

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale University
New Haven, Connecticut 06520

COWLES FOUNDATION DISCUSSION PAPER NO. 895

Note: Cowles Foundation Discussion Papers are preliminary materials circulated to stimulate discussion and critical comment. Requests for single copies of a Paper will be filled by the Cowles Foundation within the limits of the supply. References in publications to Discussion Papers (other than acknowledgment that a writer had access to such unpublished material) should be cleared with the author to protect the tentative character of these papers.

REPEATED TRADE AND THE VELOCITY OF MONEY

by

P. Dubey, S. Sahi and M. Shubik

January 6, 1989

REPEATED TRADE AND THE VELOCITY OF MONEY*

P. Dubey, S. Sahi and M. Shubik

There are two sources of inefficiency of strategic equilibria (SE) in market mechanisms. The first is the oligopolistic effect, which occurs when an agent can single-handedly influence prices. With a continuum of agents we get "perfect competition" and this effect is, of course, wiped out. But the inefficiency of SE's may nevertheless persist because agents are not "perfectly liquid," i.e., the constraints of the mechanism are such that they cannot carry out arbitrary trades at the market prices. Our main result is that, if enough repeated rounds of trade are permitted within a single utility period, then the liquidity problem is overcome: SE outcomes turn out to be not only efficient but, in fact, Walrasian.

A typical case arises when a money is used as a medium of exchange, and the purchases by an agent must be financed out of the money he has on hand at the time. Here an insufficient supply of money, or its maldistribution is often the cause of illiquidity [5]. The upshot of repeated trade is to bring about an increase in the velocity of money. This in turn restores liquidity to the economy, no matter how constraining the one-shot mechanism might have been. [See comment 1.]

Our approach is similar to that of [3] but we provide axioms which unify both Cournot and Bertrand competition. As in [3], we start with an abstract market mechanism which maps agents' moves into trades. There are

*Research supported in part by NSF Grant No. SES-8812051.

no convexity, anonymity, aggregation or continuity axioms imposed on the mechanism (all of which were needed in [3]). Instead we suppose that agents' moves produce prices which mediate trade. This is indeed a key conclusion of [3]. But it seems to us quite realistic to directly suppose this property because it holds for most mechanisms (e.g., [2], [4], [6], [7], [13], [14]).

The other key assumption we make is that commodities are "connected." This means that the mechanism does not a priori rule out the ultimate conversion of commodity i into j (over many rounds of trade for any pair i, j).

Given these assumptions and a continuum of agents, all the SE of the game with T -rounds of trade will be close to Walrasian, if T is large enough. Furthermore all Walrasian outcomes will be achieved at SE's for such T .

Let us relate this result to [3]. It was shown in [3] that all "interior" SE's are Walrasian. Interior meant that each agent could obtain trades in a full dimensional neighborhood of his final bundle in his budget set (i.e., in our parlance, was fully liquid). The question as to when interiority would obtain was left open in [3].

Our analysis sheds light on this issue. As we said, price mediation is an inference from the four axioms of [3]. Thus our theorem holds for the mechanism of [3], and shows that enough rounds of trade will guarantee interiority.

THE EXCHANGE ECONOMY

The set of agents I is the closed unit interval $[0,1]$. The set of commodities is $I_m = \{1, \dots, m\}$. We will use Greek letters α, β for agents and Roman letters i, j for commodities. Depending on the context, vectors in the non-negative orthant \mathbb{R}_+^m will represent either commodity bundles or prices.

The economy E is a pair (e, u) where $e : I \rightarrow \mathbb{R}_+^m$ is bounded and measurable with $\int e \gg 0$ where $\int e = \int_I e d\alpha$; and $u : I \times \mathbb{R}_+^m \rightarrow \mathbb{R}$ is measurable such that for each x , $u(\alpha, x)$ is bounded in α each x , and increasing in x for each α . The interpretation is that, for any α in I , $e_j(\alpha)$ is the initial endowment of commodity j of agent α , and $u(\alpha, x)$ is α 's utility for the bundle x .

