reached. In general these levels will, of course, not be unique but form a surface in M-space.

One final way in which money can be neutral can be mentioned. If the whole vector M is scaled up, if assets deliver only commodities, if money is fiat, and if the bankruptcy penalties $\bar{\lambda}$ are scaled down in the same proportion, no real effects will occur. Of course there is no reason, especially in the short run, to expect bankruptcy law to exactly track inflation rates.[9]

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[9] If money is a commodity, all the credit c^h is in real terms, and this question is moot.

9. MULTIPLICITY OF GELBI

When all assets deliver everywhere in commodities, a standard transversality argument shows that, for any choice of $(M,\tilde{\lambda},\lambda,r)$, the GELBI are finite in number for generic (u,e,A). The multiplicicty question for GELBI becomes more interesting when there are financial assets.

Consider again a nondegenerate underlying economy, and suppose that we have fiat money and have chosen levels $(M,\tilde{\lambda},\lambda,\gamma)$ so that there is a GELBI of $E^*(M,\tilde{\lambda},\lambda,\gamma,0)$ at which each agent strictly prefers not to default or go bankrupt. Theorem 3 assures us that such a choice of $(M,\tilde{\lambda},\lambda,\gamma)$ is possible if the underlying economy has a GEI. Suppose that there are fewer assets than states of nature, and that the assets promise delivery in fiat money. Putting together Theorem 3(ii) with the results of [1], [12] on the multiplicity of GEI when assets pay off in units of account (i.e. fiat money) we see that there is a continuum of GELBI, one for each GEI whose prices are neither too high nor too low, with distinct real outcomes. In all of these there will be no default, or bankruptcy, and interest rates will be zero.

The picture is dramatically different if the parameters $(M,\tilde{\lambda},\lambda,\gamma)$ are such that all interest rates are positive at any GELBI. Proposition 4 assures us that such a choice of $(M,\tilde{\lambda},\lambda,\gamma)$ is possible. Here we conjecture that an elementary argument would show that generically there will be only a finite number of GELBI.

10. GELBI WITHOUT DEFAULT OR BANKRUPTCY

Let $\gamma^+ = \max_{h,s,\ell} \gamma^h_{s\ell}$

$\gamma^- = \min_{h,s,\ell} \gamma^h_{s\ell}$.

(Recall that $\bar{\lambda}^+$, $\bar{\lambda}^-$, λ^+ , λ^- are defined similarly.)

THEOREM 4. Assume that money is a commodity in some state $s = 0, 1, \ldots, S$. Fix M and $\rho = 0$. There exist $\tilde{\lambda}(M)$, $\lambda(M,\tilde{\lambda})$, $\gamma(M,\tilde{\lambda}) > 0$ such that

(i) $\bar{\lambda}^- > \tilde{\lambda}(M) \Rightarrow$ no agent goes bankrupt on M_s at any GELBI of $E^*(M,\tilde{\lambda},\lambda,\gamma,0)$.

(ii) $\lambda^- > \lambda(M,\tilde{\lambda}) \Rightarrow$ no agent defaults on asset deliveries in state s at any GELBI of $E^*(M,\tilde{\lambda},\lambda,\gamma,0)$.

(iii) $\gamma^- > \gamma(M,\tilde{\lambda}) \Rightarrow$ no agent goes bankrupt on commodity loans in state s at any GELBI of $E^*(M,\tilde{\lambda},\lambda,\gamma,0)$.

THEOREM 5. Assume that money is fiat and that assets deliver only in commodities. Fix M, $\tilde{\lambda} > 0$ and $\rho = 0$. Conclusions (ii) and (iii) of Theorem 4 hold.

In general, for any underlying economy we could imagine gradually raising the default and bankruptcy penalties $\tilde{\lambda}$, λ , γ . At each step we are sure to find GELBI (by Theorem 2). In the beginning there will be bankruptcies and defaults, and the money rates of interest may be positive, so that liquidity is scarce. If money is commodity, then by Theorem 4, all defaults and bankruptcies will disappear once the penalties cross certain threshold levels. But this need not be a desirable progression for society. The outcomes with lower penalties permitting default and bankruptcies to

occur may Pareto dominate the GELBI with harsher penalties. (See [5].)

By concentrating on GEI one in effect restricts attention to the upper end of the spectrum of $(M,\tilde{\lambda},\lambda,\gamma)$, ignoring the more realistic, and often more efficient region lying behind.

11. THE LIQUIDITY TRAP

Take an economy in which no asset pays off in units of account, and for which GEI does not exist. Examples are well known, see e.g. [18]. Let M be a fiat money. Fix $\tilde{\lambda} > 0$. Choose $\tilde{\lambda}$, λ, γ large enough in accordance with Theorem 5 to ensure that there is no default on deliveries of assets (i.e. on λ) or on commodity borrowing (i.e. on γ) at any GELBI of $E^*(M,\tilde{\lambda},\lambda,\gamma,0) = E^*_M$, for any M.

Let $M \to \infty$ componentwise, and for each M let (η^M, σ^M) denote a GELBI of E^*_M.

Let θ^M be the vector of interest rates that occur at (η^M, σ^M). (Thus $\theta^M \in R_+ \times R_+^S$.) For no M can it happen that $\theta^M = 0$, for then, by Theorem 3, (η^M, σ^M) would correspond to a GEI of E, which we know does not exist.

Thus no matter how much bank money M is poured into the system, agents always demand more, forcing interest rates to remain positive.

Even more: $M \cdot \theta^M$ is bounded away from 0 (where $\cdot$ denotes dot product). For if $M \cdot \theta^M \to 0$ then the bankruptcy penalties incurred by agents also goes to 0. But now consider a convergent subsequence of (η^M, σ^M). The limit will constitute a GEI, a contradiction. So we see that for all M, $M \cdot \theta^M > c$ for some $c > 0$; the agents' "excess demand" for money is always a significant step ahead of the supply M, even as $M \to \infty$.

Thus our model interprets the nonexistence of GEI as a "liquidity trap." The agents are anxious to borrow M and, in fact, incur a deadweight utility loss (of the order of $\bar{\lambda}c$) on that account. When the Government responds by increasing the supply M, this anxiety is not reduced. Agents continue to put up with the same deadweight loss in order to acquire enough liquidity. No amount of M that the Government injects into the banking system can alleviate their demand for money. In the meantime the real part of the economy remains unmoved by the money explosion! (Just take a subsequence of (η^M, σ_M) with all trades convergent.)

The same analysis can be done with commodity money using Theorem 4. In this case bankruptcies will also be zero if $\bar{\lambda}$ is taken large enough. But as before we will always see $\theta^M > 0$ and $M \cdot \theta^M > c > 0$. The deadweight utility loss now does not arise from bankruptcy, but because agents must give up part of their endowment of commodity money to the bank. It is therefore of size ξc, where c is the amount of endowment lost to the bank.

$$\eta_\varepsilon(\sigma^1, \ldots, \sigma^H) = (\theta_0, p_0, p_{0L}, \pi, (\theta_s, p_s, p_{sL}, K_s)_{s\in S} \in R_+^N$$

where

$$1 + \theta_s = \frac{\varepsilon + \sum_{h=1}^{H} \mu_s^h}{\varepsilon + M_s}$$

$$p_{s\ell} = \frac{\varepsilon + \sum_{h=1}^{H} b_{s\ell}^h}{\varepsilon + \sum_{h=1}^{H} q_{s\ell}^h}$$

$$p_{0L} = p_{sL} = 1$$

$$\pi_j = \frac{\varepsilon + \sum_{h=1}^{H} a_j^h}{\varepsilon + \sum_{h=1}^{H} r_j^h}$$

$$K_{sj\ell} = \begin{cases} \dfrac{\varepsilon A_{sj\ell} + \sum_{h=1}^{H} D_{sj\ell}^h}{\varepsilon A_{sj\ell} + \sum_{h=1}^{H} r_j^h A_{sj\ell}} & \text{if } A_{sj\ell} > 0 \\ 1 & \text{if } A_{sj\ell} = 0 \end{cases}$$

We will say that (η,σ) is an ε_1-ε-GELBI if

(i) $\eta = \eta_\varepsilon(\sigma)$ for $\sigma \in (\Sigma_{\varepsilon_1})^H$ and

(ii) σ^h maximizes $U^h \cdot F_\eta^h$ on $\Sigma_\eta^h \cap \Sigma_{\varepsilon_1}$

An ε-GELBI is a 0-ε-GELBI.

