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NONPARAMETRIC TESTS OF MAXIMIZING BEHAVIOR SUBJECT TO NONLINEAR SETS

By

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This paper extends the axiomatic theory of revealed preference to choices that are generated by the maximization of a strictly concave and strictly monotone function subject to nonlinear constraint sets. I characterize finite sets of observations on choice behavior that are consistent with the maximization of a strictly concave and strictly monotone objective function. Both nonconvex and convex choice sets are considered. The analysis applies, for example, to consumers who face either regressive or progressive taxes and to households that produce commodities according to either a convex or a concave production function. For choice sets that possess convex and monotone complements, my characterization provides a nonparametric test for the maximization hypothesis. For choice sets that can be supported by unique hyperplanes at the chosen elements, the result provides a nonparametric test for the strict concavity and strict monotonicity of the maximized function.

KEYWORDS: nonparametric tests, axioms of revealed preference, nonlinear choice sets, representations, rationalizations, convexity, concavity, monotonicity.

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1. INTRODUCTION

Recently, there has been increasing interest in the development of non-parametric tests to determine whether observed consumer behavior is consistent with the existence of a utility function that belongs to a certain class. This paper contributes to the development of this theory by extending the axiomatic theory of revealed preference to choices that are generated by the maximization of a strictly concave and strictly monotone function subject to nonlinear constraint sets.

The nonparametric revealed preference conditions can be employed to test for utility-maximizing behavior and to provide nonparametric inferences about the parametric or nonparametric functional form that can be imposed on the utility function. For example, demand data may be determined to be inconsistent with the existence of any utility function. Or, the data may be consistent with the existence of a utility function but inconsistent with the existence of a strictly quasi-concave utility function. In the latter case, it would be inappropriate to specify, for example, a Cobb Douglas utility function. In contrast to parametric methods, which obtain this information by testing conditions on estimated parameters, the revealed preference methods do not require a parametric specification of the objective function.

Richter (1966, 1971), Afriat (1967a, 1967b, 1972, 1973, 1981), Varian (1982, 1983), Diewert and Parkan (1985), Yatchew (1985), Chiappori and Rochet (1987), and Matzkin and Richter (1987) are some of the studies that have presented algebraic, nonparametric, revealed preference tests of consumer behavior.

These nonparametric tests have not been fully extended to choice situations involving nonlinear choice sets. This extension is desirable because many choice models frequently encountered in economics involve such sets. Consumers or firms with monopsony power, consumers facing either regressive or progressive taxes, households producing commodities according to either a convex or a concave production function, and social planners facing a production possibilities set exemplify economic situations in which the set of feasible alternatives faced by an economic agent is determined by a nonlinear function.

In this paper, I extend the theory of nonparametric revealed preference tests to apply to these kinds of nonlinear choice sets. The results determine whether, given a finite number of observations on the choice sets and the alternatives chosen by economic agents, the observed choice behavior is consistent with the maximization of an objective function within a certain class. This function is interpreted as the utility or production function of the economic agent in the examples described above. The observations may be generated by a single economic agent or by identical economic agents. In the former case, our results concern the existence of a utility or production function for the agent; in the latter case, they relate to the existence of a function that is common to all the observed agents.

Currently, there are two methods of specifying nonparametric revealed preference conditions, the method of inequalities and the axiomatic method. The former method starts by specifying a system of inequalities determined by the observations and tests whether the system has a solution. The latter method proceeds by testing whether or not the observations satisfy certain axioms (i.e., combinatorial conditions).

In previous work, the system of inequalities has been derived for data from demand behavior subject to budget sets determined by either unions or intersections of linear sets. The inequalities are linear when budget sets are intersections of linear sets and nonlinear when the budget sets are unions of linear sets. The solution of the system yields a set of values and supergradients of a utility function that could have generated the observations.

The method of inequalities is not practical because of the large number of equations and unknown variables required for a typical problem. For example, for a data set of size n, to test for the existence of a concave utility function requires one to determine whether there exists a solution of dimension $2n$ for a system of n^2 inequalities.

The axiomatic method, in contrast, is simpler and faster to implement. For example, the test for the existence of a concave utility function only requires one to check for certain symmetries in an $n \times n$ matrix of 0's and 1's.

Presently, the use of axiomatic conditions has been limited to the analysis of data that arise when the budget set is determined by one linear function. In this paper I extend the study of axiomatic conditions to the analysis of data generated subject to choice sets determined by multiple nonlinear functions.

The choice sets studied in this paper belong to the class of sets that possess convex and monotone complements and to the class of sets that can be supported by a hyperplane. I call the first type of sets *co-convex* and the second type of sets *supportable*.

In Section 2 I describe the basic model and define the main concepts. Section 3 presents the results for data generated subject to co-convex sets. Section 4 presents the results for data generated subject to supportable sets and for data generated subject to both co-convex and supportable sets. I summarize the main results in Section 5. Appendix A contains the notation and presents the definitions of the terms employed in the paper. Appendix B contains the proofs of the main lemmas that are employed in the proofs of the theorems in the paper. Appendix C contains statements and proofs of auxiliary lemmas.

2. THE MODEL

Our objective is to characterize finite sets of observations on choice behavior that could have been generated from the maximization of a common objective function. An observation consists of a set of alternatives and a chosen alternative from the set. If a maximized function exists, we call it a representation for the observations.

Formally, we will represent each observable choice by a pair $(B,h(B))$, where the choice set B is a subset of alternatives in a set X and $h(B) \in B$ is the chosen element [1]. For example, B may be the budget set that a consumer faces and $h(B)$ the demand of the consumer. We will be concerned with two types of choices, co-convex choices and supportable choices. Topological properties of the set B and its complement B^c are meant to be relative to X.

DEFINITION 1: *A pair $(B, h(B))$ will be called a* <u>*co-convex choice*</u> *in $X \subset R^K$ if*

(i) $B \subset X$,

(ii) B^c is an open, convex, and monotone subset in X, and

(iii) for all $e > 0$ such that $h(B) + e \in X$, $h(B) + e \in B^c$.

DEFINITION 2: *A pair $(B,h(B))$ will be called a* <u>*supportable choice*</u> *in $X \subset R^K$ if*

(i) $B \subset X$ and if

(ii) there exists a neighborhood N of $h(B)$ and a unique $s \in R^K_{++}$ such that

(ii.1) $B \subset \{ x \in X \mid s\, x \leq 1 \}$,

(ii.2) $s\, h(B) = 1$,

(ii.3) $N \subset X$,

(ii.4) $B \cap cl(N)$ is closed and convex, and

(ii.5) $[(B - R^K_+) \cap N] \subset B$.

In other words, a choice $(B,h(B))$ is co-convex if the complement of the alternatives set is an open, convex, and monotone set and if the sum of the chosen element with any nonnegative vector different from 0 does not belong to the choice set. A choice $(B,h(B))$ is supportable if the set of alternatives can be supported at $h(B)$ by a unique hyperplane with a strictly positive normal, and if B is closed, convex, and decreasing in a neighborhood of $h(B)$.

Suppose for example that B is characterized by a function $g : X \to R$ according to

$$(1) \qquad B = \{x \in X \mid g(x) \leq 0\}$$

and that $g(h(B)) = 0$. Then, if g is a monotone increasing, continuous, and quasi-concave function, $(B,h(B))$ is a co-convex choice; if g is monotone increasing, convex, and differentiable at $h(B)$, $(B,h(B))$ is a supportable choice. Figures 1, 2, and 3 show co-convex choices and Figures 1 and 4 show supportable choices. Figures 2 and 3 show choices that are not supportable, while Figure 4 shows a choice that is not co-convex.

Many choice problems frequently studied in economics involve co-convex and supportable choices. We next present a few simple examples and thereafter describe how our results can be applied to each of these examples.

(a) *perfectly competitive consumers*

Consider the problem of a consumer who chooses a consumption bundle of K commodities from a set of affordable bundles. Suppose that the price of the k^{th} commodity is p_k and that the income of the consumer is I . Then, the set of affordable bundles for this consumer is defined by

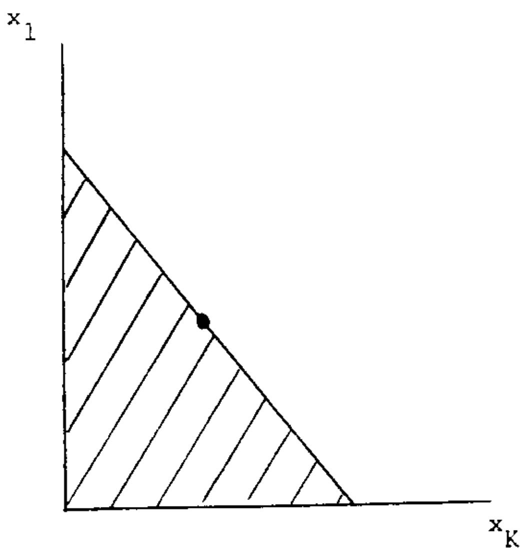

FIGURE 1

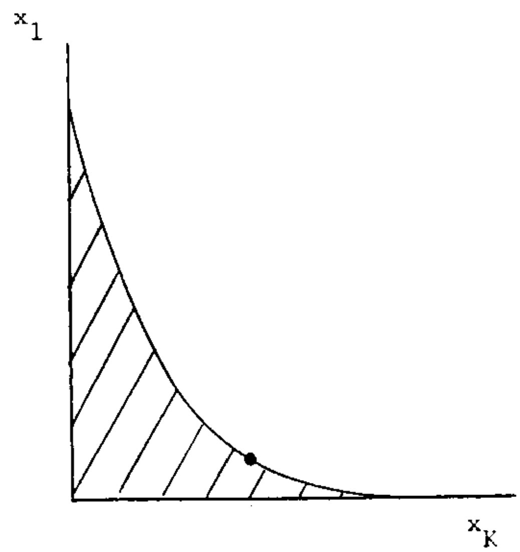

FIGURE 2

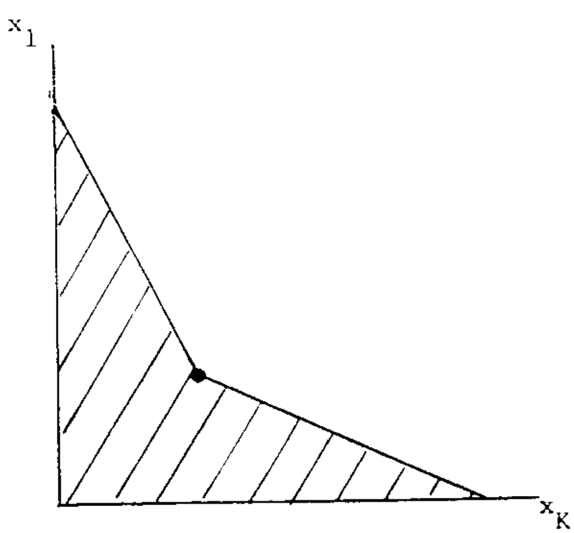

FIGURE 3

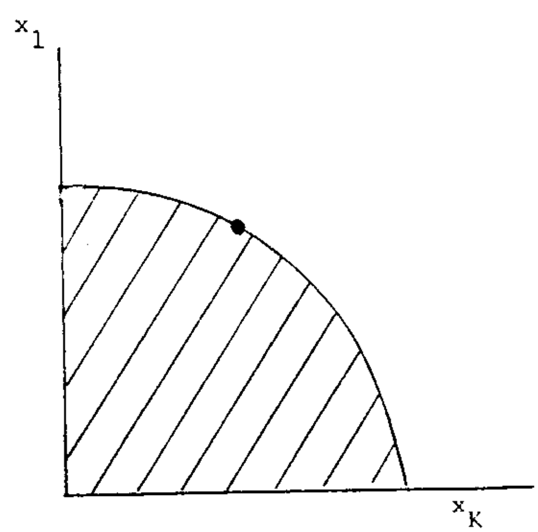

FIGURE 4

CO-CONVEX AND CONVEX CHOICES

$$B = \{ x \in R_+^K \mid \Sigma_{k=1}^K \; p_k \, x_k \le I \}.$$

Figure 1 shows the graph of this set.

(b) *household production*

Consider the problem of a household that has income I and chooses a consumption bundle of K commodities from a set of feasible and affordable bundles. Suppose that each commodity $k \in \{1,\ldots,K\text{-}1\}$ can be purchased in the market at a price p_k. The Kth commodity can only be produced by a household technology, which is characterized by a strictly increasing production function $f : R_+ \to R_+$. The input z for the production of the K^{th} commodity can be purchased in the market at a price q. In this case, the set of affordable consumption bundles for the household is given by

$$B = \{ x \in R_+^K \mid \Sigma_{k=1}^{K-1} \; p_k \, x_k + q \; f^{-1}(x_K) \le I \} .$$

Figure 2 presents a possible graph of this set B for the case in which the production function f is strictly convex; and Figure 4 presents a possible graph of B for the case in which f is strictly concave.

