COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut 06520

COWLES FOUNDATION DISCUSSION PAPER NO. 836

A MODELING SYSTEM FOR APPLIED GENERAL EQUILIBRIUM ANALYSIS

by

Thomas F. Rutherford

May 1987

A MODELING SYSTEM FOR APPLIED GENERAL EQUILIBRIUM ANALYSIS

Thomas F. Rutherford

April 1987

Cowles Foundation for Economic Research
Box 2125 Yale Station
Yale University
New Haven, CT. 06520

I am grateful to Alan Manne and Lars Mathiesen for their encouragement and guidance. I have also had useful suggestions from Glenn Harrison, Victor Norman, Michael Saunders, Herb Scarf, T.N. Srinivasan, John Stone and John Whalley. I remain responsible for the contents of the paper.

1. INTRODUCTION

Applied general equilibrium (AGE) models are typically employed to study systems involving more than one economic agent, each of which may have a separate objective function. Models of international trade and single-country models which focus on public finance issues are commonly formulated in an AGE format. They are characterized by four principal sets: commodities, consumers, producers, and institutional constraints. Typically, agents behave according to the competitive paradigm: consumers allocate their incomes to maximize the utility of consumption, and producers adjust their production plans to maximize profit. Both groups take market prices as given. The number and definition of agents and commodities, together with the characteristics of production and utility functions, constitute the formulation of an AGE model. "Institutional constraints" are extensions of the standard Walrasian model. They may include features such as taxation, tariffs, quotas and constraints on financial flows or relative prices.

Scarf (with Hansen) [1973] demonstrated the feasibility and potential of numerical modeling in the Arrow-Debreu general equilibrium framework. In Scarf's work constructive proofs of existence provided the building blocks for solution methods. Subsequent research has produced new approaches to model formulation and methods of solution. We apply a technique proposed by Mathiesen [1985a,b] which has proven capable of efficiently and reliably solving models with large dimensions.

As new methods for solving general equilibrium models have developed, so too have the scope of economic questions to which they have been applied. Harberger's analysis of tax incidence and economic efficiency in the late 1950's lead to subsequent research applying numerical models to study taxation and other topics in public finance. Ballard, Fullerton, Shoven and Whalley [1985] provide an introduction in presenting a model of the United States tax system. For a survey of AGE applications to taxation, see Whalley [1986].

Along a parallel line, applications have emerged from the development-planning tradition. In this setting, the acronym "CGE" (computable general equilibrium) is often employed to describe the modeling framework. (See, for example, Adelman and Robinson [1978], Ginsburgh and Waelbroeck [1980], or Dervis, De Melo and Robinson [1982]).

Still another line of applications has arisen from analysis of international trade. These models, based on multidimensional extensions of the standard 2x2x2 neoclassical trade model, have been used to analyze both North-North and North-South trade issues. (See, for example, Petri [1983] or Srinivasan and Whalley [1986].)

Despite advances in solution algorithms and applications, applied general equilibrium [AGE] modeling remains a specialized technique. This is in large-part due to the technical expertise required for representing and solving these models. This paper reports on a system which attempts to simplify this process. A major contribution of the system is that it provides a structured framework in which to think about general equilibrium models. To this end, issues of model formulation and representation are our central focus, and details of the solution procedure are only sketched.

The paper begins with an overview of the motivation for and general capabilities of the system. Subsequent sections describe the Walrasian model and its extensions, the solution method and performance of the system with several large-scale models.

2. MOTIVATION AND OVERVIEW

2.1. Model Representation

Numerical modeling of market economies involves several distinct activities. These may be viewed as follows:

```
                  Data
                    ↑
                    ↓
                  Model
                Formulation          Model            Scenario
Policy  ←-→        and        ←-→ Representation ←→  Solution
Issues        Interpretation
                    ↑
                    ↓
             Economic Theory
```

The model builder must be familiar with a variety of topics. Data, policy issues and economic theory all contribute to the formulation and analysis of AGE models. In terms of economic insights, the techniques for model representation and solution are of secondary importance, but from a practical standpoint, they are a major consideration. In applied modeling projects, technical problems with model representation and solution can lead to more effort than the time spent on formulation and interpretation. The efficiency of a solution code is essentially irrelevant compared with the time required to code a new model. When the development of a working computer program is a major task, too little time is left for using the model to investigate policy questions.

Perhaps these difficulties arise because economists see themselves working on one side of the model representation activity while algorithmic specialists see themselves working on the other. Model representation is seen by both groups as an uninspiring technical problem.[1]

The "best" representation of a model depends on a number of factors. A major determinant is the model's size, but the choice may also depend on the aptitude of the model builder for computer programming, the availability of hardware and considerations regarding the audience to which the model is directed. Large, complex models with many parameters and features may be represented more easily in a machine-friendly rather than person-friendly format. On the other hand, small-scale expository examples are best written out in a symbolic form so that understanding is not hindered by the vagueries of computerese.

---

[1] A notable exception is the work of Alexander Meeraus and his colleagues at the World Bank. Their system, GAMS (see Meeraus [1983]), is a higher level modeling language for formulating and solving optimization models.