APPENDIX

PROOF OF THEOREM 1

Note that Σ_η^h is a set of the form

$$\{z \in R_+^m : L_i(z) \leq 0,\ i = 1, \ldots, n\}$$

where n and m are suitable integers, and each L_i is a linear function from R^m to R. Therefore Σ_η^h is clearly convex.

Also note that the map from individual choices to outcomes (given a fixed η)

$$\Sigma_\eta^h \xrightarrow{F_\eta^h} R_+^{L(S+1)} \times R^S \times R^S \times R^{S\times J\times L}$$

is linear.

Finally note that the payoff to agent h is concave on the range of F_η^h.

Recall that Σ and R_+^N are the ambient Euclidean spaces in which individual choices σ^h and macrovariables η exist.

For any ε_1, $\varepsilon > 0$ define

$$\Sigma_{\varepsilon_1} = \{x \in \Sigma : x_j \leq 1/\varepsilon_1\} ;$$

and define

$$(\Sigma_{\varepsilon_1})^H \xrightarrow{\eta_\varepsilon} R_+^N \text{ by}$$

STEP 1. An ε_1-ε-GELBI exists for any $\varepsilon_1, \varepsilon > 0$. For any $\sigma \in (\Sigma_{\varepsilon_1})^H$ define $\psi^h(\sigma) \subset \Sigma_{\varepsilon_1}$ by $\psi^h(\sigma) = \underset{\hat{\sigma}^h \in \Sigma^h_{\eta_\varepsilon(\sigma)} \cap \Sigma_{\varepsilon_1}}{\operatorname{argmax}} \{U^h \cdot F^h_{\eta_\varepsilon(\sigma)}(\hat{\sigma}^h)\}$. The correspondence $\sigma \to \psi^1(\sigma) \times \ldots \times \psi^H(\sigma)$ satisfies all the conditions of Kakutani's fixed point theorem, and any such fixed point is easily seen to be an ε_1-ε-GELBI.

STEP 2. For sufficiently small ε, every ε-ε-GELBI is an ε-GELBI.

Proof. Denote a ε-ε-GELBI by $(\eta^\varepsilon, \sigma(\varepsilon))$ where $\eta^\varepsilon = (\theta^\varepsilon, \ldots)$ and $\sigma(\varepsilon) = \{\sigma^h(\varepsilon)\}_{h=1}^H$ with $\sigma^h(\varepsilon) = (\mu_0^h(\varepsilon), \ldots, (x^h_{sL}(\varepsilon))_{s \in S})$.

Put $\bar{M} = M_0 + \max_{1 \le s \le S} \{M_s + \sum_{h=1}^H [e^h_{0L} + e^h_{sL}]\}$. Take ε small enough so that $\varepsilon \sum_{j=1}^J A_{sjL} < \bar{M}$ for all $s = 1, \ldots, S$, and $\varepsilon \cdot 2(L-1) < \bar{M}$. The total amount of money spent in any state is clearly at most $5\bar{M}$. Thus, $a^h(\varepsilon)$ and $b^h(\varepsilon)$ are bounded by $5\bar{M}$ independent of ε.

Similarly, put $\bar{e} = \max_{\substack{0 \le s \le S \\ 1 \le \ell \le L-1}} \sum_{h=1}^H e^h_{s\ell}$. Let ε be small enough that $\varepsilon \sum_{j=1}^J A_{sj\ell} < \bar{e}$ for all $s = 1, \ldots, S$ and $\ell = 1, \ldots, L-1$; and also $\varepsilon < \bar{e}$. Clearly at most $3\bar{e}$ of any commodity can be sold at any stage, and once again $q^h(\varepsilon)$ is bounded by $3\bar{e}$, independent of ε.

It remains to check that the bound of $1/\varepsilon$ will not be binding on $\mu^h_s(\varepsilon)$, $r^h_j(\varepsilon)$. Note that $\sum_h \mu^h_s(\varepsilon) < (M_s + \varepsilon)(1 + \theta^\varepsilon_s)$, so to show that the constraint of $1/\varepsilon$ on $\mu^h_s(\varepsilon)$ is not binding, it suffices to show that θ^ε_s is bounded. Now, if $(M_s + \varepsilon)(1 + \theta^\varepsilon_s) > 6\bar{M}$, then the agents owe $(M_s + \varepsilon)(1 + \theta^\varepsilon_s) - \varepsilon > 5\bar{M}$, since $\varepsilon < \bar{M}$. Thus at least one agent is going bankrupt in the amount $[(M_s + \varepsilon)(1 + \theta^\varepsilon_s) - \varepsilon - 5\bar{M}]/H$ or more, with dis-

utility of $\bar{\lambda}^{-}[((M_s + \varepsilon)(1 + \theta_s^{\varepsilon}) - \varepsilon - 5\bar{M})/H]$ or more, where we have defined

$$\left.\begin{matrix}\tilde{\lambda}^{+} \\ \tilde{\lambda}^{-}\end{matrix}\right\} = \left.\begin{matrix}\max \\ \min\end{matrix}\right\} \left\{ \bar{\lambda}_{0s}^{h}, \bar{\lambda}_{s}^{h}, : s = 1, \ldots, S; h = 1, \ldots, H \right\} .$$

$$\left.\begin{matrix}\lambda^{+} \\ \lambda^{-}\end{matrix}\right\} = \left.\begin{matrix}\max \\ \min\end{matrix}\right\} \left\{ \lambda_{sj\ell} , \quad s = 1,\ldots, S, \; j = 1,\ldots, J , \; \ell = 1,\ldots, L \right\}$$

Suppose

$$\tilde{\lambda}^{-}\left[\frac{(M_s + \varepsilon)(1 + \theta_s^{\varepsilon}) - \varepsilon - 5\bar{M}}{H}\right] > u^{*} = \max_{1\le h\le H} (u^{h}(z^{*}) - u^{h}(e^{h})) ,$$

where $z^{*}_{s\ell} = 3\bar{e} + 5\bar{M}$

(for $s = 0, 1, \ldots, S$ and $\ell = 1, \ldots, L$). Then the bankrupt agent would do better by not trading and getting $u^{h}(e^{h})$. This contradicts that we are at an ε-ε-GELBI. So each θ_s^{ε}, hence also $\mu_s^{h}(\varepsilon)$, is bounded from above independently of ε. The bound on $1 + \theta_s^{\varepsilon}$ is

$$1 + G^{*} = \left[\frac{u^{*}H}{\tilde{\lambda}^{-}} + 6\bar{M}\right]\frac{1}{\min\limits_{0\le s\le S} M_s} .$$

Finally we need to show that the constraint of $1/\varepsilon$ on $r_j^{h}(\varepsilon)$ is not binding. First consider the case that asset j promises money in some state s (i.e. $A_{sjL} > 0$). The most that agent h could deliver is $5\bar{M}$. Hence

$$\lambda^{-}(r_j^{h}(\varepsilon)A_{sjL} - 5\bar{M}) \le u^{*} ,$$

otherwise h does better by not trading, again contradicting the ε-ε-GELBI. If asset j never promises money in period 1, then $A_{sj\ell} > 0$ for some commodity $s\ell$. Hence again $\lambda^{-}(r_j^h(\varepsilon)A_{sj\ell} - 3\bar{e}) \leq u^*$. The two inequalities gives a bound on each $r_j^h(\varepsilon)$, independently of ε, and completes the proof of STEP 2.