An allocation is an integrable function $\chi : I \rightarrow \mathbb{R}_+^m$ satisfying $\int \chi = \int e$. Given a (price) vector p in \mathbb{R}_+^m , define α 's budget set $B^\alpha(p)$ by

$$B^\alpha(p) = \{x \in \mathbb{R}_+^m : p \cdot x = p \cdot e(\alpha)\} .$$

A competitive equilibrium (CE) is a pair $\langle p, \chi \rangle$ where $p \in \mathbb{R}_+^m$ is a price vector and χ is an allocation such that $\chi(\alpha)$ is optimal in α 's budget set for (almost) all α , i.e.,

$$\chi(\alpha) \in B^\alpha(p) ,$$

and

$$u(\alpha, \chi(\alpha)) = \max\{u(\alpha, x) : x \in B^\alpha(p)\} .$$

An allocation χ (resp. a price p) is called competitive if there is price p (resp. an allocation χ) such that $\langle p, \chi \rangle$ is a CE.

THE MARKET MECHANISM

For each trader α there is a set M_α of "moves" to be thought as potentially available to α .

Let M be a set of maps f from I to $\coprod_{\alpha \in I} M_\alpha = M$ such that $f(\alpha) \in M_\alpha$ from each α . For any such f and $a \in M_\alpha$, denote by $f|a$ the same map as f except that $f(\alpha)$ is replaced by a . We assume that M has the property: $f|a \in M$ for any $f \in M$, $\alpha \in I$, $a \in M_\alpha$. This will permit us to consider unilateral deviations by a single trader. In many cases all the M_α are a fixed measurable space, and M is the set of all measurable maps from I to this space. Here the property above is immediate.

A move-selection f results in an assignment of trades $t(f, \alpha) \in \mathbb{R}^m$ to the agents α such that $\int t(f, \alpha) d\alpha = 0$. Here a positive (resp. negative) component of $t(f, \alpha)$ represents purchase (resp. sale) of the corresponding commodity by α .

Most models ([1], [2], [4], [5], [6], [8], [13], [14]), in addition, incorporate the formation of price $p(f)$ which mediate trade. However it may be the case ([1], [5]) that for certain choices of f some markets may fail to form (consider the situation when all agents decide to do "nothing!"), leaving prices undefined. This motivates the construct below.

There is a subset $N \subset M$ of (nice) move-selections such that each f in N produces prices $p(f)$ in \mathbb{R}_+^m , satisfying

Assumption I (Price Mediation)

For each α in I , there is a function

$$\varphi^\alpha : M_\alpha \times \mathbb{R}_+^m \rightarrow \mathbb{R}^m$$

such that¹

$$(a) \quad \varphi^\alpha(a, p) \cdot p = 0, \quad \text{for } a \text{ in } M_\alpha$$

and

$$(b) \quad t(f, \alpha) = \varphi^\alpha(f(\alpha), p(f)), \quad \text{for } f \text{ in } N.$$

Before proceeding it is convenient to introduce the map

$$\Psi : I \times \mathbb{R}_+^m \times M \rightarrow (\text{subsets of } M)$$

where $\Psi(\alpha, x, f) \subset M_\alpha$ for each α , and is interpreted as the set of moves available to α , when he has the bundle x on hand and others play in accordance with f .

While Assumption I holds equally for models with finitely many agents (see [1], [3], [5], [7], [9]), the next one is special to the continuum models.

Assumption II (Non-Atomicity)

If $f = g$ almost everywhere, then

$$(a) \quad f \in N \Rightarrow g \in N, \quad \text{and} \quad p(f) = p(g)$$

$$(b) \quad f(\alpha) = g(\alpha) \Rightarrow t(f, \alpha) = t(g, \alpha).$$

$$(c) \quad \Psi(\alpha, x, f) = \Psi(\alpha, x, g) \quad \text{for all } \alpha.$$

¹Condition (a) is analogous to the budget constraint in Section 2 and is a property that holds across all the models we have referenced.

Taken together, these assumptions imply that, given f in N , the trade that accrues to an agent depends only on his individual move and on the prices (which he does not affect).

The next assumption simply says that it is always possible for a trader to obtain the 0-trade vector.

Assumption III (No-Trade Option)

For any (α, x, f) , there is an element a in $\Psi(\alpha, x, f)$ such that $t(f|a, \alpha) = 0$. Where $f|a$ is the same as f except that $f(\alpha)$ is replaced by a .