2.2. MPS/GE

The mathematical programming system for general equilibrium analysis (MPS/GE) is a micro-computer system which facilitates the formulation and analysis of AGE models. Solution codes for AGE models[2] typically require that the model structure be provided in the form of a computer subroutine. With this arrangement, AGE modeling requires close familiarity with the solution algorithm. In contrast, MPS/GE separates the tasks of model formulation and model solution. This frees model builders from the tedious task of writing model-specific function evaluation subroutines. All features of a particular model are communicated to MPS/GE either interactively or through an input data file.

To simplify coding, utility and production functions are restricted to the "nested" constant elasticity of substitution (CES) family. Two special cases: Leontief (fixed coefficient) and Cobb-Douglas are included. These nested CES functions are characterized by different trade-off possibilities within each aggregate as well as between aggregates. Because functions are entered in a data file, revisions are simplified.

In contrast to the nonlinear programming language GAMS[3],

---

[2] See, for example, Broadie [1983a,b], Todd [1980], Kimball and Harrison [1985], Merrill [1972], and Mathiesen [1985a].

[3] See Meeraus [1983]. For a limited class of market structures, an extension of GAMS (HERCULES) can solve economywide models. See Drud, Kendrick and Meeraus [1986].

MPS/GE does not altogether eliminate the need for programming. In large-scale projects, model-specific programs may be used in order to avoid the tedium of entering function coefficients one element at a time. When "model-generator programs" are required, they be written in whatever language is most familiar. Furthermore, these programs operate "stand-alone", thereby simplifying development and debugging.

Downstream of the model-specific software MPS/GE provides a "model editor" and a "case generator". These two programs may be used with any MPS/GE model, whether generated by hand or with a computer program. The model editor provides interactive access to model structure and coefficients together with facilities for checking model structure and benchmark consistency. The editor can be used to great advantage when implementing a new model or developing variations of an existing one. The case generator simplifies scenario analysis, giving the user complete control over parametric variations of input data and the comparison of resulting equilibrium values.

MPS/GE may be run an IBM-PC with at least 512K of random access memory. It requires a numeric data coprocessor (8087 or 80287). In addition, it is useful to have a hard disk, although the system can be operated on a computer with two diskette drives.

Pearson [1986] has also developed a higher-level language for economic modeling in the Johansen framework.

3. THE WALRASIAN MODEL

Applied general equilibrium models are distinguished by the following features:

a. The set of commodities which are traded.

b. The set of production sectors which convert certain commodities into others.

c. The set of consumers who allocate earnings from initial endowments to final consumption.

d. The set of auxiliary variables such as balance of payment premia, unemployment rates and taxes.

e. Production functions which define the technical characteristics of sectors in the economy.

f. Utility functions which summarize the preferences of consumers in the economy.

g. Institutional assumptions about the motives and behavior of producers, consumers and the government sector.

This list accomodates a wider class of models than can be treated using MPS/GE. For example, MPS/GE assumes that all agents are price-takers.[4] This implies that producers maximize profit subject to technical constraints, and

---

[4] Formally, there are two other restrictions. Production functions exhibit constant returns to scale and utility functions are homothetic. In practice, however, it is straightforward to introduce non-homotheticity (non-unitary income elasticities) and decreasing returns to scale.

consumers maximize utility subject to budget constraints, and all agents take market prices as given. This rules out models such as Harris [1984] based on increasing returns to scale and monopolistic competition.

In the AGE model, money is neutral, and the model determines relative prices only. Fiat money can be introduced as a *commodity*, either explicitly entering the consumer's utility function, as in Malinvaud [1977], or entering through transactions demand, as in Feltenstein [1984].

To begin we consider the pure Walrasian formulation. In this class of models, there are no institutional constraints and auxiliary variables do not appear. Equilibrium concepts from the simple model provide a point of reference for extensions and more complicated market structures. Throughout, we focus on numerical solutions rather than on existence or uniqueness issues.

3.1. Supply and Demand

Suppose that there are n commodities, indexed by i; m production sectors, indexed by j; and p consumers, indexed by k. The variables determining a Walrasian equilibrium include:

$\pi \in \Re^n$, a vector of commodity prices;

$y \in \Re^m$, a vector of activity levels;

$Y \in \Re^p$, a vector of consumer incomes.

Production inputs and outputs are determined by perfectly-competitive profit maximization. The system accomodates multiple outputs. Sector j produces a vector of commodity outputs, $v \in \Re^n$, using inputs of a vector of commodity inputs, $x \in \Re^n$, when these vectors satisfy technical constraints of the form:

$$H_j(v,x) = y_j$$

where y_j is the activity level. In MPS/GE, restrictions are placed on the functional form H_j: it is separable and homogeneous of degree one. That is:

$$H_j(v,x) = g_j(v) - f_j(x)$$

and

$$H_j(\lambda v, \lambda x) = \lambda \; H_j(v,x) \quad \forall \; \lambda > 0.$$

The second of these conditions implies that all production sectors exhibit constant returns to scale. f_j is a convex function which aggregates production inputs, and g_j is a concave function which aggregates production outputs. In this treatment, single commodity outputs and vectors of joint products in fixed proportion are special cases.

