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INCREASES IN RISK AVERSION AND PORTFOLIO CHOICE IN A COMPLETE MARKET

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1. Introduction

This note examines the effect of changes in risk aversion on the optimal portfolio choice in a complete market. It is shown that an agent who is less risk averse in the Pratt [1964] sense than another will choose a portfolio whose payoff is distributed as the other's payoff plus a nonnegative random variable plus conditional-mean-zero noise. Therefore, when markets are complete we can avoid using the more complex Ross [1981] concept of more risk averse. The proof of the result uses simple first order conditions and basic results from stochastic dominance.

Another result is that if either agent has decreasing absolute risk aversion, then the non-negative random variable can be chosen to be a constant. In other words, the less risk averse agent will choose a payoff that is equal to the more risk averse payoff, plus a constant, plus random noise. This condition (less risk averse plus decreasing absolute risk aversion) is the sense of more risk averse that Kihlstrom, Romer, and Williams [1981] have used as a sufficient condition for one agent to have a larger risk premium than another when base wealth is random and the noise is independent. As in that paper, the condition is not very tight, i.e., there seems to be no natural sense in which it is necessary as well.

2. Framework and Results

We will compare the actions of two agents, A and B, who possess von Neumann–Morgenstern utility functions $u_A(\cdot)$ and $u_B(\cdot)$, respectively. We will take these utility functions to be twice differentiable, with strictly positive first derivatives and strictly negative second derivatives. Markets are complete, and each agent faces a maximization problem of the following form.

Problem 1:

Choose $\tilde{c}$ *to*
maximize $E[u(\tilde{c})]$
s.t. $E[\tilde{\lambda}\tilde{c}] = w_0$.

In Problem 1, $\tilde{c}$ is the random consumption, $u(\cdot)$ is the utility function $u_A(\cdot)$ or $u_B(\cdot)$, w_0 is initial wealth (which is the same for both agents), $E[\cdot]$ is the expectation operator, and the *state price density* $\tilde{\lambda} > 0$ is the ratio of the state price to the state probability.[1] We will assume that both agents have optimal random consumptions, called $\tilde{c}_A$ and $\tilde{c}_B$, and that these optimal consumptions have finite variances (so we can use standard results from stochastic dominance).

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[1]For example, in discrete states, if π_i is the probability that state i will occur and p_i is the price of obtaining 1 in state i, then the state price density is $\lambda_i = p_i/\pi_i$. Then $E[\tilde{\lambda}\tilde{c}] = \sum\pi_i\lambda_i c_i = \sum\pi_i(p_i/\pi_i)c_i = \sum p_i c_i$, which may be more familiar. The advantage of using $\tilde{\lambda}$ instead of p_i's is that the p_i's make no sense in a continuous state space. (There is also a slight simplification of the form of first order conditions and some other expressions.)

The first order conditions for an optimum in Problem 1 specify that the marginal utility $u'(\cdot)$ is proportional to the state price density $\tilde{\lambda}$. For convenience, we will choose representations for the preferences such that the constant of proportionality is 1 — we can do so because the von Neumann–Morgenstern utility function is defined only up to an increasing affine transform. Therefore, the first order conditions for the two agents' optima are given by

$$\tilde{\lambda} = u_A'(\tilde{c}_A) = u_B'(\tilde{c}_B). \tag{1}$$

By negativity of the second derivatives, $\tilde{c}_A$ and $\tilde{c}_B$ are decreasing functions of $\tilde{\lambda}$.

In our first result, we will assume that agent B is more risk averse than agent A (in the sense of Pratt [1964]). Recall that by definition, B is more risk averse than A if B's risk premium is never smaller, that is, if B is always willing to pay at least as much as A to avoid a fair gamble. By Pratt [1964], the following three conditions are equivalent.

(i) B is more risk averse than A in the sense of Pratt. (That is, if $E[u_A(w+\tilde{\varepsilon})] = u_A(w-\Pi_A)$ and $E[u_B(w+\tilde{\varepsilon})] = u_B(w-\Pi_B)$ where w is not stochastic and $E(\tilde{\varepsilon}) = 0$, then $\Pi_B \geq \Pi_A$.)

(ii) B's coefficient of risk aversion is never smaller than A's.

(iii) $u_B(c) = G(u_A(c))$ where $G'(\cdot) > 0$ and $G''(\cdot) \leq 0$.

The third condition, that B's utility function is an increasing concave transform of A's utility function, is the condition we will use.

