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1. INTRODUCTION

A stock market is a mechanism by which the ownership and control of firms is determined through the trading of securities. It is on this market that many of the major risks faced by society are shared through the exchange of securities and the production decisions that influence the present and future supply of resources are determined. *If the overall structure of markets is incomplete can the stock market be expected to perform its role of exchanging risks and allocating investment efficiently?* It is this question that we seek to answer.

The efficiency properties of an equilibrium depend upon the structure of the markets employed. If the markets are *incomplete* then generically equilibrium allocations are Pareto inefficient. The reason is clear: the criterion of Pareto efficiency gives the planner more freedom in allocating resources than is provided by the system of incomplete markets. The planner in essence achieves Pareto efficiency by reintroducing the missing markets. However the important economic question is not whether a new structure with more markets can be better, but *whether the existing one performs efficiently relative to the set of allocations achievable with this structure.* This key observation, first made by Diamond (1967) in the context of a one good economy, leads to the concept of *constrained efficiency.*

We study this concept in a general equilibrium model with a finite number of consumers, firms and goods in which there are two dates $(t = 0, 1)$ and uncertainty about which state of nature will occur at date 1. There is a spot market for each good in each state and hence perfect freedom to exchange goods within each state. There are security markets for the equity of each of the J firms. However the number of firms is assumed to be less than the number of states $(J < S)$, so that consumers by trading in the equity of

firms have only limited ability to redistribute their income among the spot markets. In short spot markets are complete, but security markets are incomplete. *This incompleteness of the security markets is a basic hypothesis of our model: we do not attempt to explain it.* In such a model Diamond (1967) showed that *if there is only one good and if firms have multiplicative uncertainty then every equilibrium allocation is constrained efficient.* In short in such a one good economy a stock market allocates investment efficiently. Our object is to show that this is a fortuitous circumstance of the one good model. *If there are two or more goods then the allocation of investment induced by the stock market is generically constrained inefficient.* The government can, by redirecting the investment decisions of firms and by offering lump sum transfers to consumers, make all consumers better off.

The qualitative change that occurs in the transition to a two good economy has been studied in the context of examples by Diamond (1980), Loong–Zeckhauser (1982) and Stiglitz (1982). Stiglitz stressed the possibility that this is a general phenomenon. For a marginal change in the allocation of consumers portfolios and firms production decisions changes supply and demand on the spot markets and hence relative prices. When markets are *complete* the rates of substitution of agents are equalised and such a relative price change has no effect on welfare. Similarly when there is only *one good* there is no relative price effect and hence no effect on welfare. But when there are *two or more* goods and markets are *incomplete*, agents vectors of marginal utilities of income are generically not collinear and such a relative price change has an effect on welfare. It is as if a planner by foreseeing these price changes had an additional instrument for redistributing income across the states which is not available to the more myopic competitive system.

To exhibit this pecuniary externality as a general phenomenon we need a concept of *equilibrium* for an economy with production in which the structure of markets is incomplete. Since shareholder's vectors of marginal utilities of income are not collinear, the concept of profit maximisation is, in the case of general technology sets, ambiguous. Several criteria have been proposed. Since we are interested in normative properties we

2

adopt the criterion introduced by Drèze (1974). In the one good case this leads to equilibria which satisfy the first order conditions for constrained efficiency. It will be clear from our analysis however that the generic inefficiency result can be expected to hold for a much broader class of objective functions provided that firms behave as price takers on the spot markets. The analysis should include for example the objective function proposed by Duffie-Shafer (1986). It should also be noted that the inefficiency result holds even if securities other than the equity of firms are introduced provided the overall asset structure remains incomplete.

Intuitive as the basic economic result may appear establishing it in a general equilibrium framework is technically demanding. This was already clear from the earlier analysis of Geanakoplos-Polemarchakis (1986) who studied the case of an exchange economy (exogenously given asset structure). They showed that under certain conditions *for a generic choice of utility functions and endowments all equilibrium allocations are constrained inefficient.* The basic idea that underlies their proof is that changes in portfolio holdings lead to changes in relative prices if agents have different marginal propensities to consume. In this paper we do not rely on differences in the propensities to consume (and hence perturbations of the utility functions) since changes in production alter the supplies of commodities and thereby induce changes in relative prices. Our results thus depend only on genericity in endowments.

In section 2 we lay out the basic stock market economy. In section 3 in addition to existence (theorem 1) we establish two important structural properties of equilibria: first that the equilibria are generically smooth functions of the endowment parameters (theorem 2) and second that the present value coefficients of consumers (the normalised vectors of marginal utilities of income) are generically distinct (proposition 3). These two results are basic to the proofs of the two generic inefficiency results (theorems 3, 4) of section 4. In section 5 the nature of the inefficiency is illustrated through an example. The proofs of the main results are given in section 6 and the appendix.

3

2. THE STOCK MARKET ECONOMY

In this section we outline a general equilibrium model of a stock market economy in which the security markets are incomplete. As indicated above our objective is to show that when the markets are incomplete serious questions are raised about the ability of the stock market to induce an appropriate allocation of investment. To analyse the problem in a framework that is at the same time simple but general we consider a two-period ($t = 0, 1$) economy in which there is uncertainty about the state of nature at date 1.

2.1 Utility Functions, Technology Sets and Initial Endowments.

There are $I \geq 1$ consumers ($i = 1, \ldots, I$), $J \geq 1$ firms ($j = 1, \ldots, J$) and $S \geq 1$ states of nature ($s = 1, \ldots, S$) at date 1. For convenience we include $t = 0$ as a state and write $s = 0, 1 \ldots, S$. In each state s there are L goods ($\ell = 1, \ldots, L$); we let $N = L(S + 1)$ denote the *total number of goods*, so that R^N is the basic real commodity space in the model.

Each consumer has an *initial endowment* of goods $w^i = (w^i_s)^S_{s=0} \in R^N_+$, where $w^i_s = (w^i_{s\ell})^L_{\ell=1} \in R^L_+$ is the vector of goods in state s, and chooses a vector of consumption $x^i = (x^i_s)^S_{s=0} \in R^N_+$. It is convenient to write $x^i = (x^i_0, x^i_1) \in R^L_+ \times R^{SL}_+$ where x^i_0 is date 0 consumption and $x^i_1 = (x^i_s)^S_{s=1}$ is the vector of date 1 consumption across the states. Without loss of generality we assume that each consumer's preference ordering over consumption bundles can be represented by a utility function $u^i : R^N_+ \longrightarrow R$. We make the following set of assumptions on each agent's utility function and endowment: the first part is used to establish existence of an equilibrium, the second part is added to analyse the generic properties of an equilibrium, the third is used to establish generic inefficiency.

ASSUMPTION A (UTILITY FUNCTIONS): (1) Each consumer's utility function $u^i : R^N_+ \longrightarrow R$ is continuous, quasi-concave and strictly monotone in good 1 in each state $s = 0, \ldots, S$ and each agent's initial endowment w^i is strictly positive, $w^i \in R^N_{++}$.

(2) (i) $u^i \in C^2(R^N_{++})$, $Du^i(x) \in R^N_{++} \ \forall \ x \in R^N_{++}$. (ii) $h^T D^2 u^i(x) h < 0 \ \forall \ h \neq 0$ such

4

that $Du^i(x)h = 0$, $\forall\ x \in R^N_{++}$ (strictly positive Gaussian curvature). (iii) If $U^i(\xi) = \{x \in R^N_+ | u^i(x) \geqq u^i(\xi)\}$ then $\overline{U^i(\xi)} \subset R^N_{++}$, $\forall\ \xi \in R^N_{++}$.

(3) Each utility function u^i is separable in date 0 and date 1 consumption, that is, there exist utility functions $u^i_0 : R^L_+ \longrightarrow R$, $u^i_1 : R^{SL}_+ \longrightarrow R$ such that $u^i(x) = u^i_0(x_0) + u^i_1(x_1)$ $\forall\ x \in R^N_+$ $i = 1,\ldots,I$.

On the production side of the economy, each firm j is characterised by a technology set $Y^j \subset R^N$ and the directors of the firm choose a production plan $y^j \in Y^j$. We use the standard convention for the production vector $y^j = (y^j_s)^S_{s=0}$ where $y^j_s = (y^j_{s\ell})^L_{\ell=1}$ is the vector of goods produced in state s: if $y_{s\ell} < 0(> 0)$ then good ℓ is used in state s as an input (is produced in state s as an output). In the subsequent analysis we will often find it convenient to decompose the production activity of firm j into period 0 and period 1 components, $y^j = (y^j_0, y^j_1) = \left(y^j_0, (y^j_s)^S_{s=1}\right)$. In addition to the standard closure and convexity assumptions on the technology sets we need two additional properties to be able to analyse the generic properties of an equilibrium: the first is an appropriate parameterisation of these sets, the second is the assumption that they have smooth boundaries.

To obtain genericity results we need a natural way of *parametrising* the decisions of agents in the economy. The consumption-portfolio decision of consumers is naturally parametrised by the vector of initial endowments $\omega = (w^1,\ldots,w^I) \in R^{NI}_{++}$ of the I consumers. To parametrise the production activity of each firm we assume that the production of each firm consists of two components, the *endogenously* chosen $y^j \in Y^j$ and an *exogenously* given vector of outputs $\eta^j \in R^N_+$, so that the total production of firm j is $y^j + \eta^j$. We call η^j the *initial endowment* vector of firm j and let $\eta = (\eta^1,\ldots,\eta^J)$. *To obtain genericity results we parametrise the decisions of consumers and producers by initial endowment vectors (ω,η) in the open set $R^{NI}_{++} \times R^{NJ}_{++}$.*

In order to obtain a smooth supply function we assume that the boundary ∂Y^j of Y^j is a smooth submanifold of a subspace $K^j \subset R^N$. The introduction of the subspace K^j avoids the otherwise restrictive assumption that all goods are involved in the production

5

activity of firm j. The two sets of assumptions on the technology sets of firms that we will use are the following.

ASSUMPTION B (TECHNOLOGY SETS): (1) (i) Each firm's technology set $Y^j \subset R^N$ is closed, convex and $0 \in Y^j$. (ii) $\left(\sum_{i=1}^{I} w^i + \sum_{j=1}^{J} (Y^j + \eta^j) \right) \bigcap R_+^N$ is compact.

(2) (i) Let K^j be a k_j dimensional subspace of R^N with $1 \leq k_j \leq N$ for $j = 1, \ldots, J$. $Y^j \subset K^j$ is a k_j dimensional manifold and its boundary ∂Y^j is a C^2 manifold with strictly positive Gaussian curvature at each point, (ii) $\eta^j \in R_{++}^N$, $j = 1, \ldots, J$.

If we choose utility functions (u^i), production sets (Y^j) and a vector of initial endowments (ω, η), satisfying Assumptions (A,B), then we obtain an economy $\mathcal{E}\left((u^i, Y^j), \omega, \eta\right)$. In the analysis that follows we think of the utility functions and technology sets as being *fixed* and allow the parameters (ω, η) to be free to vary in the *parameter space* $R_{++}^{N(I+J)}$.

2.2 Market Structure: Spot and Security Markets.

There are a variety of *market structures* that can be added to any such economy $\mathcal{E}\left((u^i, Y^j), \omega, \eta\right)$ to induce an allocation of resources. The classical one is the Arrow-Debreu set of *complete contingent markets* and with such a market structure equilibrium allocations are Pareto optimal. Our basic tenet is that such a market structure is essentially not observed in the real world, certainly not for the important aggregate risks to which the production activity of firms is exposed. It presupposes much too refined a system of markets.

A much more realistic market structure can be described generally as follows. There are two types of markets, *spot markets* for the trading of real goods and *financial markets* for trading financial assets. A rich class of market structures can be analysed depending on the type of financial assets introduced. In this paper we consider only one type of financial asset, namely the *security* or *equity* issued by a firm. We focus our attention on this case for several reasons. First and foremost in terms of the sheer magnitude of trade involved, equity markets are perhaps the most significant risk-sharing markets that exist in a modern economy and they are of central importance in influencing the production (investment)

6

decisions of firms. Second, adding additional asset markets would not alter our main result, provided the resulting asset structure remains incomplete. Finally adding additional asset markets substantially complicates the analysis and our interest is in obtaining the simplest framework for proving the main result.

Consider therefore a market structure consisting of spot and equity markets. In each state $s = 0, \ldots, S$ there is a spot market for each of the L goods; let $p = (p_0, p_1) = (p_0, (p_s)_{s=1}^S)$ denote the associated vector of *spot prices*. There are security markets for the shares of each firm j. Implicit in the ownership share $0 \leq \theta^{ij} \leq 1$ of firm j by agent i is the right to receive the share $\theta^{ij} y^j$ of the production plan of firm j. However since the security market is viewed as a financial market on which income is delivered rather than as a real market on which goods are delivered, we assume that what agent i receives as a result of purchasing at date 0 the proportion θ^{ij} of firm j's shares is the income $\theta^{ij} \, p \square y^j$ where $p \square y^j = (p_s y_s^j)_{s=0}^S$. The price (market value) of firm j is q_j so that the cost of purchasing θ^{ij} is $\theta^{ij} q_j$. Each agent i has an *initial ownership share* $0 \leq \delta^{ij} \leq 1$ of firm j; he thus receives $\delta^{ij} q_j$ from the sale of his initial shares and spends $\theta^{ij} q_j$ for the purchase of new shares. Since we are interested in the idea that ownership also implies some control over firm j's production decision, we assume that *securities cannot be short-sold* ($\theta^{ij} \geq 0$). Trading in the shares of the J firms by the I consumers thus gives rise to an $I \times J$ nonnegative matrix $\theta = (\theta^{ij}, i = 1, \ldots, I, j = 1, \ldots, J)$ each of whose columns sum to 1, $\sum_{i=1}^I \theta^{ij} = 1$, $j = 1, \ldots .J$. Let $q = (q_1, \ldots, q_J)$ denote the vector of *security prices*, then given the market prices (p, q) the *budget set* of consumer i is given by

$$
B^i(p, q, y; \omega, \eta) = \left\{ x \in R_+^N \;\middle|\; \begin{array}{l} \text{there exists } \theta^i \in [0,1]^J \text{ such that} \\[4pt] \begin{bmatrix} p_0(x_0 - w_0^i) - q\delta^i \\ p_1 \square (x_1 - w_1^i) \end{bmatrix} = W(p, q; y + \eta)\theta^i \end{array} \right\}
$$

where $y = (y^1, \ldots, y^J)$ and W is the $(S+1) \times J$ matrix of *security returns*

$$W(p, q, y + \eta) = \begin{bmatrix} p_0(y_0^1 + \eta_0^1) - q_1 \cdots p_0(y_0^J + \eta_0^J) - q_J \\ p_1 \,\square\, (y_1^1 + \eta_1^1) \cdots \quad p_1 \,\square\, (y_1^J + \eta_1^J) \end{bmatrix}$$

For this budget set to be well-defined we need to assume that *each agent correctly antici-pates at date 0, the date 1 spot prices* p_1 *and the outputs of firms*

$$y_1 + \eta_1 = [y_1^1 + \eta_1^1 \cdots y_1^J + \eta_1^J]$$

2.3 Present Value Coefficients.

Consumer i seeks a consumption bundle which solves the constrained maximum problem

$$\max_{x^i \in B^i(p,q,y;\omega,\eta)} u^i(x^i) \qquad i = 1, \ldots, I$$

This is a standard Kuhn-Tucker problem in which the $(S + 1)$ spot market expenditure constraints give rise to a vector of marginal utilities of income (Lagrange multipliers)

$$\lambda^i = (\lambda_0^i, \lambda_1^i, \ldots, \lambda_S^i) \qquad i = 1, \ldots, I$$

across the states of nature. If we take θ^i as fixed at its optimal level then λ^i may be characterised more directly as follows.

PROPOSITION 1 (PRESENT VALUE COEFFICIENT OF AGENT i): *Let* $A_{(1)}$ *hold and let* $(\bar{x}^i, \bar{p})$ *satisfy*

$$u^i(\bar{x}^i) \geqq u^i(\bar{x}^i + \xi^i) \quad \text{for all } \xi^i \quad \text{such that} \quad \bar{p} \,\square\, \xi^i \leqq 0$$

then there exists $\lambda^i \in R_{++}^{S+1}$ *such that*

$$u^i(\bar{x}^i) \geqq u^i(\bar{x}^i + \xi^i) \quad \text{for all } \xi^i \quad \text{such that} \quad \lambda^i \cdot (\bar{p} \,\square\, \xi^i) \leqq 0$$

If in addition $A_{(2)}$ *holds, then* λ^i *is unique up to scalar multiplication.*

PROOF: Consider the preferred and affordable sets

$$P^i = \{\xi^i \in R^{L(S+1)} \mid u^i(\bar{x}^i + \xi^i) > u^i(\bar{x}^i)\}, \quad B^i = \{\xi^i \in R^{L(S+1)} \mid \bar{p} \,\square\, \xi^i \leqq 0\}$$

8

since $P^i \bigcap B^i = \emptyset$ by the standard separation theorem there exists $v^i \in R^{L(S+1)}, v^i \neq 0$ such that $\sup_{\xi^i \in B^i} v^i \cdot \xi^i \leq \inf_{\xi^i \in P^i} v^i \cdot \xi^i$. Since $0 \in \bar{P}^i \bigcap B^i$

$$\sup_{\xi^i \in B^i} v^i \cdot u^i = 0 = \inf_{u^i \in P^i} v^i \cdot \xi^i \qquad (1)$$

Let $\bar{p}_s = (0, \ldots, \bar{p}_s, \ldots, 0)$, $s = 0, \ldots, S$. The first half of (1) is equivalent to $\bar{p}_s \xi^i \leq 0$ $s = 0, \ldots, S \implies v^i \cdot \xi^i \leq 0$. Thus there exists $\lambda^i \in R_+^{S+1}$, $\lambda^i \neq 0$ such that $q = \sum_{s=0}^{S} \lambda_s^i \bar{p}_s$. By the second half of (1), $\sum_{s=0}^{S} \lambda_s^i \bar{p}_s \xi_s^i \leq \implies \xi^i \notin P_i$, The monotonicity of u^i implies $\lambda^i \in R_{++}^{S+1}$. ∎

DEFINITION 1: We call the normalised vector of marginal utility of income $\pi^i = (\pi_s^i)_{s=0}^{S} = \left(\frac{\lambda_s^i}{\lambda_0^i}\right)_{s=0}^{S}$ the **present value coefficient** of consumer i.