Let $(\eta^\varepsilon, \sigma(\varepsilon))$ now denote an ε-GELBI in all the steps below (with $\eta^\varepsilon = (\theta^\varepsilon, p^\varepsilon, \ldots)$), and $\sigma(\varepsilon) = \{\sigma^h(\varepsilon) = (\mu_0^h(\varepsilon), \ldots, (x_{sL}^h(\varepsilon)))_{s=1}^S\}_{h=1}^H$.

STEP 3. $\theta_s^\varepsilon \geq 0$.

If $\theta_s^\varepsilon < 0$ an agent can borrow Δ of money return it without going further bankrupt, and have money left over to purchase commodities in state s. By monotonicity of utilities, we contradict being at an ε-GELBI.

STEP 4. There exist $C > 0$ such that for all ε sufficiently small

$$p^\varepsilon > Ce .$$

(Here e is the unit vector $(1, \ldots, 1)$ of suitable dimension).

To check this suppose $p_{s\ell}^\varepsilon \to 0$ as $\varepsilon \to 0$ for some $s\ell$ $(s = 0, 1, \ldots, S\ ;\ \ell = 1, \ldots, L-1)$. Choose any agent h. He can borrow Δ of bank money M_s to buy $\Delta/p_{s\ell}^\varepsilon$ of commodity $s\ell$ and incur bankruptcy of at most $(1+G^*)\Delta$.

Let

$$\left.\begin{matrix}\xi^+ \\ \xi^-\end{matrix}\right\} = \left.\begin{matrix}\max \\ \min\end{matrix}\right\} \left\{ \frac{\partial u^h}{\partial x_{s\ell}}(y) : y_{s\ell} \leq 3\bar{e} + 5\bar{M} = z_{s\ell}^*; \begin{array}{l}\ell = 1, \ldots, L-1 \text{ if money} \\ \text{is fiat in state s for h;} \\ \ell = 1, \ldots, L \text{ if money is} \\ \text{commodity in state s for} \\ \text{h; } s = 0, 1, \ldots, S\end{array} \right\} .$$

(See Step 2 for the definition of $\bar{e}$, $\bar{M}$, G^*.) Since nobody can get

more than z^* at an ε-GELBI, we see that the net gain in utility of the agent is at least

$$\left[\frac{\xi^-}{p^\varepsilon_{s\ell}} - \tilde{\lambda}^+(1+G^*)\right]\Delta$$

for small Δ. This must be nonpositive, so

$$p^\varepsilon_{s\ell} \geq \frac{\xi^-}{\tilde{\lambda}^+(1+G^*)}$$

concluding the proof of Step 4.

We will construct a GELBI as a limit of ε-GELBI $(\eta^\varepsilon, \sigma(\varepsilon))$. For this purpose, we take a sequence of ε, which define a limiting pair (η,σ).

STEP 5. Select a sequence of ε and subsequences of subsequences so that

(i) $\sigma(\varepsilon)$ converges to σ (by Step 2, choices are bounded independent of ε, so this is feasible).

(ii) Choose a subsequence so that each sum $E_s(\varepsilon) = \sum_{g=1}^{L-1} p^\varepsilon_{sg}$ $s = 1, \ldots, S$ and $E_0(\varepsilon) = \left[\sum_{g=1}^{L-1} p_{0g} + \sum_{j=1}^{J} \pi_j\right]$ is convergent (including possibly to infinity).

(iii) $\dfrac{p^\varepsilon_{s\ell}}{\sum_{g=1}^{L-1} p^\varepsilon_{sg}}$ converges for $\ell = 1, \ldots, L-1$; $s = 1, \ldots, S$.

If $\lim E_s(\varepsilon) < \infty$, define $p_{sL} = 1$ and $p_{s\ell} = \lim p^\varepsilon_{s\ell}$, $\ell = 1, \ldots, L-1$, $s = 1, \ldots, S$. If $\lim E_s(\varepsilon) = \infty$, define $p_{sL} = 0$, and $p_{s\ell} = \lim p^\varepsilon_{s\ell}/E_s(\varepsilon)$, $\ell = 1, \ldots, L-1$, $s = 1, \ldots, S$.

(iv) (a) $\dfrac{p^{\varepsilon}_{0\ell}}{\sum_{g=1}^{L-1} p^{\varepsilon}_{0g} + \sum_{j=1}^{J} \pi^{\varepsilon}_{j}} = p^{\varepsilon}_{0\ell}/E(\varepsilon)$ converges for $\ell = 1, \ldots, L-1$

(b) $\dfrac{\pi^{\varepsilon}_{j}}{\sum_{g=1}^{L-1} p^{\varepsilon}_{0g} + \sum_{j=1}^{J} \pi^{\varepsilon}_{j}} = \pi^{\varepsilon}_{j}/E(\varepsilon)$ converges for $j = 1, \ldots, J$.

If $\lim E_0(\varepsilon) < \infty$, define $p_{0L} = 1$ and $p_{s\ell} = \lim p^{\varepsilon}_{s\ell}$, $\ell = 1, \ldots, L-1$, and $\pi_j = \lim \pi^{\varepsilon}_{j}$, $j = 1, \ldots, J$. If $\lim E_0(\varepsilon) = \infty$, define $p_{0L} = 0$, and $p_{s\ell} = \lim p^{\varepsilon}_{s\ell}/E_0(\varepsilon)$, $\ell = 1, \ldots, L-1$ and $\pi_j = \lim \pi^{\varepsilon}_{j}/E_0(\varepsilon)$, $j = 1, \ldots, J$.

(v) $x^h(\varepsilon)$ converges to x^h for $h = 1, \ldots, H$.

(vi) $\theta^{\varepsilon}_{s} \to \theta_s$, $s = 0, 1, \ldots, S$.

This is possible since $0 \le \theta^{\varepsilon}_{s} \le G^*$.

(vii) (a) $\lim K^{\varepsilon}_{sj\ell} = K_{sj\ell}$ exists.

(b) If $\sum_{h=1}^{H} r^h_j(\varepsilon) \to 0$ and, $K_{sj\ell} < 1$ for some $s\ell$, then

(b1) $\lim \dfrac{r^h_j(\varepsilon)}{\sum_{h'=1}^{H} r^{h'}_j(\varepsilon)} = t^h_j$ exists and

(b2) If $t^h_j > 0$, then for all $s\ell$ with $A_{sj\ell} > 0$,

$$\lim \frac{D^h_{sj\ell}(\varepsilon)}{r^h_j(\varepsilon)A_{sj\ell}} = \rho_{sj\ell} \text{ exists.}$$

Note that $D^h_{sj\ell}(\varepsilon) \le r^h_j(\varepsilon)A_{sj\ell}$ at an ε-GELBI, so $0 \le K^{\varepsilon}_{sj\ell} \le 1$, so (a) is possible. If for some $s\ell$, $K_{sj\ell} < 1$, then we can suppose throughout the subsequence that $\sum_{h'=1}^{H} r^{h'}_j(\varepsilon) > 0$, so (b1) makes sense. Note finally that in this case $K_{sj\ell} \ge \sum_{h=1}^{H} t^h_j \rho^h_{sj\ell}$ since $\varepsilon A_{sj\ell}$ is present in both the numer-

ator and denominator of $K^{\varepsilon}_{sj\ell}$.