Given market prices π, separability implies that the cost-minimizing vector of unit inputs, $h_j(\pi)$, and the profit-maximizing vector of unit outputs, $q_j(\pi)$, can be computed independently and combined to form a price-responsive production plan, represented by a vector of net outputs at prices π. That is:

$$a_j(\pi) = q_j(\pi) - h_j(\pi) \qquad (1)$$

where:

$$q_j(\pi) \in \underset{x \in \Re^n}{\mathrm{argmax}}(\pi' x \mid g_j(x) = 1) \qquad (2.a)$$

and

$$h_j(\pi) \in \underset{x \in \Re^n}{\mathrm{argmin}}(\pi' x \mid f_j(x) = 1) . \qquad (2.b)$$

Production possibilties for the economy as a whole can then be summarized by a matrix of price-responsive input and output coefficients, $A(\pi) \in \Re^{n \times m}$, where the jth column of $A(\pi)$ is $a_j(\pi)$.

An alternative derivation[5] of $A(\pi)$ begins with unit profit functions, $\Pi_j(\pi):\Re^n \rightarrow \Re^1$. These are defined by:

$$\Pi_j(\pi) = \max \; (\pi'(y-x) \mid g(y) = 1, \; f(x) = 1). \qquad (3)$$

The jth column of $A(\pi)$, by Shepherd's lemma, equals the gradient of the profit function:

$$a_j(\pi) = \nabla \, \Pi_j(\pi). \qquad (4)$$

Note: $\Pi_j(.)$ is homogeneous of degree one in π. Hence, by Euler's law for homogenous functions, $\Pi_j(\pi) = a_j(\pi)' \, \pi$.

Consumer demands are summarized by a vector of aggregate excess demands, $\xi(\pi) \in \Re^n$. This vector sums the excess demands from each of the consumers, as follows:

---

[5] This "dual approach" is presented in Dixit and Norman [1980], chapter 2.

$$\xi(\pi) = \sum_k (d_k(\pi) - b_k). \qquad (5)$$

In this sum, $d_k(\pi) \in \Re^n$ is a vector of price-dependent final demands, and $b_k \in \Re^n$ is a fixed vector of the commodity endowments associated with consumer k. Final demands are determined through utility maximization, i.e.:

$$d_k(\pi) \in \underset{x \in \Re^n}{\text{argmax}} \left(U_k(x) \mid \pi' x \le \pi' b_k \right). \qquad (6)$$

where $U_k(.):\Re^n \to \Re^1$ is the utility function for consumer k.

Because the budget contraint is linear in π, the aggregate market excess demand vector is homogenous of degree zero in π. That is, $\xi(\pi) = \xi(\lambda\pi)$ for all positive λ.

3.2. Market Equilibrium

Market equilibrium in the Walrasian model is defined by non-negative price-activity pairs (π, y) which satisfy the following complementarity conditions:[6]

(a) Every sector in the economy earns non-positive profits. In sectors operated at positive levels, the value of outputs equals the cost of inputs.

(b) Supply minus demand for every commodity is nonnegative, and a positive price implies equality of supply and demand.

---

[6] This representation of the Walrasian model follows that of Mathiesen [1985a].

These conditions can be stated in matrix notation as follows:

Non-positive unit profits:	$-A(\pi)'\,\pi \geq 0$	(7.a)
Non-positive excess demands:	$A(\pi)\,y \geq \xi(\pi)$	(7.b)
Non-negative variables:	$\pi \geq 0\ ;\ \ y \geq 0$	(7.c)
Complementary slackness:	$y'\,(\,A(\pi)'\,\pi\,) = 0$	(7.d)
	$\pi'(\xi(\pi) - A(\pi)\,y) = 0$	(7.e)

Several observations:

• The excess profit constraint, $A'\,\pi \leq 0$, is equivalent to $\Pi_j(\pi) \leq 0\ \ \forall\ j$.

• Walras' law states that, under non-satiation, consumer's expenditures exhaust their budget. Algebraically, this is written:

$$\xi(\pi)\,'\,\pi = 0 \quad \forall\ \pi \tag{8}$$

When excess demands satisfy Walras' law and conditions (7.a,b,c) are satisfied, then (7.d,e) are automatically satisfied.

• Homogeneity of $\xi(\pi)$ and $A(\pi)$ imply that if (π,y) is an equilibrium, then so is $(\lambda\,\pi,\ y)$ for all $\lambda > 0$. In other words, *relative* rather than *absolute* prices determine an equilibrium. For the solution algorithm, it is typical to define a single good as numeraire and set its price to unity; but equilibrium prices may be scaled arbitrarily.

• When computing equilibria, exact solutions are never obtained, and one must be satisfied with a "close" point, identified as such by an error function. In MPS/GE, the error measures both infeasibility and complemetarity.

• Income levels can be added as independent variables. The equilibrium conditions are then written in extended form:

$$- \sum_i a_{ij} \pi_i \geq 0 \qquad \perp \; y_j \geq 0 \qquad (9.a)$$

$$\sum_i a_{ij} y_j \geq \xi_i(\pi, Y) \qquad \perp \; \pi_i \geq 0 \qquad (9.b)$$

$$\sum_i b_{ik} \pi_i = Y_k \qquad (9.c)$$

where the symbol "⊥" implies complementary slackness, and the aggregate excess demand function is redefined by:

$$\xi(\pi, Y) = \sum_k (d_k(\pi, Y_k) - b_{\cdot k}) \qquad (10.a)$$

and

$$d_k(\pi, Y_k) \in \underset{x \in \mathfrak{R}^n}{\mathrm{argmax}}(U_k(x) \mid \pi' x \leq Y_k) \qquad (10.b)$$

3.3. <u>A Simple Example</u>

Consider an economy with four commodities, three producers and two consumers. The commodities include two output goods (manufactures and services) and two factors of production (labor and capital).