(c) *social planners*

Consider the problem of a social planner that chooses an aggregate production plan from the set of aggregate production possibilities set. Suppose that the economy produces K commodities from a fixed quantity $\bar{X}$ of an input. Each commodity $k \in \{1,\ldots,K\}$ is produced according to a strictly increasing production function $f_k\colon R_+ \to R_+$. Then, the set of feasible aggregate production possibilities faced by the social planner is defined by

$$B = \{ x \in R_+^K \mid \Sigma_{k=1}^K \; f_k^{-1}(y_k) \le \bar{X} \} .$$

Figure 2 presents a possible graph of this set B for the case in which $K = 2$, f_1 is linear, and f_2 is strictly convex. Figure 4 presents a possible graph of this set for the case in which $K = 2$, f_1 is linear, and f_2 is strictly concave.

(d) *charitable organizations*

Consider the problem of an organization that chooses a bundle of K inputs to produce a single commodity, which it supplies at no cost. Suppose that the organization has a fixed amount C of funds assigned for the production of the commodity. Each input $k \in \{1,\ldots,K-1\}$ can be purchased at a wage w_k . The K^{th} input is internally produced according to a strictly increasing production function $f : R_+ \to R_+$, which employs an input z whose price is q . Then, the set of input bundles that are affordable and feasible is

$$B = \{ x \in R_+^K \mid \Sigma_{k=1}^{K-1} \; p_k \, x_k \; + q \; f^{-1}(x_K) \le C \} .$$

Figure 2 presents a possible graph of this set for the case in which the production function f is strictly convex; and Figure 4 presents a possible graph of B for the case in which f is strictly concave.

(e) *quantity discounts*

Consider the problem of a consumer that has income I and chooses a consumption bundle of K commodities from a set of affordable bundles. Each commodity $k \in \{1,\ldots,K-1\}$ can be purchased in the market at a price p_k . The price of the K^{th} commodity is p_K^1 if the quantity purchased is smaller than x_K^* and the price is p_K^2 if the quantity purchased is larger than x_K^*, where $p_K^1 > p_K^2$ and $p_K^1 x_K^* = p_K^2 x_K^*$. Then, the set of affordable consumption bundles for this consumer is given by

$$B = \{ x \in R_+^K \mid \Sigma_{k=1}^{K-1} p_k x_k + p_K^1 x_K \le I \quad \text{if} \quad x_K \le x_K^* \text{ and}$$
$$\Sigma_{k=1}^{K-1} p_k x_k + p_K^2 x_K \le I \quad \text{if} \quad x_K \ge x_K^* \}.$$

Figure 3 presents a graph of this set.

(f) *monopsonists*

Consider the problem of a consumer who has income I and chooses a consumption bundle of K commodities. Each commodity $k \in \{1,\ldots,K-1\}$ can be purchased in the market at a constant price p_k. The consumer is the only buyer of the K^{th} commodity. The suppliers of the Kth commodity behave in a perfectly competitive way and their aggregate supply function is strictly increasing and strictly convex. Then, in this case, the set of affordable consumption bundles for the consumer is

$$B = \{ x \in R_+^K \mid \Sigma_{k=1}^{K-1} p_k x_k + t(x_K) x_K \le I \},$$

where $t: R_+ \to R_+$ is the strictly increasing and strictly convex function determined by the aggregate supply function of the K^{th} commodity. Figure 4 presents a possible graph of this set.

Given a finite number of observations $\{B_i, h(B_i)\}_{i=1}^n$, on any of the choices described in the above examples, we will be able to determine whether each $h(B_i)$ could have been chosen to maximize a common strictly concave and strictly monotone function. Such a function will typically be interpreted as the (common) utility function of the consumer(s) in examples (a), (e), and (f), the (common) utility function of the household(s) in example (b), the utility function of the social planner in example (c), and the production function of the charitable organization in example (d).

We next provide formal definitions of the sets of observations that we will consider and of the objective functions we will be concerned with.

DEFINITION 3: *A finite set* $(B_i, h(B_i))_{i=1}^n$ *will be called a* *__co-convex choice space__* *in* X *if for all* $i=1,\ldots,n$, $(B_i,h(B_i))$ *is a co-convex choice in* X.

DEFINITION 4: *A finite set* $(B_i, h(B_i))_{i=1}^n$ *will be called a* *__supportable choice space__* *in* X *if for all* $i=1,\ldots,n$, $(B_i,h(B_i))$ *is a supportable choice in* X.

DEFINITION 5: *A finite set* $(B_i, h(B_i))_{i=1}^n$ *will be called a* *__mixed choice space__* *in* X *if for all* $i=1,\ldots,n$, $(B_i,h(B_i))$ *is either a co-convex choice in* X *or a supportable choice in* X.

DEFINITION 6: *A function* $V : X \to R$ *is a* *__representation__*[2] *for a choice space* $(B_i, h(B_i))_{i=1}^n$ *in* X *if for all* $i=1,2,\ldots,n$ *and all* $y \in B_i$ *such that* $y \neq h(B_i)$, $V(h(B_i)) > V(y)$.

Hence, a choice space is a finite set of choices; and a representation for a choice space is a function that assigns to each chosen bundle the highest value it attains on the corresponding choice set.

The above definition of a representation is stronger than the definition of rationalization employed by Afriat (1967a, 1967b, 1972, 1973, 1981), Varian (1982, 1983), Diewert and Parkan (1985), and Chiappori and Rochet (1987). These authors define a rationalization for $\{B_i, h(B_i)\}_{i=1}^n$ to be any function $V^* : X \to R$ such that for all $i=1,2,\ldots,n$ and all $y \in B_i$, $V^*(h(B_i)) \geq V^*(y)$. Matzkin and Richter (1987) have called this function V^* a sub-semirationalization, because the observed chosen element is included in a set generated from such V^*. While rationalizations in the sense of Afriat, Varian, Diewert and Parkan, and Chiappori and Rochet are maximized over each choice set on a subset that contains the observed chosen bundle, representations, in the sense of this paper, are uniquely maximized over each observed choice set at the observed chosen bundle. (See Matzkin and Richter (1987, Section 5) for further discussion about this topic.)

We will employ revealed preference conditions to determine the existence of representations. The conditions are imposed on a preference relation inferred from the observed choices. Following Samuelson and Houthakker, we define the direct and indirect revealed preference binary relations S and H on X by[3]:

DEFINITION 7: *$x \, S \, y$ iff $x \neq y$ and for some B, $y \in B$ and $x = h(B)$.*

DEFINITION 8: *$x \, H \, y$ iff for some (possibly empty) sequence $w^1, \ldots, w^r \in X$, $\quad x \, S \, w^1 \, S \ldots S \, w^r \, S \, y$.*

In other words, x is directly revealed preferred to y (x S y) if y is different from x and if x is chosen when y can be chosen; and x is indirectly revealed preferred to y (x H y) if there exists a sequence of directly revealed preferred choices such that x is directly revealed preferred to the first element in the sequence and the last element in the sequence is directly revealed preferred to y .

DEFINITION 9: *A budget space $\{B_i, h(B_i)\}_{i=1}^n$ is said to satisfy the <u>SARP</u> (Houthaker's Strong Axiom of Revealed Preference) iff for all x, y in X, $\quad x \, H \, y$ implies $\text{not}(y \, S \, x)$.*

According to this definition, $\{B_i, h(B_i)\}_{i=1}^n$ satisfies the SARP if the binary relation H inferred from $\{B_i, h(B_i)\}_{i=1}^n$ is asymmetric.

Richter (1966) showed that the SARP is a necessary and sufficient condition for the existence of a reflexive, transitive, and total binary relation generating choice behavior subject to abstract choice sets. For a finite number of observations, this is equivalent to the existence of a representation. This result, however, does not guarantee the existence of a concave, continuous, or even monotone representation. Matzkin and Richter (1987) showed that the SARP is a necessary and sufficient condition for the existence of a strictly monotone and strictly concave representation, when each budget set B_i is determined by one single hyperplane and a finite number of observations are available. In the next section we will show that for co-convex choices the SARP is still a necessary and sufficient condition for the existence of a strictly concave and strictly monotone representation. We will show that for supportable choices the SARP is not sufficient for the existence of such a representation. A characterization of supportable choice spaces that are consistent with the existence of a strictly concave and strictly monotone objective function will be provided.

Tests for strictly concave and strictly monotone representations are important because of their wide applicability. In empirical applications, it is typically assumed that the maximizing function is strictly monotone and strictly quasi-concave, in order to guarantee that the constraint is binding and that the maximizing bundle is unique. A test for the consistency of data with a strictly monotone and strictly quasi-concave function is therefore desirable. Strict concavity implies not only strict quasi-concavity but also concavity. Concavity is an important property by itself, since it guarantees that the function is continuous and that the own-price partial derivatives of perfectly competitive demand functions are bounded from above (cf. Jordan (1982) and Hurwicz, Jordan, and Kannai (1984)) .

In Section 3 we will prove the main representation theorem for co-convex choice spaces. Supportable and mixed choice spaces will be analyzed in Section 4.

3. CO-CONVEX CHOICE SPACES

In this section we show that a co–convex choice space possesses a strictly concave and strictly monotone representation if and only if it satisfies the SARP .

This result extends the axiomatic theory of representations to apply to co–convex budget sets. Existing axiomatic results about representations apply only to demand data generated subject to budget sets determined by one linear hyperplane. Under those conditions, the existence of concave representations has been shown by employing the observed linear hyperplanes to delimit the upper contour sets of the representations. This delimiting technique cannot be employed when the budget sets are co–convex, however, since in this case any hyperplane containing the chosen bundle may lie strictly inside the budget set. Hence, to prove that the SARP implies the existence of a strictly concave representation when budget sets are co–convex, it is necessary to devise a new, completely different approach to constructing a representation.

This new method, which extends ideas in Matzkin (1986), proceeds by estimating the upper contour set of any given observed bundle from information about the bundles that were revealed to be "not worse" than the chosen bundle. This approach eliminates the dependence of the estimation of the upper contour sets on hyperplanes determined by a price or a marginal price. Strictly convex upper contour sets are obtained by constructing strictly convex approximations of convex sets, in the spirit of Eggleston (1969). A utility function with the desired properties is then obtained by employing a result of Kannai (1974).

We next state and prove Theorem 1. All notations and definitions are specified in Appendix A. The proofs of Lemmas 1 - 3 are presented in Appendix B.

THEOREM 1: *Suppose that* $\{B_i, h(B_i)\}_{i=1}^{n}$ *is a co-convex choice space in a convex and compact subset* X *of* R^K. *Then,* $\{B_i, h(B_i)\}_{i=1}^{n}$ *satisfies the SARP if and only if there exists a strictly concave and strictly monotone representation for* $\{B_i, h(B_i)\}_{i=1}^{n}$ [4].

The main part of the proof of Theorem 1 consists in showing that the SARP implies the existence of a strictly concave and strictly monotone representation. This is performed by three lemmas.

Lemma 1 shows that if $\{B_i, h(B_i)\}_{i=1}^{n}$ satisfies the SARP there exist "indifference classes" C(1),...,C(T) and corresponding "upper-contour" sets Z(1),...,Z(T) for C(1),...,C(T), which are concave and monotone subsets of a bounded subset H of R^K.

LEMMA 1: *Let* $(B_i,\ h(B_i))_{i=1}^{n}$ *be a co-convex choice space in a compact and convex subset* X *of* R^K. *Let* $D = \{\ h(B_i)\ |\ i=1,\ldots,n\ \}$ *be the set of chosen bundles. Suppose that* $(B_i,\ h(B_i))_{i=1}^{n}$ *satisfies the SARP . Then, there exist a partition* $C(1),\ \ldots,\ C(T)$ *of* D *and sets* $Z(1),\ \ldots,$ $Z(T)$ *in* R^K *satisfying*

(L1.1) $Z(1),\ \ldots,\ Z(T)$ *are convex and monotone polyhedrons in a compact subset* H^0 *of* R^K ,

(L1.2) $\forall t \in \{1,\ \ldots,\ T\}$ *and* $\forall i \in \{1,\ \ldots,\ n\}$ *such that*

$$h(B_i) \in C(t)\ , \qquad B_i \cap Z(t) = \{h(B_i)\}\ ,$$

(L1.3) $\forall t \in \{1,\ \ldots,\ T-1\} \qquad Z(t+1) \subset \text{int}\ Z(t)$,

(L1.4) $\forall t \in \{1,\ \ldots,\ T\}$ *and* $\forall i \in \{1,\ \ldots,\ n\}$ *such that*

$$h(B_i) \in C(t)\ , \qquad h(B_i) \in \text{T}(Z(t))\ .$$

Lemma 2 shows that if there exist convex and monotone "upper-contour" sets $Z(1),\ldots,Z(T)$ of the "indifference-classes" $C(1),\ldots,C(T)$, then there also exist *strictly* convex and *strictly* monotone "upper-contour" sets $Y(1),\ldots,Y(T)$ of the "indifference-classes" $C(1),\ldots,C(T)$.