STEP 6. Consider $s = 1, \ldots, S$. If $E_s(\varepsilon) \to \infty$, we claim:

(i) $p^{\varepsilon}_{sg} \to \infty$ for all $g = 1, \ldots, L-1$.

(ii) $q^h_{sg}(\varepsilon) \to 0$ and $b^h_{sg}(\varepsilon)/p^{\varepsilon}_{sg} \to 0$ for all $h = 1, \ldots, H$; $g = 1, \ldots, L-1$.

(iii) money must be fiat in state s .

(iv) No agent goes bankrupt on either M_0 or M_s , or defaults on A_{sjL} in state s (in η^{ε}, $\sigma(\varepsilon)$) for ε small enough.

(v) $\theta^{\varepsilon}_s \to 0$ as $\varepsilon \to 0$.

(vi) $$\frac{\partial u^h(x^h)}{\partial x_{sg}} \begin{cases} = \lambda^h p_{sg} & \text{if } x^h_{sg} > 0 \\ \leq \lambda^h p_{sg} & \text{if } x^h_{sg} = 0 \end{cases}$$

for some $\lambda^h > 0$, and all $h = 1, \ldots, H$, $g = 1, \ldots, L-1$.

Proof of (i). Clearly $p^{\varepsilon}_{s\ell} \to \infty$ for some ℓ . Let p^{ε}_{sg} be bounded for $g \neq \ell$. Take any agent who has a positive amount of ℓ after the delivery of assets (but prior to the trade of commodities) in state s . He can borrow Δ of money M_s in state s , sell $(1 + \theta_s)\Delta/p^{\varepsilon}_{s\ell}$ of commodity ℓ (for small Δ) , buy $\Delta/p^{\varepsilon}_{sg}$ of commodity g , return $(1 + \theta^{\varepsilon}_s)\Delta$ to the bank, leaving the rest of his outcome unaltered. So his change in utility is $\Delta(\xi^-/p^{\varepsilon}_{sg} - \xi^+(1 + \theta^{\varepsilon}_s)/p^{\varepsilon}_{s\ell}) \geq \Delta(\xi^-/p^{\varepsilon}_{sg} - \xi^+(1+G^*)/p^{\varepsilon}_{s\ell})$, which becomes positive, contradicting that $(\eta^{\varepsilon}, \sigma^{\varepsilon})$ is an ε-GELBI.

Proof of (ii). Note $p^{\varepsilon}_{sg} \leq (5\bar{M} + \varepsilon)/[\sum_{h=1}^{H} q^h_{sg}(\varepsilon) + \varepsilon]$, and $b^h_{sg}(\varepsilon) \leq 5\bar{M}$. So, from (i), $q^h_{sg}(\varepsilon) \to 0$ and $b^h_{sg}(\varepsilon)/p^{\varepsilon}_{sg} \to 0$.

Proof of (iii) and (iv). For any commodity g with $e^h_{sg} > 0$, either $\sum_{j=1}^{J} D^h_{sjg}(\varepsilon) > 0$, or else $e^h_{sg} - q^h_{sg}(\varepsilon) - \sum_{j=1}^{J} D^h_{sjg}(\varepsilon) > 0$ (since by (ii) $q^h_{sg}(\varepsilon) \to 0$) for small ε . By delivering Δ less or by consuming Δ

less of commodity g (whichever is feasible), and selling Δ more at p^{ε}_{sg}, and finally using the money so obtained either as consumption (if money is commodity for h) or if h is going bankrupt to defray $\Delta p^{\varepsilon}_{sg}$, against M_0 or M_s, he gains at least

$$\Delta\left[p^{\varepsilon}_{sg} \min\{\tilde{\lambda}^{h}_{0s}, \tilde{\lambda}^{h}_{s}, \xi^{-}\lambda^{-}\} - \max\{\xi^{+}, \lambda^{+}\}\right]$$

utility which becomes positive as $p^{\varepsilon}_{sg} \to \infty$, contradicting that η^{ε}, $\sigma(\varepsilon)$ is an ε-GELBI. This proves (iii) and (iv), as far as bankruptcy is concerned.

It remains to show that $c^{h}_{sjL}(\varepsilon) = 0$ for small enough ε. If $c^{h}_{sjL}(\varepsilon) > 0$ for some h, then h can borrow $c^{h}_{sjL}(\varepsilon)$ of bank money M_s, deliver fully on A_{sjL}, and repay the bank loan $(1 + \theta^{\varepsilon}_{s})c^{h}_{sjL}(\varepsilon)$ by foregoing consumption or delivery of $(1 + \theta^{\varepsilon}_{s})c^{h}_{sjL}(\varepsilon)/p^{\varepsilon}_{s\ell}$ quantity of some good $s\ell$ as before. Then his net change in utility is at least

$$c^{h}_{sjL}(\varepsilon)\left[\lambda^{h}_{sjL} - \frac{1 + \theta^{\varepsilon}_{s}}{p^{\varepsilon}_{s\ell}} \max\{\xi^{+}, \lambda^{+}\}\right]$$

which becomes positive since $\theta^{\varepsilon}_{s} \le G^{*}$ and $p^{\varepsilon}_{s\ell} \to \infty$, a contradiction.

Proof of (v). By (iii) money is fiat for all h in state s. Hence if θ^{ε}_{s} is bounded away from 0, at least one agent h is going bankrupt (for small enough ε), either against M_0 or M_s. The reason is that at least $(1 + \theta^{\varepsilon}_{0})M_0 + (1 + \theta^{\varepsilon}_{s})M_s$ is owed against M_0 and M_s. Since θ^{ε}_{0}, $\theta^{\varepsilon}_{s} \ge 0$ by Step 3, and θ^{ε}_{s} is bounded away from 0, we get

$$(1 + \theta^{\varepsilon}_{0})M_0 + (1 + \theta^{\varepsilon}_{s})M_s > M_0 + M_s + 2\varepsilon(L-1) + \varepsilon \sum_{j=1}^{J} A_{sjL} + 2\varepsilon$$

for small ε . The LHS is less than the money owed by agents against M_0 and M_s , and the RHS is the total amount of money in the state s . Hence some agent must be going bankrupt, contradicting (iv).

Proof of (vi). Since $\theta_s^\varepsilon \to 0$, the "wedge" between buying prices $(1 + \theta_s^\varepsilon)p_s^\varepsilon$ and selling prices p_s^ε is going to 0. So in the limit, the problem is Walrasian, and (vi) follows.

STEP 7. If $p_{sL} = 0$ for some $s = 1, \ldots, S$, then by (vi) of Step 6 no reallocation of final consumption bundles in state s Pareto-improves the agents (with their choices in the tree fixed everywhere else according to σ), hence money is essential at (η,σ) in every state $s = 1, \ldots, S$.

STEP 8. Suppose $\lim E_0(\varepsilon)$ is finite. Then by Step 4, $p_{0\ell} > 0$ for $\ell = 1, \ldots, L-1$. Moreover

Claim 1. $\pi_j = 0 \Rightarrow \left\{ \begin{array}{l} A_{sj\ell} = 0 \quad \text{if} \quad \ell \neq L \\ A_{sjL} > 0 \Rightarrow \text{money is fiat in state } s \text{ , and } p_{sL} = 0 \end{array} \right\}$

Claim 2. (η,σ) is a preGELBI.

