

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
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COWLES FOUNDATION DISCUSSION PAPER NO. 864

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COMMON KNOWLEDGE OF SUMMARY STATISTICS

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February 1988

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October 1986

Revised: May 1987

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^{*}We wish to thank Robert Aumann and Eddie Dekel for helpful comments. Aumann and one of the above authors discovered part of the main argument used here one night in Berkeley.

Consider a group of people who are asked to offer their opinions on some issue. "Business confidence" surveys are an example: groups of businessmen are often asked for their predictions of economic indicators such as growth or inflation rates. Each member of the group makes a prediction based on his or her private information, and the average prediction is then publicly announced. If the members of the group are then allowed to revise their opinions, based on whatever information they glean from the public announcement, is there any tendency for the opinions in the group to converge on a common, consensus opinion?

In this note we show that under certain conditions the answer to this question is yes. Specifically, we first prove the following proposition: if the members of the group start with a common prior, and the average of their conditional expectations of some random variable is common knowledge, then everyone must have the same expectation. From this "equilibrium" result it then follows (by the argument of Geanakoplos and Polemarchakis [1982]) that even if the average expectation is not at first common knowledge, an iterative procedure of public announcements must lead eventually to consensus. Moreover the same theorem remains true even if all individuals report (different) monotonically increasing distortions of their true opinions, which are then averaged for the public announcement. Observe that before consensus is reached, it may be impossible to recover the average of the opinions from the average of the reported distortions.

The result in this note builds on work by McKelvey and Page [1986]. First, we extend Theorem 1 in their paper by covering conditional expectations of a random variable rather than only conditional probabilities of an event. Second, we extend our first proposition by allowing individuals to take actions that are not merely functions of the

expectations of some random variable. In this more abstract setting our proof and its intuitions become very brief and transparent.

An action plan for an individual is a function from what he might know (his information field) into the reals. Action plans do not rely on prior probabilities for their definition. Our second proposition shows that if all the action plans are co-correlated at ω , then whenever the average of the actions is common knowledge at ω , each action is also common knowledge at ω . If in addition each action plan satisfies the sure-thing-principle, then from the argument developed in Aumann [1976], Cave [1983], Becharach [1985], and Geanakoplos [1986-87] it can further be deduced that each action could have been generated from the same information across all individuals. When the actions can differ only on account of informational differences (as when all the individuals are estimating conditional expectations of the same random variable, with respect to the same prior) this means that the actions themselves must agree.

The issue of reconciliation of differing expert opinions is not a new one. A number of methods have been proposed in the statistics literature, of which the so-called "Delphi technique" is closely related to the procedure discussed in this note (see e.g. Dalkey [1969, 1972]).

There are n members of the group, indexed by $i = 1, \dots, n$, and a finite set Ω of states of the world. Each individual i receives private information about the true state of the world according to a partition Π_i of Ω . So if $\omega \in \Omega$ is the true state, i is informed of the member $\Pi_i(\omega)$ of Π_i that contains ω . Let $\bar{\Pi}$ be the join (coarsest common refinement) of Π_1, \dots, Π_n . Let Π be the meet (finest common coarsening) of Π_1, \dots, Π_n , and $\Pi(\omega)$ be the member of Π that contains ω . For any partition M of Ω , define M^* to be the field generated by M leaving aside the empty set, that is the set of all unions of the elements in M . Then $\Pi^* = \bigcap_i \Pi_i^*$. Following Aumann [1976], an event $A \subset \Omega$ is common knowledge at ω if $\Pi(\omega) \subset A$.

We begin by supposing that the members of the group are interested in the expectation of a random variable $X : \Omega \rightarrow R$, and that they share a common prior P that assigns positive probability to each state. For any individual i , his conditional expectation e_i^* is a function from Π_i^* to R defined on each $\pi \in \Pi_i^*$ by the formula

$$e_i^*(\pi) = \frac{1}{P(\pi)} \sum_{\omega \in \pi} X(\omega)P(\omega) .$$

That is, $e_i^*(\pi)$ is the conditional expectation of X given π , under the prior P . More generally we shall call any function $f_i^* : \Pi_i^* \rightarrow R$ an action plan. In our first proposition we consider monotonic functions $F_i : R \rightarrow R$ and action plans given by $f_i^* = F_i \circ e_i^*$.

Every action plan f^* defined on the field M^* induces a random variable $f : \Omega \rightarrow R$ given by the formula $f(\omega) = f^*(M(\omega))$. Given the action plans $f_i^* : \Pi_i^* \rightarrow R$, the average action is a random variable $\phi : \Omega \rightarrow R$ defined by $\phi(\omega) = \frac{1}{n} \sum_{i=1}^n f_i(\omega) = \frac{1}{n} \sum_{i=1}^n f_i^*(\Pi_i(\omega))$. The

average action is common knowledge at some $\bar{\omega} \in \Omega$ if the function $\phi(\omega)$ is a constant on $\Pi(\bar{\omega})$. We are now ready for our first proposition.