LEMMA 2: *Let* $(B_i,\ h(B_i))_{i=1}^{n}$ *be a co-convex choice space in a convex and compact subset* X *of* R^K. *Let* $D = \{\ h(B_i)\ |\ i=1,\ldots,n\ \}$ *be the set of chosen bundles. Suppose that* $C(1),\ \ldots,\ C(T)$ *is a partition of* D *and that subsets* $Z(1),\ \ldots,\ Z(T)$ *of* R^K *satisfy (L1.1) - (L1.4). Then, there exist subsets* $Y(1),\ldots,Y(T)$ *of* R^K *satisfying*

(L2.1) $\forall t \in \{1,\ \ldots,\ T\} \qquad Y(t)$ *is compact* ,

(L2.2) $\forall t \in \{1, \ldots, T\}$ $Y(t)$ *is strictly convex and strictly monotone in the compact subset* H^0 *of* R^K,

(L2.3) $\forall t \in \{1, \ldots, T-1\}$ $Y(t+1) \subset \text{int } Y(t)$,

(L2.4) $\forall t \in \{1, \ldots, T\}$ *and* $\forall i \in \{1, \ldots, T\}$ *such that* $h(B_i) \in C(t)$, $Y(t) \cap B_i = \{h(B_i)\}$, *and*

(L2.5) $\forall t \in \{1, \ldots, T\}$ *and* $\forall i \in \{1, \ldots, T\}$ *such that* $h(B_i) \in C(t)$, $h(B_i) \in \partial Y(t)$.

Lemma 3 shows that if the "upper-contour" sets Y(t) satisfy (L2.1)-(L2.5), there exists a strictly concave and strictly monotone representation for $\{B_i, h(B_i)\}_{i=1}^n$.

LEMMA 3: *Let* $\{B_i, h(B_i)\}_{i=1}^n$ *be a co-convex choice space in a convex and compact subset* X *of* R^K. *Let* $D = \{ h(B_i) \mid i=1,\ldots,n \}$ *be the set of chosen bundles. Suppose that* $C(1), \ldots, C(T)$ *is a partition of* D *and that subsets* $Y(1), \ldots, Y(T)$ *of* R^K *satisfy (L2.1) - (L2.5). Then, there exists a strictly monotone and strictly concave representation for* $\{B_i, h(B_i)\}_{i=1}^n$.

PROOF OF THEOREM 1: By Lemmas 1, 2, and 3 it follows that if $\{B_i, h(B_i)\}_{i=1}^n$ satisfies the SARP, there exists a strictly concave and strictly monotone representation for $\{B_i, h(B_i)\}_{i=1}^n$. It is well known that the existence of such a representation implies that $\{B_i, h(B_i)\}_{i=1}^n$

satisfies the SARP (cf. Richter (1966, 1971)).

Q.E.D.

Theorem 1 provides a nonparametric test for the consistency of co-convex choices with the existence of a strictly concave and strictly monotone objective function that generates them. Moreover, the result of Theorem 1 together with the results in Richter (1966) imply that, for finite sets of observations on co-convex choices, the existence of a strictly concave and strictly monotone representation is observationally equivalent to the existence of a reflexive, transitive, and total rationalization.

We next state formally the equivalence result.

DEFINITION 10: *A binary relation T is a* _*reflexive, transitive, and total rationalization*_ *for a choice space $(B_i, h(B_i))_{i=1}^n$ in X if*

(i) T is reflexive, transitive, and total, and

(ii) for all $i=1,2,\ldots,n$ and all $y \in B_i$ such that $y \neq h(B_i)$,

$h(B_i)\ T\ y$ and not$(\ y\ T\ h(B_i)\ )$.

COROLLARY 1: *Suppose that $(B_i, h(B_i))_{i=1}^n$ is a co-convex choice space in a convex and compact subset X of R^K. Then, there exists a strictly concave and strictly monotone representation for $(B_i, h(B_i))_{i=1}^n$ if and only if there exists a reflexive, transitive, and total rationalization for $(B_i, h(B_i))_{i=1}^n$.*

The proof of this corollary is immediate from Theorem 1 and Richter (1966).

Remark 1 in Matzkin and Richter (1987) shows that, without additional assumptions, it is not always possible to obtain a differentiable representation for a co-convex choice space that satisfies the SARP.

4. SUPPORTABLE AND MIXED CHOICE SPACES

In this section we present necessary and sufficient conditions under which supportable choice spaces and mixed choice spaces possess a strictly concave and strictly monotone representation. For supportable choice spaces, these conditions are expressed in terms of their *supporting choice spaces*, the definition of which follows:

DEFINITION 10: *Let* $\{B_i,\ h(B_i)\}_{i=1}^n$ *be a supportable choice space in a subset* X *of* R^K. *For each* $i=1,\ldots,n$, *let* $s^i \in R^K_{++}$ *be such that for all* $y \in B_i$, $s^i y \le 1$ and $s^i h(B_i) = 1$. *Define* C_i *and* $h(C_i)$ *by*

$$C_i = \{x \in X \mid s^i x \le 1\} \quad \text{and} \quad h(C_i) = h(B_i) \quad .$$

Then, the <u>*supporting choice space*</u>[5] *of* $\{B_i,\ h(B_i)\}_{i=1}^n$ *is* $\{C_i,\ h(C_i)\}_{i=1}^n$.

This definition implies that the supporting choice space of $\{B_i,\ h(B_i)\}_{i=1}^n$ is a set of supporting choices $(C_i, h(C_i))$ such that for each i, C_i is a choice set determined by a hyperplane that supports B_i at $h(B_i)$.

The following theorem characterizes supportable choice spaces for which there exists a strictly concave and strictly monotone representation.

THEOREM 2: *Suppose that* $\{B_i, h(B_i)\}_{i=1}^{n}$ *is a supportable choice space in a convex and compact subset* X *of* R^K. *Let* $\{C_i, h(C_i)\}_{i=1}^{n}$ *be the supporting choice space of* $\{B_i, h(B_i)\}_{i=1}^{n}$. *Then, there exists a strictly monotone and strictly concave representation for* $\{B_i, h(B_i)\}_{i=1}^{n}$ *if and only if* $\{C_i, h(C_i)\}_{i=1}^{n}$ *satisfies the SARP*.

The main part in the proof of Theorem 2 consists in showing that a strictly concave and strictly monotone representation for $\{B_i, h(B_i)\}_{i=1}^{n}$ is a strictly concave and strictly monotone representation for $\{C_i, h(C_i)\}_{i=1}^{n}$. This is shown by means of the next lemma:

LEMMA 4: *Let* $\{C_i, h(C_i)\}_{i=1}^{n}$ *be the supporting choice space of a supportable choice space* $\{B_i, h(B_i)\}_{i=1}^{n}$ *in a convex and compact subset* X *on* R^K, *and let* $V: X \to R$ *be a strictly concave and strictly monotone representation for* $\{B_i, h(B_i)\}_{i=1}^{n}$. *Suppose that* $h(C_j)\ S\ h(C_k)$ *for some* $j,k \in \{1,\ldots,n\}$. *Then,* $V(h(C_k)) < V(h(C_j))$.

PROOF OF THEOREM 2: Suppose first that the supporting choice space $\{C_i, h(C_i)\}_{i=1}^n$ of $\{B_i, h(B_i)\}_{i=1}^n$ satisfies the SARP . Then, by Theorem 1 it follows that there exists a strictly concave and strictly monotone representation $V: X \to R$ for $\{C_i, h(C_i)\}_{i=1}^n$, since $\{C_i, h(C_i)\}_{i=1}^n$ satisfies the conditions of that Theorem. Since $B_i \subset C_i$ and $h(B_i) = h(C_i)$, it follows that V is a representation for $\{B_i, h(B_i)\}_{i=1}^n$.

We show next that if V is a strictly concave and strictly monotone representation for $\{B_i, h(B_i)\}_{i=1}^n$, $\{C_i, h(C_i)\}_{i=1}^n$ satisfies the SARP .

Suppose that $\{C_i, h(C_i)\}_{i=1}^n$ does not satisfy the SARP . Then, there exist $r,t \in \{1, \ldots, n\}$ such that $h(C_r)\ H\ h(C_t)$ and $h(C_t)\ S\ h(C_r)$. By the definition of H it follows that for some (possibly empty) sequence $\{q, \ldots, v\} \subset \{1, \ldots, m\}$

(T2.1) $h(C_r)\ S\ h(C_q)\ S \ldots S\ h(C_v)\ S\ h(C_t)\ S\ h(C_r)$.

By Lemma 4,

(T2.2) $h(C_j)\ S\ h(C_k) \Rightarrow V(h(C_k)) < V(h(C_j))$.

Then, since (T2.1) and (T2.2) imply a contradiction, we can conclude that if there exists a strictly concave and strictly monotone representation for $\{B_i, h(B_i)\}_{i=1}^n$, $\{C_i, h(C_i)\}_{i=1}^n$ satisfies the SARP .

This completes the proof of Theorem 2.

Q.E.D.

The observational equivalence between the maximization of any function and the maximization of a strictly concave function, which was shown to hold for co-convex choice spaces, can not be obtained for supportable choice spaces. Figure 5 shows an example of a supportable choice space that satisfies the SARP , while its supporting choice space does not satisfies the SARP. Then, by the results in Richter (1966) and Theorem 2 above, this supportable choice space possesses a representation but it does not possess a strictly concave and strictly monotone representation. This argument implies that the result of Theorem 2 can be employed to test nonparametrically the strict concavity and strict monotonicity of representations, given that a representation exists.

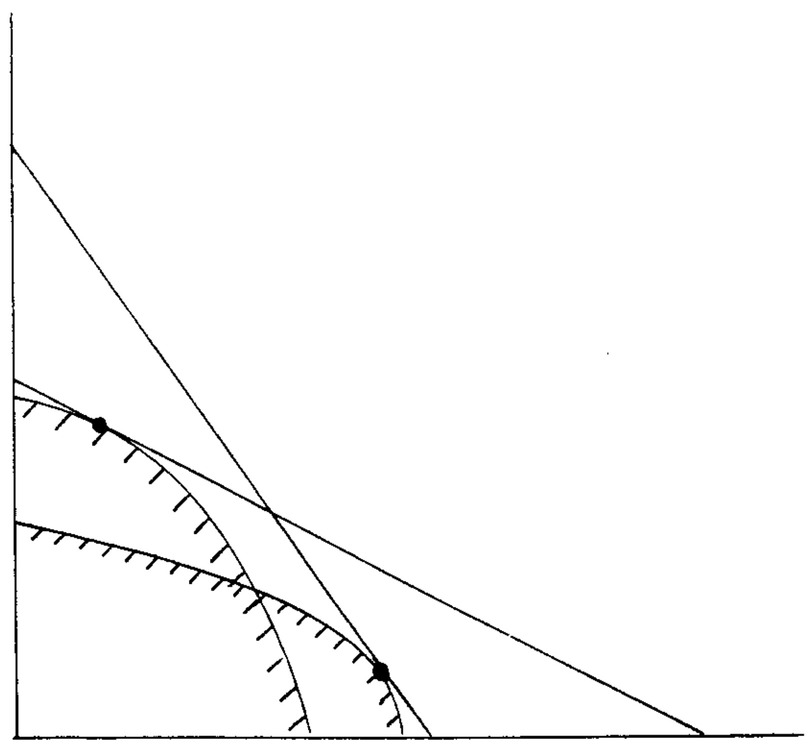

FIGURE 5

Note that to test for the existence of a strictly concave and strictly monotone representation it is sufficient to observe only the chosen element and the normal of the supporting hyperplane; we do not need to observe each choice set in its entirety.

For mixed choice spaces, the proofs of Theorems 1 and 2 immediately imply the following Corollary:

COROLLARY 2: *Suppose that* $\{B_i, h(B_i)\}_{i=1}^n$ *is a mixed choice space in a convex and compact subset* X *of* R^K. *For each* $i=,\ldots,n$, *let* $(D_i,h(D_i))$ *be equal to* $(B_i,h(B_i))$ *if* $(B_i,h(B_i))$ *is a co-convex choice in* X, *and let* $(D_i,h(D_i))$ *be equal to the supporting choice of* $(B_i,h(B_i))$ *if* $(B_i,h(B_i))$ *is a supportable choice in* X. *Then, there exists a strictly concave and strictly monotone representation for* $\{B_i, h(B_i)\}_{i=1}^n$ *if and only if* $\{D_i, h(D_i)\}_{i=1}^n$ *satisfies the SARP*.

5. CONCLUSION

We have extended the axiomatic theory of Revealed Preference to apply to finite sets of choice data that are generated subject to nonlinear choice sets. Our results apply to choices over sets that possess convex and monotone complements, called *co-convex* choices, and to choices over sets that can be supported by a unique hyperplane at the chosen element, called *supportable* choices. Examples of economic situations involving these type of choices are consumers or firms with monopsony power, consumers facing either regressive or progressive taxes, households producing commodities with either a convex or a co-convex technology, and social planners facing a production possibilities set.

We have shown that the SARP characterizes finite sets of co-convex choices generated by the maximization of a common strictly concave and strictly monotone objective function. This provides a nonparametric test for the consistency of co-convex choices with the maximization of a strictly concave and strictly monotone function. This also implies that maximization of such a function is observationally equivalent to the maximization of a reflexive, transitive, and total binary relation.

We have also provided a characterization of finite sets of supportable choices generated by the maximization of a common strictly concave and strictly monotone objective function. This characterization has been expressed in terms of the supporting choices. We have shown that a finite set of supportable choices is consistent with the maximization of a strictly concave and strictly monotone objective function if and only if the corresponding supporting choices satisfy the SARP . An example has established that a set of supportable choices may be consistent with the maximization of an objective function but inconsistent with the maximization of a strictly concave and strictly monotone function. Therefore, our result provides a nonparametric test for the strict concavity and strict monotonicity of the objective function, when one exists.