Proof of Claim 1. Suppose asset j promises delivery of something other than fiat money, i.e. $A_{sj\ell} > 0$ for at least one commodity $s\ell$ (including commodity money). Then we must have $\pi_j > 0$. Indeed, if $\sum_h r_j^h(\varepsilon) = 0$ for all ε (in the sequence chosen) then $\pi_j^\varepsilon \geq 1$. Otherwise (by going to a subsequence) take h such that $r_j^h(\varepsilon) > 0$ for all ε . Let h reduce $r_j^h(\varepsilon)$ by Δ and borrow $\Delta\pi_j^\varepsilon$ of M_0 . If he was defaulting on $A_{sj\ell}$, leave the rest of his actions as before. If not, let him deliver fully on $A_{sj\ell}$ and consume Δ more of $s\ell$. In any case his gain in utility is at

least

$$\Delta[A_{sj\ell} \cdot \min(\xi^-, \lambda^h_{sj\ell}) - \pi^\varepsilon_j(1+G^*)\bar{\lambda}^+ S]$$

(where G^* is the upper bound on θ^ε_0 for all ε). Since we are at an ε-GELBI, the term within [ ] must be nonpositive, which gives

$$\pi^\varepsilon_j \geq \frac{A_{sj\ell} \min(\xi^-, \lambda^h_{sj\ell})}{(1+G^*)\bar{\lambda}^+ S}$$

proving the first implication.

Next suppose $A_{sjL} > 0$ and $p_{sL} > 0$. Then by Step 5, $p_{sL} = 1$. Repeat the argument above, except that h consumes $\Delta/p_{s\ell}$ of $s\ell$ for any $s\ell$ (if he was not defaulting on the delivery of A_{sjL}). Then we obtain

$$\pi^\varepsilon_j \geq A_{sjL} \frac{\min(\xi^-/p^\varepsilon_{s\ell}, \lambda^h_{sjL})}{(1 + G^*)\bar{\lambda}^+ S}$$

or $\pi^\varepsilon_j \geq 1$. But $\lim \pi^\varepsilon_j = \pi_j = 0$, hence we have shown $p_{sL} = 0$. But, by Step 5, $E_s(\varepsilon) \to \infty$, and then by Step 6(iii) money is fiat in state s.

Proof of Claim 2. Let $S^* = \{s \in S : p_{sL} = 0\}$. It follows from Step 5 that $w^h = \lim_\varepsilon w^h(\varepsilon)$, where $w^h(\varepsilon) = (x^h(\varepsilon), c^h(\varepsilon))$ is the outcome of the ε-GELBI $(\eta^\varepsilon, \sigma(\varepsilon))$, exists. From Step 6 we know that $w^h = F_\eta(\hat{\sigma}^h)$ where

$$\hat{\sigma}^h = \lim_{\varepsilon \to 0} \sigma^h(\varepsilon)$$

except that for $s \in S^*$, $\hat{\sigma}^h$ prescribes borrowing M_s/H on M_s, delivering nothing on A_{sjL}, buying and selling nothing of commodities $s\ell$,

$\ell = 1, \ldots, L-1$, and paying back nothing on M_0 , and M_s/H on M_s . It is also clear from Step 6 that w^h maximizes U^h on $F_\eta(\Sigma^h_\eta)$.

STEP 9. Suppose $\lim E_0(\varepsilon) < \infty$, i.e. $p_{0L} = 1$. If for any asset j , $\Sigma_h r^h_j = 0$, raise π_j to the largest $\pi^*_j \geq \pi_j$ at which no agent would want to sell asset j , holding the rest of η fixed. Let η^* be the same as η with all such π_j replaced by the corresponding π^*_j . Then K is rationalizable at (η^*, σ) . In fact, (η^*, σ) is a GELBI.

Proof of Step 9. For any j with $\sum_{h=1}^{H} r^h_j = 0$, the set $H(j)$ of agents on the verge of selling asset j at (η^*, σ) is obviously nonempty. If $\pi^*_j > \pi_j$, then for small ε it must have been that $\sum_{h=1}^{H} r^h_j(\varepsilon) = 0$. Hence $K^\varepsilon_{sj\ell} = 1$ for all $s\ell$, hence $K_{sj\ell} = 1$. For such j , expectations are clearly sufficiently optimistic. If $\pi^*_j = \pi_j$ and for some $s\ell$, $K_{sj\ell} < 1$, then for small enough ε , $\sum_{h=1}^{H} r^h_j(\varepsilon) > 0$. But then as pointed out in (vii) of Step 5, $K_{sj\ell} > \sum_{h=1}^{H} t^h_j \rho^h_{sj\ell}$. Obviously $t^h_j > 0$ only if $h \in H(j)$. Finally, we claim that if h is completely reliable at (η^*, σ) in state s and commodity ℓ on asset j , then $\rho^h_{sj\ell} = 1$. The reason is that if h is completely reliable at (η^*, σ) , the marginal increase in utility that h could gain by defaulting on $sj\ell$ is strictly negative. By continuity, this must also hold true in $(\eta^\varepsilon, \sigma(\varepsilon))$. Hence for small ε $D^h_{sj\ell}(\varepsilon) = r^h_j(\varepsilon) A_{sj\ell}$ if $A_{sj\ell} > 0$, so $\rho^h_{sj\ell} = \lim[D^h_{sj\ell}(\varepsilon)/r^h_j(\varepsilon)A_{sj\ell}] = 1$. Therefore K is rationalizable at (η^*, σ) . Since (η, σ) is a pre-GELBI (Step 8) in which money is essential (Step 7 and the hypotheses that $p_{0L} = 1$), so is (η^*, σ) since raising π_j to π^*_j does not affect any choices.

STEP 10. Suppose that the sum $E_0(\varepsilon) \to \infty$. Then

(i) $p^{\varepsilon}_{s\ell} \to \infty$ for all $\ell = 1, \ldots, L-1$; $s = 0, 1, \ldots, S$, so $p_{sL} = 0$ for all $s = 1, \ldots, S$.

(ii) money is fiat in each state $s = 0, 1, \ldots, S$.

(iii) $q^{h}_{s\ell}(\varepsilon) \to 0$ for all $h = 1, \ldots, H$; $s = 0, 1, \ldots, S$; $\ell = 1, \ldots, L-1$.

(iv) no agent goes bankrupt against M_s for any $s = 0, 1, \ldots, S$ in $(\eta^{\varepsilon}, \sigma(\varepsilon))$, for ε small enough.

(v) $\theta^{\varepsilon}_{s} \to 0$ for all $s = 0, 1, \ldots, S$.

(vi) $r^{h}_{j}(\varepsilon) \to 0$ for all $h = 1, \ldots, H$, or else asset j only delivers fiat money.

(vii) $x^{h}(\varepsilon) \to e^{h}$ for all $h = 1, \ldots, H$.

(viii) Let σ_0 prescribe zero purchases and sales of all commodities and assets and arbitrary μ^{h}_{s} , $\sum_{h=1}^{H} \mu^{h}_{s} = M_s$, $s = 0, 1, \ldots, S$. Then (η, σ_0) is a GELBI.

Proof of (i). If some $\pi^{\varepsilon}_{j} \to \infty$, then all $p^{\varepsilon}_{0\ell} \to \infty$ for otherwise, by borrowing $\Delta\pi^{\varepsilon}_{j}/(1 + \theta^{\varepsilon}_{0})$, selling Δ of asset j , and purchasing $\Delta\pi^{\varepsilon}_{j}/[(1 + \theta^{\varepsilon}_{0})p^{\varepsilon}_{0\ell}]$ of commodity 0ℓ , any agent h can increase his utility for ε small enough. If $\lim \pi^{\varepsilon}_{j} < \infty$ for all j , then $p^{\varepsilon}_{0\ell} \to \infty$ for at least one ℓ . But by the same argument as in (i) of Step 6, $p^{\varepsilon}_{0g} \to \infty$, for all $g = 1, \ldots, L-1$. We further deduce that $p^{\varepsilon}_{sg} \to \infty$ for all $s = 1, \ldots, S$ and $g = 1, \ldots, L-1$, since any agent h can sell a little of some commodity in period 0, inventory the money until period 1 and purchase commodities in all states $s = 1, \ldots, S$. This concludes the proof of (i).