The final assumption is the most significant. It is best stated in terms of the following notions.

Definition 1: Commodity i is linked to j ($i \rightarrow j$) in the mechanism, if for all η sufficiently small, x in \mathbb{R}_+^{ℓ} , with $x_i > \eta$, any α in I and f in M : there is an a in $\Psi(\alpha, x, f)$ such that

$$t_k(f|a, \alpha) = \begin{cases} 0, & k \neq i, j \\ -\eta, & \text{if } k = i \end{cases}.$$

Simply put, $i \rightarrow j$ means that it is always possible for a trader, who has i , to give up a small amount (η) of it in exchange for only j .

Definition 2: Commodity i is connected to j (in the mechanism), if there is a chain $i = i_0 \rightarrow i_1 \rightarrow \dots \rightarrow i_k = j$.

Assumption IV (Connectivity)

For each pair (i,j) of commodities in $I_m \times I_m$, i is connected to j .

THE GENERALIZED GAME

The game is played for T periods as follows. In period 1, all agents select moves f_1 such that

$$f_1(\alpha) \in \Psi(\alpha, e(\alpha), f_1) .$$

In period $\tau+1$, the agents observe f_1, \dots, f_τ (modulo null sets) and their own past actions and choose $f_{\tau+1}$ such that

$$f_{\tau+1}(\alpha) \in \Psi\left(\alpha, e(\alpha) + \sum_{\sigma=1}^{\tau} t(f_\sigma, \alpha), f_{\tau+1}\right) .$$

Note that $\Psi(\dots)$ is the set of moves available to α . His strategy is to select a move at each node that he can distinguish. In keeping with the non-atomicity assumption, we require that he observes others' moves only modulo null sets.

Finally, a choice of strategies is said to be feasible if, for the induced play (g_1, g_2, \dots, g_T) in the game tree, we have

$$e(\alpha) + \sum_{\sigma=1}^{\tau} t(g_\sigma, \alpha) \in \mathbb{R}_+^L$$

for each $\alpha \in I$, and $\tau = 1, \dots, T$. The payoff to α of this play is

$$u\left(\alpha, e(\alpha) + \sum_{\sigma=1}^T t(g_\sigma, \alpha)\right) .$$

A strategic equilibrium (SE) is a choice of strategies such that no individual can improve his payoff by unilateral (permitted) deviations.

Let g_1, \dots, g_T (where each $g_\sigma \in M$) be the moves made along an SE play which give rise to the allocation χ . Suppose we can find prices p^1, \dots, p^T in R_+^m such that

- (i) $p^\sigma = p(g_\sigma)$ if $g_\sigma \in N$, $\sigma = 1, \dots, T$;
- (ii) for almost all α in I , α 's strategy is a best response even when α assumes that he could trade in period σ at the prices p^σ , i.e., that any choice $a \in \psi_M(\alpha, \chi(\alpha), f)$ yields the trade $\varphi^\alpha(a, p^\sigma)$.

In this case we will say that the SE is open.

The idea is that when the mechanism fails to produce prices for some commodities, we can nevertheless announce virtual prices which will sustain the same actions by the agents.

Finally an allocation χ will be called ϵ -competitive if there is a price vector $p \in R_+^m$ such that for all z in $B^\alpha(p)$, $u^\alpha(\chi(\alpha)) \geq u^\alpha((1-\epsilon)z)$ for almost all α . (Note: 0-competitive is competitive.)

Theorem 1: Suppose Assumptions I-IV hold. Further suppose for each pair of commodities (i,j) , there is a non-null set of traders such that for all α in this set we have

$$e_i(\alpha) > 0 \tag{1}$$

$$\partial u^\alpha / \partial x_j \text{ is bounded away from 0 in } R_+^m. \tag{2}$$

Then for each $\epsilon > 0$, there a T^* such that, if $T \geq T^*$, any open

SE of the T-period generalized game gives ε -competitive allocations.

We begin with

Lemma 1: Let S be a (measurable) subset of I and let A be a fixed vector in \mathbb{R}_+^n . Then there is a constant $C > 0$, such that if χ is any function from S to \mathbb{R}_+^n with

$$\int_S \chi \leq A, \text{ then}$$

$$2\mu(\{\alpha \in S : u(\alpha, \chi(\alpha)) > C\}) < \mu(S)$$

(where μ is the Lebesgue measure).