As we shall see in the analysis that follows, these present value coefficients summarise the essentially new aspects of the problem of resource allocation that arise on the *consumer* side of the economy when markets are incomplete. We shall now show how these coefficients can be used on the *production* side of the economy to define an objective function for the firm in the presence of incomplete markets.

2.4 The Problem of Defining Present Value of Profit.

Given the prices of real goods determined on the spot markets and the market value of each firm determined on the equity markets, column j of the matrix W gives the vector of returns across the states $s = 0, 1, \ldots, S$ obtained from the ownership of firm j when its production decision and initial endowment are $y^j + \eta^j$. It is easy to check that the present value coefficient of agent i is a normalised vector $\pi^i = (1, \pi_1^i, \ldots, \pi_s^i) \in R_{++}^{S+1}$ which must satisfy the *no-arbitrage equation*

$$\pi^i W(p, q; y + \eta) = 0 \iff \pi_1^i V(p_1; y_1 + \eta_1) = q - p_0(y_0 + \eta_0) \qquad (2)$$

where $V(p_1; y_1 + \eta_1) = [p_1 \square (y_1^j + \eta_1^j)]_{j=1}^{J}$ is the $S \times J$ submatrix of period 1 returns from the J firms. When $J \leq S$ then rank $V \leq J$ and the dimension of the set of solutions of (2) is at least $S - J$. If $J < S$ then we say that the security markets are *incomplete*. If

9

$J \geq S$ and rank $V = S$ we say the security markets are *complete*. When the equity markets are incomplete any two consumers will typically have different present value coefficients, so that their induced preference orderings over profit streams will differ. *When the equity markets are complete there is a unique solution $\pi_1^i = \beta_1$ to equation (2); the present value coefficients of all agents coincide $\pi^i = (1, \beta_1) = \beta$, $i = 1, \ldots, I$ and lead to a well-defined present value for each firm's stream of profit $p \square (y^j + \eta^j)$ which coincides with its market value $q_j = \beta \cdot \left(p \square (y^j + \eta^j)\right)$, $j = 1, \ldots, J$.* The objective of each firm is well-defined and consists in maximising the present value of its stream of profit.

The spanning literature has shown that *even if markets are incomplete if suitable restrictions are placed on the technology sets Y^j of firms then a well-defined objective function can be assigned to each firm* [see Ekern-Wilson (1974) and Radner (1974)]. The idea of spanning is that firms find themselves in a market environment in which no single firm can by itself alter the spanning opportunities available on the market as a whole. To express this idea let the initial endowment vector of each firm be zero so that $\eta = (\eta^1, \ldots, \eta^J) = 0$ and let $\langle V(\bar{p}_1, \bar{y}_1) \rangle$ denote the subspace of R^S spanned by the columns of the matrix V. Then the spanning condition requires that for any *new* production plan $y^j \in Y^j$ of firm j, the profit stream that it generates at date 1, $\bar{p}_1 \square y_1^j$ can be written as a linear combination of the *existing* profit streams $(\bar{p}_1 \square \bar{y}_1^k)_{k=1}^J$ of all firms so that

$$\bar{p}_1 \square y_1^j \in \langle V(\bar{p}_1, \bar{y}_1) \rangle$$

With this assumption firm j cannot create any new spanning opportunities for investors by altering its production plan y^j. In this case the market value of the firm for each alternative production plan can be evaluated in terms of the market values of all firms at the existing production plan $\bar{y}$ and *market value maximisation* gives a well-defined objective function for the firm.

As soon as a firm can create a date 1 profit stream $\bar{p}_1 \square y_1^j$ which does not lie in the subspace $\langle V(\bar{p}_1, \bar{y}_1) \rangle$, investors will not in general agree on the value to be assigned to this production plan and the spanning approach breaks down. Whether the spanning

condition is a reasonable approximation to what we observe on security markets is an *empirical* question that we shall not enter into here. It should be noted however that if the number of firms (J) is small relative to the number of states of nature (S) then the spanning assumption is likely to be very restrictive. The problem is thus to extend the definition of the objective function of a firm to a market environment in which the spanning condition no longer holds. We will examine a variety of ways in which this can be done by considering a class of objective functions which reduce to market value maximisation when the spanning condition holds.

What needs to be added to determine an objective function for firm j is a *present value coefficient* $\beta^j = (1, \beta_1^j, \ldots, \beta_S^j)$ satisfying the no-arbitrage equation (2). Thus one way of arriving at an equilibrium concept is to choose a sequence of such β^j coefficients, one for each firm

$$(\beta^1, \ldots, \beta^J) \quad \text{with} \quad \beta^j W = 0, \quad j = 1, \ldots, J \tag{3}$$

and have each firm maximise the present value of its profit

$$\max_{y^j \in Y^j} \beta^j \cdot (p \square y^j) \quad j = 1, \ldots, J \tag{4}$$

The procedure adopted by Duffie and Shafer (1986) involves choosing the same present value coefficient for each firm, $\beta^1 = \cdots = \beta^J = \beta$ with $\beta W = 0$. If no further criteria are involved in the choice of the β^j coefficients in (3), such a procedure can present some conceptual difficulties. First, since the dimension of the set of solutions of (2) is at least $S - J$, the dimension of the set of equilibrium allocations is of the order of $S - J$; in short the ambiguity in the definition of the objective function of each firm introduces a *multiplicity* of equilibria. Second since there is no market for determining the β^j coefficient, where does the information come from to determine β^j? The third criticism is perhaps the most fundamental and hinges on the whole issue of what a firm really is and in whose interests it acts. *If we view a firm as an entity that makes decisions in the interests of its shareholders then if the present value coefficient β^j chosen by the directors of firm j*

11

does not reflect some kind of average of the present value coefficients of its shareholders then its production decision may be "rejected" by the shareholders. The problem here is the breakdown of the relation between ownership and control. An extreme form of this problem arises in the utility of profits approach $v^j(p\square(y^j + \eta^j))$ originally considered by Radner (1972) — in equilibrium the firm may be owned by a single consumer whose preference ordering over profit streams is far removed from that of the manager.[1]

Two criteria have been suggested for choosing a β^j coefficient for firm j which are free from these difficulties. These two criteria reflect the fact that in the two period economy outlined above there are two groups of shareholders, the *new* (θ^{ij}) shareholders and the *original* (δ^{ij}) shareholders. In the context of a one good $(L = 1)$ economy Drèze (1974) proposed that β^j should be the average present value coefficient of the firm's *new* shareholders

$$\beta^j = \sum_{i=1}^{I} \theta^{ij} \pi^i, \quad j = 1, \ldots, J \tag{5}$$

Grossman and Hart (1979) proposed that β^j be the average present value coefficient of the firm's *original* shareholders

$$\beta^j = \sum_{i=1}^{I} \delta^{ij} \pi^i, \quad j = 1, \ldots, J$$

The distinction between these two criteria is far from trivial and a discussion of the issues would take us beyond the confines of this paper. Suffice it to say that Grossman and Hart came up with their criterion in attempting to resolve the difficulties that appear when the Dreze criterion is applied to an economy with three or more periods. For then the new shareholders at date 1 may differ from the new shareholders at date 0: shareholders planning to sell at date 1 are concerned with the *selling price* of the shares, while those who will hang on to their shares are interested in the *dividends* (profits) that will accrue. The criterion of Grossman-Hart focuses all attention on the *original* shareholders at date

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[1] The utility function approach also has the problem that it does not reduce to maximising the present value of profit when the equity markets are complete.

0. This however is reasonable only if the investment decisions made at date 0 are *not reversible* by subsequent shareholders, an assumption that ultimately becomes untenable.

In a two-period economy the *new* shareholders criterion has better normative properties since it takes into account the interests of the shareholders who will receive its stream of profits. As we shall find, *in the one good case, it is the only criterion which satisfies the first order conditions for constrained Pareto optimality.* It seems reasonable therefore in extending the analysis of the normative properties of equilibrium to the multigood case to adopt the *new* as opposed to the *original* shareholder criterion. Indeed it should be clear more generally that if in the multigood case the equilibrium allocations induced by the *new* shareholder criterion (5) are generically constrained suboptimal, then the same should be true for any criterion satisfying (3) and (4).

2.5 Shareholder Constrained Efficiency.

If each firm adopts the new shareholder criterion outlined above in what sense is the resulting collective set of decisions of firms made in the best interests of shareholders in the economy? Is the production sector induced to act in the best interests of the group of shareholders?

DEFINITION 2: Let $(\bar{x}, \bar{\theta}, \bar{y}; \bar{p}) = (\bar{x}^1, \ldots, \bar{x}^I, \bar{\theta}^1, \ldots, \bar{\theta}^I, \bar{y}^1, \ldots, \bar{y}^J; \bar{p})$ denote actions of the agents and a vector of spot prices such that $\bar{x}^i$ is optimal given $(\bar{\theta}^i, \bar{y}, \bar{p})$, $i = 1, \ldots, I$. The vector of production plans $y = (y^1, \ldots, y^J)$, where $y^j \in Y^j$, $j = 1, \ldots, J$, is preferred to $\bar{y}$ by the (new) shareholders of all firms if there exist transfers and changes in consumption $(\tau, \xi) = \left((\tau^{ij}, \xi^i), \ j = 1, \ldots, J, \ i = 1, \ldots, I \right) \in R^{IJ} \times R^{NI}$ such that

$$\sum_{j=1}^{J} \sum_{i=1}^{I} \tau^{ij} = 0 \quad \text{with} \quad \tau^{ij} = 0 \quad \text{if} \quad \bar{\theta}^{ij} = 0 \tag{6}$$

$$\bar{p}_0 \xi_0^i = \sum_{j=1}^{J} [\bar{p}_o \bar{\theta}^{ij} (y_0^j - \bar{y}_0^j) + \tau^{ij}] \quad i = 1, \ldots, I \tag{7}$$

$$\bar{p}_1 \square \xi_1^i = \sum_{j=1}^{J} \bar{p}_1 \square \bar{\theta}^{ij} (y_1^j - \bar{y}_1^j) \quad i = 1, \ldots, I \tag{8}$$

13

$$u^i(x^i + \xi^i) > u^i(x^i), \quad i = 1, \ldots, I \tag{9}$$

DEFINITION 3: Let $(\bar{x}, \bar{\theta}, \bar{y}; \bar{p})$ denote actions of the agents and a vector of spot prices such that x^i is optimal given $(\bar{\theta}^i, \bar{y}, \bar{p})$, $i = 1, \ldots, I$. The vector of production plans $\bar{y}$ is shareholder constrained efficient if there does not exist another vector of production plans $y = (y^1, \ldots, y^J)$ with $y^j \in Y^j$, $j = 1, \ldots, J$, which is preferred to $\bar{y}$ by the shareholders of all firms.

REMARK: The idea behind this definition is that if $\bar{y}$ were not optimal for the shareholders then a meeting of shareholders could be convened in which those interested in changing production plans could "buy" the votes of others to obtain unanimity for a change from $\bar{y}$ to some y. If we restrict the change in production plans to a *single* firm, say firm j', holding the production plans of all other firms fixed and setting $\tau^{ij} = 0$, $j \neq j'$, $i = 1, \ldots, I$, then with the above definitions we may say that $\bar{y}^{j'}$ is *firm j' shareholder constrained efficient*. In this case the above meeting of shareholders is restricted to the shareholders of firm j'. We want to emphasise however that under the shareholder criterion that we adopt the vector of production plans $\bar{y}$ is optimal in the broader sense of definition 3, where *simultaneous changes in the production decisions of all firms are permitted*, as we now show.

PROPOSITION 2 (OPTIMALITY OF SHAREHOLDER PRESENT VALUE CRITERION):
Let $(\bar{x}, \bar{\theta}, \bar{y}; \bar{p})$ *denote the actions of the agents and a vector of spot prices such that* x^i *is optimal given* $(\bar{\theta}^i, \bar{y}, \bar{p})$, $i = 1, \ldots, I$ *and let* π^i, $i = 1, \ldots, I$ *denote the present value coefficients of definition 1 with* $\bar{\beta}^j = \sum_{i=1}^{I} \bar{\theta}^{ij} \pi^i$ *the average present value coefficient of firm j's shareholders. If each firm maximises the present value of its profit*

$$\bar{\beta}^j \cdot (\bar{p} \square \bar{y}^j) \geqq \bar{\beta}^j \cdot (\bar{p} \square y^j) \quad \text{for all } y^j \in Y^j, \quad j = 1, \ldots, J \tag{10}$$

then $\bar{y} = (\bar{y}^1, \ldots, \bar{y}^J)$ *is shareholder constrained efficient.*

PROOF: Suppose $\bar{y} = (\bar{y}^1, \ldots, \bar{y}^J)$ is not shareholder constrained efficient, then there exists a vector of transfers and changes in consumption $(\tau, \xi) = ((\tau^{ij}, \xi^i), j = 1, \ldots, J, i = 1, \ldots, I) \in R^{IJ} \times R^{NIJ}$ such that (6)-(9) are satisfied. Since x^i is optimal given $(\bar{\theta}^i, \bar{y}, \bar{p})$, $i = 1, \ldots, I$, proposition 1 implies that there exist present value coefficients π^i, $i = 1, \ldots, I$ such that

$$\pi^i \cdot (\bar{p} \square \xi^i) > 0, \quad i = 1, \ldots, I \tag{11}$$

If we multiply (7) and (8) by π^i, substitute into (11), sum over i and use (6) we obtain

$$\sum_{i=1}^{I} \pi^i \cdot (\bar{p} \square \xi^i) = \sum_{j=1}^{J} \bar{\beta}^j \cdot (\bar{p} \square (y^j - \bar{y}^j)) > 0$$

which contradicts (10).

REMARK: *The assumption that each firm uses the profit maximising criterion (10) implies that each firm behaves competitively in that it takes spot prices as given and independent of its production decision. When markets are incomplete this has important consequences as we shall see in section 4.*

3. EQUILIBRIUM

With the objective function (10) assigned to each firm our model becomes closed and we are led to the following concept of equilibrium for a stock market economy.

DEFINITION 4: An equilibrium for the economy $\mathcal{E}\left((u^i, Y^j), \omega, \eta\right)$ is a pair of actions and prices $\left((\bar{x}, \bar{\theta}, \bar{y}), (\bar{p}, \bar{q}, \bar{\pi})\right)$ such that

(i) $(\bar{x}^i, \bar{\theta}^i, \bar{\pi}^i)$, $i = 1, \ldots, I$ satisfy

$$x^i \in \operatorname*{arg\,max}_{x^i \in B^i(\bar{p}, \bar{q}, \bar{y}; \omega, \eta)} u^i(x^i), \quad \begin{bmatrix} \bar{p}_0(\bar{x}_0^i - w_0^i) - \bar{q}\delta^i \\ \bar{p}_1 \square (\bar{x}_1^i - w_1^i) \end{bmatrix} = W(\bar{p}, \bar{q}; \bar{y} + \bar{\eta})\bar{\theta}^i$$

and $\bar{\pi}^i$ is the present value coefficient of consumer i

(ii) $$\bar{y}^j \in \operatorname*{arg\,max}_{y^j \in Y^j} \bar{\beta}^j \cdot (\bar{p} \square y^j), \quad j = 1, \ldots, J$$

with $\bar{\beta}^j = \sum\limits_{i=1}^{I} \bar{\theta}^{ij}\pi^i$ the average present value coefficient of firm j's shareholders

(iii) $\sum\limits_{i=1}^{I}(x^i - w^i) = \sum\limits_{j=1}^{J}(\bar{y}^j + \eta^j)$ (iv) $\sum\limits_{i=1}^{I}\bar{\theta}^{ij} = 1$ $j = 1,\ldots,J$

We now establish some properties of this concept of equilibrium which besides their intrinsic interest are necessary to prove the generic inefficiency of equilibrium. The proofs of all the theorems that follow are collected in section 6.

THEOREM 1 (EXISTENCE OF EQUILIBRIUM): *Under assumptions* $(A_{(1)}, B_{(1)})$ *the economy* $\mathcal{E}\big((u^i, Y^j), (\omega, \eta)\big)$ *has an equilibrium.*

To carry-out a qualitative analysis we need equilibria to be smooth functions of the parameters (ω, η). It is at this point that the smoothness assumptions $(A_{(2)}, B_{(2)})$ on preferences and technology are introduced. Even with these assumptions there are three types of *degeneracy* that can prevent an equilibrium from being a smooth function of the parameters $(\omega, \eta) \in R_{++}^{N(I+J)}$: if

(a) the matrix of security returns $W = \begin{bmatrix} p_0(y_0 + \eta_0) - q \\ p_1 \square (y_1 + \eta_1) \end{bmatrix}$ is degenerate in that its rank is $\rho < J$

(b) equilibrium prices are such that for some agent i the no short sales constraint begins to be binding and the portfolio choice θ^i is not a differentiable function of the prices

(c) the parameter value (ω, η) is a *critical value* of the projection from the equilibrium manifold onto the parameter space.

Theorem 2 below shows that the parameter values for which such non–smoothness can occur are *exceptional* in the sense that they form a closed set of measure zero in the parameter space $R_{++}^{N(I+J)}$.

DEFINITION 5: We say that an equilibrium $\big((\bar{x}, \bar{\theta}, \bar{y}), (\bar{p}, \bar{q}, \bar{\pi})\big)$ is a *rank ρ equilibrium* if rank $W(\bar{p}, \bar{q}; \bar{y} + \bar{\eta}) = \rho$, $1 \le \rho \le J$.

THEOREM 2 (FINITENESS AND FULL RANK): *Under assumptions* $(A_{(1),(2)}; B_{(1),(2)})$,

if $I + J \leqq S + 1$, then there exists an open set of full measure $\Omega \subset R_{++}^{N(I+J)}$ such that an economy $\mathcal{E}\left((u^i, Y^j), \omega, \eta\right)$ with $(\omega, \eta) \in \Omega$ has a finite number of equilibria, each of full rank. Furthermore for each $(\bar{\omega}, \bar{\eta}) \in \Omega$ there exists a neighborhood $\mathcal{N}_{(\bar{\omega}, \bar{\eta})}$ such that each equilibrium is a smooth function of (ω, η) for all $(\omega, \eta) \in \mathcal{N}_{(\bar{\omega}, \bar{\eta})}$.