Proofs of (ii)-(v). The proof that money is fiat proceeds as (iii) of Step 6. The proofs of (iii), (iv) and (v) are identical to the proofs of the (ii), (iv) and (v) in Step 6.

Proof of (vi). Assume $r_j^h(\varepsilon)$ stays bounded away from zero. If asset j delivers some commodity $g = 1, \ldots, L-1$ in some state $s = 1, \ldots, S$ then $\pi_j^\varepsilon \to \infty$, for otherwise if h reduces $r_j^h(\varepsilon)$ by Δ, borrows $\Delta\pi_j^\varepsilon$ of fiat money in period 0, and sells $\Delta\pi_j^\varepsilon(1 + \theta_0^\varepsilon)/p_{0\ell}^\varepsilon$ of any commodity ℓ in period 0 (such an ℓ exists since $q_{0\ell}^h(\varepsilon) \to 0$ and $e_0^h \neq 0$) then (further deviating as in the proof of Claim 1) his change in utility is at least

$$\Delta\left[\min(\xi^-, \lambda^-)A_{sjg} - \xi^+ \frac{\pi_j^\varepsilon(1 + \theta_0^\varepsilon)}{p_{0\ell}^\varepsilon}\right]$$

which becomes positive since $p_{0\ell}^\varepsilon \to \infty$. Hence $\pi_j^\varepsilon \to \infty$. Since money is bounded, this can only happen if $\sum_{h=1}^{H} r_j^h(\varepsilon) \to 0$. This concludes the proof of (vi).

Proof of (vii). This follows immediately from (iii)-(vi).

Proof of (viii). Since $\theta_s^\varepsilon \to 0$, the wedge between buying and selling prices for commodities and assets goes to zero in each state $s = 0, 1, \ldots, S$. The optimality conditions for each h in an ε-GELBI imply, taking limits, that (vi) of Step 6 holds for each s. This shows that money is essential at (η, σ_0). Q.E.D.

REMARK 1. Since no trade is permitted when $p_{0L} = 0$ we have not insisted that K be rationalizable in this case. (Indeed H(j) will be empty since, no matter how high $\hat{\pi}_j$ becomes, asset sales are ruled out.) But it turns out that K (obtained at the limit of ε-GELBI) is "rationalizable" in a somewhat different sense. Redefine " h is completely reliable in commodity $s\ell$ on asset j " to mean:

$$\text{"} \max_{1 \le g \le L-1} \left\{ \frac{\partial u^h}{\partial x_{sg}} \cdot \frac{p_{s\ell}}{p_{sg}} \right\} < \lambda^h_{sj\ell} \text{ ."}$$

This says that the marginal gain to h of defaulting on $A_{sj\ell}$ (given by the LHS) is less than the penalty of default. Now define " K is rationalizable at (η,σ) " exactly as before. Finally define " (η,σ) is a GEI_λ" exactly as GELBI except that liquidity constraints and bankruptcies are eliminated, and the revised notion of rationalizable is applied (see [5]). By taking limits of ε-GELBI carefully, one can check that when $p_{0L} = 0$ at the limit, no trade is a GEI_λ outcome. We leave this verification to the reader.

PROOF OF THEOREM 2. Define ε-ε-GELBI and ε-GELBI as before. Clearly as before ε-ε-GELBI exists.

Next we observe that at any ε-ε-GELBI the final consumption x^h of commodity bundles is bounded (independent of ε). Let $\sum_{h=1}^{H} x^h_{s\ell} - \sum_{h=1}^{H} e^h_{s\ell} = \delta > 0$, then some agent defaults on commodity $s\ell$ in at least the amount δ/H and incurs a penalty of at least $\nu(\delta/H)$, where $\nu = \min\{\gamma^-, \bar{\lambda}^-\}$, $\gamma^- = \min\{\gamma^h_{s\ell} : h = 1, \ldots, H;\ s = 1, \ldots, S;\ \ell = 1, \ldots, L-1\}$. But $u^{**} - \gamma^-(\delta/H) > 0$ at an ε-ε-GELBI, otherwise no trade would be an improve-

ment for the agent over his current choice. Thus $\delta < Hu^{**}/\nu$. Finally, since consumption bundles are uniformly bounded above, the marginal utility of consumption of any commodity is bounded away from zero (independent of small ε), so ξ^- can be defined and used as before.

The rest of the argument to show that an ε-ε-GELBI is also an ε-GELBI for small ε goes through with some obvious modifications (corresponding to the change in the model) as for Theorem 1. We give a sketch.

As in Step 2, there is never more than $5\bar{M}$ of money in the system. Moreover, at any ε-ε-GELBI, the money rates of interest θ_s^ε, and hence the money bids $\mu_s^h(\varepsilon)$ are bounded from above by the same argument as in Step 2, replacing u^* by u^{**}.

Now we show that the $r_j^h(\varepsilon)$ are bounded. If asset j promises fiat money in some state s, the proof in Step 2 of Theorem 1 applies with u^{**} in place of u^*, i.e. $\bar{\lambda}^-(r_j^h(\varepsilon)A_{sjL} - 5\bar{M}) \le u^{**}$ otherwise h does better by not trading. Next suppose that asset j never promises fiat money. Then $A_{sj\ell} > 0$ for some commodity $s\ell$. Let $r_j^h(\varepsilon) \to \infty$. Then $D_{sj\ell}^h(\varepsilon)/r_j^h(\varepsilon)A_{sj\ell} \to 1$ for all $s\ell$ with $A_{sj\ell} > 0$, otherwise the amount of default on $sj\ell$ by h will go to ∞, hence so will his default or bankruptcy penalty. But when his penalty exceeds u^{**} we have a contradiction as before to an ε-ε-GELBI. Hence $r_j^h(\varepsilon) \to \infty$ implies that $K_{sj\ell}^\varepsilon \to 1$, for all $s\ell$ with $A_{sj\ell} > 0$. This in turn implies that an extra unit of asset j yields utility at least $\xi^- \sum_{s=1}^{S} \sum_{\ell=1}^{L} \frac{1}{2}A_{sj\ell}$ for small ε. Yet $r_j^h(\varepsilon) \to \infty$ implies that $\pi_j(\varepsilon) \to 0$, since the total money bids on asset j are at most $5\bar{M}$. This is a contradiction, since any agent could borrow Δ money in period 0, incurring at most a bankruptcy penalty of $\bar{\lambda}^+\Delta(1 + G^*)$, but gaining $(\Delta/\pi_j(\varepsilon))\xi^- \sum_{s=1}^{S} \sum_{\ell=1}^{L} \frac{1}{2}A_{sj\ell}$ which goes to ∞ as $\pi_j(\varepsilon) \to 0$.

This shows that the $1/\varepsilon$ constraint on $r_j^h(\varepsilon)$ is not binding for small ε.

Exactly as in Step 4 and the proof of Claim 1 of Step 8 (of the proof of Theorem 1) we establish lower bounds for the prices p^ε, and for π_j^ε in any ε-ε-GELBI when asset j promises delivery of some commodity (independent of ε). Since p^ε is bounded from below, and the money stocks are bounded from above, the commodity offers $q_{s\ell}^h(\varepsilon)$ are bounded from above.