Proposition 1: Suppose each individual's knowledge is described by a partition Π_i of the finite state space Ω , and each person forms his conditional expectation e_i^* of the same random variable X , given his knowledge and a common strictly positive prior P . Suppose also that each person reports $f_i^* = F_i \circ e_i^*$, where F_i is a monotonic function. If at some $\bar{\omega} \in \Omega$ the average report is common knowledge, then each opinion $e_i(\bar{\omega})$ is common knowledge, and in fact all the opinions are equal.

Proposition 1 was first proved in McKelvey-Page [1986], for the special case where X is the characteristic function of some event $A \subset \Omega$. We shall derive the above proposition from a more general proposition, which in turn is based on the following definitions of co-correlation, and the sure-thing property.

Given random variables f and g on Ω , a probability Q on Ω , and an event $S \subset \Omega$, we say that f and g are positively Q -correlated on S if $Q(S) > 0$ and $\sum_{\omega \in S} [f(\omega) - \bar{f}][g(\omega) - \bar{g}]Q(\omega) > 0$, where $\bar{f} = \frac{1}{Q(S)} \sum_{\omega \in S} f(\omega)Q(\omega)$ and $\bar{g} = \frac{1}{Q(S)} \sum_{\omega \in S} g(\omega)Q(\omega)$. Equivalently, we could have written $\sum_{\omega \in S} f(\omega)[g(\omega) - \bar{g}]Q(\omega) > 0$. If f is the random variable induced by some action plan f^* , then we also say f^* and g are positively Q -correlated on S . Given a group of action plans $f_i^* : \Pi_i^* \rightarrow \mathbb{R}$ we say that they are co-correlated at $\bar{\omega}$ iff there exists a probability Q and a random variable Z such that each f_i that is not a constant on $\Pi(\bar{\omega})$ is positively Q -correlated with Z on $\Pi(\bar{\omega})$. Note that the hypothesis does not rule out the possibility that each pair of

action plans is negatively Q-correlated on $\Pi(\bar{\omega})$. Intuitively, co-correlation suggests that there is some common cause that similarly affects all the action plans, at least under one view of the world that it is logically possible to hold at $\bar{\omega}$.

We say that an action plan $f^* : M^* \rightarrow R$ satisfies the sure-thing principle if whenever π and π' are disjoint elements of M^* such that $f^*(\pi) = f^*(\pi')$, it is also true that $f^*(\pi \cup \pi')$ is equal to both of them.

Proposition 2: Let $f_i^* : \Pi_i^* \rightarrow R$ be action plans for individuals on a finite state space Ω . Suppose that at $\bar{\omega} \in \Omega$ the average action is common knowledge. If the action plans are co-correlated at $\bar{\omega}$ then each action is common knowledge. If in addition each action plan satisfies the sure-thing principle, then it is common knowledge that each action is $f_i^*[\Pi(\bar{\omega})]$.

Remark: Note that the conclusion implies that it is common knowledge that all the actions could have been produced by the same information.

Proof: Let the co-correlation be with respect to the random variable Z and the probability Q , let $\bar{Z} = \frac{1}{Q[\Pi(\bar{\omega})]} \sum_{\omega \in \Pi(\bar{\omega})} Z(\omega)Q(\omega)$ and $\phi(\omega) = \frac{1}{n} \sum_{i=1}^n f_i(\omega)$. If ϕ is a constant function on $\Pi(\bar{\omega})$, then

$$0 = \sum_{\omega \in \Pi(\bar{\omega})} \phi(\omega)[Z(\omega) - \bar{Z}]Q(\omega) = \frac{1}{n} \sum_{i=1}^n \sum_{\omega \in \Pi(\bar{\omega})} f_i(\omega)[Z(\omega) - \bar{Z}]Q(\omega) > 0$$

unless f_i is a constant function on $\Pi(\bar{\omega})$ for each i . If each f_i is constant on $\Pi(\bar{\omega})$, it follows at once from the sure-thing hypothesis that $f_i^*(\pi) = f_i^*[\Pi(\bar{\omega})]$ for each $\pi \in \Pi_i$, $\pi \subset \Pi(\bar{\omega})$. \square

Observe that conditional expectations satisfy the sure-thing principle. Moreover, if each person forms an opinion about the same random variable, with respect to the same prior, and if there is symmetric information, all the opinions must agree. Thus Proposition 1 follows at once from Proposition 2 if we can find the appropriate Z and probability Q . It is worthwhile noting that the opinions e_i^* are not generally positively correlated with themselves. Take for example $\Omega = \{1,2,3,4\}$, $\Pi_1 = \{\{1,2\}, \{3,4\}\}$, $\Pi_2 = \{\{1,3\}, \{2,4\}\}$, $P(1) = P(4) = 2/8$, $P(2) = 3/8$, $P(3) = 1/8$, $X(1) = X(4) = 0$, $X(2) = X(3) = 1$. Then e_1^* and e_2^* are negatively P -correlated on Ω .