REMARK: A comment is in order regarding the assumption $I + J \leqq S + 1$. For general technology sets (Y^j) the coefficient $\beta^j = \sum_{i=1}^{I} \theta^{ij} \pi^i$ of each firm j depends nontrivially on the portfolios $(\theta^{ij})_{i=1}^{I}$ of its shareholders. When rank $W = \rho < J$ agents portfolio choices (θ^i) are typically indeterminate and this can lead to indeterminacy of the equilibrium. To be sure that this happens only exceptionally we need to limit the number of agents $(I + J)$ relative to the number of states $(S + 1)$. We will show in section 4 that if the technology sets are restricted to *ray* technologies then this assumption is not necessary, since the production decisions of the firms no longer depend on the distribution of ownership. In this case the matrix W is generically of maximal rank independent of the number of agents.

The key characteristic of equilibria with incomplete markets is that the dimension of the set of solutions of the no-arbitrage equation $\pi^i W = 0$ is $S - J > 0$. This suggests the likelihood that in equilibrium agents present value coefficients will be distinct: the next proposition asserts that this property is generic. This result is a basic step in establishing the generic inefficiency theorem of section 4. Note that when $I + J \leqq S + 1$ if we require that $I \geqq 2$ then $J < S$ must hold.

PROPOSITION 3 (DISTINCT PRESENT VALUE COEFFICIENTS): *Let the assumptions of Theorem 2 hold. If $I \geqq 2$ then there exists an open set of full measure $\Omega' \subset R_{++}^{N(I+J)}$ such that for every economy $\mathcal{E}\left((u^i, Y^j), \omega, \eta\right)$ with $(\omega, \eta) \in \Omega'$, in each shareholder equilibrium the present value coefficients of all consumers $\pi^1, \ldots, \pi^I$ are distinct.*

4. INEFFICIENCY OF EQUILIBRIUM

In this section we will examine the efficiency properties of the stock market equilibrium introduced in section 3. It is clear that there are many market structures that can be

adjoined to the basic production economy of section 2.1 to induce an allocation of resources and the efficiency properties of equilibrium allocations will depend upon the market structure introduced. *With a set of complete contingent markets every equilibrium allocation is Pareto optimal.* If instead the market structure consists of a system of spot and financial markets, in which spot markets are complete and the asset markets are *potentially* complete (in that $J \geqq S$) then an equilibrium allocation is not necessarily Pareto optimal: the problem here is that the matrix W can be of less than full rank in an equilibrium.[2] However as Magill-Shafer (1985) have shown, in an exchange economy this situation is exceptional: *when asset markets are potentially complete then equilibrium allocations are generically efficient.*

The situtation changes dramatically when asset markets are incomplete $(J < S)$ for then equilibrium allocations are generically constrained inefficient. This result was established by Geanakoplos-Polemarchakis (1986) for the case of an exchange economy. The object of this section is to show how this result can be extended to the case of a production economy. Note that we do not attempt to answer the important question of *why asset markets are incomplete:* we leave the explanation of this for subsequent research.

The basic objective behind theorems of this kind is to determine whether decentralised decision making based on *prices* leads to an efficient co-ordination of decisions by the agents in the economy. *If a decentralised price system does not lead to an efficient co-ordination of decisions then presumably some form of government intervention is called for:* this is a basic theme that underlies our analysis of equilibrium in a production economy with incomplete markets. In this paper we will not however attempt to explore what form such government intervention should take.

4.1 The Problem of Defining Efficiency.

Government intervention suggests the idea of a *planner* running the economy. If we allow the planner access to the standard "feasible allocations" then we arrive at the concept

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[2]See the well-known example of Hart (1975).

18

of Pareto efficiency. When markets are incomplete it is clear (at least intuitively) that equilibrium allocations are generically Pareto inefficient: a planner allocating resources with access to the standard "feasible allocations" is given much more freedom to allocate resources across states than is provided by the system of spot and financial markets. When markets are incomplete the concept of Pareto efficiency is essentially irrelevant: it does not allow us to determine whether the existing structure of incomplete markets is used efficiently.

What is needed is clear. The planner must only be permitted access to a constrained set of feasible allocations which mimics the opportunities that a system of spot and financial markets offers for redistributing goods across the states of nature. The first concept of constrained efficiency was that introduced by Diamond (1967) for the case of a one good ($L = 1$) economy. This concept was extended by Grossman (1977) and Grossman-Hart (1979) to the case of a multigood economy. They showed that their concept of efficiency, which we refer to as *weak constrained efficiency* characterises equilibrium allocations with spot and financial markets. This is both its strength and its weakness: its strength in that it identifies all those allocations which can be attained as equilibrium allocations; its weakness in that every equilibrium allocation is efficient, even one which is Pareto dominated by another and such an equilibrium is clearly not making best use of the existing structure of markets.

The basic idea behind weak constrained efficiency is to consider a restricted set of reallocations about an existing equilibrium $\big((\bar{x}, \bar{\theta}, \bar{y}), (\bar{p}, \bar{q}, \bar{\pi})\big)$. Reallocations on the financial markets and spot markets are kept completely separate. Thus when shareholders portfolios are changed $(d\theta^i)$ each agent is obliged to "consume" the bundle of commodities $\sum_{j=1}^{J} d\theta^{ij} \bar{y}^j$ to which the new portfolio holdings give him the right: agent i is not allowed to sell this newly acquired bundle of goods on the spot markets. The reason is clear: if they allowed such newly acquired bundles induced by portfolio changes to be exchanged on the spot markets *then spot prices would in general need to be changed.* This is precisely

the effect that weak constrained effic.ency eliminates but *constrained efficiency* (as defined below) introduces. Constrained efficiency differs in another important respect from weak constrained efficiency in that *simultaneous changes in the portfolios and production plans* are permitted.

We have argued that Pareto efficiency is too demanding a criterion and weak constrained efficiency is not sufficiently demanding to determine whether the existing structure of incomplete markets is used efficiently. Afterall, the fact that equilibrium allocations are generically efficient when asset markets are potentially complete suggests that equilibrium allocations should have a better efficiency property than that of weak efficiency. We now come to the appropriate intermediate concept. We shall find that it reduces to Diamond's (1967) concept of constrained efficiency when there is only one good, for then there are no relative prices that need to be determined on spot markets. It generalises the concept introduced by Stiglitz (1982) and that studied by Geanakoplos-Polemarchakis (1986) for the case of a pure exchange asset market economy. The key idea is to give the planner the same *spanning opportunities* across the states of nature as those offered by a joint system of spot and financial markets. More precisely, a planner can determine portfolios (θ), production plans (y) and income transfers (τ) at date 0; consumption (x) is then determined through spot markets at an appropriate market clearing price (p).

DEFINITION 6: A plan $((x,p),(\tau,\theta,y))$ is *constrained feasible* (or feasible for a constrained planner) if

(i) $\tau = (\tau^i, \, i = 1,\ldots,I) \in R^I, \; \sum_{i=1}^{I} \tau^i = 0$

(ii) $\theta^i \in [0,1]^J, \, i = 1,\ldots,I, \; \sum_{i=1}^{I} \theta^i = e$ where $e = (1,\ldots,1)$

(iii) $y^j \in Y^j, \, j = 1,\ldots,J$

(iv) the spot prices $p \in (\Delta_{++}^{L-1})^{S+1}$ are such that for $i = 1,\ldots,I$

$$x^i = \arg\max_{x^i \in B(p,m^i)} u^i(x^i) \quad B(p,m^i) = \{x \in R_+^N \mid p \,\square\, x^i = m^i\}$$

$$m^i = p \,\square\, \big(w^i + (y + \eta)\theta^i\big) + \tau^i \epsilon_0, \quad \epsilon_0 = (1,0,\ldots,0) \in R^{S+1}$$

and spot markets clear $\sum_{i=1}^{I}(x^i - w^i) = \sum_{j=1}^{J}(y^j + \eta^j)$.

A plan $((\bar{x}, \bar{p}), (\bar{\tau}, \bar{\theta}, \bar{y}))$ is *constrained efficient* if it is constrained feasible and there does not exist a constrained feasible plan $((x, p), (\tau, \theta, y))$ such that $u^i(x^i) > u^i(\bar{x}^i)$, $i = 1, \ldots, I$. An allocation which is not constrained efficient is called *constrained inefficient*.

4.2 First Order Conditions for Efficiency.

We now introduce a fundamental simplifying idea. Since spot prices at date 0 are normalised so that $p_{01} = 1$, the transfer payment τ^i can be considered as a transfer of good 1. Thus when a planner chooses a triple (τ, θ, y) this is equivalent to choosing a *virtual endowment* in goods

$$\underset{\sim}{w}^i = w^i + (y + \eta)\theta^i + \tau^i e_0, \quad i = 1, \ldots, I \tag{12}$$

for each consumer, where $e_0 = (1, 0, \ldots, 0) \in R^N$. (12) can be viewed as the initial endowments of an exchange economy $\mathcal{E}((u^i), \underset{\sim}{w})$ where $\underset{\sim}{w} = (\underset{\sim}{w}^i, \ldots, \underset{\sim}{w}^I)$, for which consumption is allocated through a system of spot markets.

DEFINITION 7: A *spot market equilibrium* for the economy $\mathcal{E}((u^i), \underset{\sim}{w})$ is a pair $(\bar{x}, \bar{p})$ such that

(i)
$$\bar{x}^i = \underset{x^i \in B(\bar{p}, \bar{p} \,\square\, \underset{\sim}{w}^i)}{\arg\max}\; u^i(x^i) \qquad i = 1, \ldots, I \tag{13}$$

(ii)
$$\sum_{i=1}^{I}(\bar{x}^i - \underset{\sim}{w}^i) = 0$$

Let $\tilde{x}^i(p, m^i)$ denote agent i's solution to (i) when income is $m^i \in R_{++}^{S+1}$, then a spot market equilibrium price for the economy $\mathcal{E}((u^i), \underset{\sim}{w})$ satisfies

$$\sum_{i=1}^{I}\left(\tilde{x}(p, p \,\square\, \underset{\sim}{w}^i) - \underset{\sim}{w}^i\right) = 0$$

If the economy $\mathcal{E}((u^i), \underset{\sim}{w})$ is regular at $\underset{\sim}{w}$ then the price system $\tilde{p}(\underset{\sim}{w})$ is locally a differentiable function of the parameter ω.

Consider a constrained feasible plan $\big((\bar{x},\bar{p}),(\tau,\theta,y)\big)$ then $x^i = \tilde{x}^i(\bar{p},\bar{p}\,\square\,\psi^i)$, $i = 1,\ldots,I$, $\bar{p} = \tilde{p}(\psi)$ with $\psi = (\psi^1,\ldots,\psi^I)$ defined by (12). We want to examine the effect of a marginal change $(d\tau, d\theta, dy)$ in the plan $i.e.$, a change satisfying

$$
\begin{aligned}
&\sum_{i=1}^{I} d\tau^i = 0, \quad dy^j \in T_{y^j}\partial Y^j, \; j = 1,\ldots,J \\
&\sum_{i=1}^{I} d\theta^i = 0, \quad d\theta^{ij} \geqq 0 \text{ if } \theta^{ij} = 0, \; i = 1,\ldots,I, \; j = 1,\ldots,J
\end{aligned}
\tag{14}
$$

Such a change induces a marginal change in the virtual endowments

$$
d\psi^i = (y + \eta)d\theta^i + dy\theta^i + d\tau^i e_0 \quad i = 1,\ldots,I
$$

which in turn leads to a marginal change in consumption and spot prices $(d\tilde{x}, d\tilde{p})$. The latter adjust so that

$$
\bar{p}\,\square\, d\tilde{x}^i = \bar{p}\,\square\, d\psi^i - d\tilde{p}\,\square\,(\bar{x}^i - \psi^i) + d\tau^i \epsilon_0 \quad i = 1,\ldots,I
$$

The resulting change in utility for agent i is given by

$$
du^i = \frac{\partial u^i(\bar{x}^i)}{\partial x^i}\cdot d\tilde{x}^i, \text{ where } \frac{\partial u^i(\bar{x}^i)}{\partial x^i} = \left(\frac{\partial u^i(\bar{x}^i)}{\partial x_{01}^i},\ldots,\frac{\partial u^i(\bar{x}^i)}{\partial x_{SL}^i}\right) \in R^N
$$

is the gradient of u^i at $\bar{x}^i$. The first order conditions for (13) imply that there exist $\bar{\lambda}^i = (\bar{\lambda}_0^i,\ldots,\bar{\lambda}_S^i) \in R_{++}^{S+1}$ such that $\frac{\partial u^i(\bar{x}^i)}{\partial x^i} = \bar{\lambda}^i \square \bar{p}$, $i = 1,\ldots,I$. Thus

$$
du^i = \bar{\lambda}^i\left[\bar{p}\,\square\,\big((y+\eta)d\theta^i + dy\theta^i\big) - d\tilde{p}\,\square\,(\bar{x}^i - \psi^i)\right] + \bar{\lambda}_0^i d\tau^i \quad i = 1,\ldots,I
\tag{15}
$$

The following lemma is now evident.

LEMMA: Let $\big((\bar{x},\bar{p}),(\tau,\theta,y)\big)$ be a constrained feasible plan, then there exists a marginal change $(d\tau, d\theta, dy)$ satisfying (14) such that $du^i > 0$, $i = 1,\ldots,I$ if and only if $\sum_{i=1}^{I}\frac{du^i}{\bar{\lambda}_0^i} > 0$. Since $\frac{\bar{\lambda}^i}{\bar{\lambda}_0^i} = \pi^i$, dividing (15) by $\bar{\lambda}_0^i$ and summing over i gives the marginal change in social welfare arising from the change $(d\tau, d\theta, dy)$

$$
\sum_{i=1}^{I}\frac{du^i}{\bar{\lambda}_0^i} = \sum_{i=1}^{I}\pi^i\cdot\big(\bar{p}\,\square\,(y+\eta)\big)d\theta^i + \sum_{i=1}^{I}\bar{\pi}^i\cdot(\bar{p}\,\square\,dy)\theta^i
$$
$$
- \sum_{i=1}^{I}\bar{\pi}_1^i\cdot[d\tilde{p}_1\,\square\,(\bar{x}_1^i - \psi_1^i)]
\tag{16}
$$

The term $\sum_{i=1}^{I} \pi_0^i d\tilde{p}_0(x_0^i - \psi_0^i)$ vanishes since $\pi_0^i = 1$ for all i and $\sum_{i=1}^{I}(x_0^i - \psi_0^i) = 0$. The first two terms in (16) represent the *direct income* effect of the change $(d\theta, dy)$, the last term is the *indirect price* effect.

A shareholder equilibrium $\left((\bar{x}, \bar{\theta}, \bar{y}), (\bar{p}, \bar{q}, \bar{\pi})\right)$ is clearly a constrained feasible plan corresponding to $(\tau, \theta, y) = \left((\delta - \bar{\theta})\bar{q}, \bar{\theta}, \bar{y}\right)$. Let us examine the marginal effect on social welfare (16) arising from a marginal change $(d\tau, d\theta, dy)$ around such an equilibrium. Evaluating such marginal change is legitimate since we prove in proposition 5 in the appendix that the virtual endowments induced by an equilibrium are generically regular. The first order conditions for the portfolio choice $\bar{\theta}^i$ of agent i implies that there exist $\bar{\rho}^{ij} \geqq 0$ such that

$$\bar{\pi}^i \cdot \left(\bar{p} \square (\bar{y}^j + \eta^j)\right) = \bar{q}^j - \bar{\rho}^{ij} \quad \text{with } \bar{\rho}^{ij} = 0 \text{ if } \bar{\theta}^{ij} > 0$$

Multiplying by $d\theta^{ij}$ and summing over i and j gives

$$\sum_{i=1}^{I} \bar{\pi}^i \cdot \left(\bar{p} \square (\bar{y} + \eta)\right) d\theta^i = -\sum_{i=1}^{I}\sum_{j=1}^{J} \bar{\rho}^{ij} d\theta^{ij} = -\bar{\rho} d\theta$$

where $\bar{\rho}$ denotes the matrix $\bar{\rho} = (\bar{\rho}^{ij})$. The first order conditions for profit maximisation by firm j imply

$$\sum_{i=1}^{I} \bar{\theta}^{ij} \bar{\pi}^i \cdot (\bar{p} \square dy^j) = 0 \quad \forall \quad dy^j \in T_{\bar{y}^j} \partial Y^j$$

Thus at an equilibrium the social welfare change (16) reduces to

$$\sum_{i=1}^{I} \frac{du^i}{\bar{\lambda}_0^i} = -\bar{\rho} d\theta - \sum_{i=1}^{I} \bar{\pi}_1^i \cdot [d\tilde{p}_1 \square (x_1^i - \psi_1^i)] \tag{17}$$

The first term $-\bar{\rho} d\theta$ measures the cost of the no-short-sales constraints $\theta^{ij} \geqq 0$. This term is zero in an equilibrium where $\bar{\theta}^{ij} > 0 \; \forall \; i, j$. The second term is the effect on welfare of a change in the equilibrium spot prices.

The price function $\tilde{p}$ is a function of ω, which is in turn a function of the planners action (τ, θ, y). With a slight abuse of notation we let $\tilde{p}$ also denote the composite function $(\tau, \theta, y) \longrightarrow \omega \longrightarrow \tilde{p}$. If we make the *separability* assumptions $A_{(3)}$ then the period 1

23

spot price function $\tilde{p}_1$ depends only on $\underset{\sim}{\omega}_1$ and hence only on (θ, y_1). Let $\frac{\partial \tilde{p}_1}{\partial \theta^{ij}}$ and $\frac{\partial \tilde{p}_1}{\partial y_{st}^j}$ denote the partial derivatives of the vector valued function $\tilde{p}_1$ with respect to θ^{ij} and y_{st}^j respectively. These are both column vectors. Thus $\frac{\partial \tilde{p}_1}{\partial y_1^j}$ denotes the $SL \times SL$ matrix $\left[\frac{\partial \tilde{p}_1}{\partial y_{1i}^j}, \ldots, \frac{\partial \tilde{p}_1}{\partial y_{sL}^j} \right]$. Since the price effect $d\tilde{p}_1$ decomposes into the change induced by $d\theta$ and the change induced by dy, applying the lemma to (17) gives the following result.