The consumers, capital owners and workers, are distinguished by both factor endowments and preferences. The capital owner's utility function is Cobb-Douglas while the worker's is nested CES. In the nested function, leisure and consumption enter at the top level.

Excess demand functions are determined by the following utility maximization problems:

<u>Owners</u>

$$\max_{d} \quad A\, d_m^{\gamma} \; d_s^{1-\gamma} \tag{11}$$

subject to:

$$\pi_m \, d_m + \pi_s \, d_s \le \pi_k \, K$$

<u>Workers</u>

$$\max_{d} \quad [\; \alpha_0 \, d_\ell^{\rho} \; + \; \alpha_1 \, (d_m^{\beta} \quad d_s^{1-\beta})^{\rho} \;]^{1/\rho} \tag{12}$$

subject to:

$$\pi_1 \, d_\ell \; + \; \pi_m \, d_m \; + \; \pi_s \, d_s \; \le \; \pi_1 \, L$$

There is one producer associated with each of the two output goods, and a third producer who produces these two goods as joint products. Production functions are nested CES.

Manufactured goods requires inputs of capital and labor. Services requires intermediate inputs of manufactured goods in addition to the primary factors. The joint production sector, which we will call "hi-tech", requires inputs of services in addition to primary factors. The production input functions are written:[7]

$$f_m \, (\ell, k) \quad = \quad [\; \gamma_0 \, \ell^{\rho} \; + \; \gamma_1 \, k^{\rho} \;]^{1/\rho} \tag{13}$$

---

[7] α, β, ρ and γ are used as "generic" coefficients, bearing no relation to the coefficients in (11) and (12).

$$f_s(m,\ell,k) = [\ \alpha_0\ m^{\rho} + \alpha_1\ (\ \beta_0\ \ell^{\rho_1} + \beta_1\ k^{\rho_1})^{\rho/\rho_1}\ ]^{1/\rho} \tag{14}$$

$$f_h(s,\ell,k) = A\ s^{\theta}\ [\ \min(\frac{\ell}{\eta_0}, \frac{k}{\eta_1})\ ]^{1-\theta} \tag{15}$$

The first two sectors are conventional single product technologies. The third sector, however, is assumed to produce manufactured goods and services as fixed-coefficient joint products. The output transformation function is:

$$g_h(m,s) = \max\ [\ \frac{m}{\kappa_0}\ ,\ \frac{s}{\kappa_1}\ ] \tag{16}$$

Following from constant returns to scale, profit maximization determines unit outpus and inputs. Producers solve the following constrained maximization problems:

Goods

$$\max_{h}\quad \pi_m - \pi_1\ h_{\ell m} - \pi_k\ h_{km} \tag{17}$$

subject to:

$$f_m(\ h_{\ell m},\ h_{km}) = 1$$

Services

$$\max_{h}\quad \pi_s - \pi_m\ h_{ms} - \pi_1\ h_{\ell s} - \pi_k\ h_{ks} \tag{18}$$

subject to:

$$f_s(\ h_{ms},\ h_{\ell s},\ h_{ks}) = 1$$

Hi-Tech

$$\max_{q,h}\quad \pi_s\ q_{sh} + \pi_m\ q_{mh} - \pi_s\ h_{sh} - \pi_1\ h_{\ell h} - \pi_k\ h_{kh} \tag{19}$$

subject to:

$$g_h(q_{sh}, q_{mh}) = 1$$

$$f_h(h_{sh}, h_{\ell h}, h_{kh}) = 1$$

Subproblems summarized in (11) through (19) complete an algebraic description of the model. Before solving the model, we consider the process by which function coefficients might be specified.

From an operational standpoint, the CES function offers the advantage of requiring relatively few data for its specification. It can be determined from (i) a set of micro-consistent data for the economy, and (ii) estimates of key elasticities of substitution and transformation. MPS/GE assists in this benchmarking procedure. Function coefficients entering a MPS/GE data file can, for the most part be read directly from benchmark data.

Figure 1: Benchmark Data

	Net Supplies			Excess Demands	
	service	goods	hi-tech	workers	owners
svcs.	4	0	0	2	2
mfrs.	-1	6	0	4	1
labor	-2	-4	0	2-8	0
capital	-1	-2	0	0	-3

The "labor" entry in the column "workers" contains both positive and negative components. This entry is consistent with the assumption that labor supply is price-responsive, with leisure entering the workers' utility function. In the benchmark, there are 8 units of labor available, of which 2 are "consumed" in the form of leisure.

The financial flows alone are insufficient to determine the function coefficients. To complete the model specification, it is necessary to provide slope information in the form of elasticities of substitution. These numbers, shown in Figures 2 and 3, would presumeably be determined by econometric estimation or by literature search.