Lemma: Under the hypothesis of Proposition 1, each $f_i^* = F_i \circ e_i^*$ is positively P -correlated with the random variable X on $\Pi(\bar{\omega})$ unless e_i is constant on $\Pi(\bar{\omega})$.

Proof:

Let

$$\bar{X} \equiv \frac{1}{P[\Pi(\bar{\omega})]} \sum_{\omega \in \Pi(\bar{\omega})} X(\omega)P(\omega) = \frac{1}{P[\Pi(\bar{\omega})]} \sum_{\substack{\pi \in \Pi_i \\ \pi \subset \Pi(\bar{\omega})}} e_i^*(\pi)P(\pi).$$

Then

$$\sum_{\omega \in \Pi(\bar{\omega})} [F_i \circ e_i(\omega)][X(\omega) - \bar{X}]P(\omega) = \sum_{\substack{\pi \in \Pi_i \\ \pi \subset \Pi(\bar{\omega})}} [F_i \circ e_i^*(\pi)][e_i^*(\pi) - \bar{X}]P(\pi).$$

But now we have an expression which is greater than zero if and only if $F_i \circ e_i^*$ and e_i^* are positively P-correlated on $\{\pi \in \Pi_i : \pi \subset \Pi(\bar{\omega})\}$. And we know that every nonconstant random variable is positively correlated with a monotonic transformation. To see this, break the sum into two parts:

$$\begin{aligned} & \sum_{\substack{\pi \in \Pi_i \\ \pi \subset \Pi(\bar{\omega}) \\ e_i^*(\pi) \leq \bar{X}}} [F_i \circ e_i^*(\pi)] [e_i^*(\pi) - \bar{X}] P(\pi) + \sum_{\substack{\pi \in \Pi_i \\ \pi \subset \Pi(\bar{\omega}) \\ e_i^*(\pi) > \bar{X}}} [F_i \circ e_i^*(\pi)] [e_i^*(\pi) - \bar{X}] P(\pi) \\ & \geq \sum_{\substack{\pi \in \Pi_i \\ \pi \subset \Pi(\bar{\omega})}} F_i(\bar{X}) [e_i^*(\pi) - \bar{X}] P(\pi) = 0, \end{aligned}$$

where the inequality is strict unless e_i^* is a constant function on $\{\pi \in \Pi_i : \pi \subset \Pi(\bar{\omega})\}$. \square

In order to answer the question posed at the beginning of this paper about the convergence to consensus, we must assume that the action plans f_i^* can be extended to the whole join-field $\bar{\Pi}^*$. Such an extended action plan satisfies the sure-thing principle if whenever it agrees on disjoint subsets of Ω , it agrees on their union. If $\bar{\Pi}_i \subset \bar{\Pi}^*$ is a refinement of Π_i , let $f_i^*|_{\bar{\Pi}_i}$ be the restriction of the extended action plan f_i^* to $\bar{\Pi}_i$. We say that the extended action plans (f_1^*, \dots, f_n^*) are co-correlated if for any refinements $\bar{\Pi}_i \subset \bar{\Pi}^*$ of Π_i , and any $\omega \in \Omega$, the action plans $f_1^*|_{\bar{\Pi}_1}, \dots, f_n^*|_{\bar{\Pi}_n}$ are co-correlated at ω . The lemma assures us that conditional expectations of the same random variable calculated according to a common prior are co-correlated, since the partitions and ω were chosen arbitrarily.

For co-correlated extended action plans, consider the following iterative procedure, based on Geanakoplos-Polemarchakis [1982]. The average action is publicly announced. Each individual i refines his partition on the basis of the information contained in the public statistic, and computes a new action. The average of the new actions is announced. And so on. Since Ω is finite, there is a finite number of rounds after which the individuals' partitions cease to be refined. At this point the partition generated by the public statistic must be a coarsening of everyone's partition, that is the statistic must be common knowledge. Hence Proposition 2 applies to tell us that everyone's action could be produced by the same information. Under the conditions of Proposition 1, it tells us that consensus will be reached.

Proposition 1 can be extended in different directions, with different proofs. Bergin [1986] extended McKelvey and Page's theorem for indicator functions X to the case of an infinite state space with information represented by σ -algebras, and Nielsen [1987] further extended the infinite theory to allow for arbitrary random variables X .

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