PROPOSITION 4 (EFFICIENCY CONDITIONS): *If an equilibrium $((\bar{x}, \bar{\theta}, \bar{y}), (\bar{p}, \bar{q}, \bar{\pi}))$ is constrained efficient then*

(i)
$$\sum_{i=1}^{I} \bar{\pi}_1^i \cdot \left[\left(\frac{\partial \tilde{p}_1}{\partial \theta^{kj}} - \frac{\partial \tilde{p}_1}{\partial \theta^{k'j}} \right) \square (\bar{x}_1^i - \underset{\sim}{\omega}_1^i) \right] = 0, \quad j = 1, \ldots, J$$

for all k, k' such that $\bar{\theta}^{kj} > 0$, $\bar{\theta}^{k'j} > 0$

(ii)
$$\sum_{i=1}^{I} \bar{\pi}_1^i \cdot \left[\left(\frac{\partial \tilde{p}_1}{\partial y_1^j} dy_1^j \right) \square (\bar{x}_1^i - \underset{\sim}{\omega}_1^i) \right] = 0$$

for all $dy^j \in T_{\bar{y}^j} \partial Y^j$, $j = 1, \ldots, J$.

We refer to (i) as the *portfolio efficiency* condition and (ii) as the *production efficiency* condition. Two important cases where both efficiency conditions are satisfied are the following:

(a) *There is only one good in each state $(L = 1)$.* (i) and (ii) hold since the price effects vanish. This explains the result of Diamond (1967), for in the case of *ray* technologies the set of feasible allocations is *convex* and the first order conditions are sufficient. In the case of general technology sets studied by Drèze (1974) the set of feasible allocations is *non-convex* and the first order conditions are no longer sufficient. Drèze gives examples of equilibria with one good which are not constrained efficient.

(b) *The present value coefficients are all the same $(\bar{\pi}_1^i = \pi_1, i = 1, \ldots, I)$.* This happens if asset markets are *complete* $(J \geq S$ and rank $W = S)$ and the constraints $\bar{\theta}^{ij} \geq 0$ are not binding. Proposition 3 asserts that when asset markets are incomplete this case will not be observed. In addition there are two special cases where (i) and (ii) will hold.

24

(c) *There is no exchange at equilibrium* $(x_1^i - w_1^i = 0, \; i = 1,\ldots,I)$. In the pure exchange case this occurs if initial endowments are Pareto optimal, a situation which is not generic.

(d) *There is no production and all agents have identical income effects (identical homothetic preferences)*. The price effects disappear in (i).

(a) and (b) suggest the possibility that if there are at least two goods in each state $(L \geqq 2)$ and if markets are incomplete $(J < S)$ then equilibria are generically constrained inefficient. In the case of an economy without production $(Y^j = \{0\}, \; j = 1,\ldots,J)$ Geanakoplos-Polemarchakis (1986) have shown that this is indeed true. In order to eliminate case (d) the genericity is with respect to *utility functions* as well as endowments. Our object here is to extend the result to the case of an economy with production: in particular we will show that the production efficiency condition (ii) is generically not satisfied, the genericity being with respect to the endowments (ω, η). Thus even if portfolios were efficiently allocated, it is unlikely that production decisions are efficient.

From a policy point of view (*i.e.*, should the government intervene or not) the significance of the inefficiency theorems which follow depends upon the *magnitude of the distortions* which they assert are generically present at an equilibrium. We do not attempt to provide estimates of these magnitudes even though the analysis makes clear how such estimates can be calculated.

4.3 Inefficiency of Equilibrium.

We give two polar conditions on the technology sets (Y^j) which imply that at an equilibrium the production efficiency condition (ii) is generically violated. The first requires that for *some* firm j the dimension of its production set Y^j be $L(S + 1)$: this means that the firm uses as an input or produces as an output each of the $L(S + 1)$ commodities. From a technical point of view this assumption is similar in spirit to the requirement that an agent have a positive endowment of each good.

THEOREM 3 (INEFFICIENCY): *If the assumptions (i)* $(A_{(1)-(3)}, B_{(1)-(2)})$; *(ii)* $L \geqq 2$;

25

(iii) $I \geqq 2$; *(iv)* $I + J \leqq S + 1$; *(v)* $K^j = R^N$ for some $j \in \{1, \ldots, J\}$, are satisfied, then there exists an open set of full measure $\Omega^* \subset R_{++}^{N(I+J)}$ such that for every $(\omega, \eta) \in \Omega^*$ each equilibrium is constrained inefficient.

When $K^j = R^N$ firm j can completely control its date 1 production vector y_1^j by suitably changing its production decision y_0^j at date 0. The polar case is to assume that K^j is a subspace of R^N generated by the requirement that the *composition* of date 1 production y_1^j is *fixed* and only its scale can be influenced by changing the date 0 production decision y_0^j. This leads to the following concept which reduces to the *multiplicative uncertainty* of Diamond (1967) when $L = 1$.

DEFINITION 8: Firm j has a *ray* technology set if there exists a non-zero vector of date 1 commodities (the ray) $\eta_1^i \in R^{SL}$ and a function $h^j : R^L \longrightarrow R$ such that

$$Y^j = \{(-y_0^j, y_1^j) \in R_-^L \times R^{LS} \mid y_1^j = h^j(y_0^j)\eta_1^j\}$$

In all genericity arguments of the paper we need a parameter of dimension $L(S + 1)$ for each firm to perturb its supply function out of non-generic situations. *Provided parameters are introduced which permit sufficient controllability of the supply functions, the particular parameterisation used is not of importance in our analysis.* With a general technology set the simplest way to perturb Y^j is by introducing an *additive* parameter $(Y^j + \eta^j)$. When Y^j has the additional structure of being a ray technology set it is natural to replace the additive parameter at date 1 by the ray parameter η_1^j in definition 8, leaving the date 0 parameter η_0^j to enter additively as before. We will follow this convention in all cases where ray technology sets are introduced.

THEOREM 4 (INEFFICIENCY): *If the assumptions (i)* $(A_{(1)-(3)}, B_{(1)-(2)})$; *(ii)* $L \geqq 2$; *(iii)* $I \geqq 2$; *(iv)* $I + J \leqq S + 1$; *(v) firm j has a ray technology set for some $j \in \{1, \ldots, J\}$, are satisfied, then there exists an open set of full measure $\Omega^{**} \subset R_{++}^{N(I+J)}$ such that for every $(\omega, \eta) \in \Omega^{**}$ each equilibrium is constrained inefficient. Furthermore if all firms have ray technology sets then the result holds without assumption (iv).*

REMARK: When all firms have ray technology sets the assumption $I + J \leq S + 1$ can be dropped because the objective function of each firm does not depend on the distribution of its ownership among its shareholders; thus indeterminacy in the portfolios (θ^i) of the agents does not translate into indeterminacy of the equilibrium. The second result in Theorem 4 is important since it shows that the inefficiency result does not depend on the upperbound on the number of agents.

5. EXAMPLE

The simplest class of economies in which the production inefficiencies of the previous section can be explored are those in which production activity consists of using a single input (investment) at date 0 and producing outputs at date 1. In such an economy there are two causes of production inefficiency: inappropriate use of inputs $i.e.$, under or overinvestment at date 0 and inappropriate production of outputs $i.e.$, production of outputs in the wrong proportions across the states at date 1. We shall consider an economy in which only the former type of inefficiency can arise by assuming that there is only one input at date 0 and that the production sector consists of a single firm with a ray technology set.

The basic data of the economy are as follows. There are two consumers ($I = 2$), one firm ($J = 1$) and two states at date 1 ($S = 2$). There is one good (the input) at date 0 and the two outputs ($L = 2$) in each state at date 1. The characteristics of the two consumers (called α and β) are given by separable utility functions $u^i(x^i) = u_0^i(x_0^i) + \sum_{s=1,2} \rho_s u_1^i(x_s^i)$ and endowments w^i, $i = 1, \beta$ with $\rho_1 = \rho_2 = \frac{1}{2}$

$$u_0^\alpha(x_0) = \left(\frac{1+b}{4}\right) \log x_0, \quad u_1^\alpha(x_s) = (x_{s1} x_{s2})^{\frac{1}{2}}$$

$$u_0^\beta(x_0) = \left(1 + \frac{b}{a}\right) \log x_0, \quad u_1^\beta(x_s) = \log x_{s1} + \log x_{s2} \quad a > 0, \ b > 0$$

$$w^\alpha = \left(3, (0, 2b - a), (0, 1)\right), \quad w^\beta = \left(1, (0, a), (0, 1)\right), \quad 2b - a > 0$$

Since $u_1^\beta = 2 \log u_1^\alpha$, agent β is more risk averse than agent α. As we shall see, this induces agent α to become the sole owner of the firm. Thus we can think of agent α as the

entrepreneur and agent β as the *worker*. We assume that the firm has a ray technology set with constant returns to scale

$$h(y_0) = y_0, \quad \eta_1 = \big((b,0),(1,0)\big)$$

Equilibrium $\big((\bar{x},\bar{\theta},\bar{y}),(\bar{p},\bar{q},\bar{\pi})\big)$ is given by

$$\bar{x}^\alpha = \Big(1,(2b - \frac{a}{2}, 2b - \frac{a}{2}),(\frac{3}{2},\frac{3}{2})\Big), \quad \bar{x}^\beta = \Big(1,(\frac{a}{2},\frac{a}{2}),(\frac{1}{2},\frac{1}{2})\Big)$$

$$\bar{\theta}^\alpha = 1, \quad \bar{\theta}^\beta = 0, \quad \bar{y} = \big(-2,(2b,0),(2,0)\big)$$

$$\bar{p} = \Big(1,(1,\frac{y_0}{2}),(1,\frac{y_0}{2})\Big), \quad \bar{q} = 0, \quad \bar{\pi}^\alpha = \Big(1,\frac{1}{1+b},\frac{1}{1+b}\Big), \quad \bar{\pi}^\beta = \Big(1,\frac{1}{a+b},\frac{a}{a+b}\Big)$$

Note that

$$\bar{x}^\alpha_{12} - \underaccent{\sim}{w}^\alpha_{12} = \frac{a}{2} = -(\bar{x}^\beta_{12} - \underaccent{\sim}{w}^\beta_{12}), \quad \frac{dp_{s2}}{dy_0} = \frac{1}{2}, \quad s = 1,2$$

$$\bar{x}^\alpha_{22} - \underaccent{\sim}{w}^\alpha_{22} = \frac{1}{2} = -(\bar{x}^\beta_{22} - \underaccent{\sim}{w}^\beta_{22})$$

and that $\bar{\theta}^\alpha = 1$, $\bar{\theta}^\beta = 0$ are also the *unconstrained* optimal choices of α and β. Thus the expression for the social welfare change (17) becomes

$$\frac{du^\alpha}{\lambda^\alpha_0} + \frac{du^\beta}{\lambda^\beta_0} = -dp_{12}(\bar{x}^\alpha_{12} - \underaccent{\sim}{w}^\alpha_{12})(\bar{\pi}^\alpha_1 - \bar{\pi}^\beta_1) - dp_{22}(\bar{x}^\alpha_{22} - \underaccent{\sim}{w}^\alpha_{22})(\bar{\pi}^\alpha_2 - \bar{\pi}^\beta_2)$$

$$= -dy_0 k\big(a(a-1) - b(a-1)\big) = -k dy_0 \Delta(a,b) \qquad (17')$$

where $k = \frac{1}{4(1+b)(a+b)} > 0$, $\Delta(a,b) = (a-b)(a-1)$. Thus $dy_0 \Delta(a,b) < 0$ leads to a marginal gain in social welfare.

The parameter space $P = \{(a,b) \in R^2_{++} \mid 2b - a > 0\}$ is thus partitioned into four disjoint open sets $\Delta(a,b) \lessgtr 0$ in which equilibrium is constrained inefficient, two of over investment $(a > b, a > 1$ and $a < b, a < 1)$ and two of underinvestment $(a < b, a > 1$ and $a > b, a < 1)$, and two closed sets of measure zero $\Delta(a,b) = 0$ in which equilibrium satisfies the first order conditions for constrained efficiency $(a = b$ and $a = 1)$. When $a = 1$, since $\bar{\pi}^\alpha = \bar{\pi}^\beta$, the equilibrium is a Pareto optimum.

28

How do we explain the welfare improving change in investment dy_0 that a planner can undertake for a given economy $(a, b) \in P$? A marginal change in investment always helps one agent and hurts the other. Thus determining the sign of dy_0 which leads to a welfare improvement amounts to determining which agent stands to obtain the largest net gain from a change in investment. Reducing (increasing) investment raises (lowers) the price of good 1 relative to good 2 in each state and it is good 1 that the entrepreneur α sells (worker β buys): thus reducing (increasing) investment helps the entrepreneur (worker). If $a \neq 1$ the two terms in $(17')$ have the opposite sign. Thus if the gain to α exceeds the loss to β in state 1 then the converse is true in state 2. The sign of $a - b$ determines which of these two terms dominates and hence which agent should be helped. Thus if $a > 1$ then the gain to α exceeds the loss to β in state 1 and if $a > b$ $(a < b)$ this term dominates (is dominated by) the net gain to β in state 2. Thus social welfare is improved by helping the entrepreneur (the worker), namely by reducing (increasing) investment.

6. PROOFS

6.1 Proof of Theorem 1.

Following Debreu (1952) we exhibit a shareholder equilibrium as an equilibrium of a game in which consumers and producers choose actions $(\bar{x}, \bar{\theta}, \bar{y})$ and "market players" choose prices $(\bar{p}, \bar{q}, \bar{\pi})$. The n-player game is defined by a set of triples for each player

$$\{X^i, \; A^i, \; U^i, \quad i = 1, \ldots, n\} \tag{a}$$

consisting of a *choice set* X^i, a *constraint correspondence* A^i and a *pay-off function* U^i, $i = 1, \ldots, n$ where $X^i \subset R^m$, $X^i \neq \emptyset$ (with $X = \prod_{i=1}^{m} X^i$), $A^i : X \longrightarrow X^i$ is continuous, convex valued, $A^i(\xi) \neq \emptyset$ for all $\xi \in X$, $U^i : X \longrightarrow R$, $U^i(\xi) = U^i(\xi^1, \ldots, \xi^i, \ldots, \xi^m)$ is continuous and quasi-concave in ξ^i. An action $\bar{\xi} = (\bar{\xi}^1, \ldots, \bar{\xi}^n)$ is an equilibrium if

$$\bar{\xi}^i \in \arg\max_{\xi^i \in A^i(\bar{\xi})} U^i(\bar{\xi}^1, \ldots, \bar{\xi}^{i-1}, \xi^i, \bar{\xi}^{i+1}, \ldots, \bar{\xi}^m), \quad i = 1, \ldots, n \tag{b}$$

It is convenient to replace π by $\lambda = (\lambda^1, \ldots, \lambda^I)$. We thus let $\xi = (x, \theta, y, p, q, \lambda)$. Consider (a) and (b) for consumers, producers and market players respectively.

Step 1 Consumers and Producers. By assumption B (1)(ii) there exists $M > 0$ such that the truncated commodity choice sets of consumers and producers

$$\hat{C}^i = \{x^i \in R_+^N \mid x_{s\ell}^i \leqq M, \quad s = 1, \ldots, S, \quad \ell = 1, \ldots, L\} \quad i = 1, \ldots, I$$

$$\hat{Y}^j = \{y^j \in Y^j \mid -M \leqq y_{s\ell}^j \leqq M, \quad s = 1, \ldots, S, \quad \ell = 1, \ldots, L\} \quad j = 1, \ldots, J$$

satisfy $\hat{C}^i \subset \mathrm{proj}_{C^i} \mathcal{F}$, $i = 1, \ldots, I$, $\hat{Y}^j \subset \mathrm{proj}_{Y^j} \mathcal{F}$, $j = 1, \ldots, J$ where

$$\mathcal{F} = \left\{ (x, y) \in \prod_{i=1}^I C^i \times \prod_{j=1}^J Y^j \,\Big|\, \sum_{i=1}^I (x^i - w^i) - \sum_{j=1}^J (y^j + \eta^j) \leqq 0 \right\}$$

Let $\mu = \min\{w_{s\ell}^i, \ s = 1, \ldots, S, \ \ell = 1, \ldots, L, \ i = 1, \ldots, I\}$, $\epsilon < \dfrac{\mu}{MJ}$. To avoid $\sum_{i=1}^I \bar{\theta}^{ij} = 0$ we constrain $\theta^{ij} \geqq \epsilon \delta^{ij} \ \forall \ i, j$ and let $\epsilon \longrightarrow 0$. Let $[\epsilon \delta^i, e] \subset [0, 1]^J$ then the choice set of consumer i is $\hat{C}^i \times [\epsilon \delta^i, e]$. Let

$$B^i(\xi) = \left\{ (x^i, \theta^i) \in \hat{C}^i \times [\epsilon \delta^i, e] \,\big|\, \lambda^i \cdot p \,\square\, \left(x^i - w^i - (y + \eta)\theta^i\right) + \lambda_0^i q(\theta^i - \delta^i) \leqq 0 \right\} \quad i = 1, \ldots, I$$

Let $F^i(\bar{\xi}) = \underset{x^i \in B^i(\xi)}{\arg \max} \ u^i(x^i) \quad i = 1, \ldots, I$. Note that $\epsilon < \dfrac{\mu}{MJ}$ implies $B^i(\xi) \neq \emptyset$ for all $\xi \in X$ defined by (c) below. Let

$$G^j(\bar{\xi}) = \underset{y^j \in \hat{Y}^j}{\arg \max} \ \beta^j(\bar{\theta}, \bar{\lambda}) \cdot (\bar{p} \,\square\, y^j) \quad j = 1, \ldots, J$$

where $\beta_0^j(\theta, \lambda) = \left(\sum_{i=1}^I \bar{\theta}^{ij} \right) \left(\prod_{i=1}^I \lambda_0^i \right)$, $\quad \beta_s^j(\theta, \lambda) = \sum_{i=1}^I \theta^{ij} \lambda_s^i \left(\prod_{i' \neq i} \lambda_0^{i'} \right)$.