In ε-ε-GELBI there are new choice variables, namely how much commodity $1, \ldots, L-1$, an agent borrows (and returns to the "commodity bank"). We must show that these too are bounded, independent of ε. Let us begin with $\rho >> 0$. All bounds up until now are unaffected by ρ. Since $r_j^h(\varepsilon)$ is bounded from above, so is the total delivery any agent makes at an ε-ε-GELBI. Since consumptions of commodities are also bounded from above, as are the sales $q_{s\ell}^h(\varepsilon)$, no consumer will have need to borrow more than a bounded amount of commodities with any $\rho >> 0$. Note that this bound is independent not only of ε, but <u>also</u> of ρ. This establishes that an ε-ε-GELBI is also an ε-GELBI for small ε and for $\rho >> 0$ and, by letting $\rho \to 0$, also for $\rho = 0$.

The proof that we can obtain a GELBI of $E^*(M,\bar{\lambda},\lambda,r,\rho)$ as a limit of ε-GELBI goes exactly as in the proof of Theorem 1. Q.E.D.

<u>PROOF OF PROPOSITION 3</u>. Consider the end of the tree in state $s = 1, \ldots, S$ at a GELBI. If $p_{sL} \neq 0$, no agent h has positive money x_{sL}^h left over, otherwise he can borrow $x_{sL}^h/(1 + \theta_s)$ from M_s, consume (since $p_{sL} > 0$) some commodities in state s, and then return x_{sL}^h on M_s. Such a deviation world increase the utility of h. Thus, if

$p_{sL} > 0$, $x^h_{sL} = 0$ for all h . It follows that the money returned on $M_0 + M_s$ is $M_0 + M_s + \Sigma_h(e^h_{0L} + e^h_{sL}) > M_0 + M_s$.

An identical argument shows that no agent will return more than he owes on M_0 or M_s .

But the money owed is exactly $(1 + \theta_0)M_0 + (1 + \theta_s)M_s$, showing that $\theta_0 + \theta_s > 0$.

Let $p_{sL} = 0$ for $s = 1, \ldots, S$, then clearly no agent will carry any money into period 1, and since $p_{0L} > 0$ (by the nondegeneracy assumption) we show that $\theta_0 > 0$ by a similar argument as before. Q.E.D.

PROOF OF PROPOSITION 4. For simplicity we give it for the model E only.

Suppose we have a sequence of GELBI of $E(M, \bar{\lambda}(n), \lambda)$ with interest rates $\theta(n)$ such that $\bar{\lambda}(n) \to 0$ and $\theta(n) \to 0$. We need to reach a contradiction.

By nondegeneracy of the economy, we take $p_{0L}(n) = 1$ for all n . Note first that $p_{0\ell}(n) \to \infty$ for $\ell = 1, \ldots, L-1$; otherwise an agent h could borrow Δ of M_0 , get at least $[\Delta/p_{0\ell}(n)]\bar{\xi}$ utility of consumption, and suffer $\sum_{s=1}^{S}(1 + \theta_0(n))\Delta\bar{\lambda}^h_{0s}(n)$ of bankruptcy penalty, and be better off. Consequently, since the total money spent on purchases in period 0 is at most $M_0 + \Sigma_h e^h_{0L}$, the quantities sold $q^h_{0\ell}(n) \to 0$ for all h and all $\ell = 1, \ldots, L-1$.

Next we claim that if $A_{sj\ell} > 0$ for some $\ell = 1, \ldots, L-1$ and some $s = 1, \ldots, S$, then $r^h_j(n) \to 0$ for all h . If not, then $\pi_j(n)$ is bounded from above. Note that no seller of asset j is defaulting on the delivery of $A_{sj\ell}$, otherwise he could sell Δ less of the asset, borrow $\pi_j(n)\Delta$ of M_0 , save $\lambda^h_{sj\ell}\Delta A_{sj\ell}$ of default penalty and suffer only

$\sum_{s=1}^{S} \tilde{\lambda}^h_{0s}(n)\pi_j(n)\Delta(1 + \theta_0(n))$ of bankruptcy penalty. This would result in an increase in his payoff for small enough $\tilde{\lambda}^h(n)$, a contradiction. Thus $K_{sj\ell}(n) = 1$ for large enough n. But then, as in the previous paragraph, any agent h could increase his utility by borrowing a little of M_0, and using the money to buy asset j. To sum up, $r^h_j(n) \to 0$ for all h.

Finally we claim that $q^h_{s\ell}(n) \to 0$ for all h and $\ell = 1, \ldots, L-1$. If $p_{sL}(n) = 0$, this is so by definition at a GELBI. If $p_{sL}(n) > 0$, repeat the argument of the first paragraph replacing 0 by s throughout.

Thus we have shown that all trades in assets and commodities go to 0. Since $\theta(n) \to 0$, the wedge between buying and selling prices also goes to zero, and then (taking limits) the initial endowments are Pareto-optimal in state s for all $s = 0, 1, \ldots, S$. This cannot happen since the economy is nondegenerate.

PROOF OF THEOREM 4. Let (η,σ) be a GELBI, in which h chooses σ^h. Let h scale down all his choices by $1-\delta$, choosing $(1-\delta)\sigma^h$. Because the choice to net outcome map is linear, h will now consume $e^h + (1-\delta)z^h$, where z^h is his net trade of commodities that arise from the choice σ^h. In state s under σ^h, h devoted some of his endowment to consumption, some to asset delivery, and the rest to sales. The last two have been scaled back by $(1-\delta)$, so that a fraction δ is available for other uses. Let h also withhold a fraction δ of the part of endowment he directly consumed before. Thus in all h has available δe^h_s in state s for other purposes. The decrease in consumption by h in any commodity $s'\ell$ is bounded above by $\delta\bar{e}$, where $\bar{e}$ is the maximum aggregate consumption of the commodities. (Recall that $\bar{e}$ was derived in the proof of Theorem 2,

and shown to decrease with γ^-, and be independent of λ and $\tilde{\lambda}$.) Thus the total loss of utility of consumption from scaling down σ^h is at most $\xi^+L(S+1)\bar{e}\delta$.

Suppose, at σ^h, h was going bankrupt against M_0 or was defaulting on some asset or commodity loan in state s or going bankrupt against M_s. All of these have been reduced by a fraction δ, and also defaults and bankruptcies in the other states have been reduced by the fraction δ. Let h now apply the newly available resources δe^h_s in state s to paying off part of the remaining bankruptcy (on commodity borrowing) or default (on asset deliveries) in state s. Since money is commodity in state s, we know that $p_{sL} = 1$, so $p_{s\ell}/p_{sL} = p_{s\ell} > 1/R$, where $R = \xi^+(1 + G^*)/\xi^-$. Moreover, we know that G^* is bounded above by a number which grows smaller as $\tilde{\lambda}^-$ increases, for fixed M (see Step 2 of Theorem 1 with u** in place of u*). Thus without incurring any more penalties, agent h can devote the income[1] $\delta p_s \cdot e^h_s$ to reducing either default on any commodity or bank loans of commodities, or bank loans of money on M_0 or M_s by at least $\frac{1}{R}\delta p_s \cdot e^h_s$. The saving in penalties is at least $\lambda^- \frac{1}{R}\delta p_s \cdot e^h_s$, $\gamma^- \frac{1}{R}\delta p_s \cdot e^h_s$, or $\tilde{\lambda}^-\delta p_s \cdot e^h_s$, respectively. Note that

$$\begin{aligned} p_s \cdot e^h_s &\geq p_{s1} \cdot e^h_{s1} + \ldots + p_{s(L-1)} e^h_{s(L-1)} \\ &\geq \frac{1}{R}[e^h_{s1} + \ldots + e^h_{s(L-1)}] \\ &\geq \frac{1}{R}\, e^- \end{aligned}$$

where $e^- = \min\{ \sum_{\ell=1}^{L-1} e^h_{s\ell} : 1 \leq h \leq H,\ 0 \leq s \leq S \}$.