Proof: Let $C = \sup\{u(\alpha, 2mA/\mu(S)) : \alpha \in S\}$, and let $S' = \{\alpha : u(\alpha, \chi(\alpha)) > C\}$. Then $\alpha \in S'$ implies $\chi_i(\alpha) > 2mA_i/\mu(S)$ for some i . Consequently, there is at least one fixed commodity i and a subset S'' of S' such that

$$\mu(S'') \geq \mu(S')/m$$

and $\chi_i(\alpha) > (2mA_i)/\mu(S)$ for all α in S'' .

Now $A_i \geq \int_{S''} \chi_i > \mu(S'')(2mA_i)/\mu(S) \geq 2\mu(S')A_i/\mu(S)$ which yields the desired result. Q.E.D.

Lemma 2: There is an integer T_0 and a positive real number R (both independent of T) such that if (p^1, \dots, p^T) are the prices generated at any open SE of the T-period game, we have, for each pair (i, j) ,

$$|(\tau : p_i^\tau/p_j^\tau > R)| < T_0. \quad (3)$$

(In other words, for all but a finite number (T_0) of time periods, all price ratios lie in the fixed compact set $[1/R, R]$.)

Proof: It suffices to establish the lemma for a fixed, linked pair of commodities (i,j) . (The general case then follows by the connectivity Assumption IV.)

By the assumption (1) in Proposition 1 we can find a positive number δ and a (non-null) set S such that $e_i(\alpha) > \delta$ for all α in S . Let $A = \int_I e$ and let C be the constant described in Lemma 1.

Now by assumption (2) of the proposition, there is a number E such that for any vector $z \geq 0$ with $z_j > E$, we have $u(\alpha, z) \geq C$ for all α in S .

Let $R = E/\delta$ and let T_0 be an integer greater than δ/η , where η is as in Definition 1; and suppose, by way of contradiction, that (3) fails to hold. Then by the openness of the SE, and by Assumptions I, II, III, IV, each α in S can convert δ units of commodity i into at least $R\delta = E$ units of commodity j . Consequently, if χ is the SE allocation, we have $u(\alpha, \chi(\alpha)) \geq C$ for all α in S . But since $\int_S \chi \leq \int_I \chi = \int_I e = A$, this is not possible in view of Lemma 1; and so (3) must hold. Q.E.D.

We are now ready to prove Theorem 1. The main point is that Lemma 2 implies that as $T \rightarrow \infty$, the price ratios in the T-period game will have a non-zero limit point, which will be a candidate for a competitive price. The actual proof is a more careful, finite version of this idea.

Fix R as in Lemma 2, and define the sets

$$\Delta = \{p \in R_+^n : \sum_i p_i = 1 \text{ and } p_i/p_j \leq R \text{ for all pairs } (i,j)\} \quad (4)$$

$$\square = \{y \in \mathbb{R}_+^m : 1/R \leq y_{ij} \leq R \text{ for } (i,j) \in I_m \times I_m\} .$$

Also, for each commodity pair (i,j) , let us fix a chain (as in Definition 2) such that

$$i = i_0 \rightarrow i_1 \rightarrow \dots \rightarrow i_k = j ,$$

and define the functions on \square given by

$$F(y;i,j) = y_{i_0 i_1} y_{i_1 i_2} \dots y_{i_{k-1} i_k}$$

$$\text{and } F(y) = \max_{i,j} F(y;i,j) . \quad (5)$$

Note that F is continuous, and hence uniformly continuous on the (compact) set \square .

Proof of Theorem 1: Given $\varepsilon > 0$ (as in the Theorem), we can find a $\zeta > 0$, such that for y, y' in \square ,

$$|y - y'| < \zeta \text{ implies } |F(y) - F(y')| < \varepsilon ,$$

where $|\cdot|$ denotes the ℓ_1 -norm.