Step 2 Market Players. Since the original budget set B^i of each consumer is homogeneous in (p_0, q) and p_s, $s = 1, \ldots, S$ we may choose $(p_0, q) \in \Delta^{L+J-1}$ and $p_s \in \Delta^{L-1}$, $s = 1, \ldots, S$ where $\Delta^{k-1} = \{x \in R_+^k \mid \sum_{i=1}^k x_i = 1\}$ is the standard simplex. Similarly we may choose $\lambda^i \in \Delta^S$, $i = 1, \ldots, I$. The choice of prices by the market players is made as follows.

$$P_s(\bar{\xi}) = \underset{p_s \in \Delta^{L-1}}{\arg \max} \ p_s \sum_{i=1}^I \left(x_s^i - w_s^i - \sum_{j=1}^J \bar{\theta}^{ij} \bar{y}_s^j \right), \quad s = 1, \ldots, S$$

$$P_0(\xi) = \underset{(p_0,q)\in\Delta^{L+J-1}}{\arg\max} \; p_0 \sum_{i=1}^{I}\left(\bar{x}_0^i - w_0^i - \sum_{j=1}^{J}\bar{\theta}^{ij}\bar{y}_0^j\right) + \sum_{j=1}^{J}q_j\left(\sum_{i=1}^{I}\bar{\theta}^{ij}-1\right)$$

$$\Lambda^i(\bar{\xi}) = \underset{\lambda^i\in\Delta^S}{\arg\max}\,\lambda^i\cdot\bar{p}\,\square\left(\bar{x}^i - w^i - (\bar{y}+\eta)\bar{\theta}^i\right) + \lambda_0^i q(\bar{\theta}^i-\delta^i)\quad i=1,\dots,I$$

$$\text{Let}\,X = \prod_{i=1}^{I}(\hat{C}^i\times[\epsilon\delta^i,e])\times\prod_{j=1}^{J}\hat{Y}^j\times\Delta^{(L-1)S}\times\Delta^{S+J-1}\times\Delta^{SI} \tag{c}$$

and let $\psi : X \longrightarrow X$ denote the correspondence

$$\psi(\xi) = \big(F(\xi),\ G(\xi),\ P(\xi),\ \Lambda(\xi)\big),\quad \xi\in X$$

where $F = (F^1,\dots,F^I)$, $G = (G^1,\dots,G^J)$, $P = (P_0,P_1,\dots,P_S)$, $\Lambda = (\Lambda^1,\dots,\Lambda^I)$. Since $X\neq\emptyset$ is a compact convex subset of R^m and $\psi : X \longrightarrow X$ is an upper semicontinuous, convex valued correspondence such that $\psi(\xi)\neq\emptyset$ for all $\xi\in X$, the conditions of Kakutani's Fixed Point theorem hold [Berge (1963), p. 174]. *Thus ψ has a fixed point.*

Step 3. A fixed point $\bar{\xi}\in\psi(\bar{\xi})$ is a shareholder equilibrium. First check that $\bar{\xi}$ is an equilibrium for the truncated economy $\mathcal{E}_{(M,\epsilon)}$. To obtain an equilibrium for the original economy $\mathcal{E}$ let $M\longrightarrow\infty$ and $\epsilon\longrightarrow 0$. If $\big((x^*,\theta^*,y^*),(p_0^*,p_1^*,q^*,\lambda^*)\big)$ is a limit of a convergent subsequence of equilibria for $\mathcal{E}_{(M,\epsilon)}$ then a standard argument proves that this is an equilibrium for $\mathcal{E}$. It suffices to verify that each consumer i has a positive income $\lambda^{*i}(p^*\square w^i) + \lambda_0^{*i}q^*\delta^i > 0\; i=1,\dots,I.$ ∎

6.2 Proof of Theorem 2.

Step 1. We begin by introducing a procedure for handling the no short sales restrictions on portfolios. We consider an *artificially restricted* portfolio choice θ_A^i arising when agent i is not permitted to invest in a subset of firms indicated by A and is unconstrained in his investment in the remaining firms. More precisely let $\mathcal{A}$ denote the set of all subsets of pairs $(i,j)\in\{1,\dots,I\}\times\{1,\dots,J\}$ (including the empty set $\emptyset$) such that $A\in\mathcal{A}$ implies that for each $j\in\{1,\dots,J\}$ there exists $i\in\{1,\dots,I\}$ such that $(i,j)\notin A$ (for each firm j, some consumer i must be allowed to invest in firm j). The set $\{j|(i,j)\in A\}$ is the set

of firms that consumer i may not invest in: A thus consists of a list of the firms that each consumer $i = 1, \ldots, I$ is forbidden to invest in. Let $A \longrightarrow \theta_A^i$ denote a map defined on A with values in R^J such that $\theta^{ij} = 0$ if $(i,j) \in A$. Let $\theta_A^i > 0$ mean $\theta^{ij} > 0$ for all $(i,j) \notin A$. We place no short sales restrictions on the portfolios θ_A^i other than those implied by A. Define the *A-restricted budget set* for consumer i by

$$B_A^i(p,q,y;\omega,\eta) = \left\{ x \in R_+^N \;\middle|\; \begin{array}{c} \text{there exists } \theta_A^i \in R^J \text{ such that} \\ \begin{bmatrix} p_0(x_0 - w_0^i) - q\delta^i \\ p_1 \circ (x_1 - w_1^i) \end{bmatrix} = W(p,q;y+\eta)\theta_A^i \end{array} \right\}$$

DEFINITION 9: An A-equilibrium for the economy $\mathcal{E}\big((u^i, Y^j), \omega, \eta\big)$, for $A \in \mathcal{A}$, is a pair of actions and prices $\big((\bar{x}_A, \bar{\theta}_A, \bar{y}_A), (\bar{p}, \bar{q}, \bar{\pi}_A)\big)$ such that (i) $(\bar{x}_A^i, \bar{\theta}_A^i, \bar{\pi}_A^i), i = 1, \ldots, I$ satisfy (i) in definition 4 with the budget set $B^i(\bar{p}, \bar{q}, \bar{y}; \omega, \eta)$ replaced by the A-restricted budget set $B_A^i(\bar{p}, \bar{q}, \bar{y}; \omega, \eta)$, (ii) $\bar{y}_A^j = \arg \max_{y^j \in Y^j} \bar{\beta}_A^j \cdot (\bar{p} \circ y^j)$, $j = 1, \ldots, J$ with $\bar{\beta}_A^j = \sum\limits_{i=1}^{I} \bar{\theta}_A^{ij} \bar{\pi}_A^i$, (iii) $\sum\limits_{i=1}^{I} (\bar{x}_A^i - w^i) = \sum\limits_{j=1}^{J} (\bar{y}_A^j + \eta^j)$, (iv) $\sum\limits_{j=1}^{I} \bar{\theta}_A^{ij} = 1$, $j = 1, \ldots, J$. An A-equilibrium is a *positive (non-negative)* A-equilibrium if $\theta_A^i > 0$ $(\theta_A^i \geq 0)$, $i = 1, \ldots, I$.

REMARK 1: If $\xi = \big((\bar{x}, \bar{\theta}, \bar{y}), (\bar{p}, \bar{q}, \bar{\pi})\big)$ is an equilibrium then there exists $A \in \mathcal{A}$ such that ξ is a positive A-equilibrium. It suffices to let $A = \{(i,j) \mid \bar{\theta}^{ij} = 0\}$. The equilibria for which the constraints $\theta^{ij} \geq 0$ begin to be binding for some (i,j) are equilibria ξ for which there exist A and $\tilde{A}$ with $A \subsetneq \tilde{A}$ such that ξ is a non-negative A-equilibrium and a positive $\tilde{A}$-equilibrium. If we let ρ^{ij} denote the multiplier for the constraint $\theta^{ij} \geq 0$ then $A = \{(i,j) \mid \bar{\theta}^{ij} = 0, \; \bar{\rho}^{ij} > 0\}$, $\tilde{A} = \{(i,j) \mid \bar{\theta}^{ij} = 0\}$.

Step 2. For any compact set $K \subset R_{++}^{N(I+J)}$ the set of non-negative A-equilibria

$$E_A(K) = \{\xi_A = (x_A, \theta_A, y_A, p_A, q_A, \pi_A) \mid \xi_A \text{ is a non-negative } A\text{-equilibrium for } (\omega, \eta) \in K\}$$

is bounded. The boundedness of the actions (x_A, θ_A, y_A) follows from assumption $B_{1(ii)}$ and the no-short sales restriction on portfolios. We assume that spot prices are normalised

within each state so that the price of the first good is 1. Thus if $p = (p_s)_{s=0}^{S}$ then $p_{s1} = 1$, $s = 0, \ldots, S$. A standard argument based on the monotonicity of the utility function shows that the A-equilibrium spot prices p_A are bounded. The fact that each present value coefficient π^i is bounded follows from the fact that $\pi_s^i = \frac{(\partial u^i / \partial x_{s1}^i)}{(\partial u^i / \partial x_{01}^i)}$, $s = 0, \ldots, S$ is bounded, since $u^i(x^i) \geq \min_{(\omega, \eta) \in K} u^i(w^i)$ and since x^i is bounded above. Since for each firm j there is some agent i with $\theta^{ij} > 0$ and since $q^j = \sum_{s=0}^{S} \pi_s^i p_s (y_s^j + \eta_s^j)$, q^j is bounded.

Step 3. In step 1 we have fixed $\theta^{ij} = 0$ for all $(i,j) \in A$ and allowed agents to choose the remaining portfolio components freely, $\theta^{ij} \in (-\infty, \infty)$ for $(i,j) \notin A$. Thus $\beta_A^j = \sum_{i=1}^{I} \theta_A^{ij} \pi_A^i$ can have negative components. In order to ensure that the supply function of each firm j remains well-defined and smooth we *compactify* the production set Y^j as follows. Let $U \subset R_{++}^{N(I+J)}$ be a bounded open subset and let $\bar{U}$ denote its closure. The set of attainable production plans given $\bar{U}$ is defined by

$$Y_{\bar{U}} = \left\{ y \in \prod_{j=1}^{J} Y^j \;\middle|\; \begin{array}{c} (\omega, \eta) \in \bar{U} \\ \left(\sum_{j=1}^{J} (y^j + \eta^j) + \sum_{i=1}^{I} w^i \right) \bigcap R_+^N \neq \emptyset \end{array} \right\}$$

Let $C^j = \pi_{Y^j} Y_{\bar{U}}$ where π_{Y^j} denotes the projection onto Y_j and let $\tilde{Y}^j$ be a compact convex set such that $\tilde{Y}^j \supset C^j$, the boundary $\partial \tilde{Y}^j$ is a $k_j - 1$ dimensional C^2 manifold with strictly positive Gaussian curvature at each point and $\partial \tilde{Y}^j \subset \partial Y^j \bigcap C^j$ (*i.e.*, the efficiency frontier of $\tilde{Y}^j$ coincides with the efficiency frontier of the truncation C^j).

Step 4. We now show how to handle the problem of degeneracies or changes in the rank of the matrix of security returns W. We employ a technique similar to that introduced by Magill-Shafer (1985) which consists in writing out equations characterising equilibria of all possible ranks $\rho \leq J$ and showing that with the assumption $I + J \leq S + 1$ equilibria of rank $\rho < J$ are exceptional. The first step is as follows. Consider an A-equilibrium in which the rank of W is ρ. Let $\mathcal{P}$ denote the set of permutations of $\{1, \ldots, J\}$, then there exists a permutation $\sigma \in \mathcal{P}$ (a relabelling of the firms) such that if W_σ denotes the matrix

33

obtained from W by permuting the columns according to the permutation σ, then the first ρ columns $[W^1_\sigma, \ldots, W^\rho_\sigma]$ of W_σ are linearly independent. Thus there exists a $\rho \times J - \rho$ matrix $E = (E_{kj}, \; k = 1, \ldots, \rho, \; j = 1, \ldots, J - \rho)$ such that

$$W^{\rho+j}_\sigma = \sum_{k=1}^{\rho} E_{kj} W^k_\sigma, \quad j = 1, \ldots, J - \rho$$

Since the return on each of the last $J - \rho$ firms of the permutation σ is a linear combination of the returns of the first ρ firms, the choice of a portfolio θ^i_A which gives rise to the vector of returns across the states

$$\sum_{k=1}^{\rho} \theta^{ik}_{A,\sigma} W^k_\sigma + \sum_{j=1}^{J-\rho} \theta^{i(\rho+j)}_{A,\sigma} \left(\sum_{k=1}^{\rho} E_{kj} W^k_\sigma \right), \quad \theta^{ik}_{A,\sigma} = \theta^{i\sigma(k)}_A$$

is equivalent to the choice of a portfolio $\gamma^i_{A,\rho,\sigma} = (\gamma^{i,k}_{A,\rho,\sigma})^\rho_{k=1}$ in the first ρ firms of the permutation σ

$$\gamma^{ik}_{A,\rho,\sigma} = \theta^{ik}_{A,\sigma} + \sum_{j=1}^{J-\rho} \theta^{i(\rho+j)}_{A,\sigma} E_{kj}, \quad k = 1, \ldots, \rho$$

Of course if we let $M^i_{A,\sigma} = \{\theta^i \in R^J \mid \theta^{ij} = 0 \text{ if } (i, \sigma(j)) \in A\}$ denote the portfolio space of agent i implied by A, then the transformed portfolio $\gamma^i_{A,\rho,\sigma}$ must lie in the image of the original portfolio set, $\gamma^i_{A,\rho,\sigma} \in [I_\rho \mid E]M^i_{A,\sigma}$ where I_ρ is a $\rho \times \rho$ identity matrix. When W has rank $\rho < J$ while there are infinitely many θ^i_A that give rise to a given revenue stream $m^i = (m^i_s)^S_{s=0}$, there is a *unique* $\gamma^I_{A,\rho,\sigma}$ that generates m^i. Thus agent i's optimal portfolio is well-defined. Note that $\gamma^{ik}_{A,\rho,\sigma} \equiv 0$ is equivalent to $(i, \sigma(k)) \in A$ and $(i, \sigma(\rho + j)) \in A$, $j = 1, \ldots, J - \rho$. We are thus led to introduce the (A, ρ, σ)-*restricted budget set* for consumer i.

$$B^i_{A,\rho,\sigma}(p, q, y, E; \omega, \eta) = \left\{ x \in R^N_+ \;\middle|\; \begin{array}{l} \text{there exists } \gamma^i_{A,\rho,\sigma} \in [I_\rho \mid E]M^i_{A,\sigma} \text{ such that} \\[6pt] \begin{bmatrix} p_0(x_0 - w^i_0) - q\delta^i \\ p_1 \square (x_1 - w^i_1) \end{bmatrix} = [W^1_\sigma \cdots W^\rho_\sigma]\gamma^i_{A,\rho,\sigma} \end{array} \right\}$$

and for each $A \in \mathcal{A}$, $1 \le \rho \le J$, $\sigma \in P$ the open sets

$$\Delta_{A,\rho,\sigma} = \{(p, q, y, \theta_A, E, \omega, \eta) \in \Gamma \times R^J_{++} \times R^{NJ} \times R^{IJ - \#A} \times R^{\rho(J-\rho)} \times U \mid$$

$$\text{rank}[W^1_\sigma, \ldots, W^\rho_\sigma] = \rho\}$$

34

where $\Gamma = (\Delta_{++}^{L-1})^{S+1}$, $\Delta_{++}^{L-1} = \{(v_1, \ldots, v_L) \in R_{++}^L \mid v_1 = 1\}$. We let

$$(f_{A,\rho,\sigma}^i, \tilde{\gamma}_{A,\rho,\sigma}^i, \tilde{\pi}_{A,\rho,\sigma}^i) : \Lambda_{A,\rho,\sigma} \longrightarrow R^N \times R^\rho \times \Delta_{++}^S \quad i = 1, \ldots, I$$

denote consumers i's solution to maximising utility u^i over the budget set $B_{A,\rho,\sigma}^i$. Similarly we define the *supply function* of firm j

$$\tilde{g}^j : \Gamma \times \Delta_{++}^S \longrightarrow R^N \text{ by } \tilde{g}^j(p, \beta^j) = \arg\max_{y^j \in \tilde{Y}^j} \beta^j \cdot (p \square y^j), \quad j = 1, \ldots, J$$

LEMMA 1: *The decision functions* $(f_{A,\rho,\sigma}^i, \tilde{\gamma}_{A,\rho,\sigma}^i, \tilde{\pi}_{A,\rho,\sigma}^i) : \Lambda_{A,\rho,\sigma} \longrightarrow R^N \times R^\rho \times \Delta_{++}^S$, $i = 1, \ldots, I$ *and the supply functions* $\tilde{g}^j : \Gamma \times \Delta_{++}^s \longrightarrow R^N$, $j = 1, \ldots, J$ *are* C^1 *functions.*

PROOF: Use the fact that $\mathrm{rank}[W_\sigma^1 \ldots W_\sigma^\rho] = \rho$ and adapt the argument in Geanakoplos-Polemarchakis (1984, section 3) for the demand functions. Use Mas-Colell (1985, p. 106) to establish smoothness of the supply functions. Note that it is at this point that the assumption of strictly positive Gaussian curvature of the indifference curves and the boundaries of the production sets enters. ∎

We are now in a position to write out the equations characterising rank ρ equilibria. Since Walras' Law holds for each state we can eliminate the market clearing equation of good 1 in each state. To this end we introduce the following notation: if $x \in R^{L(S+1)}$, $\hat{x} = (x_{sl}, \ell \geq 2, s = 0, 1, \ldots, S) \in R^{(L-1)(S+1)}$ denotes the truncation of x. Let $z = (p, q, y, \theta_A, E, \omega, \eta)$ denote a typical element of $\Lambda_{A,\rho,\sigma}$. Then an A-restricted rank ρ equilibrium with permutation of the firms σ must satisfy the equations

$$\sum_{i=1}^{I} (\hat{f}_{A,\rho,\sigma}^i(z) - \hat{w}^i) - \sum_{j=1}^{J} (\hat{y}^j + \hat{\eta}^j) = 0 \tag{i}$$

$$\tilde{g}^j \left(p, \sum_{i=1}^{I} \theta_{A,\sigma}^{ij} \tilde{\pi}_{A,\rho,\sigma}^i(z) \right) - y^j = 0, \quad j = 1, \ldots, J \tag{ii}$$

$$\tilde{\gamma}_{A,\rho,\sigma}^i(z) - [I_\rho \mid E]\theta_{A,\sigma}^i = 0, \quad i = 1, \ldots, I \tag{iii} \tag{18}$$

$$W_\sigma^{\rho+j} - \sum_{k=1}^{\rho} E_{kj} W_\sigma^j = 0, \quad j = 1, \ldots, J - \rho \tag{iv}$$

35

$$\sum_{i=1}^{I} \theta^i_{A,\sigma} - e = 0 \qquad\qquad (v)$$

where $e = (1,\ldots,1) \in R^J$. We may write the system of equations (18) as

$$F_{A,\rho,\sigma}(z) = 0, \quad A \in \mathcal{A}, \; 1 \le \rho \le J, \; \sigma \in \mathcal{P} \qquad\qquad (18')$$

where $F_{A,\rho,\sigma} : \Lambda_{A,\rho,\sigma} \longrightarrow \mathcal{M}$ with $\mathcal{M} = R^{(L-1)(S+1)} \times R^{NJ} \times R^{I\rho-\alpha} \times R^{(S+1)(J-\rho)} \times R^J$.
In view of lemma 1, $F_{A,\rho,\sigma}$ is a C^1 function on $\Lambda_{A,\rho,\sigma}$. In (18)(iii) it is assumed that if the jth equation reduces to $\tilde{\gamma}^i_{A,\rho,\sigma} \equiv 0$ then it is eliminated and we let α denote the number of equations eliminated in this way. Clearly $\alpha \le \#A$ and when $\rho = J$, $\alpha = \#A$.