Finally, we have also characterized mixed choice spaces that possess a strictly concave and strictly monotone representation.

APPENDIX A

In this appendix we present the notation and definitions employed in the paper.

If $x = (x_1, \ldots, x_K)$ and $y = (y_1, \ldots, y_K)$ are vectors in R^K then $\mathbf{x} \gg \mathbf{y}$ iff $\forall k \in \{1, \ldots, K\}$ $x_k > y_k$; $\mathbf{x} > \mathbf{y}$ iff $\forall k \in \{1, \ldots, K\}$, $x_k \geq y_k$, and $x \neq y$; and $\mathbf{x} \geq \mathbf{y}$ iff $\forall k \in \{1, \ldots, K\}$ $x_k \geq y_k$. $R^K_+ = \{x \in R^K | x \geq 0\}$, $R^K_{++} = \{x \in R^K | x \gg 0\}$. For all x in R^K , $\|\mathbf{x}\| = \Sigma^n_{j=1}(x_j)^2$ denotes the Euclidean norm in R^K ; $\mathbf{d}$: $R^K \times R^K \to R_+$ denotes the Euclidean metric $(d(x,y) = \|x-y\|)$.

Suppose that T is a binary relation on a set X . Then, T is <u>reflexive</u> if $\forall\, x \in X$ $x\,T\,x$; T is <u>transitive</u> if $\forall\, x, y, z \in X$ such that $x\,T\,y$ and $y\,T\,z$, $x\,T\,z$; T is <u>total</u> if $\forall\, x, y \in X$ such that $x \neq y$, either $x\,T\,y$ or $y\,T\,x$ or both $x\,T\,y$ and $y\,T\,x$; T is <u>asymmetric</u> if $\forall\, x, y \in X$ such that $x\,T\,y$, it is not the case that $y\,T\,x$.

For $\eta > 0$, $\mathbf{N}(\mathbf{x},\eta) = \{z | d(x,z) < \eta\}$ denotes the neighborhood of radius η around x ; and for $r > 0$ and $a \in R^K$, $\mathbf{P}(\mathbf{a},\mathbf{r}) = \{y | [d(y,a)]^2 \leq r^2\}$ denotes the sphere of radius r with center in a , $\mathbf{G}(\mathbf{a},\eta) = \{x \mid a-(\eta,\ldots,\eta) \leq x \leq a + (\eta,\ldots,\eta)\}$ denotes the "box" with center a and diameter $2\,\eta$, and if $p \in R^K$ and $\|p\|=1$, $\Phi(p,x) = \{\, y \mid p\,y = p\,x \,\}$ denotes the hyperplane passing through x and with normal p .

If P is any statement, $\neg(P)$ denotes the negation of P . $\emptyset$ denotes the empty set and $\setminus$ denotes set subtraction. If A is a subset of R^K , then **int(A)** denotes the interior of A , **cl(A)** denotes the closure of A, and $\partial A = A\setminus \text{int}(A)$ denotes the topological boundary of A ; **diam(A)** = sup { d(x,y) | x, y ∈ A} .

A subset A of $H \subset R^K$ is <u>convex</u> if $\forall x, y \in A$ and $\beta \in [0,1]$: $\beta x + (1-\beta)y \in A$; A is <u>strictly convex</u> iff $\forall x, y \in A$ and $x \neq y$, and all $\beta \in (0,1)$: $\beta x + (1-\beta)y \in \text{int}(A)$. The subset $A \subset H$ is <u>monotone in H</u> if $\forall x \in A$ and $\forall e \in R^K_+$ such that $x + e \in H$, $x + e \in A$; $A \subset H$ is <u>decreasing</u> in H if $\forall x \in A$ and $\forall e \in R^K_+$ such that $e \neq 0$ and $x + e \in H$, $x - e \in \text{int}(A)$; $A \subset H$ is <u>strictly monotone</u> in H if $\forall x \in A$ and $\forall e \in R^K_+$ such that $e \neq 0$ and $x + e \in H$, $x + e \in \text{int}(A)$. The <u>convex hull of</u> A , denoted conv(A) , is the set $\{x \in R^K | \text{for some } x^1, \ldots, x^s \in A, a^1, \ldots, a^s \in R_+ \ (\Sigma^s_{j=1} a^j = 1 \ \& \ x = \Sigma^s_{j=1} a^j x^j)\}$; the <u>convex monotone hull of</u> A , denoted $\text{com}^+(A)$, is the convex hull of the set $(A + R^K_+)$.

The set $A \subset R^K$ is a <u>polyhedron</u> if there exist $g_1, \ldots, g_S \in R^K$ such that $\forall x \in A$, $\exists c_1, \ldots, c_S \in R_+$ such that $\Sigma^S_{j=1} c_j = 1$ and $\Sigma^S_{j=1} c_j q_j = 1$. An element x of a polyhedron A of R^K is an <u>extreme point</u> of A if there exist $x_1, x_2 \in A$ and $\lambda \in (0,1)$ such that $x_1 \neq x_2$ and $x = \lambda x_1 + (1-\lambda)x_2$. If $A \subset R^K$ is a polyhedron, **T(A)** is the set of all extreme points q_j of A for which there does not exist another extreme point q_k of A and $e \in R^K$ such an $q_j = q_k + e$.

If G is a convex subset in R^K, and f is real valued function on G, f is <u>concave</u> if $\forall x, y \in G$ and $\beta \in [0,1]$, $f(\beta x + (1-\beta)y) \geq \beta f(x) + (1-\beta)f(y)$; f is <u>strictly concave</u> if $\forall x, y \in G$ and $\beta \in (0,1)$, $f(\beta x + (1-\beta)y) > \beta f(x) + (1-\beta)f(y)$; f is <u>convex</u> if $\forall x, y \in G$ and $\beta \in [0,1]$, $f(\beta x + (1-\beta)y) \geq \beta f(x) + (1-\beta)f(y)$; f is <u>strictly convex</u> if $\forall x, y \in G$ and $\beta \in (0,1)$, $f(\beta x + (1-\beta)y) > \beta f(x) + (1-\beta)f(y)$; f is <u>quasi-concave</u> if $\forall x \in G$ the set $\{y \in G | f(y) \geq f(x)\}$ is convex; f is <u>strictly quasi-concave</u> if $\forall x \in G$ the set $\{y \in G | f(y) \geq f(x)\}$ is strictly convex; f is <u>quasi-convex</u> if $|x \in G$ the set $\{y \in G | f(y) \leq f(x)\}$ is convex; f is strictly <u>quasi-convex</u> if $\forall x \in G$ the set $\{y \in G | f(y) \leq f(x)\}$ is strictly convex; f is <u>continuous</u> if $\forall x \in G$ the sets $\{x \in G | f(x) \geq d\}$ and $\{x \in G | f(x) \leq d\}$ are both closed in G ; f is <u>monotone</u> if $\forall x, y \in G$: $x > y \Rightarrow f(x) \geq f(y)$; and f is <u>strictly monotone</u> if $\forall x, y \in G : x > y \Rightarrow f(x) > f(y)$.

If A is a closed and bounded set in R^K and $x \in R^K$, we define the S-distance from x to A by $\mathbf{S(x,A)} = \inf\{d(x,y) | y \in A\}$.
For each $r > 0$, we define the S-neighborhood of radius r around A by

$$\mathbf{NS(A,r)} = \{x \in R^K \, | S(x,A) < r\} .$$

Then S-distance between any two closed and bounded sets A and B in R^K, is then defined with respect to the metric

$$\Delta(\mathbf{A,B}) = \delta_1 + \delta_2 ,$$

where $\delta_1 = \inf\{\delta > 0 | B \subset NS(A,\delta)\}$, and $\delta_2 = \inf\{\delta > 0 | A \subset NS(B,\delta)\}$.

APPENDIX B

In this appendix we present the proofs of Lemmas 1 - 4, which were stated in the proofs of Theorems 1 and 2 in Sections 3 and 4. The proofs of Lemmas 1 - 4 employ Lemmas A1 - A5, which are stated and proved in Appendix C.

PROOF OF LEMMA 1: Define the sequence of sets $C(1), C(2), \ldots$ and associated sets $D(0), D(1), D(2), \ldots$, recursively as follows:
Let $D(0) = D$. For $t = 1, 2, \ldots$ define $C(t)$ by for all $i=1, \ldots, n$

(1.1) $h(B_i) \in C(t)$ iff there does not exist $h(B_j) \in D(t-1)$ such that $h(B_i)\ H\ h(B_j)$.

So $C(t)$ is the set of elements in the range of h which are not revealed preferred to any elements of $D(t-1)$. For each t we define $D(t)$ by

(1.2) $D(t) = D \setminus \cup_{s=1}^{t} C(s)$.

Note that

for $t = 1, 2, \ldots,\ C(t) \neq \emptyset$ if $D(t-1) \neq \emptyset$.

Since, if $C(t) = \emptyset$, then by (1.1), for all $h(B_i) \in D(t-1)$ there exists $h(B_j) \in D(t-1)$ such that $h(B_i)\ H\ h(B_j)$. Since $D(t-1)$ has a finite number of elements, there must exist a finite sequence

$h(B_i)$, $h(B_j)$, ..., $h(B_r)$ of elements of $D(t-1)$ satisfying

$$h(B_i) \; H \; h(B_j) \; H \; \ldots \; H \; h(B_r) \; H \; h(B_i) \; .$$

But this contradicts the SARP . Hence, $C(t) \neq \emptyset$.

Hence, $\cup_{s=1}^{t} C(s)$ strictly increases as t increases, until the $D(t)$ sets become empty. Therefore, there exists a first T such that $D(T) = \emptyset$.

Clearly then, $C(1), \ldots, C(T)$ is a partition of D .

We proceed now to obtain the sets $Z(1), \ldots, Z(T)$. We first note that there exists $\eta > 0$ such that

(1.3) $\forall i, j \in \{1, \ldots, n\}$ such that $h(B_i) \neq h(B_j)$ and $\neg(h(B_j) \; S \; h(B_i))$

$$N(h(B_i), \eta) \subset B_j^c \; .$$

Since, if $h(B_i) \neq h(B_j)$ and $\neg(h(B_j) \; S \; h(B_i))$ then by the definition of S, $h(B_i) \notin B_j$. Since B_j is a closed set, there exists $\eta^{ij} > 0$ such that $N(h(B_i), \eta^{ij}) \subset B_j^c$. Hence, $\eta = \min\{\eta^{ij} | i, j = 1, \ldots, n\}$ satisfies (1.3).

Since $\eta > 0$, we can define a decreasing sequence of positive numbers $\eta(1), \eta(2), \ldots$ by:

(1.4) $\quad \eta(t) = (4)^{-t}\eta \;$ for $\; t = 1, 2, \ldots$.

Further, we can define a large enough compact set H^0 by

(1.5) $\quad H^0 = \{ x \in R^K \mid w^1 \leq x \leq w^2 \}$.

where $w^1 = (v,\ldots,v) \in R^K$ and $w^2 = (k,\ldots,k) \in R^K_+$ are such that $\forall x \in X$, $(v+2\eta,\ldots,v+2\eta) \ll x \ll (k-2\eta,\ldots, k-2\eta)$.

For $t = 1, \ldots, T$, we define

(1.6) $\quad S(t) = D(t-1)\backslash C(t)$,

(1.7) $\quad E(t) = \cup_{y\in S(t)} G(y, \eta(t))$, and

(1.8) $\quad Z(t) = \text{com}^+ [C(t) \cup E(t)] \cap H^0$,

where $G(a,r) = \{ x \in R^K \mid a - \underline{r} \le x \le a + \underline{r} \}$ for $\underline{r} = (r,\ldots,r) \in R^K$.

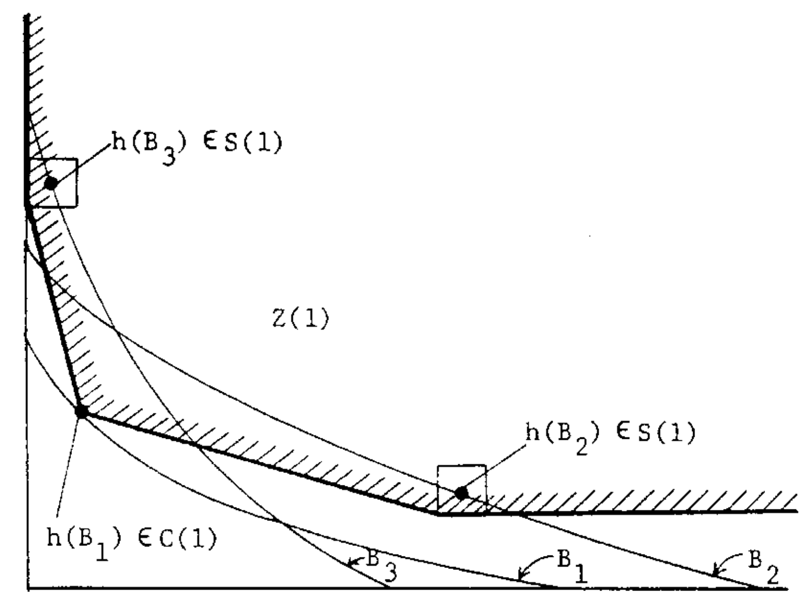

FIGURE B1

Hence for each t, $S(t)$ is the set of bundles in $D(t-1)$ that are revealed preferred to some bundle in $D(t-1)$, $E(t)$ is the union of boxes of diameter $2\eta(t)$ around each of the revealed preferred bundles in $D(t-1)$, and $Z(t)$ is the intersection of H^0 with the convex monotone hull of the not-revealed-preferred bundles in $D(t-1)$ and the "revealed-preferred boxes" in $D(t-1)$. (See Figure B1)

We prove now that these $Z(t)$ sets satisfy (L1.1)-(L1.4).