Before our main result, we need the following Lemma, which says that if B is more risk averse than A, then A's optimal portfolio has a higher expected return.

Lemma 1: Let B be more risk averse than A. Whether or not A and B have equal initial wealths, $\tilde{c}_A$ and $\tilde{c}_B$ are monotonely related, and there is some critical consumption level c^* (perhaps $\pm\infty$) such that $\tilde{c}_A \geq \tilde{c}_B$ whenever $\tilde{c}_A$ or $\tilde{c}_B \geq c^*$, and such that $\tilde{c}_A \leq \tilde{c}_B$ whenever $\tilde{c}_A$ or $\tilde{c}_B \leq c^*$. Furthermore, if A and B do have equal initial wealths, then $E[\tilde{c}_A] \geq E[\tilde{c}_B]$.

Proof: Using the concave transform characterization of more risk averse, the first order conditions (1) become

$$\tilde{\lambda} = u_A'(\tilde{c}_A) = G'(u_A(\tilde{c}_B))\cdot u_A'(\tilde{c}_B). \tag{2}$$

As noted before, $\tilde{c}_A$ and $\tilde{c}_B$ are both decreasing in $\tilde{\lambda}$. Because $G''(\cdot) \leq 0$, $G'(\cdot)$ is nonincreasing in its argument. Because $u_A(\cdot)$ is increasing, G' is nonincreasing in $\tilde{c}_B$ (and therefore in $\tilde{c}_A$), and nondecreasing in $\tilde{\lambda}$. Our assumptions on $u_A(\cdot)$ imply that $u_A'(\cdot)$ is invertible, and therefore we can rewrite the second equality in (2) as

$$c_A = u_A'^{-1}\Big(G'(u_A(c_B))\cdot u'(c_B)\Big). \tag{3}$$

Therefore, $\tilde{c}_A = \tilde{c}_B$ when $G' = 1$, and because marginal utility is decreasing, $\tilde{c}_A \gtrless \tilde{c}_B$ as $G' \lesseqgtr 1$. Choose c^* so that $G'(u_A(c^*)) = 1$ if possible, or pick $c^* = \infty$ if $G' < 1$ everywhere or $c^* = -\infty$ if $G' > 1$ everywhere. This choice is the critical consumption level c^* needed for the first part of the theorem.

Now suppose that the initial wealths are equal. Then the budget constraints for the agents are that

$$E[\tilde{\lambda}\tilde{c}_A] = E[\tilde{\lambda}\tilde{c}_B] = w_0. \tag{4}$$

Because $\tilde{\lambda}$ is inversely related to both $\tilde{c}_A$ and $\tilde{c}_B$, the expectation in the first equality of (4) weights consumption most highly in states (with $\tilde{c}_A$ and $\tilde{c}_B \le c^*$) in which $\tilde{c}_A$ may be less than $\tilde{c}_B$. Weighting states by the probability measure alone increases the relative influence of states (with $\tilde{c}_A$ and $\tilde{c}_B \ge c^*$) in which $\tilde{c}_A$ may be more than $\tilde{c}_B$. Therefore,

$$E[\tilde{c}_A] \ge E[\tilde{c}_B], \tag{5}$$

as was to be shown. ∎

Lemma 1 gives us a sense in which decreasing the agent's risk aversion takes us further from the riskless asset. In fact, we can obtain a more explicit description of how decreasing the agent's risk aversion changes the optimal portfolio choice. The description and proof are both related to second order stochastic dominance. A random variable $\tilde{x}$ second-order stochastically dominates another random variable $\tilde{y}$ if $\tilde{x}$ is weakly preferred to $\tilde{y}$ by all agents with strictly increasing and concave von Neumann-Morgenstern preferences. The following are equivalent (see Hadar and Russell [1969]). (The $F(\cdot)$'s are the distribution functions.)

(i) $\tilde{x}$ second-order stochastically dominates $\tilde{y}$. (That is, $E[u(\tilde{x})] \ge E[u(\tilde{y})]$ for all strictly increasing and concave $u(\cdot)$.)

(ii) $\tilde{y}$ has the same distribution as $\tilde{x}-\tilde{z}+\tilde{\varepsilon}$ where $\tilde{z} \ge 0$ and $E[\tilde{\varepsilon}|x-z] = 0$.

(iii) For all c, $\int_{q=-\infty}^{c}[F_y(q)-F_x(q)]dq \ge 0$.