Step 5. $F_{A,\rho,\sigma} \pitchfork 0$, for all $A \in \mathcal{A}$, $1 \le \rho \le J$, $\sigma \in \mathcal{P}$, in other words the linear transformation between tangent spaces $D_z F_{A,\rho,\sigma} : T_z \Lambda_{A,\rho,\sigma} \longrightarrow T_{F_{A,\rho,\sigma}(z)}\mathcal{M} = \mathcal{M}$ is surjective $\forall z \in F^{-1}_{A,\rho,\sigma}(0)$.

Let $m = \dim \mathcal{M}, e^j = (e^j_i)^m_{i=1}$ with $e^j_i = 0$, $i \ne j$, $e^j_j = 1$, $j = 1,\ldots,m$ denote the standard basis for $\mathcal{M} = R^m$. It suffices to show that for each e^j, $j = 1,\ldots,m$ there exists $dz^j \in T_z \Lambda_{A,\rho,\sigma}$ such that $(D_z F_{A,\rho,\sigma})dz^j = e^j$, $j = 1,\ldots,m$. Consider in turn each of the equations (18)(i)-(18)(v). (i) The equations in (18)(i) are indexed by the goods (s,ℓ), $s = 0, 1, \ldots, S$, $\ell = 2, \ldots, L$. To obtain $e^{(s,\ell)}$, let $dw^1_{s1} = p_{s\ell}$, $dw^1_{s\ell} = -1$ with all other components zero, so that $dz^{(s,\ell)} = (0,\ldots,dw^1_{s1},\ldots,dw^1_{s\ell},\ldots,0) = (0,\ldots,p_{s\ell},\ldots,-1,\ldots,0)$. By reducing the endowment of good (s,ℓ) of agent 1 by one unit and simultaneously compensating him with an amount $dw^1_{s1} = p_{s\ell}$ of good 1 in state s, so that his income and hence his demand is unchanged, the excess demand for good (s,ℓ) is increased by one unit and no other equation is affected. (ii) The equations in (18)(ii) are indexed by firms and goods, (j,s,ℓ), $j = 1,\ldots,J$, $s = 0,\ldots,s$, $\ell = 1,\ldots,L$. To obtain the change with 1 for (j,s,ℓ) let $dy^j_{s\ell} = -1$, $d\eta^j_{s\ell} = 1$, so that the profits $p(y^j + \eta^j)$ of firm j are unchanged and no other equation is affected. (iii) The α components for which $\tilde{\gamma}^i_{A,\rho,\sigma} \equiv 0$ have been removed. Component k of every other $\tilde{\gamma}^i_{A,\rho,\sigma}$ equation, can be increased by ϵ by altering

the initial resources of agent i in the following way:

$$dw_{01}^i = \epsilon(q_k - p_0(y_0^k + \eta_0^k)), \quad dw_{s1}^i = -\epsilon p_s(y_s^k + \eta_s^k), \quad s = 1, \ldots, S \qquad (19)$$

These changes induce agent i to increase $\tilde{\gamma}_{A,\rho,\sigma}^i$ by ϵ without changing his consumption. The first order conditions are still satisfied so that $d\hat{f}_{A,\rho,\sigma}^i = 0$ and no other equation is affected. (iv) Component s of $W^{\rho+j}$ can be changed by 1 unit by setting $p_{s1} d\eta_{s1}^{\rho+j} = 1$, without affecting any other equation. (v) To change equation k in (18)(v) by ϵ unit pick $(i,k) \notin A$ and set $d\theta_A^{ik} = \epsilon$. Equation (18)(iii) will not be affected if agent i can be induced to a change $d\tilde{\gamma}_{a,\rho,\sigma}^i$ in $\tilde{\gamma}_{A,\rho,\sigma}^i$, such that $d\tilde{\gamma}_{A,\rho,\sigma}^i = [I_\rho \mid E]d\theta_A^i$. If $k \le \rho$, the change in initial endowment (19) induces a change $d\tilde{\gamma}_{A,\rho,\sigma}^{ik} = \epsilon$. If $k > \rho$, $k = \rho + j$, agent i must be induced to change $\tilde{\gamma}_{A,\rho,\sigma}^{ik'}$, $1 \le k' \le \rho$, by $E_{k'j}\epsilon$. The change in initial endowments

$$dw_{01}^i = \epsilon\Big(\sum_{k'=1}^{\rho} E_{k'j}(q^{k'} - p_0(y_0^{k'} + \eta_0^{k'}))\Big) \quad dw_{s1}^i = -\epsilon\Big(\sum_{k'=1}^{\rho} E_{k'j}p_s(y_s^{k'} + \eta_s^{k'})\Big) \quad s = 1, \ldots, S$$

will induce such a change. The change $d\theta_A^i$ induces a change in (18)(ii) which is compensated by a change in output $dy^j = d\tilde{g}^j$. This will not affect the demand function of the agents if it is compensated by a change $d\eta^j$ such that $dy^j + d\eta^j = 0$. Thus $D_z F_{A,\rho,\sigma}$ is surjective.

Step 6. We show that for each $A \in \mathcal{A}$, $1 \le \rho < J$, $\sigma \in \mathcal{P}$, there exists a set of measure zero $N_{A,\rho,\sigma}$ in U (the bounded open subset of $R_{++}^{N(I+J)}$ introduced in step 3) such that for all $(\omega, \eta) \in U \backslash N_{A,\rho,\sigma}$ (18′) has no solution. Since $F_{A,\rho,\sigma} \pitchfork 0$ it follows from the Preimage theorem [Guillemin-Pollack (1974, p. 21)] that $F_{A,\rho,\sigma}^{-1}(0)$ is a C^1 manifold and

$$\dim F_{A,\rho,\sigma}^{-1}(0) = \dim \Lambda_{A,\rho,\sigma} - \dim \mathcal{M} = N(I+J) + (J-\rho)[I + \rho - (S+1)] - (\#A - \alpha)$$

Since by assumption $I + J \le S + 1$ and $\rho \le J - 1$ it follows that $I + \rho - (S+1) < 0$. Since $\#A \ge \alpha$, $\dim F_{A,\rho,\sigma}^{-1}(0) < N(I+J)$. Consider the projection $\phi : F_{A,\rho,\sigma}^{-1}(0) \longrightarrow U$ defined by $\phi(p,q,y,\theta_A,E,\omega,\eta) = (\omega,\eta)$. By Sard's theorem the set $N_{A,\rho,\sigma}$ of critical values of σ is a set of measure zero in U. Thus if $(\omega,\eta) \in U \backslash N_{A,\rho,\sigma}$ since $\dim F_{A,\rho,\sigma}^{-1}(0) <$

$\dim U$, $\phi^{-1}(\omega, \eta) = \emptyset$, so that $(18')$ has no solution. Let $\tilde{N}_{A,\rho,\sigma} \subset N_{A,\rho,\sigma}$ denote the subset for which there exists a *non-negative* A-equilibrium of rank ρ with permuation σ. We can now assemble the subset of parameter values (ω, η) in U for which *degenerate* rank non-negative A-equilibria can arise by defining

$$N' = \bigcup_{\substack{A \in \mathcal{A} \\ 1 \leq \rho < J \\ \sigma \in \mathcal{P}}} \bar{N}_{A,\rho,\sigma}$$

N' is a subset of U of measure zero. In addition N' *is closed*. To show this pick a sequence $\{(\omega^n, \eta^n)\}_{n=1}^{\infty} \subset N'$ with $(\omega^n, \eta^n) \longrightarrow (\bar{\omega}, \bar{\eta})$. For each (ω^n, η^n) there exists a non-negative (A^n, ρ^n, σ^n)-equilibrium $\xi_{A^n} = (x_{A^n}, \theta_{A^n}, y_{A^n}, p_{A^n}, q_{A^n}, \pi_{A^n})$. Since $(A^n, \rho^n, \bar{\sigma}^n) \in (\mathcal{A}, \{1, \ldots, J-1\}, \mathcal{P})$, which is a finite set, we can assume without loss of generality that $(A^n, \rho^n, \sigma^n) = (A, \rho, \sigma) \; \forall \, n$. Since $\{(\omega^n, \eta^n)\}_{n=1}^{\infty} \subset U$ by step 2 $\{\xi_A^n\}_{n=1}^{\infty}$ is bounded. Thus there exists a subsequence $\{\xi_A^m\}_{m=1}^{\infty}$ such that $\xi_A^m \longrightarrow \bar{\xi}_A$. It is easy to check that $\bar{\xi}_A$ is a non-negative A-equilibrium. Since rank $W(p_A^n, q_A^n, y_A^n + \eta_A^n) = \rho \; \forall \, n$ implies that rank $W(\bar{p}_A, \bar{q}_A, \bar{y}_A + \bar{\eta}_A) = \bar{\rho} \leq \rho$ there exists $\bar{\sigma} \in \mathcal{P}$ such that $\bar{\xi}_A$ is a non-negative $(A, \bar{\rho}, \bar{\sigma})$-equilibrium. Thus $(\bar{\omega}, \bar{\eta}) \in \tilde{N}_{A,\bar{\rho},\bar{\sigma}} \subset N'$.

Step 7. We assemble the parameter values $(\omega, \eta) \in U' = U \backslash N'$ for which equilibrium prices are such that some agent's portfolio θ^i is not a differentiable function of the prices. For each $A \in \mathcal{A}$ consider the open set

$$\Lambda_A' = \{(p, q, y, \omega, \eta) \in \Gamma \times R^J \times R^{JN} \times U' \mid \text{rank } W = J\}$$

Let $(f_A^i, \tilde{\theta}_A^i, \tilde{\pi}_A^i) : \Lambda_A' \longrightarrow R^N \times R^J \times \Delta_{++}^S$ denote consumer i's solution to maximising utility u^i over the budget set $B_A^i = B_{A,J}^i$ (the full rank budget set). Let $z = (p, q, y, \omega, \eta)$ then it follows from (18) that the equations for full-rank A-equilibria are given by

$$\sum_{i=1}^{I} (\hat{f}_A^i(z) - \hat{w}^i) - \sum_{j=1}^{J} (\hat{y}^j + \hat{\eta}^j) = 0 \tag{i}$$

$$\tilde{g}^j\left(p, \sum_{i=1}^{I} \tilde{\theta}_A^{ij}(z)\tilde{\pi}_A^i(z)\right) - y^j = 0 \quad j = 1, \ldots, J \qquad \text{(ii)} \quad (20)$$

$$\sum_{i=1}^{I} \tilde{\theta}_A^i(z) - e = 0 \qquad \text{(iii)}$$

Equations (iii)-(v) in (18) reduce to the single equation (iii) in (20) since no column of the matrix W needs to be expressed as a linear combination of linearly independent columns as in (18). (20) can be written as

$$F_A(z) = 0 \qquad (20')$$

where $F_A : \Lambda_A' \longrightarrow \mathcal{M} = R^{(L-1)(S+1)} \times R^{NJ} \times R^J$. For each $A \in \mathcal{A}$ let A' be a set of pairs (i,j) such that $A' \bigcup A \in \mathcal{A}$, $A' \bigcap A = \emptyset$ and $A' \neq \emptyset$. Since every full-rank equilibrium is a positive A-equilibrium for some $A \in \mathcal{A}$ we can find equilibria at which some agent's portfolio is not differentiable by solving the following system of equations

$$H_{A,A'}(z) = \left(F_A(z), \left(\tilde{\theta}_A^{ij}(z), (i,j) \in A'\right)\right) = 0 \qquad (21)$$

where $H_{A,A'} : \Lambda_A' \longrightarrow \mathcal{M} \times R^{\#A'}$. Let us show that $H_{A,A'} \pitchfork 0$. It suffices to show that we can control the second component of $H_{A,A'}$ without affecting F_A. To induce the change $d\theta_A^{ij} = \epsilon$ it suffices to change the endowment of i by

$$dw_{01}^i = \epsilon\left(q^j - p_0(y_0^j + \eta_0^j)\right), \quad dw_{s1}^i = -\epsilon p_s(y_s^j + \eta_s^j), \quad s = 1, \ldots, S$$

Let i' be an agent such that $\theta_A^{i'j} > 0$. By transfering the goods from agent i to agent i', $dw_{s1}^{i'} = -dw_{s1}^i$ we can induce the change $d\theta_A^{i'j} = -\epsilon$ so that (20)(iii) is re-established. The change in the supply $\tilde{g}^j$ in (20)(ii) is compensated by dy^j which in turn is compensated by $d\eta^j$ so that (20)(i) holds. Thus $H_{A,A'} \pitchfork 0$ so that $H_{A,A'}^{-1}(0)$ is a manifold of dimension $N(I + J) - \#A$. If we consider the projection $\phi : H_{A,A'}^{-1}(0) \longrightarrow U'$ defined by $\phi(p,q,y,\omega,\eta) = (\omega,\eta)$ then by Sard's theorem the set of critical values $N_{A,A'}$ is a set of measure zero and since $\dim H_{A,A'}^{-1}(0) < \dim U'$, $\phi^{-1}(\omega,\eta) = \emptyset \ \forall \ (\omega,\eta) \in U' \backslash N_{A,A'}$. Let

$\tilde{N}_{A,A'}$ denote the subset of $N_{A,A'}$ for which there is a non-negative A-equilibrium satisfying (21). Then $\tilde{N}_{A,A}$ is a closed set of measure zero in U' (the proof of this follows the approach in step 6 above). Thus the set

$$U'' = U' \backslash N'' \quad \text{where} \quad N'' = \bigcup_{\substack{A \in \mathcal{A} \\ A'}} N_{A,A'}$$

is an open set of full measure in U such that if $(\omega, \eta) \in U''$ then neither of the degeneracies (a) or (b) mentioned in section 3 above can occur.

Step 8. We eliminate the final type of degeneracy (c) mentioned in section 3, namely the parameter values (ω, η) for which there exist equilibria which are not regular. Recall the function F_A defined by (20) and (20'). We modify the domain Λ'_A by restricting (ω, η) to the subset U'', thus we define

$$\Lambda''_A = \left\{ (p, q, y, \omega, \eta) \in \Gamma \times R^J \times R^{JN} \times U'' \Big|^{\text{rank } W = J}_{\tilde{\theta}_A > 0} \right\}$$

where $\tilde{\theta}_A > 0$ means $\tilde{\theta}_A^i > 0$, $i = 1, \ldots, I$. Since by step 5 $F_A \pitchfork 0$, $F_A^{-1}(0)$ is a manifold of dimension $N(I+J)$. The projection $\phi : F_A^{-1}(0) \longrightarrow U''$ defined by $\phi(p, q, y, \omega, \eta) = (\omega, \eta)$ is *proper*. To show this we pick a compact set $K \subset U''$ and show that $\phi^{-1}(K)$ is compact. Since by step 2 $\phi^{-1}(K)$ is bounded it remains to show that it is closed. Pick a sequence $\{(p^n, q^n, y^n, \omega^n, \eta^n)\}_{n=1}^\infty \subset \phi^{-1}(K)$ with $(p^n, q^n, y^n, \omega^n, \eta^n) \longrightarrow (\bar{p}, \bar{q}, \bar{y}, \bar{\omega}, \bar{\eta})$. Since K is compact $(\bar{\omega}, \bar{\eta}) \in K \subset U$. Since a limit of non-negative A-equilibria is an A-equilibrium, $(\bar{p}, \bar{q}, \bar{y})$ is a non-negative A-equilibrium for $(\bar{\omega}, \bar{\eta})$. But since $(\bar{\omega}, \bar{\eta}) \in U$, $(\bar{p}, \bar{q}, \bar{y})$ is a full-rank equilibrium and $\theta_A^{ij} > 0$ $(i,j) \notin A$. Thus $(\bar{p}, \bar{q}, \bar{y}, \bar{\omega}.\bar{\eta}) \in F_A^{-1}(0)$ and ϕ is proper.