Figure 2: Utility Function Elasticities

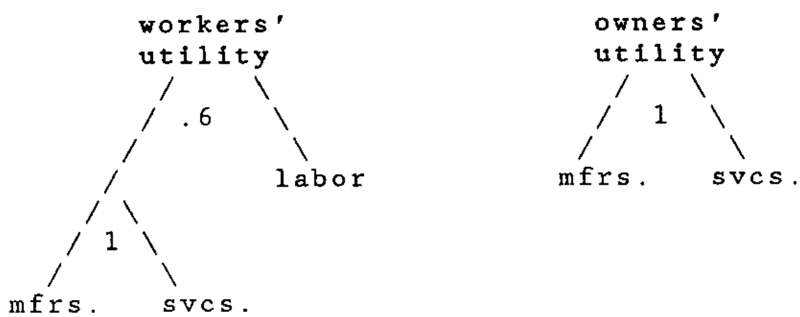
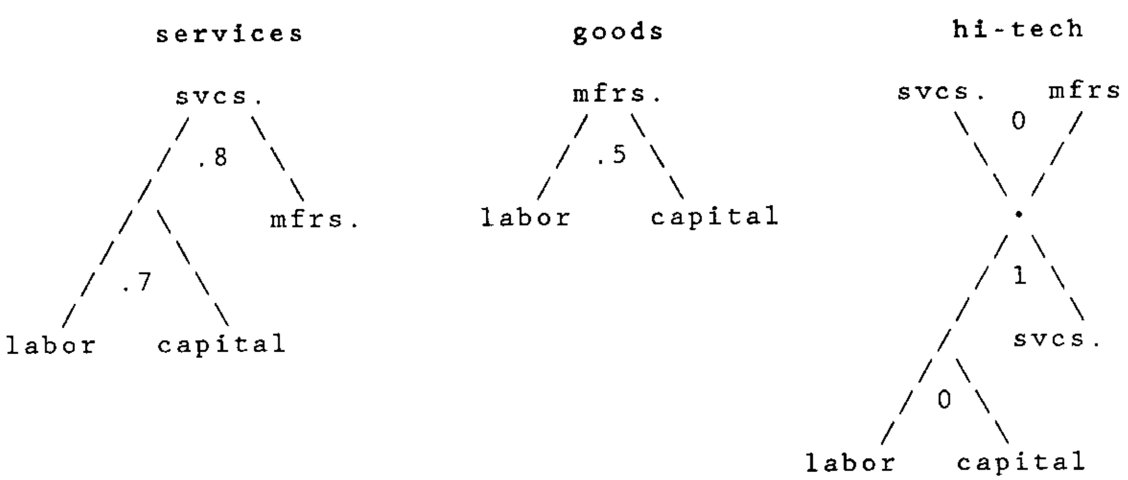


Figure 3: Production Function Elasticities



part, be read directly from the benchmark financial flows. To illustrate, suppose that data for the fictitious economy of our example are summarized by the matrix shown in Figure 1.

In Figures 2 and 3, numbers indicate elasticities of substitution between component inputs into the related aggregate. The structure and parameters of this model are described by the MPS/GE data file of Figure 4. In this file, all upper-case words are keywords. Price-responsive coefficients (identified as fields "X:" and "P:" in the PROD and DEMAND blocks) are quantity-price pairs which may be derived directly from the benchmark year's financial flows.[8] For example, the benchmark demand by owners for manufactured goods equals unity and appears as "D:mfrs. X: 1. P: 1." in the data file.

The hi-tech production function is not described by the benchmark equilibrium. We assume that when optimally adjusted to benchmark prices, the sector produces .2 units of services and .8 units of goods with inputs of .5, .5, and .1 for labor, capital and services, respectively. (Notice that the sector would incur a unit loss of .1 at benchmark prices, and therefore it is not operated.)

---

[8] Consider the Cobb-Douglas production function:

$$f(K,L) = A\ K^{\alpha}\ L^{1-\alpha}$$

This function can be equivalently described by parameters (A, α) or by inputs (L,K) for reference prices of unity.

Figure 4: Input Data File for 3 Sector Example

```
$MODEL:example

$SECTORS:
     services  4.  goods 6.  hi-tech 0.

$COMMODITIES:
     svcs.  1. mfrs. 1. labor 1.  capital .8

$CONSUMERS:
     workers owners

$DEMAND:workers     s: .6      a: 1.

          E:labor   X:  8.
          D:labor   X:  1.    P: 1.
          D:svcs.   X:  1.    P: 1.  a:
          D:mfrs.   X:  2.    P: 1.  a:

$DEMAND:owners  s: .8

          E:capital      X:  3.
          D:svcs.        X:  2.  P: 1.
          D:mfrs.        X:  1.  P: 1.

$PROD: services  s:.8    a:.7

          O:svcs.
          I:mfrs.        X: .25  P: 1.
          I:labor        X: .5   P: 1.   a:
          I:capital      X: .25  P: 1.   a:

$PROD: goods    s:.5

          O:mfrs.
          I:labor        X:  .66667  P: 1.
          I:capital      X:  .33333  P: 1.

$PROD: hi-tech     s: 1.   t: 0.     a: 0.

          O:svcs.        X:  .2
          O:mfrs.        X:  .8
          I:svcs.        X:  .1   P: 1.
          I:labor        X:  .5   P: 1.  a:
          I:capital      X:  .5   P: 1.  a:
```

Equilibrium conditions for this model are are shown in Figure 5. These correspond to conditions (9a), (9b) and (9c) for the "generic" Walrasian model.