Recall the budget set $B^\alpha(p)$ defined in Section 2 and let $B^\alpha = \cup_{p \in \Delta} B^\alpha(p)$; and $B = \cup_{\alpha \in I} B^\alpha$. Then by (4) and the boundedness of $(e(\alpha) : \alpha \in I)$ it follows that B is bounded. Let

$$X = (\max x_j : 1 \leq j \leq m \text{ and } x \in B) . \quad (6)$$

Let N_1 be the smallest integer greater than R/ζ and divide \square into N_1^m equal hypercubes each of which has edges of length $(R - 1/R)/N_1 < \zeta$.

We will show that if T is any integer greater than $T_0 + mN_1^m(X/\eta)^m = T^*$ then open SE-allocations in the T -period game are ε -competitive.

To see this, we observe that Lemma 1 and the choice of N_1 imply that for some $N \geq m(X/\eta)^m$, there are N time periods in which the price ratios lie inside a fixed hypercube of edge ζ in \square . Let these periods be (τ_1, \dots, τ_N) and to simplify notation, let us write q^ℓ for p^{τ_ℓ} , and p for q^1 .

Fix a player α and let $x \in B^\alpha(p)$. Then to finish the proof of the theorem, it suffices to show that α can achieve the bundle $(1-\varepsilon)x$ (or more) by an appropriate strategy. We describe such a strategy below.

Let $t = e(\alpha) - x$, and suppose without loss of generality that $t_1, \dots, t_k \geq 0$ and $t_{k+1}, \dots, t_m \leq 0$; i.e. α sells $1, \dots, k$ and buys $k+1, \dots, m$, in the game.

Suppose α deviates to the following strategy: He buys and sells only in periods (τ_1, \dots, τ_N) . He begins by converting t_1 units of good 1 to good m using the $(1,m)$ -chain. If (i,j) is a link in this chain, then (6) implies that the conversion of i to j may be effected in less than (X/η) steps. Consequently, in at most $(X/\eta)^m$ steps he can either finish selling t_1 units of 1 or will have bought $|(1-\varepsilon)t_m|$ units of m . In the first case he starts to use the $(2,m)$ chain, and in the second case the $(1, m-1)$ chain, etc. This procedure may be finished in N ($\geq m(X/\eta)^m$) steps.

By the choice of ζ and (5) we see that each time α converts s units of good i to good j in the (i,j) -chain he gets at least $(1-\varepsilon)sp_j/p_i$ units of good j .

Since repeated trading in the manner described above at the fixed price

vector p would have got him exactly to x , we see from the above that he will manage to buy at least $(1-\varepsilon)|t_{k+1}|, \dots, (1-\varepsilon)|t_m|$ of goods $k+1, \dots, m$ by selling at most $|t_1|, \dots, |t_k|$ units of goods $1, \dots, k$. This finishes the proof. Q.E.D.

The converse is easier.

Theorem 2: Any competitive allocation can be achieved as an open SE in finitely many rounds of trade.

Proof. This is obvious. Clearly if p is a competitive price vector, then $p > 0$. Chose R such that $p_i/p_j < R$ for all (i,j) . Construct $X = X(R)$ as in the proof of Theorem 1. Announce the prices p repeatedly in each of the $m(X/\eta)^m$ periods, and scale the competitive trades to make the sales no more than η in each period. Then define strategies in the game tree to be consistent with the above trades (i.e., they can be arbitrary off the play that gives these trades). Since the player-set is nonatomic, these strategies constitute an SE. Q.E.D.

CONCLUDING COMMENTS

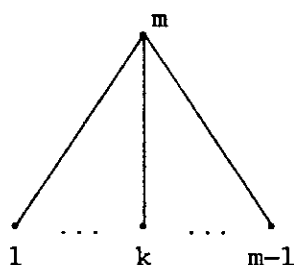
1) We consider a class of market mechanisms defined by Shapley and Shubik [13]. The central notion is that of a trading-post for a pair of commodities (i,j) . If agents send $0 \leq q_i(\alpha) \leq e_i(\alpha)$, $0 \leq q_j(\alpha) \leq e_j(\alpha)$ to the post then α gets back

$$t_j(\alpha) = q_i(\alpha) \frac{\int q_i(\alpha) d\alpha}{\int q_i(\alpha) d\alpha}$$

of j if both integrals are positive (and $q_j(\alpha)$ otherwise);

$$t_i(\alpha) = q_j(\alpha) \frac{\int q_i(\alpha) d\alpha}{\int q_j(\alpha) d\alpha}$$

of i if both integrals are positive (and $q_i(\alpha)$ otherwise). Consider a graph whose nodes are commodities and arcs are trading posts. If the graph is a spanning tree,² then the mechanism (defined in an obvious way) satisfies all our axioms. If the graph looks like



then m may be viewed as a (commodity) money. In this case we can interpret an increase in T to be tantamount to an increase in the "velocity" of money; and Theorem 1 shows that this increase is enough to overcome the initial "illiquidity."