By Sard's theorem the set N_A of critical values of ϕ is a set of measure zero in U''. Since ϕ is proper N_A is closed. Define the positive A-equilibrium price set (recall that Λ''_A is now the domain of F_A)

$$\Pi_A(\omega, \eta) = \{(p^k, q^k, y^k) | F_A(p^k, q^k, y^k, \omega, \eta) = 0\}, \quad k = 1, \ldots, n$$

Consider the map $F_A(\cdot, \omega, \eta) : \Gamma \times R^J \times R^{JN} \longrightarrow M$. The basic step in the proof of the elementary Transversality theorem [Mas-Colell (1985), p. 231] asserts that $F_A(\cdot, \omega, \eta) \pitchfork 0$ *for the parameter value $(\bar{\omega}, \bar{\eta})$ if and only if $\phi \pitchfork (\bar{\omega}, \bar{\eta})$ where ϕ is the projection $\phi : F_A^{-1}(0) \longrightarrow U''$*. Thus if $(\bar{\omega}, \bar{\eta}) \in U'' \backslash N_A$ and $\Pi_A(\bar{\omega}, \bar{\eta}) \neq \emptyset$ then by the implicit function theorem there exist a neighborhood $N_{(\bar{\omega}, \bar{\eta})}$ and n C^1 functions $\psi^k : N_{(\bar{\omega}, \bar{\eta})} \longrightarrow \Gamma \times R^j \times R^{JN}$, $k = 1, \ldots, n$ such that

$$\Pi_A(\omega, \eta) = \{\psi^k(\omega, \eta), \ k = 1, \ldots, n\}, \quad (\omega, \eta) \in N_{(\bar{\omega}, \bar{\eta})}$$

Let $\Omega_A = U'' \backslash N_A$ and let $\Omega = \bigcap_{A \in \mathcal{A}} \Omega_A$. Since an equilibrium is a positive A-equilibrium for some $A \in \mathcal{A}$, if $(\bar{\omega}, \bar{\eta}) \in \Omega$ there is a finite number of equilibria (each of full rank) and a neighborhood $N_{(\bar{\omega}, \bar{\eta})}$ such that the equilibrium prices (ψ^k) and agents decisions are smooth functions of (ω, η) for all $(\omega, \eta) \in N_{(\bar{\omega}, \bar{\eta})}$. Step 3 began with the bounded open set $U \subset R_{++}^{N(I+J)}$. Since $R_{++}^{N(I+J)}$ can be covered by a countable union (U_α) of such sets, repeating steps 3-8 leads to a family of open sets (Ω_α) whose union gives the required open set. ∎

6.3 Proof of Proposition 3.

We prove the more general property that generically A-restricted equilibria have distinct $\bar{\pi}_A^1, \ldots, \bar{\pi}_A^I$ for each $A \in \mathcal{A}$. Let Λ_A be the domain defined by

$$\Lambda_A = \left\{ z = (p, q, y, \omega, \eta) \in \Gamma \times R^J \times R^{JN} \times \Omega \ \middle| \ \begin{array}{l} \text{rank } W(z) = J \\ \tilde{\theta}_A(z) > 0 \end{array} \right\}$$

where Ω is the open set defined in theorem 2. It suffices to show that for any pair of consumers $i, k \in \{1, \ldots, I\}$, $i \neq k$ generically $\bar{\pi}_A^i \neq \bar{\pi}_A^k$ or that generically the system of equations

$$H_A(z) = (F_A(z), G_A(z)) = 0 \tag{22}$$

has no solution where $F_A : \Lambda_A \longrightarrow M$ is defined by (20), $G_A : \Lambda_A \longrightarrow R^S$ is given by $G_A(z) = \hat{\tilde{\pi}}_A^i(z) - \hat{\tilde{\pi}}_A^k(z)$ and $H_A : \Lambda_A \longrightarrow M \times R^S$.

The idea is as follows. In (20′) for fixed (ω, η) the number of independent equations $(\dim M)$ equals the number of independent unknowns: we want to show that adding the

S equations $G_A = 0$ makes the number of independent equations exceed the number of unknowns. However we cannot show that rank $D_z H_A = \dim M + S$ (*i.e.*, that 0 is regular value of H_A). Since $J < S$ we can show that rank $D_z H_A \geqq \dim M + (S - J)$ and by the following lemma this will be sufficient.

LEMMA 2: *Let X be a manifold and $\phi : X \longrightarrow R^n$ a C^1 function. If rank $D_x \phi \geqq r$ for all $x \in \phi^{-1}(0) = \{x \in X \mid \phi(x) = 0\}$ then there exist submanifolds $M_\alpha \subset X$, $\dim M_\alpha = \dim X - r$, $\alpha = 1, \ldots, m$ such that $\phi^{-1}(0) \subset \bigcup_{\alpha=1}^{m} M_\alpha$.*

PROOF: (see Corollary 1, Magill-Shafer (1985)).

Since rank $V(z) = J$ for all $z \in \Lambda_A$ it follows that

$$\dim V(z)^\perp = S - J, \quad V(z)^\perp = \{\beta_1 \in R^S \mid \beta_1 V(z) = 0\}, \quad \forall z \in \Lambda_A$$

Consider any $\bar{z} \in H_A^{-1}(0)$ we want to show that rank $D_z H_A(\bar{z}) = \dim M + (S - J)$, where $\bar{z} = (\bar{p}, \bar{q}, \bar{y}, \bar{\omega}, \bar{\eta})$. Pick any $d\pi_1 \in V(\bar{z})^\perp$. We show that there exists $dz = (0, 0, 0, dw^i, 0)$ such that $D_z H_A(\bar{z})dz = (0, d\pi_1)$. Let $\bar{\lambda}^i \in R_{++}^{S+1}$ be defined by $D_{x^i} u^i(\bar{x}^i) = \bar{\lambda}^i \Box \bar{p}$ and let $d\lambda^i = \bar{\lambda}_0^i(0, d\pi_1)$. We want to show that there exists $dx^i \in R^N$ such that

$$D_{x^i} u^i(\bar{x}^i + dx^i) = (\bar{\lambda}^i + d\lambda^i) \Box \bar{p} \tag{23}$$

Since by $A_{(2)}$ the indifference surfaces of u^i have strictly positive Gaussian curvature for all $x^i \in R_{++}^N$, it follows from a theorem in [Mas-Colell (1985), p. 80] that we may without loss of generality assume that $D_{x^i}^2 u^i(\bar{x}^i)$ is negative definite. But then there exists dx^i satisfying (23). Let us now note that if we let $dw^i = dx^i$, if $(w^i + dw^i)$ is agent i's endowment then $(\bar{x}^i + dx^i, \bar{\theta}_A^i, \bar{\lambda}^i + d\lambda^i)$ is the solution of his utility maximising problem. This follows at once from the fact that first (23) holds, second the budget equations are satisfied with $\bar{\theta}_A^i$ since $p \Box dx^i = p \Box dw^i$ and third $(\bar{\lambda}^i + d\lambda^i)W^j(\bar{z}) = 0$ for $(i, j) \notin A$, since $d\pi_1 \in V(\bar{z})^\perp$. Since the change dw^i leaves $\hat{\bar{\pi}}_A^k$ and F_A unchanged and since in the proof of theorem 2 we have already shown that rank $D_z F_A = \dim M$, it

follows that rank $D_z H_A(\bar{z}) = \dim \mathcal{M} + (S - J)$. By lemma 2 there exist submanifolds $M_\alpha \subset \Lambda_A$, $\alpha = 1, \ldots, m$ such that $H_A^{-1}(0) \subset \bigcup_{\alpha=1}^m M_\alpha$. Consider the projection $\phi :$ $M_\alpha \longrightarrow R_{++}^{N(I+J)}$, $\phi(p, q, y; \omega, \eta) = (\omega, \eta)$. By Sard's theorem the set Ω_A^α of regular values of ϕ is a set of full measure in $R_{++}^{N(I+J)}$. Since by lemma 2 $\dim M_\alpha - \dim \Omega_A^\alpha = -(S - J)$, $\phi^{-1}(\omega, \eta) = \emptyset \; \forall \; (\omega, \eta) \in \Omega_A$. Let $\Omega_{i,k}' = \bigcap_{A \in \mathcal{A}} \bigcap_{\alpha=1}^m \Omega_A^\alpha$, then $H_A(p, q, y; \omega, \eta) = 0$ has no solution for all $(\omega, \eta) \in \Omega_{i,k}'$, for all $A \in \mathcal{A}$. Thus if we let $\Omega' = \bigcap_{1 \le i < k \le I} \Omega_{i,k}'$ then for every $(\omega, \eta) \in \Omega'$ in each equilibrium the present value coefficients $(\pi^i, \ldots, \pi^I)$ are distinct. ∎

6.4 Proof of Theorem 3.

We show that the equations of equilibrium and the production efficiency equation (ii) of proposition 4 are generically not simultaneously satisfied, so that equilibrium and production efficiency are incompatible. Let $\big((\bar{x}, \bar{\theta}, \bar{y}), (\bar{p}, \bar{q}, \bar{\pi})\big)$ be an equilibrium and let $\underset{\sim}{w}^i = w^i + (\bar{y} + \eta)\bar{\theta}^i$, $\underset{\sim}{\omega} = (\underset{\sim}{w}^1, \ldots, \underset{\sim}{w}^I)$ denote the induced virtual endowment for the economy $\mathcal{E}\big((u^i), \underset{\sim}{\omega}\big)$. By proposition 5 (appendix) there exists an open set of full measure $\Omega'' \subset R_{++}^{N(I+J)}$ such that if $(\omega, \eta) \in \Omega''$ then the equilibria of $\mathcal{E}\big((u^i, Y^i), \omega, \eta\big)$ induce virtual endowments $\underset{\sim}{\omega}$ for which the economy $\mathcal{E}\big((u^i), \underset{\sim}{\omega}\big)$ is regular. Thus if $\bar{p}(\underset{\sim}{\omega})$ is a spot market equilibrium for $\mathcal{E}\big((u^i, \underset{\sim}{\omega})\big)$, namely a solution of $\hat{Z}(p, \underset{\sim}{\omega}) = \sum_{i=1}^I \big(\bar{x}^i(p, p \square \underset{\sim}{w}^i) - \underset{\sim}{w}^i\big) = 0$, then $\bar{p}$ is differentiable at $\underset{\sim}{\omega}$. Thus $\frac{\partial \bar{p}_1}{\partial \underset{\sim}{w}_1^i}$ is well-defined.

Step 1 By assumption (v) ∂Y^j is an $(N - 1)$-dimensional manifold. Its tangent space at $\bar{y}^j$, $T_{\bar{y}^j} \partial Y^j$ is thus an $(N - 1)$-dimensional hyperplane in R^N. Since firm j is maximizing profit $\bar{\beta}^j \cdot (\bar{p} \square y^j)$ over Y^j at $\bar{y}^j$, this hyperplane is orthogonal to the price vector $(\bar{p}_0, \bar{\beta}_1^j \bar{p}_1, \ldots, \bar{\beta}_A^j \bar{p}_S) \gg 0$. Thus its projection onto the date 1 commodity space R^{LS} is surjective. In view of this the efficiency condition (ii) can be written as

$$\left\langle \left[\frac{\partial \bar{p}_1}{\partial y_1^j}\right] dy_1^j, \; \sum_{i=1}^I \pi_1^i \square (\bar{x}^i - \underset{\sim}{w}^i) \right\rangle = 0 \quad \forall \; dy_1^j \in R^{LS} \tag{24}$$

where $\langle, \rangle$ denotes the inner product. The normalisation of spot prices $p_{s1} = 1$, $s =$

$1, \ldots, S$, implies $d\bar{p}_{s1} = 0$, $s = 1, \ldots, S$. Thus if we define

$$Q = \left[\frac{\partial \hat{\bar{p}}_1}{\partial y_1^j}\right] : R^{LS} \longrightarrow R^{(L-1)S}, \quad Q^T : R(L-1)S \longrightarrow R^{LS}$$

(25)

$$u = dy_1^j, \quad v = \sum_{i=1}^{I} \hat{\pi}_1^i \, \square \, (\hat{\bar{z}}_1^i - \hat{\bar{w}}_1^i)$$

Then (24) reduces to

$$\langle Qu, v \rangle = \langle u, Q^T v \rangle = 0 \quad \forall \, u \in R^{LS} \iff Q^T v = 0$$

(26)

If we show that rank $Q^T = (L-1)S$ then (26) is satisfied only if $v = 0$. The key idea is to then use proposition 3 to show that generically $v \neq 0$ at equilibrium.

Step 2. We show rank $Q^T = (L-1)S$. In view of the separability assumption $A_{(3)}$ the equation $\hat{Z}(p, \underline{\omega}) = 0$ decomposes into a pair of equations $\hat{Z}_0(p_0, \underline{\omega}_0) = 0$, $\hat{Z}_1(p_1, \underline{\omega}_1) = 0$. Differentiating the latter at $\underline{\omega} = \bar{\underline{\omega}}$ and using the fact that $\bar{p}_1(\bar{\underline{\omega}}) = \bar{p}_1$ gives

$$\left[\frac{\partial \hat{Z}_1}{\partial \hat{p}_1}\right] \frac{\partial \hat{\bar{p}}_1}{\partial y_{s\ell}^j} = -\sum_{i=1}^{I} \bar{\theta}^{ij} \frac{\partial \hat{\bar{z}}_1^i}{\partial m_s^i} \bar{p}_{s\ell} + e_1^{s\ell} \quad \begin{array}{l} s = 1, \ldots, S \\ \ell = 1, \ldots, L \end{array}$$

(27)

where $e_1^{s\ell} \in R^{LS}$ is the vector whose component (s, ℓ) is 1 and whose other components are zero. Since $\bar{\underline{\omega}}$ is regular the matrix $B = \left[\frac{\partial \hat{Z}_1}{\partial \hat{p}_1}\right]$ has $\text{rank}(L-1)S$, so that B^{-1} is well-defined. Thus (27) can be written as

$$Q = B^{-1}C$$

where the matrix C is given by (recall $\bar{p}_{s1} = 1$, $s = 1, \ldots, S$)

$$C = \left[\begin{array}{cccc} -\sum_{i=1}^{I} \bar{\theta}^{ij} \frac{\partial \bar{z}_{12}^i}{\partial m_1^i} & 1 - \bar{p}_{12} \sum_{i=1}^{I} \bar{\theta}^{ij} \frac{\partial \bar{z}_{12}^i}{\partial m_1^i} & \cdots & -\bar{p}_{SL} \sum_{i=1}^{I} \bar{\theta}^{ij} \frac{\partial \bar{z}_{12}^i}{\partial m_s^i} \\ \vdots & \vdots & & \vdots \\ -\sum_{i=1}^{I} \bar{\theta}^{ij} \frac{\partial \bar{z}_{SL}^i}{\partial m_1^i} & -\bar{p}_{12} \sum_{i=1}^{I} \bar{\theta}^{ij} \frac{\partial \bar{z}_{SL}^i}{\partial m_1^i} & \cdots & 1 - \bar{p}_{SL} \sum_{i=1}^{I} \bar{\theta}^{ij} \frac{\partial \bar{z}_{SL}^i}{\partial m_s^i} \end{array}\right]$$

44

C is a matrix with SL columns (each component $y^j_{s\ell}$ has an effect on price) and $(L-1)S$ rows (we consider truncated demand). To prove that rank $Q = (L-1)S$ it suffices to show that rank $C = (L-1)S$. Let $C_{s\ell}$ denote column (s,ℓ) of C. If we substract from each column $C_{s\ell}$, $\ell \geq 2$, the multiple $\bar{p}_{s\ell}C_{s1}$ of column C_{s1}, $s = 1,\ldots,S$ then we obtain a new matrix $D = [\ldots, C_{s1}, C_{s2} - \bar{p}_{s2}C_{s1}, \ldots, C_{sL} - \bar{p}_{sL}C_{s1}, \ldots]$ with the same rank as C

$$D = \begin{bmatrix} -\sum_{i=1}^{I} \bar{\theta}^{ij} \frac{\partial \bar{x}_1^i}{\partial m_1^i} & I & -\sum_{i=1}^{I} \bar{\theta}^{ij} \frac{\partial \bar{x}_1^i}{\partial m_2^i} & 0 & \cdots & -\sum_{i=1}^{I} \bar{\theta}^{ij} \frac{\partial \bar{x}_1^i}{\partial m_S^i} & 0 \\ -\sum_{i=1}^{I} \bar{\theta}^{ij} \frac{\partial \bar{x}_2^i}{\partial m_1^i} & 0 & -\sum_{i=1}^{I} \bar{\theta}^{ij} \frac{\partial \bar{x}_2^i}{\partial m_2^i} & I & \cdots & -\sum_{i=1}^{I} \bar{\theta}^{ij} \frac{\partial \bar{x}_2^i}{\partial m_S^i} & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \sum_{i=1}^{I} \bar{\theta}^{ij} \frac{\partial \bar{x}_S^i}{\partial m_1^i} & 0 & -\sum_{i=1}^{I} \bar{\theta}^{ij} \frac{\partial \bar{x}_S^i}{\partial m_2^i} & 0 & \cdots & -\sum_{i=1}^{I} \bar{\theta}^{ij} \frac{\partial \bar{x}_S^i}{\partial m_S^i} & I \end{bmatrix}$$

where I is an $(L-1) \times (L-1)$ identity matrix. Clearly rank $D = (L-1)S$.

Step 3. As usual, to prove a property for equilibria, we prove that it holds for all positive A-equilibria. For $A \in \mathcal{A}$ define the domain

$$\Lambda_A = \left\{ z = (p, q, y, \omega, \eta) \in \Gamma \times R^J \times R^{NJ} \times (\Omega' \bigcap \Omega'') \middle| \begin{array}{c} \text{rank } W(z) = J \\ \bar{\theta}_A(z) > 0 \end{array} \right\}$$

We add the equation $v = 0$, namely

$$G_A(z) = \sum_{i=1}^{I} \bar{\pi}^i_{A1}(z) \square \left[\bar{x}^i_1 \left(p_1, p_1 \square \tilde{w}^i_{A1}(z) \right) - \hat{\psi}^i_{A1}(z) \right] = 0 \tag{28}$$

to the A-equilibrium equations $F_A(z) = 0$. Thus if we define $H_A : \Lambda_A \longrightarrow \mathcal{M} \times R^{(L-1)S}$ by $H_A(z) = \left(F_A(z), G_A(z) \right)$ where F_A is defined by equations (20) and G_A by (28), we want to prove that generically for each $A \in \mathcal{A}$ the system of equations $H_A(z) = 0$ has no solution in Λ_A.