Proof of (L1.1): This is immediate from (1.8).

Proof of (L1.2): Let $h(B_i) \in C(t)$. Then, by hypothesis and (1.8), $h(B_i) \in Z(t) \cap B_i$. By (1.8), the definitions of $C(t)$ and $E(t)$, and the monotonicity and convexity and B_i^c it follows that for any $z \in Z(t)$ such that $z \neq h(B_i)$, $z \in B_i^c$. Hence, (L1.2) follows.

Proof of (L1.3): Let y belong to $Z(t+1)$. Then, by (1.8) and the definition of com^+,

$$y \in [\text{com}^+(C(t+1) \cup E(t+1))] \cap H^0$$
$$= [\text{conv}[(C(t+1) \cup E(t+1))] + R_+^K] \cap H^0 .$$

Let then $w \in \text{conv}[(C(t+1) \cup E(t+1))]$ and $e \in R_+^K$ be such that

(1.10.1) $\quad y = w + e \in H^0$.

To show that $w \in \text{int } Z(t)$ we proceed as follows:.

For any $x \in C(t+1)$, $x \in S(t)$ by (1.1), (1.2) and (1.6); hence, by (1.7), $x \in \text{int } E(t)$. Since by (1.8) $E(t) \subset Z(t)$, $\text{int } E(t) \subset \text{int } Z(t)$. Hence, (1.10.2) $C(t+1) \subset \text{int } Z(t)$.

For any $x \in E(t+1)$, there exists by (1.7) $y \in S(t+1)$ such that $x \in G(y, \eta(t+1))$. But then by (1.6) and (1.2), $y \in S(t)$, and by (1.6), $x \in \text{int } G(y, \eta(t))$. Hence, by (1.7), $x \in \text{int } E(t)$, which by (1.8) is included in int $Z(t)$. Therefore,

(1.10.3) $E(t+1) \subset \text{int } Z(t)$.

By (1.10.2) and (1.10.3), $w \in \text{com}^{+}[C(t+1) \cup E(t+1)] \subset \text{int } Z(t)$, where by interior we mean interior relative to H^0.

Hence, from (1.10.1), $y \in \text{int } Z(t) + e$, which implies by (1.8) that $y \in \text{int}(Z(t) + e) = \text{int } Z(t)$.

We have then shown that for $t = 1, \ldots, T-1$, $Z(t+1) \subset \text{int } Z(t)$.

Proof of (L1.4): To see that $h(B_i)$ is an extreme point of $Z(t)$, suppose that there exist $x_1, x_2 \in Z(t)$ and $\lambda \in (0,1)$ such that $h(B_i) = \lambda x_1 + (1-\lambda)x_2$. Then, by (L1.2), $x_1 \in B_i^c$ and $x_2 \in B_i^c$. Hence, $h(B_i) \in B_i^c$ by the convexity of B_i^c, which contradicts the fact that $h(B_i) \in B_i$. Moreover, if there existed an extreme point q_j of $Z(t)$ and some $e > 0$ such that $h(B_i) = q_j + e$ then, since by (L1.2) $q_j \in B_i^c$, the monotonicity of B_i^c would imply that $h(B_i) = q_j + e \in B_i^c$, which again is impossible. Hence, $h(B_i) \in T(Z(t))$.

Q.E.D.

PROOF OF LEMMA 2: For each t and each $r > \text{diam}(H^0)$, we will let ¥(Z(t),r) denote the intersection of H^0 with all spheres P(a,r') with radius $r' \leq r$ and center a that satisfy

(2.1.1) $Z(t) \subset P(a,r')$, and

(2.1.2) for all $y \in H^0$, $a \gg y$.

After proving some properties of these ¥(Z(t),r) sets, we will define Y(t) to be equal to ¥(Z(t),r) for a sufficiently large r . We will first show that for each t there exists r_t large enough such that

(2.2) $¥(Z(t),r_t)$ is compact,

(2.3) $¥(Z(t),r_t)$ is strictly convex and strictly monotone in H^0,

(2.4) $¥(Z(t),r_t) \subset \text{int } ¥(Z(t-1),r_{t-1})$, and

(2.5) $\forall$ i such that $h(B_i) \in C(t)$, $¥(Z(t),r_t) \cap B_i = \{h(B_i)\}$.

<u>Proof of (2.2):</u> Let $r > \text{diam }(H^0)$. Then, ¥(Z(t),r) satisfies (2.2) since, by its definition, ¥(Z(t),r) is the intersection of closed and bounded sets.

<u>Proof of (2.3):</u> Let $r > \text{diam}(H^0)$ be given. It is clear that ¥(Z(t),r) ; since it is the intersection of convex sets. To see that ¥(Z(t),r) is strictly convex, we follow the argument given in Eggleston (1969, Theorem 34): Suppose that x_1, $x_2 \in ¥(Z(t),r)$. Then, ¥(Z(t),r) must contain the intersection of all spheres of radius $r' \leq r$ that contain both x_1 and x_2 . Hence, all points in the segment (x_1, x_2) are interior points of ¥(Z(t),r) . It then follows that ¥(Z(t),r) is strictly convex.

To see that $\Psi(Z(t),r)$ is strictly monotone in H^0, suppose that $x \in \Psi(Z(t),r)$, $e > 0$, and $x + e \in H^0$. Since $x \in \Psi(Z(t),r)$, it follows from the definition of $\Psi(Z(t),r)$ and (2.1.2) that $\| a - x \|^2 \le r'^2$ for all $P(a,r')$ satisfying (2.1.1) and (2.1.2). Since $e > 0$ and $x + e \in H^0$, for all such a and r', $\| a - x - e \|^2 < \| a - x \|^2 \le r'^2$. Hence, $x + e \in \text{int } (\Psi(Z(t),r))$. It then follows that $\Psi(Z(t),r)$ is strictly monotone in H^0.

Proof of (2.4): Note that by (L1.1) the Z(t) sets are compact, therefore by (L1.3) and the definition of Δ there exists $\delta_t > 0$ be such that for all $x \in Z(t)$ and all $y \in Z(t-1)$, $\| x - y \| > \delta_t$. By Lemma A3 there exists r' such that for all $r \ge r'$, $\Delta(\Psi(Z(t),r), Z(t-1)) < \delta_t$. By the definition of Δ and $\Psi(t-1,r)$ this implies that for all $r \ge r'$, $\Psi(Z(t),r) \subset \text{int } Z(t-1) \subset \text{int } \Psi(Z(t-1),r)$. Hence, (2.4) follows.

Proof of (2.5): To prove (2.5) we will show that

(2.6) $\forall i$ such that $h(B_i) \in C(t)$,

there exists $\eta_i > 0$ and r_i such that $\forall\, r \ge r_i$

(2.6.1) $\Psi(Z(t),r) \cap B_i \cap J(t,i) = \{ h(B_i) \}$, and

(2.6.2) $\Psi(Z(t),r) \cap B_i \cap J(t,i)^c = \emptyset$,

where $J(t,i) = G(h(B_i),\eta_i)$.

The statement of (2.5) follows then immediately from (2.6).

<u>Proof of (2.6):</u> To prove (2.6.1) we will construct a monotone polyhedron D(i) for each i such that $h(B_i) \in C(t)$. The set D(i) will contain Z(t) and it will intersect B_i only at $h(B_i)$. (See Figure B2.1) We will then show that for a large enough r_i $\Psi(Z(t),r_i) \cap J(t,i)$ is included into $D(i) \cap J(t,i)$. (See Figure B2.2)

Let t be given and let $i \in \{1,\ldots,n\}$ be such that $h(B_i) \in C(t)$. By (L1.1) there exist $\{q_1,\ldots,q_J\} = T(Z(t))$. By (L1.4) we can assume w.l.o.g. that $q_1 = h(B_i)$. By (L1.2) and the closeness of B_i there exists $\delta > 0$ such that for $j=2,\ldots,J$

(2.7) $G(q_j,\delta) \subset B_i^c$. Let

(2.8) $D(i) = \text{com}^+ (h(B_i), G(q_2,\delta), \ldots , G(q_J,\delta)) \cap H^0$.

Then,

(2.9) D(i) is a polyhedron that is monotone in H^0.

The next step is to show that,

(2.10) $h(B_i) \in T(D(i))$,

(2.11) $Z(t) \setminus \{h(B_i)\} \subset \text{int } D(i)$, and

(2.12) $D(i) \setminus \{h(B_i)\} \subset B_i^c$.

<u>Proof of (2.10)</u>: Suppose that (2.10) is not true. Then, by (2.9) there exist $\bar{q}_1, \ldots, \bar{q}_J \in T(D(i))$, $a_1,\ldots,a_J \in R_+$, and $e \in R_+^K$ such that $\bar{q}_j \neq h(B_i)$ $(j=1,\ldots,J)$, $\Sigma_{j=1}^J a_j = 1$, and $h(B_i) = \Sigma_{j=1}^J a_j \bar{q}_j + e$. By (2.8)

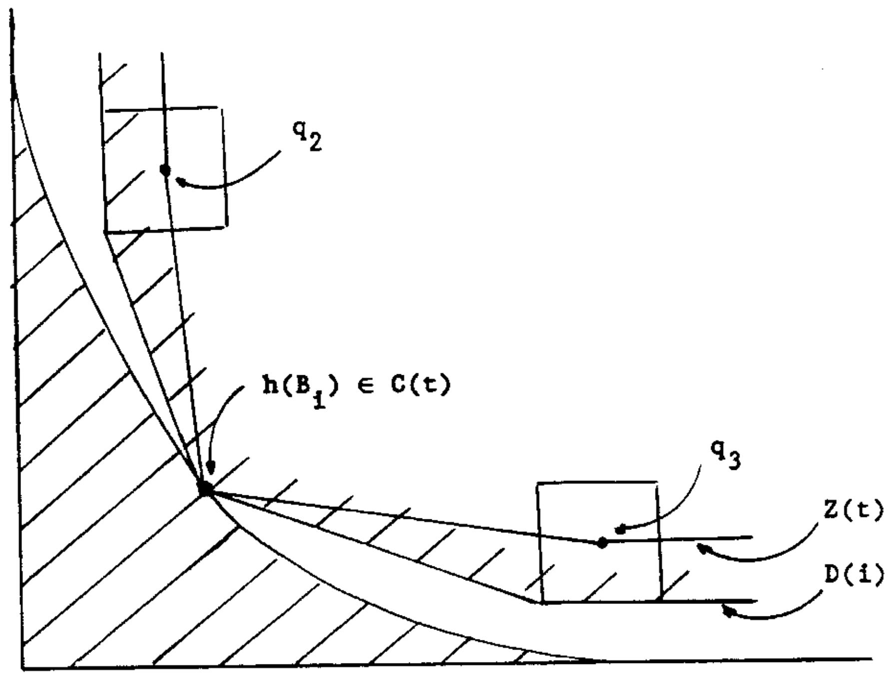

FIGURE B2.1

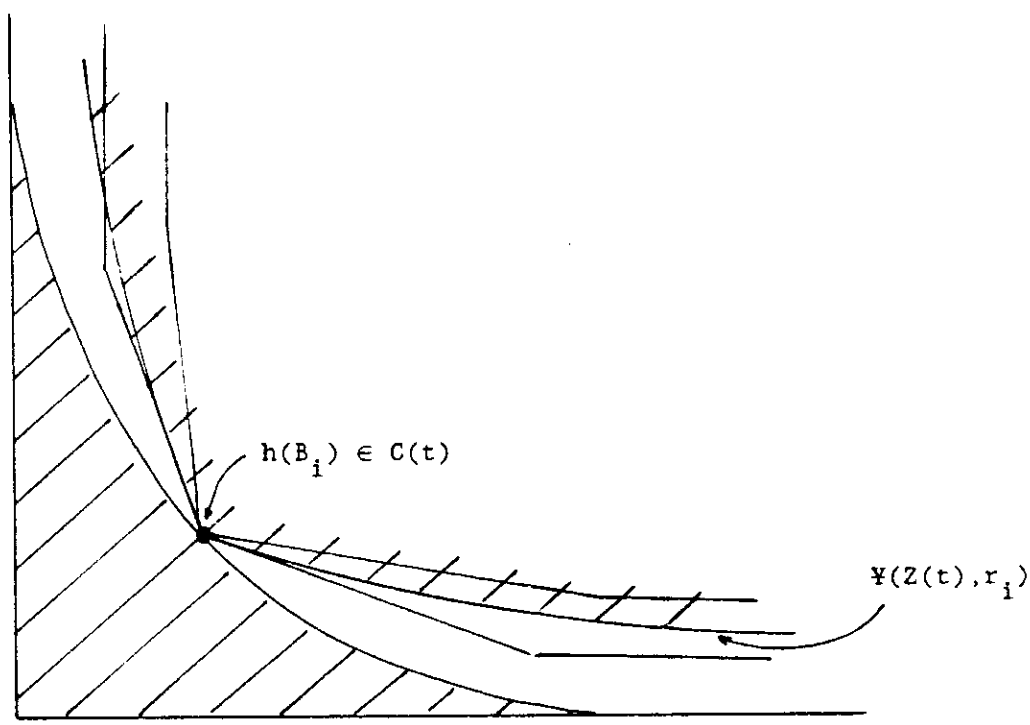

FIGURE B2.2

and (2.7) $\bar{q}_j \in B_i^c$ $(j = 1,\ldots,J)$. Then, the convexity and monotonicity of B_i^c imply that $h(B_i) \in B_i^c$, which contradicts the fact that $h(B_i) \in B_i$. Hence, (2.10) follows.