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[1] Recall $p_{sL} = 1$.

Also $e^- > 0$ by assumption. If we put this together with the earlier computation we find:

$$\lambda^- > \frac{R^2\xi^+L(S+1)\bar{e}}{e^-} \Rightarrow \text{no agent defaults on asset deliveries in state } s$$

$$\gamma^- > \frac{R^2\xi^+L(S+1)\bar{e}}{e^-} \Rightarrow \text{no agent defaults on commodity borrowing in state } s$$

$$\tilde{\lambda}^- > \frac{R^2\xi^+L(S+1)\bar{e}}{e^-} \Rightarrow \text{no agent defaults on } M_s \ .$$

The precise statements in Theorem 4 can easily be derived from these inequalities. Q.E.D.

PROOF OF THEOREM 5. Same as the proof of (ii) and (iii) of Theorem 4. The reason (i) does not carry over is that if $p \to 0$ as $\tilde{\lambda} \to \infty$, then agent h cannot defray much of his bankruptcy penalty by selling δe^h. Similarly, if an asset promises fiat money, h may not be able to significantly reduce his default on the delivery of the asset by selling δe^h. Q.E.D.

PROOF OF (iv) OF THEOREM 3

Proof of (iv) of Theorem 3

For $\tilde{p} \in R_+^{SL}$ define $\tilde{p} \square A$ as an $S \times J$ matrix, whose sj^{th} entry is $p_s \cdot A_{sj}$. Note that for all $\tilde{p} >> 0$, the dimension of the span of $\tilde{p} \square A$ is constant, so WLOG we take this constant to be J.

Let $(\tilde{\lambda},\lambda,\gamma)$ be a sequence of penalties converging to infinity. Suppose that $\lambda = \lambda(M,\tilde{\lambda})$ and $\gamma = \gamma(M,\tilde{\lambda})$ are chosen in accordance with

Theorem 5 to rule out default and bankruptcy on commodity borrowing. For each economy $E^*(M,\tilde{\lambda},\lambda,\gamma,0)$, choose any GELBI allocation $(x^h(\tilde{\lambda},\lambda,\gamma), h = 1, \ldots, H)$ with corresponding interest rates $\theta(\tilde{\lambda},\lambda,\gamma)$, and prices $p(\tilde{\lambda},\lambda,\gamma)$. We need to show that for large enough $(\tilde{\lambda},\lambda,\gamma)$, the corresponding GELBI allocation is a GEI allocation. By Theorem 5, it suffices to show that $\theta(\tilde{\lambda},\lambda,\gamma) = 0$.

We organize the proof into a sequence of claims.

Claim 1: For any $(\tilde{\lambda},\lambda,\gamma)$, and $s = 1, \ldots, S$, if $\theta_s > 0$, then $p_{s\ell} \geq \frac{\xi^-}{\xi^+} \frac{M_s}{Le_s} > 0$ for some ℓ, and $p_{sk} \geq \frac{\xi^-}{\xi^+} \frac{M_s}{Le_{s\ell}}$ for all $k \neq \ell$.

Proof: Since $\theta_s > 0$, all the money M_s must be bid on some commodity or the other. (It cannot be optimal to borrow and return the money without using it.) Thus $p_{s\ell} \geq \frac{M_s}{L \sum_{h=1}^{H} e^h_{s\ell}}$ for some commodity k. The rest follows, otherwise any agent could shift his consumption from ℓ to k by a small amount, improving his payoff.

Claim 2: For any $(\tilde{\lambda},\lambda,\gamma)$, if any agent inventories money from period 0 until period 1, then $\theta_s > 0$, $s = 1, \ldots, S$.

Proof: At a GELBI, no agent will be left with any unused money. Furthermore, no agent will return more than he owes. If any money is inventoried into state s, then more than M_s will be returned, hence more than M_s must be owed, hence $\theta_s > 0$.

Claim 3: As $(\tilde{\lambda},\lambda,\gamma) \to \infty$, all bankruptcies $\to 0$, and total inventory of money $\to 0$, and $\theta \to 0$.

Proof: Otherwise the bankruptcy penalty becomes infinitely high for at least one agent, contradicting the fact that he is optimizing. (Recall $u^h(x) < u^{**}$ for all x and h.)

Claim 4: For large enough $(\tilde{\lambda},\lambda,\gamma)$, there is no bankruptcy in any state $s = 1, \ldots, S$.

Proof: If there is bankruptcy in some state s, then $\theta_s > 0$. By Claim 1, that means $p_s(\tilde{\lambda},\lambda,\gamma)$ is bounded away from 0. By borrowing one less dollar an agent who is going bankrupt saves $(1 + \theta_s)\tilde{\lambda}_s^h$ and loses at most $(1/p_{s\ell})\xi^+$ for some $\ell = 1, \ldots, L-1$.

Claim 5: For large enough $(\tilde{\lambda},\lambda,\gamma)$, no agent inventories money between time 0 and 1.

Proof: If any money is inventoried, then some agent is going bankrupt at time 0. This agent would be better off borrowing one less dollar on M_0, and purchasing less commodities or assets at time 0, unless $(p_0, \pi) \to 0$. By Claim 2 all p_s are bounded away from 0, $s = 1, \ldots, S$. Hence any agent who inventoried a dollar would do better to spend it at time 0 on buying commodities or assets (if $p_0 \to 0$ or $\pi \to 0$).

Claim 6: For large enough $(\tilde{\lambda},\lambda,\gamma)$, $\theta_s = 0$, $s = 1, \ldots, S$.

Proof: Combine Claims 4 and 5.

From now on, consider subsequences of $(\tilde{\lambda},\lambda,\gamma)$ on which $x^h(\tilde{\lambda},\lambda,\gamma) \to x^h$, $p_s(\tilde{\lambda},\lambda,\gamma)/\|p_s(\tilde{\lambda},\lambda,\gamma)\| \to p_s$ for $h = 1, \ldots, H$ and $s = 0, 1, \ldots, S$; and for $j = 1, \ldots, J$, $\pi_j(\tilde{\lambda},\lambda,\gamma)/\|p_{01}(\tilde{\lambda},\lambda,\gamma)\| \to \pi_j/p_{01}$. Write $p = (p_0,\bar{p})$.

Claim 7: If for some agent h, and asset j, the sale of assets $r_j^h(\tilde{\lambda},\lambda,\gamma) \to \infty$ and $\theta_0(\tilde{\lambda},\lambda,\gamma) > 0$, then span$\{\tilde{p} \square A^j, j = 1, \ldots, J\}$ has dimension less than J.

Proof: If $\theta_0 > 0$, no agent will both buy and sell the same asset since defaults are zero and the asset always delivers. Wash sales lose the interest. But if the asset-money-equivalent deliveries $\tilde{p} \square A^j$ are linearly independent, then no agent can buy or sell an arbitrarily large amount without finding himself in some state with arbitrarily negative or else positive wealth, contradicting the GELBI conditions of market clearing and optimization.

Claim 8: By Claim 7 all sales $r_j^h(\tilde{\lambda},\lambda,\gamma)$ are uniformly bounded. Hence $(p_0(\tilde{\lambda},\lambda,\gamma), \pi(\tilde{\lambda},\lambda,\gamma))$ is bounded from below away from zero. Hence by the same argument as in Claim 4, for large enough $(\tilde{\lambda},\lambda,\gamma)$, there is no bankruptcy in state 0. Hence for large enough $(\tilde{\lambda},\lambda,\gamma)$, $\theta_0 = 0$. This concludes the proof. Q.E.D.

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