By lemma 2 it will suffice to show that rank $D_z H_A(z) \geq \dim \mathcal{M} + 1$, for all $z \in H_A^{-1}(0)$. By earlier arguments rank $D_z F_A(z) = \dim \mathcal{M}$ for all $z \in F_A^{-1}(0)$. Since $G_A : \Lambda_A \longrightarrow R^{(L-1)S}$, G_A has components (s,ℓ), $s = 1,\ldots,S$, $\ell = 2,\ldots,L$. It suffices to show that

for each $z \in H_A^{-1}(0)$ there exists some state s and some good $\ell \geqq 2$ such that component (s, ℓ) of G_A can be locally controlled without affecting any other component of G_A or F_A. By the proof of the proposition 3 for each $A \in \mathcal{A}$ and each $z \in H_A^{-1}(0)$ for any pair of agents i and i', $\tilde{\pi}_A^i(z) \neq \tilde{\pi}_A^{i'}(z)$, provided $(\omega, \eta) \in \Omega'$. Let $\tilde{\pi}_A^i = (\tilde{\pi}_{A1}^i, \ldots, \tilde{\pi}_{AS}^i)$. Thus there exists a state s such that, without loss of generality $\tilde{\pi}_{As}^i(z) > \tilde{\pi}_{As}^{i'}(z)$. Pick some good $\ell \geqq 2$. To obtain an increase in component (s, ℓ) of G_A consider the transfer of good ℓ from agent i to agent i'

$$ dw_{s\ell}^{i'} = -dw_{s\ell}^i, \quad dw_{s\ell}^i > 0 $$

At the same time decrease (increase) the endowment of good 1 of agent i (agent i') so as to leave his income and hence his demand unchanged, $dw_{s1}^i = -p_{s\ell} dw_{s\ell}^i$ $(dw_{s1}^{i'} = p_{s\ell} dw_{s\ell}^i)$. Component (s, ℓ) of G_A increases by

$$ \left(\pi_{As}^i(z) - \pi_{As}^{i'}(z) \right) dw_{s\ell}^i > 0 $$

No other component of $G_A(z)$ is affected and $F_A(z)$ unchanged. Thus rank $D_Z H_A(z) \geqq \dim \mathcal{M} + 1$, $\forall z \in H_A^{-1}(0)$.

By lemma 2 there exist submanifolds $M_\alpha \subset \mathbb{A}_A$, $\dim M_\alpha = N(I+J) - 1$, $\alpha = 1, \ldots, m$ such that $H_A^{-1}(0) \subset \bigcup_{\alpha=1}^m M_\alpha$. Now apply Sard's theorem to the projection onto $\Omega' \bigcap \Omega''$ as in the proof of proposition 3 to deduce the existence of an open set of full measure $\Omega^* \subset \Omega' \bigcap \Omega''$ such that for all $A \in \mathcal{A}$, $H_A(z) = 0$ has no solution for all $(\omega, \eta) \in \Omega^*$. Thus no A-equilibrium and hence no equilibrium is constrained efficient whenever $(\omega, \eta) \in \Omega^*$. ∎

6.5 Proof of Theorem 4.

Step 1. Let $T \subset \{1, \ldots, J\}$ denote the subset of firms with ray technology sets and let T^c denote the remaining firms. We need to indicate how the proof of theorem 2 can be modified when $T \neq \emptyset$. For each firm $j \in T$ the supply function

$$ \tilde{g}^j(p, \beta^j, \eta_1^j) = \underset{y_0^j \in R^L}{\arg \max} \, \beta_1^j \cdot (p_1 \square \eta_1^j) h^j(y_0^j) + p_0 y_0^j \tag{29} $$

46

where $\beta^j = \sum_{i=1}^{I} \theta^{ij}\pi^i$, now depends in addition on the ray parameter η_1^j. Since for all agents i such that $(i,j) \notin A$ the no-arbitrage equation

$$q^j = \Big(\sum_{s=1}^{S} \tilde{\pi}_{A_s}^i p_s \eta_s^j \Big) h^j(y_0^j) + p_0 y_0^j \tag{30}$$

holds, it follows that $\sum_{s=1}^{S} \tilde{\pi}_{A_s}^i p_s \eta_s^j$ is independent of i. Thus for $j \in T$ (18)(ii) reduces to

$$\tilde{g}^j(p, \tilde{\pi}_{A,\rho,\sigma}^i(z), \eta_1^j) - y^j = 0, \quad j \in T \tag{18 (ii)$'$}$$

Note that the terms $\sum_{j \in T} \hat{\eta}_1^j$ fall out from equation (18)(i). For the newly defined function $F_{A,\rho,\sigma}$ we need to show $F_{A,\rho,\sigma} \pitchfork 0$ (step 5). The main difficulty is to control equation (18)(ii)$'$ without affecting the remaining equations. We indicate the steps leaving details to the reader. Consider equation (s,ℓ) with $s \geq 1$ for firm $j \in T$. To control this equation consider $d\eta_1^j$ such that $d\eta_{\sigma k}^j = 0$ if $(\sigma, k) \neq (s, \ell)$, $d\eta_{s\ell}^j \neq 0$. This induces a change dg_0^j in goods at date 0 and a change dh^j in the scale of production. The change in the supply function at date 1 is thus $dg_{\sigma k}^j = \eta_{\sigma k}^j dh^j$ if $(\sigma, k) \neq (s, \ell)$ and $dg_{s\ell}^j = d\eta_{s\ell}^j h^j + \eta_{s\ell}^j dh^j$. Compensating by $dy^j = (dg_0^j, \eta_1^j dh^j)$ restores the equality in (ii)$'$ except for equation (s, ℓ) which changes by the amount $d\eta_{s\ell}^j h^j$. A change $dw^i = -\theta_{A,\rho,\sigma}^{ij} dy^j \ \forall \ i$ restores equality in (18)(i). Agents do not change their consumption and portfolios since by the envelope theorem the change dy^i does not affect the no-arbitrage equation. Appropriate changes in the matrix E restore equality in (iv). These changes in E are compensated by induced changes in $\tilde{\gamma}_{A,\rho,\sigma}^i$ effected as in equation (19).

Step 2. We prove that if all firms have ray technology sets we do not need the assumption $I + J \leq S + 1$ to establish step 6 in the proof of theorem 2. Since the objective functions of the firms do not depend on the portfolios (θ^{ij}), equilibria can be defined without these variables. Summing (18)(iii) over i and using (18)(v) gives the compatibility conditions

$$\sum_{i=1}^{I} \tilde{\gamma}_{A,\rho,\sigma}^{ik}(z) = 1 + \sum_{j=1}^{J-\rho} E_{kj}, \quad k = 1,\dots,\rho$$

47

Note that for fixed (ω, η) there are more equations than unknowns so the rest of step 6 can be applied as before.

Step 3. As in the proof of theorem 3 we show that equations of equilibrium and the production efficiency condition (ii) of proposition 4 are generically not compatible when $\mathcal{T} \neq \emptyset$. For a firm $j \in \mathcal{T}$ with ray technology condition (ii) takes the form

$$\left\langle \left[\frac{\partial \bar{p}_1}{\partial y_1^j} \right] \eta_1^j \alpha , \ \sum_{i=1}^{I} \bar{\pi}_1^i \, \Box \, (\bar{x}_1^i - \underline{w}_1^i) \right\rangle = 0 \quad \forall \, \alpha \in R$$

Using the notation in (25) this can be written as

$$\eta_1^{jT} Q^T v = 0 \quad \Longleftrightarrow \quad v^T Q \eta_1^j = 0$$

As usual we show that this condition is not satisfied at positive A-equilibria. Applying step 3 in the proof of theorem 3 we obtain a set $\Omega^* \subset R_{++}^{N(I+J)}$ on which $v_A(z) \neq 0$ where

$$v_A(z) = \sum_{i=1}^{I} \pi_{A1}^i(z) \, \Box \, \left[\bar{x}_1^i \left(p_1, p_1 \Box \psi_{A1}^i(z) \right) - \hat{\psi}_{A1}^i(z) \right]$$

Let

$$\Lambda_A = \left\{ z = (p, q, y, \omega, \eta) \in \Gamma \times R^J \times R^{NJ} \times \Omega^* \, \middle| \, \begin{matrix} \text{rank } W(z) = J \\ \bar{\theta}_A(z) > 0 \end{matrix} \right\}$$

The function F_A in (20) remains unchanged except that $\sum_{j=1}^{J} \eta_1^j$ in (i) is replaced by $\sum_{j \in \mathcal{T}} \eta_1^j$ and the components in (ii) are replaced by $\tilde{g}^j \left(p, \tilde{\pi}_A^i(z), \eta_1^j \right) - y^j$ for $j \in \mathcal{T}$. Let $G_A :$ $\Lambda_A \longrightarrow R$ be defined by

$$G_A(z) = \eta_1^{jT} Q(z)^T v_A(z) = 0$$

and let $H_Z : \Lambda_A \longrightarrow \mathcal{M} \times R$ be given by $H_A(z) = \left(F_A(z), G_A(z) \right)$. We show that generically $H_A(z) = 0$ has no solution for each $A \in \mathcal{A}$. The fact that $H_A \pitchfork 0$ will be established if we can show that G_A can be perturbed without affecting the equations $F_A(z) = 0$. Consider a perturbation $d\eta_1^j \in R^{LS}$ satisfying

$$\bar{p}_1 \Box d\eta_1^j = 0, \quad d\eta_1^{jT} Q^T v_A \neq 0 \tag{31}$$

48

If such a change $d\eta_1^j$ exists then the objective function of firm j in equation (29) and the no-arbitrage equation (30) are unchanged. Thus the input decision $\bar{y}_0^j$ is unchanged and $dy_1^j = d\eta_1^j h^j(\bar{y}_0^j)$ restores equality in 20(ii)', while $dw_1^i = -\bar{\theta}_A^{ij} dy_1^j$ compensates for the change in return on firm j's equity. Thus G_A is perturbed without affecting the equations $F_A(z) = 0$. Note also that this sequence of changes leaves $Q(z)$ and $v_A(z)$ unaffected.

(31) has a solution if and only if there does not exist $a_1 \in R^S$, $a_1 \neq 0$ such that $Q^T v_A = a_1 \,\square\, \bar{p}_1$. Thus it suffices to show that $a_1 \,\square\, \bar{p}_1 \notin Im\ Q^T$ for $a_1 \neq 0$. Since $Q = B^{-1}C$ and B is nonsingular, $Im\ Q^T = Im\ C^T$ and $Im\ C^T = (ker\ C)^\perp$. Note that the vectors of income effects

$$r^s = \sum_{i=1}^{I} \bar{\theta}^{ij} \frac{\partial \bar{x}_1^i}{\partial m_s^i}, \qquad s = 1, \ldots, S$$

satisfy $Cr^s = 0$ so that $r^s \in ker\ C$, $s = 1, \ldots, S$. Since $r^s \cdot (a_1 \,\square\, \bar{p}_1) = a_s$, $s = 1, \ldots, S$, $a_1 \,\square\, \bar{p}_1 \notin Im\ Q^T$ for $a_1 \neq 0$. Thus $H_A \pitchfork 0$. The existence of a set of full measure $\Omega^{**} \subset R_{++}^{N(I+J)}$ is then obtained by applying Sard's theorem to the projection of $H_A^{-1}(0)$ onto Ω^*. ∎

APPENDIX

DEFINITION 10: $\omega \in R^{NI}$ is a *regular* endowment for the spot market economy $\mathcal{E}((u^i), \omega)$ if, for all equilibria of this economy (Definition 7) the matrix of derivatives of the excess demand function with respect to prices is of rank $(L-1)(S+1)$.

PROPOSITION 5: *If the assumptions $\left(A_{(1)-(3)}, B_{(1)-(2)}\right)$ and $I + J \leq S + 1$ are satisfied then there exists an open set of full measure $\Omega'' \subset R_{++}^{N(I+J)}$ such that, for every economy $\mathcal{E}((u_i), (Y^j), (\omega, \eta))$ with $(\omega, \eta) \in \Omega''$, in each equilibrium $(\bar{x}, \bar{\theta}, \bar{y}), (\bar{p}, \bar{q}, \bar{\pi})$, the induced endowment $\bar{\omega} = (\bar{\omega}^i)_{i=1}^I$, $\bar{\omega}^i = w^i + (\bar{y} + \eta)\bar{\theta}^i$, is regular for the spot market economy $\mathcal{E}((u^i), \bar{\omega})$.*

PROOF: From the separability assumption $A_{(3)}$, a spot market economy $\mathcal{E}\left((u^i),\omega\right)$ can be decomposed into independent date 0 date 1 economies. Consider the excess demand function

$$Z(p,w) = \sum_i \left(x^i(p,p\,\square\,w^i) - w^i\right)$$

ω is regular for $\mathcal{E}\left((u^i),\omega\right)$ if and only if $\frac{\partial \hat{Z}_0}{\partial \hat{p}_0}(p_0,\omega_0)$ has rank $(L-1)$ and $\frac{\partial \hat{Z}_1}{\partial \hat{p}_1}(p_1,w_1)$ has rank $(L-1)S$. As usual we will prove that this property holds generically for all positive A equilibria.

Let A_A be as in the proof of Proposition 3. Consider the system of equations $H_A(Z) = 0$ where $H_A : A_A \longrightarrow \mathcal{M} \times R$ is defined by $H_A(Z) = \left(F_A(Z), G_A(Z)\right)$, F_A is defined by (20) and

$$G_A(Z) = \det \frac{\partial \hat{Z}_0}{\partial \hat{p}_0}(p_0,\underset{\sim}{w}_{A0}) \det \frac{\partial \hat{Z}_1}{\partial \hat{p}_1}(p_1,\underset{\sim}{w}_{A1})$$

with $\tilde{w}_A^i = w^i + (y+\eta)\theta_A^i$, $\underset{\sim}{w}_A = (w_A^i)_{i=1}^I$.

An argument, by now familiar proves that a set Ω'' with the properties described in Proposition 5 exists if $H_A \pitchfork 0$. To show this it suffices to prove that we can perturb $G_A(Z)$ without affecting the equations $F_A(Z) = 0$.

A little vector calculus shows the the Slutsky equation still holds with multiple budget constraints so that

$$\frac{\partial \hat{Z}_0}{\partial \hat{p}_0}(p_0,\underset{\sim}{w}_{A0}) = \left[\sum_{i=1}^I \frac{\partial \tilde{x}_{o\ell}^i}{\partial p_{ok}}\right]$$

$$= \left[\sum_{i=1}^I \left(\frac{\partial \tilde{x}_{o\ell}^i}{\partial p_{ok}}\right)^{u_0^i=\text{const}} - \sum_{i=1}^I (x_{ok}^i - w_{Aok}^i)\frac{\partial \tilde{x}_{o\ell}^i}{\partial m^i}\right] \begin{array}{l} \ell = 2,\dots,L \\ k = 2,\dots L \end{array}$$

$$\frac{\partial \hat{Z}_1}{\partial \hat{p}_1}(p_1,\underset{\sim}{w}_{A1}) = \left[\sum_{i=1}^I \frac{\partial \tilde{x}_{s\ell}^i}{\partial p_{\sigma k}}\right]$$

$$= \left[\sum_{i=1}^I \left(\frac{\partial \tilde{x}_{s\ell}^i}{\partial p_{\sigma k}}\right)^{u_1^i=\text{const}} - \sum_{i=1}^I (x_{\sigma k}^i - w_{A\sigma k}^i)\frac{\partial \tilde{x}_{s\ell}^i}{\partial m_\sigma^i}\right] \begin{array}{l} \ell,k = 2,\dots,L \\ s,\sigma = 1,\dots,S \end{array}$$

50

Any transfer of goods among the agents which does not affect their incomes leaves the equations of equilibrium $F_A(Z) = 0$ unchanged but affects the terms $(x^i_{\sigma k} - \psi_{A\sigma k})$. Let $Z^i = (x^i - \psi^i_A)$. Such transfers can generate any dZ_i such that $\sum_{i=1}^{I} d\hat{Z}_i = 0$. Since the matrix of substitutions terms is negative definite, the result is implied by the following lemma. ∎

LEMMA 3: *Let (a_k), $k = 1,\ldots,n$, (b^i_k), $i = 1,\ldots,I$, $k = 1,\ldots,n$ be $(I+1)n$ vectors of R^n such that $\det(a_1,\ldots,a_n) \neq 0$. Let $E = \{Z \in R^{nI} \mid \sum_{i=1}^{I} Z^i_k = 0\ \forall\, k = 1,\ldots,n\}$ and let $P : E \longrightarrow R$ be defined by $P(Z) = \det\left(u_1(Z_1),\ldots,u_n(Z_n)\right)$ where $u_k(Z_k) = a_k + \sum_{i=1}^{I} Z^i_k b_k$. If $P(\bar{Z}) = 0$ for some $\bar{Z} \in E$, then there exists $h \in E$ such that $D_Z P(\bar{Z}) \cdot h \neq 0$.*

PROOF: By linearity of the determinant with respect to each variable

$$P(\bar{Z}) = \det\left(a_1, u_2(\bar{Z}_2),\ldots, u_n(\bar{Z}_n)\right) + \sum_{i=1}^{I} \bar{Z}^i_1 \det\left(b^i_1, u_2(\bar{Z}_2),\ldots, u_n(\bar{Z}_n)\right).$$

Consider the set of vectors $h \in E$ such that $h^i_k = 0\ k = 2,\ldots,n$. Either for some of these vectors $D_Z P(\bar{Z}) \cdot h = \sum_{i=1}^{I} h^i_1 \det\left(b^i_1, u_2(\bar{Z}_2),\ldots, u_n(\bar{Z}_n)\right) \neq 0$ or $\det\left(b^i_1, u_2(\bar{Z}_2),\ldots, u_n(\bar{Z}_n)\right)$ is independent of i and $P(\bar{Z}) = \det\left(a_1, u_2(\bar{Z}_2),\ldots, u_n(\bar{Z}_n)\right)$. In this case we repeat the reasoning for the following index. Taking successively all indices, either we find a vector h such that $D_Z P(\bar{Z})\, h \neq 0$ or $P(\bar{Z}) = \det(a_1,\ldots,a_n)$. But by assumption $P(\bar{Z}) = 0$ and $\det(a_1,\ldots,a_n) \neq 0$: thus the first alternative must hold and the proof is complete. ∎

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