The distributional condition (ii) says that $\tilde{y}$ has the same distribution function as $\tilde{x}$ less something, plus noise. Our characterization of the relation between $\tilde{c}_A$ and $\tilde{c}_B$ will say that $\tilde{c}_A$ is distributed as $\tilde{c}_B$ *plus* something, plus noise, which is the same as saying that $-\tilde{c}_B$ second-order stochastically dominates $-\tilde{c}_A$. The integral condition (iii) is the condition we will use in our proof.

Our first main result says that if B is more risk averse than A in the sense of Pratt, then A's optimal consumption is distributed as B's optimal consumption plus something, plus noise.

Theorem 1: If B is more risk averse than A in the sense of Pratt, then $\tilde{c}_A$ is distributed as $\tilde{c}_B+\tilde{z}+\tilde{\varepsilon}$, where $\tilde{z} \geq 0$ and $E[\tilde{\varepsilon}|c_B+z] = 0$. Furthermore, if $\tilde{c}_A \neq \tilde{c}_B$, then neither $\tilde{z}$ nor $\tilde{\varepsilon}$ is identically zero.

Proof: The first step of the proof is to show that $-\tilde{c}_B$ second-order stochastically dominates $-\tilde{c}_A$. By Lemma 1, $\tilde{c}_A$ and $\tilde{c}_B$ are monotonely related and there is a critical value c^* above which c_A is weakly larger and below which c_B is weakly larger. In other words, the set of states $\{-\tilde{c}_A \geq q\}$ is weakly larger (and therefore weakly more probable) than the set of states $\{-\tilde{c}_B \geq q\}$ when $q \geq -c^*$ (because $\tilde{c}_B < -q$ implies $\tilde{c}_A < -q$ when $-q < c^*$), and the converse is true when $q \geq -c^*$. By the definition of the distribution functions for $-\tilde{c}_A$ and $-\tilde{c}_B$, this means precisely that

$$F_{-c_A}(q) \geq F_{-c_B}(q) \quad \text{for } q \leq -c^* \tag{6}$$

and

$$F_{-c_A}(q) \le F_{-c_B}(q) \quad \text{for } q \ge -c^*. \tag{6}$$

Now, consider the expression in the integral condition (ii) for stochastic dominance of $\tilde{c}_B$ over $\tilde{c}_A$. Define

$$I(c) \equiv \int_{q=-\infty}^{c} \left[F_{-c_A}(q) - F_{-c_B}(q)\right] dq. \tag{8}$$

Then, (6) and (7) imply that the integrand in (8) is nonnegative for $q \le -c^*$ and nonpositive for $q \ge -c^*$. Therefore, $I(c)$ starts at zero (at $-\infty$), weakly increases until the critical value $c = -c^*$, and then weakly decreases. However, $I(+\infty)$ is just $E[\tilde{c}_A-\tilde{c}_B]$ (by a simple integration by parts), and therefore $I(c)$ is never negative. This shows that $-\tilde{c}_B$ second-order stochastically dominates $-\tilde{c}_A$. By the distributional condition, this says that $-\tilde{c}_A$ is distributed as $-\tilde{c}_B-\tilde{z}+\tilde{\varepsilon}$, where $\tilde{z} \ge 0$ and $E[\tilde{\varepsilon}|-c_B-z] = 0$. But this is exactly the same as saying that c_A is distributed as $\tilde{c}_B+\tilde{z}-\tilde{\varepsilon}$, where $\tilde{z} \ge 0$ and $E[-\tilde{\varepsilon}|c_B+z] = 0$. Relabel $-\tilde{\varepsilon}$ as $\tilde{\varepsilon}$, and we have proven the first sentence of the statement of the theorem.

To prove the second sentence, note that because $\tilde{c}_A$ and $\tilde{c}_B$ are monotonely related, $\tilde{c}_A$ is distributed the same as $\tilde{c}_B$ only if $\tilde{c}_A \equiv \tilde{c}_B$. Therefore, if $\tilde{c}_A \not\equiv \tilde{c}_B$, one or the other of $\tilde{z}$ or $\tilde{\varepsilon}$ is not identically zero. Now, if $\tilde{z}$ is identically zero, then $\tilde{\varepsilon}$ must not be identically zero, and $\tilde{c}_A$ is distributed as $\tilde{c}_B+\tilde{\varepsilon}$, contradicting optimality of $\tilde{c}_A$ for A (by Jensen's inequality). Similarly, if $\tilde{\varepsilon}$ is identically zero, then $\tilde{z}$ must not be, and $\tilde{c}_A$ strictly first-order stochastically dominates $\tilde{c}_B$, contradicting optimality of $\tilde{c}_B$ for B, since $u_B(\cdot)$ is strictly increasing. ■