This example also highlights why we need to pay special attention to SE's at which no trade might occur but nevertheless markets are "open" via prices--see our definition of open SE's. If T is small, the money constraint $\sum_j q_j(\alpha) \leq e_m(\alpha)$ can be robustly³ binding at equilibrium leaving some markets inactive. Yet there are (virtual) prices that we could announce at these markets at which no agent would want to trade. It is reasonable to permit such equilibria. Note that we exclude trivial equi-

²For more general graphs [1] our analysis can be extended in an obvious manner, even though there may no longer be consistent prices that mediate trade.

³I.e. for an open set of endowments and utilities (e,u) .

libria in which an arbitrary subset of markets are closed to trade simply because no one has sent any commodities to them.

2) In the context of a single round of trade (i.e. $T = 1$), axioms were presented in [3] so that "interior" SE's were competitive. "Interior" meant that each agent could move in a full-dimensional neighborhood of his final trade vector in his budget set. The question as to when SE's would be interior was left open.

Our analysis sheds light on this issue. If agents can trade in an open neighborhood (whose size is independent of prices) of the null trade vector, then our analysis shows that enough time yields interiority. Actually we need something much weaker than this: the possibility of pairwise trade is enough, provided commodities are connected. For then time helps to achieve full span.

3) We conjecture that with extra labor and notation these results can be extended to economies with production and exchange. But an important process and institutional feature, which distinguishes our abstract presentation here from a functioning economy, is that consumption, production and exchange take a finite amount of time. A richer and more detailed model is required to study the implications of the efficiency to be gained in trading off more money against fewer rounds of exchange. Mechanistic arguments involving variations of Fisher's equation $PQ = MV$ [9, 10, 11, 12] cannot cast light on questions of optimal velocity without a sufficiently detailed process analysis of the costs involved in the time consuming activities of consumption, production and exchange.

REFERENCES

- [1] Amir, R., S. Sahi, M. Shubik and S. Yao, 1987. A strategic market game with complete markets. Cowles Foundation Discussion Paper No. 814R, Yale University, New Haven, CT.
- [2] Benassy, J. P., 1986. On competitive market mechanisms, Econometrica 54: 95-108.
- [3] Dubey, P., A. Mas-Colell, and M. Shubik, 1980. Efficiency properties of strategic market games: An axiomatic approach, Journal of Economic Theory, 22: 339-362.
- [4] Dubey, P. 1982. Price quantity strategic market games, Econometrica 50: 111-126.
- [5] Dubey, P., and L. S. Shapley, 1976. Noncooperative exchange with a continuum of traders. Cowles Foundation Discussion Paper No. 447, Yale University, New Haven, CT.
- [6] Dubey, P. and M. Shubik, 1978. The noncooperative equilibria of a closed trading economy with market supply and bidding strategies, Journal of Economic Theory, 17: 1-20.
- [7] Dubey, P. and M. Shubik, 1980, "A strategic market game with price and quantity strategies," Zeitschrift fur Nationalokonomie, 40(1-2): 25-34.
- [8] Fisher, I., 1931, The Purchasing Power of Money (2nd ed.). New York: MacMillan, p. 26.
- [9] Friedman, M., 1969, The Optimum Quantity of Money and Other Essays. Chicago: Aldine.
- [10] Keynes J. M., 1936, The General Theory of Employment, Interest and Money. London: MacMillan.
- [11] Keynes, J. M., 1958, A Treatise on Money, Vol. I. London: MacMillan.
- [12] Keynes, J. M., 1960, A Treatise on Money, Vol II. London: MacMillan (First printing 1930).
- [13] J. Peck and K. Shell,
- [14] Shapley, L. S., and M. Shubik, 1977, Trade using one commodity as a means of payment, Journal of Political Economy, 85(5): 937-968.