Figure 5: Equilibrium Conditions: 3 Sector Model

Excess profit:

$$-\pi_s + h_{ms}\pi_m + h_{ls}\pi_l + h_{ks}\pi_k \geq 0 \quad \perp \; y_s \geq 0$$
$$-\pi_m + h_{lm}\pi_l + h_{km}\pi_k \geq 0 \quad \perp \; y_m \geq 0$$
$$(h_{sh} - q_{sh})\pi_s - q_{mh}\pi_m + h_{lh}\pi_l + h_{kh}\pi_k \geq 0 \quad \perp \; y_h \geq 0$$

Market clearance:

$$y_s + (q_{sh} - h_{sh})\, y_h \geq \xi_s(\pi, Y) \quad \perp \; \pi_s \geq 0$$
$$-h_{ms}\, y_s + y_m + q_{mh}\, y_h \geq \xi_m(\pi, Y) \quad \perp \; \pi_m \geq 0$$
$$-h_{ls}\, y_s - h_{lm}\, y_m - h_{lh}\, y_h \geq \xi_l(\pi, Y) \quad \perp \; \pi_l \geq 0$$
$$-h_{ks}\, y_s - h_{km}\, y_m - h_{kh}\, y_h \geq \xi_k(\pi, Y) \quad \perp \; \pi_k \geq 0$$

Income:

$$L\,\pi_l = Y_w$$
$$K\,\pi_k = Y_o$$

In Figure 5, prices, activities and intermediate input coefficients are identified by indices s, m, n, k and l standing, respectively, for services, manufactures, hi-tech, capital and labor. For example, $h_{ms}\, y_s$ represents the inputs of manufactured goods into the production of services. Y_o and Y_w stand for owners' and workers' income levels. Excess demands functions are given by:

$$\xi_s(\pi, Y) = d_{sw}(\pi, Y_w) + d_{so}(\pi, Y_o) \qquad (19)$$
$$\xi_m(\pi, Y) = d_{mw}(\pi, Y_w) + d_{mo}(\pi, Y_o)$$
$$\xi_l(\pi, Y) = d_{lw}(\pi, Y_w) - L$$
$$\xi_k(\pi, Y) = -K$$

For the most part, intermediate and final demand functions are tedious algebraic expressions.[9] An exception is owners' final demand. These Cobb-Douglas functions are:

$$d_{so}(\pi, Y_o) = \frac{Y_o}{3\ \pi_s} \qquad d_{mo}(\pi, Y_o) = \frac{2\ Y_o}{3\ \pi_m} \tag{20}$$

The input file reproduces the benchmark year's equilibrium. To perform a counterfactual numerical experiment, one or more coefficients must be changed from its benchmark value. Consider, for example, the effect of increasing efficiency in the hi-tech sector. Suppose that the output coefficient for mfrs. in that sector increases from .8 to 1. The resulting equilibrium is shown in the Figure 6.[10]

Figure 6: Increased Hi-Tech Efficiency

		Net Supplies			Excess Demands	
	Price	service	goods	hi-tech	workers	owners
svcs.	1.	4.239	0	.207	1.717	2.728
mfrs.	1.004	-1.056	3.555	2.282	3.421	1.360
labor	.844	-2.386	-2.584	-1.131	-6.101	0
capital	1.364	-.853	-1.016	-1.131	0	-3.000

[9] They are analytic solutions to the constrained optimization problems (11) through (18).

[10] The underlined terms represent supply and demand of the same commodity by a single agent. The hi-tech sector produces .456 units of services and consumes .250. The worker is endowed with 8 units of labor, of which s/he retains 1.899 units as leisure.

In this hypothetical example, the improvement in hi-tech productivity causes a substantial shift in factor prices, raising the cost of capital and lowering the wage rate. The aggregate output of services increases by over 10%, and industrial production remains roughly constant. In the process, workers lose and owners gain from technical change.

4. EXTENSIONS

MPS/GE accomodates AGE models with exogenous ad-valorem taxes on production and lump-sum transfers. It also permits constraints on relative prices with associated endogenous subsidy rationing of producers or endowment rationing of consumers. These features extend the neoclassical model and are essential for many empirical applications of AGE models.

4.1. Ad-Valorem Taxes

Taxes on production create an additional link between production level and income. Tax payments typically accrue to a "government consumer". In MPS/GE framework, Walras' law applies to this consumer, and it is essential to assume that government expenditures include discretionary demands in addition to subsidy payments and lump-sum transfers. The former then adjust so that the value of net expenditures always equals the net value of tax revenues. In the static, counterfactual modeling format permits government budget deficits only when they are specified as ex-ante.

Because the producer is free to select input and output coefficients (subject to technical constraints), the imposition of an input tax can cause a shift in production technique. The "user cost" of commodity i in sector j reflects the imposed taxes. That is, producer j determines input and output coefficients using the factor prices inclusive of taxes:

$$h_j(\pi) \in \underset{x \in \mathbb{R}^n}{\operatorname{argmin}} \left(\sum_i \pi_i \; (1 + t_{ij}) \; x_i \;\middle|\; f_j(x) = 1 \right) \qquad (21.a)$$

$$q_j(\pi) \in \underset{x \in \mathbb{R}^n}{\operatorname{argmax}} \left(\sum_i \pi_i \; (1 + s_{ij}) \; x_i \;\middle|\; g_j(x) = 1 \right) \qquad (21.b)$$

When a complete set of taxes are imposed, the equilibrium conditions for the economy as a whole are represented by Figure 7. In this representation, tax payments appear in the income constraint of the government agent.