Proof of (2.11): Let $\bar{D} = cl(D(i)^c)$. Then $\bar{D}$ is a closed subset of H and by (2.9) $\bar{D}^c = int\,(D(i))$ is a monotone and convex subset in H. By (L1.2) $h(B_i) \in Z(t)$; by (2.8) $h(B_i) \in \bar{D}$; by (L1.1), (L1.4), and (2.8) $h(B_i) + R_+^K \subset \bar{D}^c$; and by (2.8) and (2.7) for all $y \in T(Z(t))$ such that $y \neq h(B_i)$, $y \in \bar{D}^c$. Then, letting B, A, and x in Lemma A1 be respectively $\bar{D}$, Z(t), and $h(B_i)$, we obtain from Lemma A1 that $Z(t) \setminus \{h(B_i)\} \subset \bar{D}^c =$ int D(i).

Proof of (2.12): By hypothesis, B_i is closed and B_i^c is monotone and convex in H. Moreover, $h(B_i) \in B_i$ and $h(B_i) + R_+^K \subset B_i^c$. By (2.8), $h(B_i) \in D(i)$. And, by (L1.2), it follows that for all $y \in Z(t)$ such that $y \neq h(B_i)$, $y \in B_i^c$. Then, letting A, B, and x in Lemma A1 be respectively D(i), B_i, and $h(B_i)$, we can conclude that $D(i) \setminus \{h(B_i)\} \subset B_i^c$. Hence, (2.12) follows.

These results imply that D(i) is a monotone polyhedron that intersects B_i and Z(t) at $h(B_i)$, it includes Z(t), and it is included in B_i^c. We will now employ (2.10)-(2.12) to show that for a large enough r and a small enough box J(t,i) that contains $h(B_i)$, $¥(Z(t),r) \cap J(t,i)$ is included into $D(i) \cap J(t,i)$. From this it will follow that $¥(Z(t),r) \cap J(t,i)$ intersects $B_i \cap J(t,i)$ only at $h(B_i)$. Hence, we will next show that

(2.13) there exists $\eta_i > 0$ and r_i such that $\forall\, r \geq r_i$

$$(\Psi(Z(t),r) \cap J(t,i)) \subset (D(i) \cap J(t,i)) ,$$

where $J(t,i) = G(h(B_i),\eta_i)$.

<u>Proof of (2.13)</u>: By (2.9) and (2.10) $D(i)$ has a finite number of faces $A_1, \ldots, A_Q$ adjacent to $h(B_i)$. Let $p_1, \ldots, p_Q$ be their normals. By (2.9) $p_j \in R^K_+$ $(j=1,\ldots,Q)$. By (L1.1), (2.9), and (2.11), for all j in $\{1,\ldots,Q\}$ and all $x \in Z(t)$ such that $x \neq h(B_i)$, $p_j\, x > p_j\, h(B_i)$. It then follows by (L1.1), (L1.4), and Lemma A2 that for each $j \in \{ 1,\ldots,Q \}$ there exists a sphere $P(a_j,r_j)$ such that (see Figure B2.3)

(2.14) $a_j \gg y$ for all $y \in H^0$,

(2.15) $Z(t) \subset P(a_j,r_j)$, and

(2.16) $P(a_j,r_j)$ is supported by the hyperplane $\Phi(p_j,h(B_i))$ at $h(B_i)$.

Let

(2.17) $P_i = \cap_{j=1}^{Q} P(a_j,r_j)$.

Then, by (2.14), (2.15), (2.17), and the definition of $\Psi(Z(t),r)$ (see Figure B2.4)

(2.18) $\Psi(Z(t),r) \subset P_i$ for $r > r_i = \max \{r_j \mid j=1,\ldots,Q\}$,

and by (2.16) and (2.17)

(2.19) $P_i \subset \{ x \in R^K \mid p_j\, x \geq p_j\, h(B_i) \;\; j=1,\ldots,Q \}$.

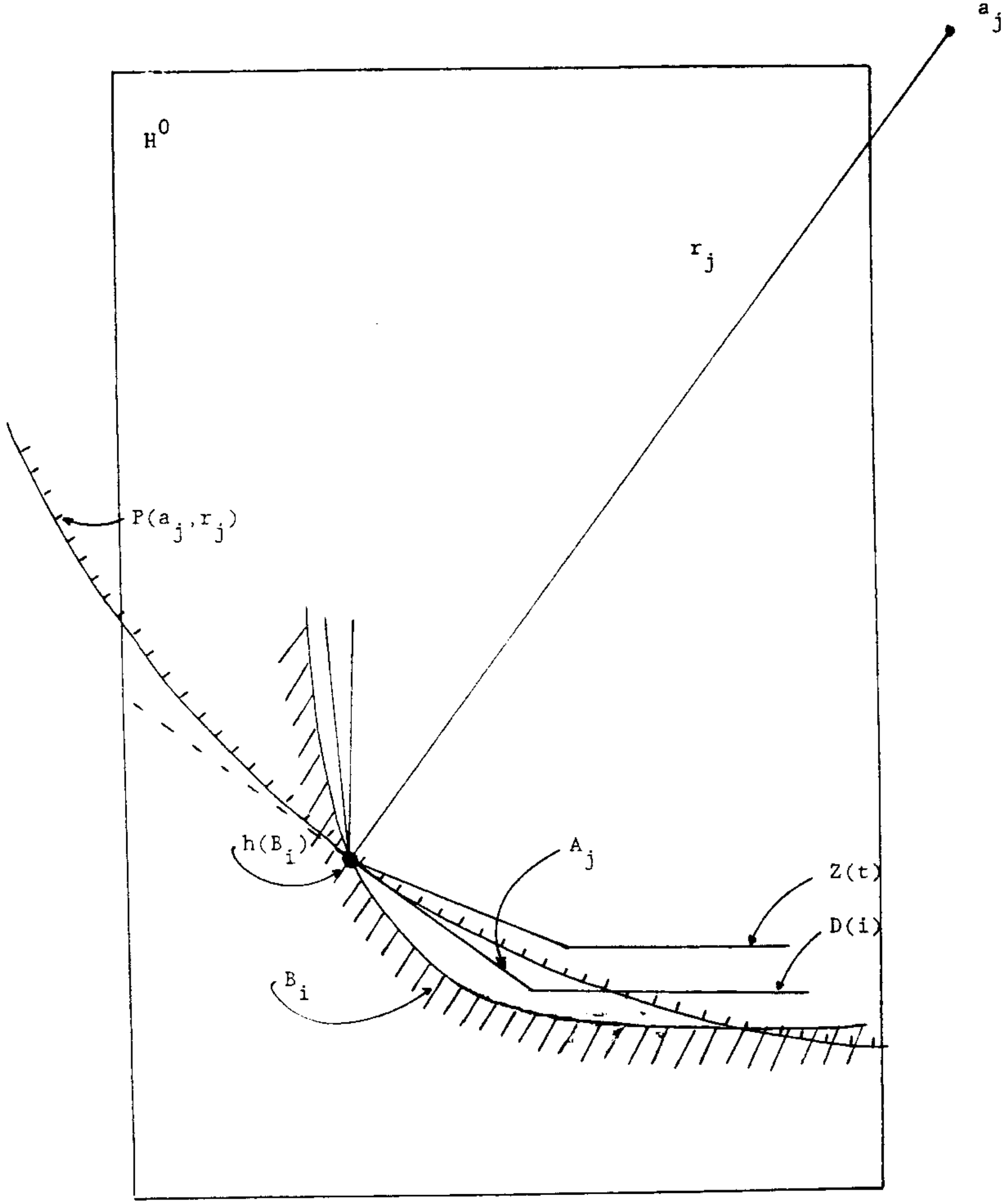

FIGURE B2.3

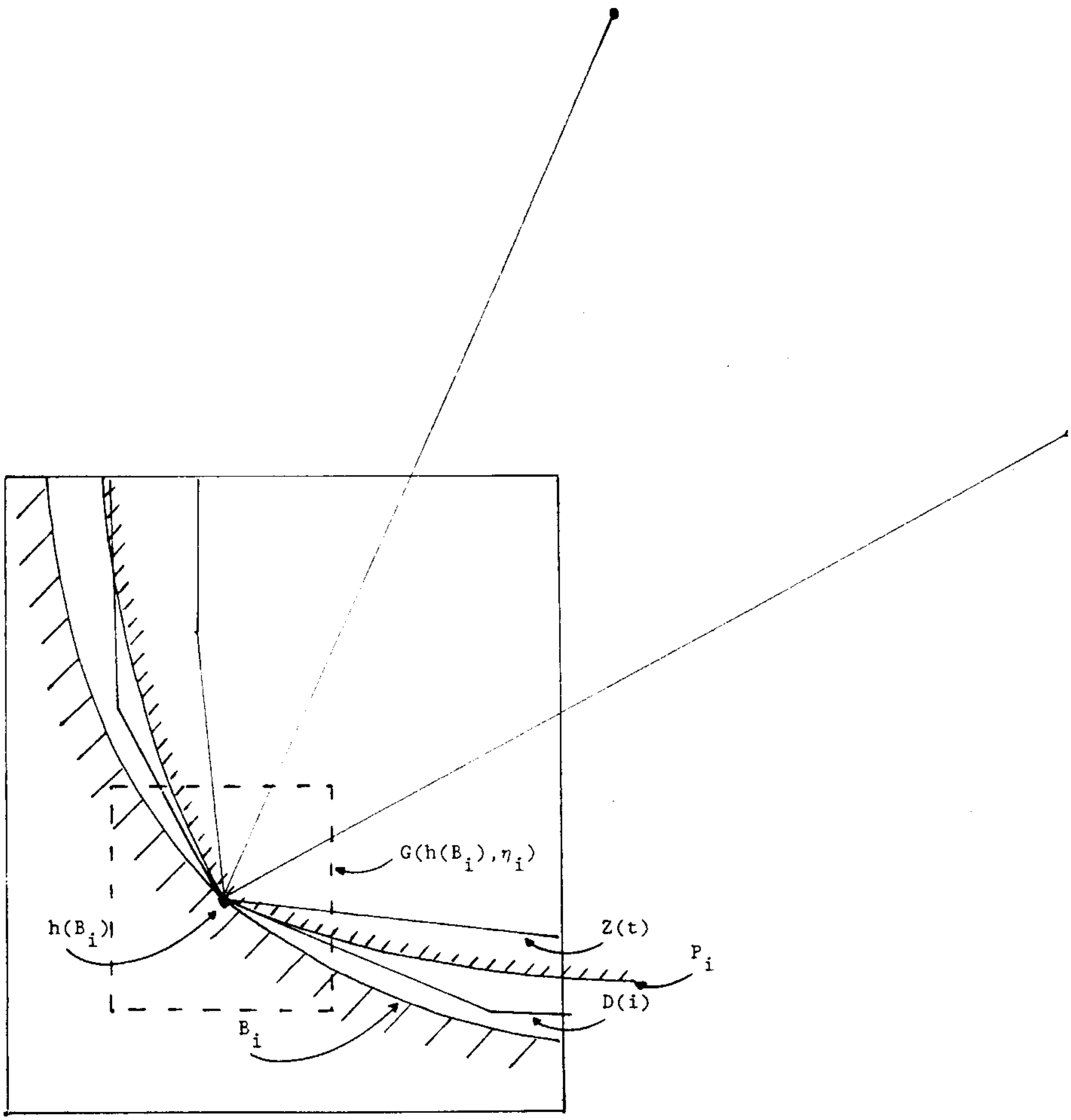

FIGURE B2.4

Let $\eta_i > 0$ be small enough such that

$G(h(B_i),\eta_i) \cap T(D(i)) \cap T(Z(t)) = \{h(B_i)\}$.

Then, by (2.19) and the definition of $p_1,\ldots,p_Q$

(2.20) $(P_i \cap G(h(B_i),\eta_i)) \subset (D(i) \cap G(h(B_i),\eta_i))$.

The statement of (2.13) then follows from (2.18) and (2.20).