Theorem 1 shows that if B is more risk averse than A in the sense of Pratt, $\tilde{c}_A$ is distributed as $\tilde{c}_B$ plus a risk premium plus random noise. Except that the risk premium has mean equal to the difference of the mean consumptions, the distributions of the risk premium and the noise term are typically not uniquely determined. This is as in the theory of second order stochastic dominance. For example, let $\tilde{x}$ be uniform on $[1,2]$ and let $\tilde{y}$ be uniform on $[0,1]$. Then $\tilde{x}$ second-order stochastically dominates $\tilde{y}$, because $\tilde{y}$ is distributed as $\tilde{x}-1$. But $\tilde{y}$ is also distributed as $\tilde{x}-(x-\frac{1}{2})+\tilde{\varepsilon}$, where $\tilde{x}-1/2 > 0$ and $\tilde{\varepsilon}$ is uniform on $[-\frac{1}{2},\frac{1}{2}]$ and is chosen independent of $\tilde{x}$.

Our second main result says that when agent A has nonincreasing relative risk aversion, we can choose $\tilde{z}$ to be nonstochastic.

Theorem 2: If B is more risk averse than A in the sense of Pratt and one of the two agents has nonincreasing absolute risk aversion, then $\tilde{c}_A$ is distributed as $\tilde{c}_B+z+\tilde{\varepsilon}$, where $z = E[\tilde{c}_A-\tilde{c}_B] \geq 0$ and $E[\tilde{\varepsilon}|c_B] = 0$.

Proof: The proof is essentially the same as the proof of Theorem 1, except that we will show that the single-crossing property of the integrand still holds when we add the difference in the means to $\tilde{c}_B$ (as we will shortly). Once we have shown that, we will have that $-(\tilde{c}_B+E[\tilde{c}_A-\tilde{c}_B])$ second-order stochastically dominates $-\tilde{c}_A$. This is to say that $-\tilde{c}_A$ is distributed as $-(\tilde{c}_B+E[\tilde{c}_A-\tilde{c}_B])+\tilde{z}+\tilde{\varepsilon}$, where $\tilde{z} \geq 0$ and $E[\tilde{\varepsilon}|-(c_B+E[c_A-c_B])+z] = 0$. But equating the means implies that $\tilde{z} = 0$, and therefore we must have that $-\tilde{c}_A$ is distributed as $-(\tilde{c}_B+E[\tilde{c}_A-\tilde{c}_B])+\tilde{\varepsilon}$, where $E[\tilde{\varepsilon}|-c_B] = 0$, or equivalently that $\tilde{c}_A$ is distributed as $\tilde{c}_B+E[\tilde{c}_A-\tilde{c}_B]+\tilde{\varepsilon}$, where $E[\tilde{\varepsilon}|c_B] = 0$ (where we have taken $\tilde{\varepsilon}=-\tilde{\varepsilon}$).

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To complete the proof, consider first the case when A has nonincreasing absolute risk aversion. Define the utility function $u_A^*(w) = u_A(w+E[\tilde{c}_A-\tilde{c}_B])$. A hypothetical agent A^* with this utility function will optimally hold $\tilde{c}_A-E[\tilde{c}_A-\tilde{c}_B]$ given initial wealth for which this satisfies the budget constraint, because the first order conditions are the same as for optimality of $\tilde{c}_A$ for A. Also, A^* is less risk averse than B because A is less risk averse than B and nonincreasing risk aversion of A implies that A^* is weakly less risk averse than A. Therefore, Lemma 1 (with $u_A^*(\cdot)$ in place of $u_A(\cdot)$) again implies the sign pattern for the integral condition corresponding to (8) but using the demeaned variables. The only difference is that the integral now asymptotes to zero because we have subtracted the means.

In the case that B has nonincreasing absolute risk aversion, the proof is similar except that we define a utility function $u_B^*(w) = u_B(w-E[\tilde{c}_A-\tilde{c}_B])$ that holds $\tilde{c}_B+E[\tilde{c}_A-\tilde{c}_B]$. This agent is weakly more risk averse than B and is therefore more risk averse than A. The rest of the argument is similar. ■