To introduce ad-valorem taxes on consumer demands, a production function producing an aggregate consumable must be introduced. Taxes may then be levied on this producer. This is always possible because production and utility functions are both nested CES.

Figure 7: Market Equilibrium with Ad-Valorem Taxes

Excess profit:

$$\sum_i \pi_i \; [\; h_{ij} \; (1 + t_{ij}) \; - \; q_{ij} \; (1 + s_{ij}) \;] \geq 0 \qquad \perp \; y_j \geq 0$$

Market clearance:

$$\sum_j a_{ij} \; y_j \qquad \geq \xi_i(\pi, Y) \quad \perp \; \pi_i \geq 0$$

Income:

$$\sum_i \pi_i \quad b_{ik} \quad = \quad Y_k \qquad \text{(Household k)}$$

$$\sum_j y_j \quad (h_{ij} \; t_{ij} \; - \; q_{ij} \; s_{ij}) \quad = \quad Y_G \qquad \text{(Government)}$$

4.2. <u>Price constraints</u>

There are several situations in which it is convenient to impose bounds on relative prices or value flows. Within MPS/GE, these constraints are supported by auxiliary variables which either ration consumer endowments or adjust ad-valorem tax rates.

We represent auxiliary variables by $\mu \in \Re^L$ and index them with ℓ and k. These variables are non-negative and complementary with an inequality constraint.[11] Constraints may be composed of four types of terms. These are:

---

[11] In the present version of MPS/GE, any equality constraint must be represented by a pair of inequality constraints. Corresponding free variables μ are thereby represented as the difference of two nonnegative variables: $\mu = \mu^+ - \mu^-$.

(i) **Simple terms**, consisting of a commodity price multiplied by a constant: $\pi_i \alpha_{i1}$.

(ii) **Bilinear terms**, consisting of a commodity price times an auxiliary variable times a constant: $\pi_i \mu_k \beta_{ik}$.

(iii) **Production value terms**, consisting of the value of a production input or output multiplied times a constant: $y_j \pi_i x_{ij} \gamma_{ij1}$;[12] and

(iv) **Bilinear production values**, consisting of the value of a production input or output times an auxiliary variable times a constant: $y_j \pi_i x_{ij} \mu_k \tau_{ijkl}$.

An auxiliary constraint may contain any combination of these terms. Because they are linear in commodity prices, the constraints are invariant under scaling of the numeraire price. Furthermore, complementary slackness obtains between the associated non-negative auxiliary variable and the auxiliary constraint. Only when the constraint is binding does the associated variable increase from zero.

The general form of auxiliary constraint ℓ is given by:

$$\sum_i \pi_i \left(\alpha_{i\ell} + \sum_k \beta_{ik\ell} \mu_k + \sum_j y_j x_{ij} \left(\gamma_{ij\ell} + \sum_k \mu_k \tau_{ijk\ell} \right) \right) \geq 0 \qquad (22)$$

To illustrate the application of these constraints, consider a model similar to that of Malinvaud [1977]. There

12 In (iii) and (iv), x_{ij} is an input or output coefficient for commodity i in sector j. In either case, it is a positive quantity.

are three commodities (labor, capital and output), and one production sector (macro output). Macro production is Cobb-Douglas, and leisure enters a Cobb-Douglas utility function for employed consumers.

For concreteness, assume that the following "stylized facts" represent a benchmark (dis)equilibrium for the economy:

Gross output	10
Capital's value share of GNP	20%
Capital:output ratio	2
Unemployment rate	10%
Leisure value-share	20%

In this model, a distinction is made between *voluntary* and *involuntary* unemployment. The former arises from leisure demand, while the later is associated with rationing to support the real wage. Let L denote the total labor supply, μ the endogenously-determined unemployment rate, K the capital stock, p the output price, w the nominal wage rate and r the rental rate for capital services.

Assuming no difference in capital ownership between employed and unemployed households, the employed consumers' consumption demands (C_e) and leisure demands (ℓ) are determined by solving the following utility maximization problem:

Employed Households

$$\max \; C_e^{.8} \; \ell^{.2} \qquad (23)$$

subject to:

$$p \, C_e + w \, \ell \;\leq\; (1 - \mu) \; (w \, L + r \, K)$$

The unemployed households have the same utility function, but their leisure demand is fixed and income is devoted entirely to consumption (C_u). They solve the utility maximization problem:

<u>Unemployed Households</u>

$$\max \; C_u^{.8} \, \ell_u^{.2} \qquad (24)$$

subject to:

$$p \, C_u \le \mu \, r \, K$$

$$\ell_u \le \mu \, L$$

Figure 8: Classical Unemployment

Excess profit:			
	$- p + a_k r + a_\ell w$	≥ 0	$\perp \; y \ge 0$
Market clearance:			
	y	$\ge C_e + C_u$	$\perp \; p \ge 0$
	$- a_k y$	$\ge -K^e$	$\perp \; r \ge 0$
	$- a_\ell y \qquad\qquad - L\mu$	$\ge \ell - L$	$\perp \; w \ge 0$
Wage constraint:			
	$- \bar{w} p + w$	≥ 0	$\perp \; \mu \ge 0$
Income:			
	$(1-\mu)(L w + K r)$	$= Y^e$	
	$\mu K r$	$= Y_u$	