We next employ (2.13) and (2.12) to complete the proof of (2.6.1):

It is clear that $h(B_i) \in ¥(Z(t),r) \cap B_i \cap J(t,i)$, since, by hypothesis, $h(B_i) \in B_i$ and, by definition, $h(B_i) \in Z(t) \subset ¥(Z(t),r)$ and $h(B_i) \in J(t,i)$. Let $x \in J(t,i)$ and suppose that $x \in B_i$ and $x \neq h(B_i)$. Then, by (2.12) $x \notin D(i)$, which then implies by (2.13) that $x \notin ¥(Z(t),r) \cap J(t,i)$. Hence, (2.6.1) follows.

To show now that (2.6.2) is satisfied, let $J(t) = \cup \{ J(t,i) \mid h(B_i) \in C(t) \}$. Let d^* be the minimum distance between any $x \in (Z(t) \cap cl(J(t)^c) \cap H^0)$ and any $y \in (B_i \cap cl(J(t)^c) \cap H^0)$. The compactness of $B_i \cap H^0$ and $Z(t)$ imply by (L1.2) that $d^* > 0$. By Lemma A3, there exists $\bar{r}$ such that for all $r > \bar{r}$, $\Delta(¥(Z(t),r), Z(t)) < d$. It then follows that for all $r > \bar{r}$ and all i such that $h(B_i) \in C(t)$,

$¥(Z(t),r) \cap B_i \cap cl(J(t)^c) = \emptyset$.

This proves (2.6.2) .

We have then shown that (2.6) holds. Then, also (2.5) holds.

We now define, for each $t = 1,\ldots,T$,

$Y(t) = \Psi(Z(t), r_t)$.

where r_t is large enough so that (2.2)-(2.5) are satisfied. Then, the $Y(t)$ sets satisfy (L2.1)-(L2.4).

To see that (L2.5) is satisfied, suppose that $h(B_i) \in C(t)$. Then, by (L2.4), $h(B_i) \in Y(t)$. If $h(B_i) \in \text{int } Y(t)$, then, since by (L2.4) $Y(t) \setminus \{h(B_i)\} \subset B_i^c$, the convexity of B_i^c implies that $h(B_i) \in B_i^c$, which is a contradiction. Then, (L2.5) must hold.

Q.E.D.

PROOF OF LEMMA 3: Let P^* be any sphere containing H^0 and with center $a \in R^K$ such that $\forall z \in H^0$, $z \ll a$. We define the compact set H by

$$H = \{x \in R^K \mid x \in P^* \text{ and } x \le w^2\}.$$

Clearly, $\forall t \in \{1, \ldots, T\}$, $Y(t) \cap H = Y(t)$.

Define the sets $Y(T+1)$ and $Y(0)$ by:

$$Y(T+1) = \{w^2\} \text{ and } Y(0) = H.$$

Then, it is clear that

(3.1) **$Y(T+1)$ and $Y(0)$ are strictly convex and compact sets, and**

(3.2) **$Y(T+1) \subset \text{int}(Y(T))$, and $Y(1) \subset \text{int}(Y(0))$.**

Then, from (3.1), (3.2), and (L2.1)-(L2.3) it follows that

(3.3) $\forall t \in \{0, 1, \ldots, T, T+1\}$, **$Y(t)$ is a strictly convex and strictly monotone subset in H.**

(3.4) $\forall t \in \{0, 1, \ldots, T, T+1\}$, **$Y(t)$ is compact, and**

(3.5) $\forall t \in \{1, \ldots, T, T+1\}$, **$Y(t) \subset \text{int } Y(t-1)$.**

From (3.3), (3.4), (3.5), **and Lemma A4 it follows that there exist numbers**

(3.6) $z_0 < z_1 < \ldots < z_T < z_{T+1}$,

such that (see Figure B3):

(3.7) $\partial_n Y(s) \times \{z_s\} \subset \partial_{n+1} \text{conv}(\cup_{s=0}^{T+1} Y(s) \times \{z_s\})$.

We let $Y = \text{conv}(\cup_{s=0}^{T+1} Y(s) \times \{z_s\})$. Following Kannai (1974) and Mas-Colell (1974), we define $f : X \to R$ by

(3.8) $f(x) = \max\{z \mid (x,z) \in Y\}$ **for** $x \in X$.

Note that f is well defined, since $X \subset H$. Then,

(3.9) **$f(\ )$ is a continuous, concave, strictly quasi-concave and strictly monotone function on X .**

The concavity and continuity of f follows from (3.7) and (3.8), and the strict quasi-concavity and strict monotonicity of f follows from (3.3), (3.7), and (3.8). From (3.7) and (3.8) it is clear that $\partial Y(1), \ldots, \partial Y(T+1)$ are level sets of the function f .

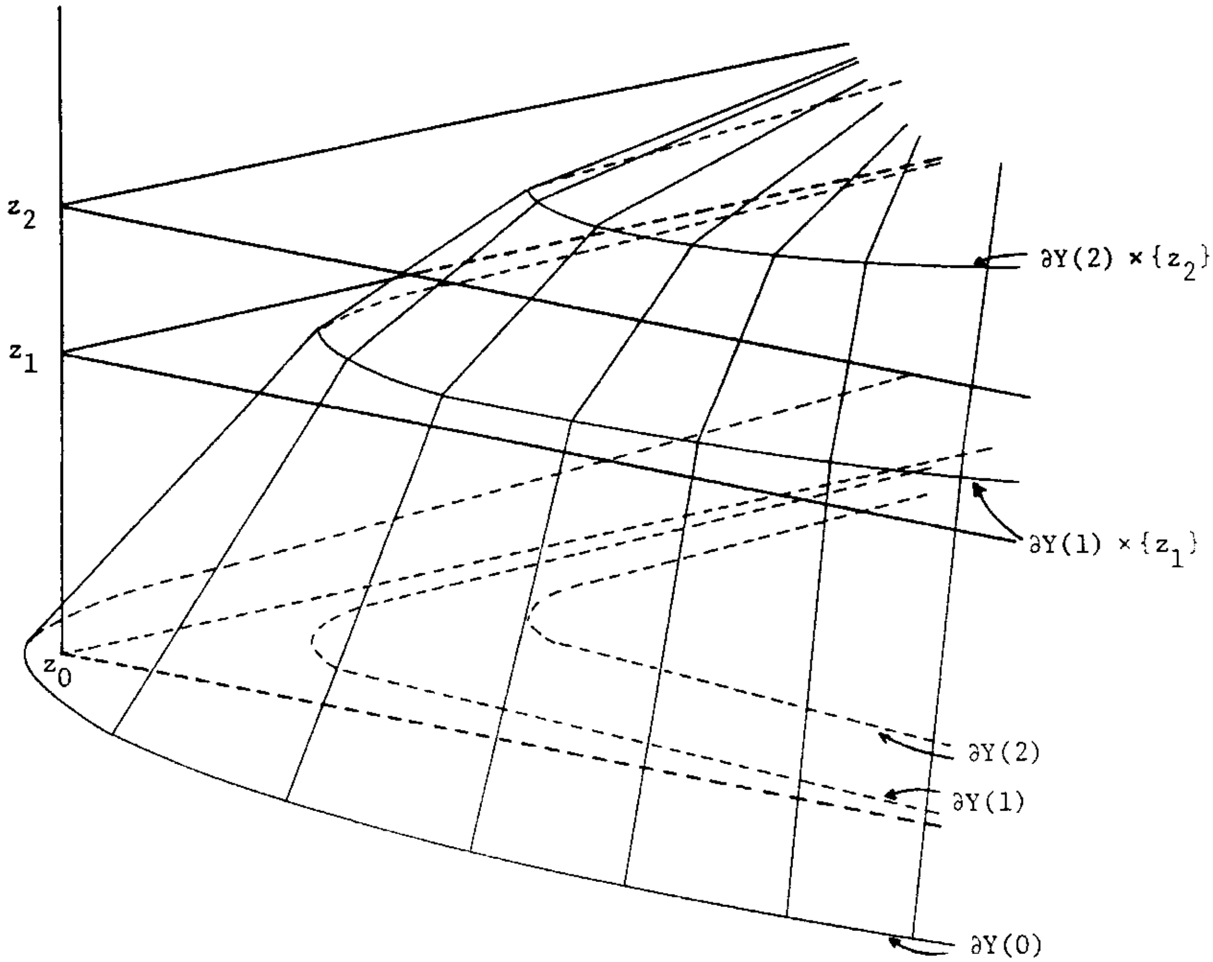

FIGURE B3

Let now $s : R \to R$ be any strictly monotone and strictly concave function, and let $v = s \circ f$. By (3.9)

(3.10) v is a strictly monotone and strictly concave function.

We will next show that

(3.11) v is a representation for $(B_i, h(B_i))_{i=1}^{n}$ on X .

Let t belong to $0, \ldots, T+1$, let i be such that $h(B_i)$ belongs to $C(t)$, and let $y \in B_i$ be such that $y \neq h(B_i)$. By (L2.5) and (L2.4), $h(B_i) \in \partial Y(t)$ and $B_i \cap Y(t) = \{h(B_i)\}$. Hence by (3.7) and (3.9) $f(h(B_i)) = z_t$ and for all $y \in B_i$ such that $y \neq h(B_i)$, $f(y) < f(h(B_i))$. Since v is strictly monotone, $v(y) < v(h(B_i))$. Hence, (3.11) follows.

This completes the proof of Lemma 3.

Q.E.D.

PROOF OF LEMMA 4: Let $x^j = h(C_j)$ and $x^k = h(C_k)$. By the definition of supportable choices it follows that there exists a neighborhood N of x^j such that $B_j \cap cl(N)$ is a nonempty, convex, and closed set. Let $A = (B_j \cap N) - R^K_+$. Then, A is also a nonempty, convex, and closed set. Moreover, $x^j \in \partial A$, and s^j is the unique vector in R^K_{++} satisfying $A \subset \{ x \in R^K \mid s^j x \leq 1 \}$, and $s^j x^j = 1$. By Lemma A5 and the definition of A it then follows that

(T2.3) for all $y \in C_j$ such that $s^j y < 1$,
there exists $\lambda \in (0,1)$ such that $\lambda x^j + (1 - \lambda) y \in B_j$.

Suppose that for some $y \in C_j$, $y \neq x^j$ and $V(y) \geq V(x^j)$.
If $s^j y < 1$, (T2.3) and the strict concavity of V imply that for

some $\lambda \in (0,1)$, $\lambda x^j + (1 - \lambda) y \in B_j$ and $V(\lambda x^j + (1 - \lambda)) > V(x^j)$. This contradicts the hypothesis that V is a utility rationalization for $\{B_i, h(B_i)\}_{i=1}^n$.

If $s^j y = 1$, then, since $s^j x^j = 1$, $V(y) \geq V(x^j)$, V is strictly concave, $y \in X$, $x^j \in N^j \subset X$, for some neighborhood N^j of x^j, and X is convex, there exists $\alpha \in (0,1)$ such that

$(\alpha y + (1-\alpha) x^j) \in \text{int}(X)$, $s^j (\alpha y + (1-\alpha) x^j) = 1$, and

$V(\alpha y + (1-\alpha) x^j) > V(x^j)$. Since V is continuous on $\text{int}(X)$, there exists $w \in C_j$, $w \ll (\alpha y + (1-\alpha) x^j)$, such that $s^j w < 1$ and $V(w) > V(x^j)$. Again by (T2.3), this contradicts the assumption that V is a representation for $\{B_i, h(B_i)\}_{i=1}^n$. Hence, the case in which $s^j y = 1$ is also not possible.

It follows that for all $y \in C_j$ such that $y \neq x^j$, $V(y) < V(x^j)$.

Hence, since $x^j \; S \; x^k$, $V(x^k) < V(x^j)$.

Q.E.D.

APPENDIX C

In this appendix we state and prove Lemmas A1 - A5, which have been employed in the proof of Lemmas 1 - 4.

LEMMA A1: *Suppose that $A \subset H \subset R^K$ is a polyhedron that is monotone in H, $B \subset H$ is a closed set, and $B^c \subset H$ is a monotone and convex set. If there exists $x \in A$ such that $x \in B$ and $((x + R^K_+) \cap H) \subset B^c$, and for all $y \in T(A)$ such that $y \neq x$, $y \in B^c$, then $A \setminus \{x\} \subset B^c$.*

PROOF: Let $z \in A$ be such that $z \neq x$. Since A is a polyhedron that is monotone in H and $x \in A$ there exist $q_1,\dots,q_J \in T(A) \setminus \{x\}$, $c_0, c_1, \dots, c_J \in R_+$, and $e \in R^K_+$ such that

(A1.1) $c_0 + \Sigma_{j=1}^J c_j = 1$,

(A1.2) $c_0\, x + \Sigma_{j=1}^J c_j\, q_j + e = z$, and

(A1.3) $c_0 \neq 1$.

Since B is a closed set and $q_j \in A \setminus \{x\}$ for all $j = 1,\dots,J$,

$q_j \in \text{int } B^c$.

If $c_0 = 0$, the monotonicity and convexity of B^c and (A1.1)-(A1.2) imply that $z \in B^c$.

If $c_0 \neq 0$, then since $q_j \in \text{int } B^c$ for $j \geq 1$, there exists $\varepsilon > 0$ such that for all $j \geq 1$ $N(q_j, \varepsilon) \subset B^c$. Let $\underline{\varepsilon} = (\varepsilon/2,\dots,\varepsilon/2) \in R^K$. Since $(\Sigma_{j=1}^J c_j\, \underline{\varepsilon}/ c_0) \in R^K_+$, $x + R^K_+ \subset B^c$, $q_j - \underline{\varepsilon} \in B^c$, and B^c is convex and monotone, it follows from (A1.1)-(A1.3) that

$$z = c_0\, x + \Sigma_{j=1}^J c_j\, q_j + e$$

$= c_0 (x + (\Sigma_{j=1}^{J} c_j \varepsilon / c_0)) + \Sigma_{j=1}^{J} c_j (q_j - \varepsilon)) + e$

$\in B^c$.

Hence, $A \setminus \{x\} \subset B^c$.

Q.E.D.

LEMMA A2: *Suppose that $A \subset H \subset R^K$ is a polyhedron that is monotone in H. Let $x \in T(A)$ and suppose that for some $p \in R^K_+$ and all $y \in A$ such that $y \neq x$, $p x < p y$. Then, there exists a sphere $P(a,r)$ such that*

A2.1. $a >> y$ for all $y \in H$,

A2.2. $A \subset P(a,r)$, and

A2.3. $P(a,r)$ is supported by the hyperplane $\Phi(p,x)$ at x.

PROOF: Let $\lambda \in R_+$ be large enough such that

(A2.4) for all $y \in H$, $x + \lambda p >> y$, and

(A2.5) for all extreme points q_s of A such that $q_s \neq x$

$$\lambda > \| x - q_j \|^2 / (2 (p q_j - p x)) .$$

Let $P(a,r)$ be the sphere with $a = x + \lambda p$ and $r = \| \lambda p\|$. Then, $P(a,r)$ satisfies A2.3 by construction, and it satisfies A2.1 by (A2.4). To see that $P(a,r)$ satisfies A2.2, suppose that $y \in A$. Then, $y = \Sigma_{s=1}^{J} b_s q_s + e$ for some $b_1,\ldots,b_J \in R_+$ such that $\Sigma_{s=1}^{J} b_s = 1$, some $e \geq 0$, and some extreme points $q_1,\ldots,q_J$ of A. Since $y \in A$ it follows from the definition of a , (A2.5), and the assumption that $y \neq x$, that

$\| y - a \|^2$

$$= \| \Sigma_{s=1}^{J} b_s q_s + e - a \|^2$$
$$\leq \| \Sigma_{s=1}^{J} b_s q_s - a \|^2$$
$$\leq \Sigma_{s=1}^{J} b_s \| q_s - x - \lambda p \|^2$$
$$= \Sigma_{s=1}^{J} b_s (\| q_s - x \|^2 + \| \lambda p \|^2 - 2 \lambda p (q_s - x))$$
$$< \Sigma_{s=1}^{J} b_s (2 \lambda p (q_s - x) + \| \lambda p \|^2 - 2 \lambda p (q_s - x))$$
$$= \Sigma_{s=1}^{J} b_s \| \lambda p \|^2$$
$$= \| \lambda p \|^2$$
$$= r^2 .$$

Hence $y \in P(a,r)$ and then $P(a,r)$ satisfies A2.2 .

Q.E.D.

LEMMA A3: Suppose that *$A \subset R^K$ is a polyhedron that is monotone in a compact set $H \subset R^K$. Let $¥(A,r)$ denote, for $r > diam(H)$, the intersection of H with all spheres $P(a,r')$ of radius $r' \leq r$ and center a such that $A \subset P(a,r')$ and for all $y \in H$, $a >> y$. Then*

$$lim_{r \to \infty} \Delta(¥(A,r), A) = 0 .$$

PROOF: Suppose that the lemma is not true. Then, by the definition of Δ, there exists $y \in H$ such that $y \notin A$ and $y \in ¥(A,r)$ for all sufficiently large r.

Let z denote the element of A that is closest to y. Since A is a monotone polyhedron in H, $y = z-e$ for some $e > 0$, $z \in \partial A$, and there exists an hyperplane $\Phi(p,z)$ with $p \in R_+^K$ that separates A from y. Let q_j be an element of T(A) that is closest to z. If $q_j = z$, then any sphere $P(a, r')$ with $z \in \partial P(a, r')$ and with $a >> y$ for all $y \in H$, $r' \leq$

r, and $A \subset P(a, r')$ will not contain y, contradicting the hypothesis that $y \in ¥(A,r)$ for all r. Hence, we can suppose that z lies in a face A_s of A that is adjacent to q_j and that $p = p_s$ is the normal of A_s. Since $\Phi(p,z)$ separates A from y,

(A3.1) $p_s q_j = p_s z > p_s y$.

Let $A_1,\ldots,A_Q$ be the faces of A that are adjacent to q_j. Let $p_1,\ldots,p_Q$ denote respectively, the normals to $A_1,\ldots,A_Q$. Then, for $\varepsilon > 0$ small enough, $a_s = 1-(\varepsilon/(Q-1))$, $a_k = \varepsilon$ $(k \neq s)$, and $p' = \Sigma_{k=1}^{Q} a_k p_k$, we have that from (A3.1)

(A3.2) $$p'y = \Sigma_{k=1}^{Q} a_k p_k y < \Sigma_{k=1}^{Q} a_k p_k q_j = p' q_j .$$

Moreover, for all extreme points q_s of A for which $q_s \neq q_j$,

(A3.3) $$p'q_s = \Sigma_{k=1}^{Q} a_k p_k q_s > \Sigma_{k=1}^{Q} a_k p_k q_j = p' q_j .$$

since for all k $a_k > 0$ and for some k' $q_s \notin A_{k'}$.

Suppose that $w \in A$ and $w \neq q_j$. Then, $w = \Sigma_{s=1}^{J} b_s q_s + e$ for some $b_1,\ldots,b_J \in R_+$ such that $\Sigma_{s=1}^{J} b_s = 1$, some $e \geq 0$, and some extreme points $q_1,\ldots,q_J$ of A. It then follows from (A3.3) and the assumption that $w \neq q_j$ that

$$\begin{aligned}
&p'w \\
&= \Sigma_{k=1}^{Q} a_k p_k w \\
&= \Sigma_{k=1}^{Q} a_k p_k (\Sigma_{s=1}^{J} b_s q_s + e) \\
&\geq \Sigma_{k=1}^{Q} \Sigma_{s=1}^{J} b_s a_k p_k q_s \\
&> \Sigma_{k=1}^{Q} \Sigma_{s=1}^{J} b_s a_k p_k q_j \\
&= \Sigma_{k=1}^{Q} a_k p_k q_j \\
&= p' q_j .
\end{aligned}$$

Hence,

(A3.4) for all $w \in A$ such that $w \neq q_j$ $p' w > p' q_j$.

From the assumption that $q_j \in A$ and (A3.4) it follows that there exists $P(a,r)$ such that for all $y \in H$ $a >> y$, $A \subset P(a,r)$, and $P(a,r)$ is supported by $\Phi(p',q_j)$ at q_j. But then, since by (A3.2) $p'y < p'q_j$, it follows that $y \notin P(a,r)$ and hence, $y \notin \Psi(A,r)$. This contradiction completes the proof.

Q.E.D.

LEMMA A4 (Kannai, 1974): *If* T^i $(i = 1, \ldots, m)$ *are compact and convex subsets of a set* H *in* R^K, *and*

$$T^{i+1} \subset \text{int } T^i \quad (i = 1, \ldots, m-1),$$

there exists real numbers $t^1 < t^2 < \ldots < t^m$ *such that*

$$\partial_K T^i \times \{t^i\} \subset \partial_{K+1} \text{conv}(\cup_{i=1}^m T^i \times \{t^i\})$$

($\partial_j A$ *denotes the boundary of* A *in the topology of* R^j).

We quote Kannai (1974):

> 'The proof of this lemma is a formal elaboration of the geometrically obvious fact that the slopes of the supporting hyperplanes of $\text{conv}(\cup_{i=1}^m T^i \times \{t^i\})$-which join $\partial_K T^i \times \{t^i\}$ to $\partial_K T^{i+1} \times \{t^{i+1}\}$ form a decreasing sequence of positive numbers, so that the next convex set (which is smaller) can be "squeezed in.'

LEMMA A5: *Suppose that* $A \subset R^K$ *is a nonempty, convex, and closed set such that* $\forall\, x \in A$, $(x - R^K_+) \subset A$. *Let* $x^* \in \partial A$ *and* $s \in R^K_{++}$ *be the unique vector in* R^K_{++} *such that*

$A \subset \{\, x \in R^K \mid s\, x \le 1 \,\}$ *and* $s\, x^* = 1$.

Then, if y *is such that* $s\, y < 1$, *there exists* $\lambda \in (0,1)$ *such that*

$\lambda\, x^* + (1 - \lambda)\, y \in A$.

PROOF: Let $y \in R^K$ be such that $s\, y < 1$. Suppose that $\forall\, \lambda \in (0,1)$, $\lambda\, x^* + (1 - \lambda)\, y \notin A$. Then, by the separating hyperplane theorem there exists $p_\lambda \in R^K$, $p_\lambda \neq 0$, $\|p_\lambda\| < \infty$, such that

(A5.1) $\forall\, x \in A \qquad p_\lambda\, x < p_\lambda\, (\lambda\, x^* + (1 - \lambda)\, y)$.

Suppose that a coordinate $p_{\lambda,1}$ of p_λ is negative. Then, since for any $x \in A$, $(x - R^K_+) \subset A$, there exists a large enough K such that, $w = x - (K,0,\dots,0) \in A$ and $p_\lambda\, w = p_\lambda\, x - p_{\lambda,1} > p_\lambda\, (\lambda x^* + (1\text{-}\lambda)\, y)$, contradicting (A5.1). Hence,

(A5.2) $\forall\, \lambda \in (0,1) \qquad p_\lambda \ge 0$.

Take a sequence $\{\lambda_n\}_{n=1}^{\infty}$ such that $\lambda_n \to 1$. We can assume w.l.o.g. that $\|p_\lambda\| = 1$. Hence, there exists a subsequence $\{\lambda_{n_i}\}_{i=1}^{\infty}$ of $\{\lambda_n\}_{n=1}^{\infty}$ and $p \in R^K$ such that

$\|p\| = 1$, $p \ge 0$, $p_{n_i} \to p$, and $\lambda_{n_i} \to 1$.

By (A5.1) it follows that

(A5.3) $\forall\, x \in A$ and large enough $i \qquad p\, x \le p\, (\lambda_{n_i}\, x^* + (1\text{-}\lambda_{n_i})\, y)$, and

(A5.4) $\forall\, x \in A \quad p\, x \le p\, x^*$, where (A5.4) is obtained from (A5.1) by letting $\lambda_{n_i} \to 1$.

Let c be such that $p x^* = c$. By translating x^* if necessary, we can assume w.l.o.g. that $c > 0$. Let $p' = p/c$. Then,

(A5.5) $p' x^* = 1$

and from (A5.3) and (A5.4),

(A5.6) $p' x \le p' (\lambda_{n_i} x^* + (1-\lambda_{n_i}) y) \quad \forall x \in A$ and large enough i, and

(A5.7) $p' x \le p' x^* \quad \forall x \in A$.

By (A5.5) and (A5.6),

(A5.8) $p' x^* = 1 \le p' (\lambda_{n_i} x^* + (1-\lambda_{n_i}) y)$.

Since by hypothesis $s y < 1 = s x^*$, $s (\lambda_{n_i} x^* + (1-\lambda_{n_i}) y) < 1$; hence, it follows from (A5.8) that

(A5.9) $p' \ne s$.

Moreover, by (A5.5) and (A5.7),

(A5.10) $A \subset \{ x \in R^K \mid p' x \le 1 \}$ and $p' x^* = 1$.

Let $p^* = \alpha p' + (1-\alpha) s$ for some $0 < \alpha < 1$. Then, since $s \in R^K_{++}$, $p' \ge 0$, $A \subset \{ x \in R^K \mid s x \le 1 \}$ and $s x^* = 1$, it follows from (A5.10) that

(A5.11) $p^* \in R^K_{++}$, $p^* \ne s$, $A \subset \{ x \in R^K \mid p^* x \le 1 \}$, and $p^* x^* = 1$.

This contradicts our hypotheses about the vector s. Hence, if $s y < 1$, there exists $\lambda \in (0,1)$ such that $\lambda x^* + (1 - \lambda) y \in A$.

Q.E.D.

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NOTES

1. Similar frameworks were introduced by Uzawa (1956), Arrow (1959), and Richter (1966).

2. This definition was employed in Richter (1966) and it is equivalent to the definition of a utility-rationalization employed in Matzkin and Richter (1987).

3. The relations S and R were introduced respectively, by Samuelson (1938) and Houthakker (1950) for perfectly competitive budget spaces. Both relations were extended to abstract spaces in Richter (1966).

4. Note that the case in which any B_i is determined by a single hyperplane is included in the hypothesis of the theorem. Hence, the proof of Theorem 1 also provides a different proof of Theorem 2 in Matzkin and Richter (1987).

5. Note that from the definition of supportable choices it follows that for each supportable choice $(B_i, h(B_i))$ there exists a unique supporting choice $(C_i, h(C_i))$ of $(B_i, h(B_i))$. Hence, the supporting choice $(C_i, h(C_i))$ is well defined.