The schema in Figure 8 summarizes equilibrium conditions, including the auxiliary constraint on relative prices. This constraint maintains the real wage (w/p) above $\bar{w}$. A decline in the real wage (due to w decreasing or p increasing) causes the wage constraint to bind and μ to increase from zero and reducing the supply of labor. This supports the wage of workers who remain employed. The

MPS/GE data file which reproduces the benchmark equilibrium is displayed in Figure 9.[13]

Whenever bounds are introduced on relative prices, the model formulation specifies which side of the market is restricted. The classical unemployment example *rations consumer endowments while producers' notional demands are realized.* Under different institutional assumptions, demand rather than supply can adjust to accomodate the wage constraint. One such market structure involves an auxiliary variable, s, interpreted as the endogenously-determined rate of subsidy on labor inputs to production. Figure 10 displays a schema representing this alternative market structure which can also be represented in MPS/GE.

---

[13] The labor supply (L = 11.61) derives from the benchmark data. Assuming equilibrium, labor demand (8) must equal labor supply (.9 L - ℓ). Also, the value of leisure demand (ℓ) must equal 20% of employed household expenditures ($(C_e + \ell)$).

Figure 9: Input Data File: Unemployment Model

```
$MODEL:unemplmt

$SECTORS:
         macro  6.

$COMMODITIES:
         output  labor  capital  .10

$AUXILIARY:
         unemplmt

$CONSUMERS:
         employed   idle

$DEMAND:employed    s: 1.

         D:output      X:  .8
         D:labor       X:  .2

         E:labor       X:  11.61
         E:capital     X:  20.

         E:labor       X: -11.61  R:unemplmt
         E:capital     X: -20.    R:unemplmt

$DEMAND:idle    s:1.

         D:output
         E:capital     X:  20.       R:unemplmt

$PROD:macro    s:  1.

         O:output
         I:labor       X:    .8
         I:capital     X:   2.        P:.1

$CONSTRAINT:unemplmt

         C:labor
         C:output      X:-1.1
```

Figure 10: Subsidy Rationing of Labor Demand

Excess profit:		
$-p + a_k\, r + a_\ell\, (1-s)\, w$	≥ 0	$\perp\ y \geq 0$
Market clearance:		
y	$\geq C$	$\perp\ p \geq 0$
$-a_k\, y$	$\geq -K$	$\perp\ r \geq 0$
$-a_\ell\, y$	$\geq -L$	$\perp\ w \geq 0$
Wage constraint:		
$-\bar{w}\, p + w$	≥ 0	$\perp\ s \geq 0$
Income:		
$-a_\ell\, s\, w\, y + L\, w + K\, r$	$= Y$	

4.3. <u>Other Extensions</u>

• Multi-level nesting.

In MPS/GE, production input and utility functions may be one or two level CES. Functions with more than two levels may be handled in a straightforward way by decomposition into two or more sectors. For example, consider the following function:

```
                GNP
               /   \
              •     energy
             / \
            •   imports
           / \
     capital   labor
```

This three-level production function can be introduced in an MPS/GE model when two additional dimensions are added to the model. The first is a commodity named "GDP" and the second an activity producing GDP with capital and labor inputs. This decomposes the three-level function into one two-level function and one single-level function.

```
              GNP                      GDP
             /   \                    /   \
            •     energy      capital    labor
           /  \
        GDP    imports
```

• Decreasing Returns to Scale

A production function exhibits decreasing returns to scale (DRS) if factor demands increase superlinearly with output (i.e., $f(\lambda x) < \lambda f(x)$). In most cases, DRS production functions can be represented by constant returns to scale (CRS) functions in which a sector-specific input is in fixed supply. Ownership of this input determines allocation of rents associated with the DRS sector.

• Non-Homothetic Demand

Commodity endowments may be either positive or negative in MPS/GE. They may thereby be used to shift the origin of the excess demand functions, producing non-unitary income elasticities. Addition (or subtraction) of equal quantities from commodity endowments and reference demands changes the price elasticity of final demand for a commodity, but it does not alter the aggregate excess demand at reference prices.

5. IMPLEMENTATION AND PERFORMANCE

Mathiesen [1985a] proposed a modeling format and sequential method for solving market equilibrium models. The method is named SLCP because it involves a "Sequence of Linear Complementarity Problems". Preckel [1982] reported favorable numerical experience with SLCP compared to other solution methods. The MPS/GE implementation of SLCP confirms earlier results. This section provides a brief description of the algorithm and then reports on computational experience with several large-scale models.

5.1. <u>The SLCP Algorithm</u>

We begin by observing that the equilibrium conditions for the Walrasian model, (9a)-(9c), represent a nonlinear complementarity problem (NLCP). The general NLCP is the following:

Given $F: \Re^N \rightarrow \Re^N$

Find $z \in \Re^N$, $z \geq 0$ such that $F(z) \geq 0$ $z' F(z) = 0$

In the MPS/GE representation, the unknown vector z represents activity levels (y), prices (π), auxiliary variables (μ) and income levels (Y), so $N = m + n + L + p$. The nonlinear function $F(y, \pi, \mu, Y)$ correspondingly represents unit profit-, market excess demand, auxiliary constraint and income-functions.

To solve the nonlinear complementarity problem, SLCP repeatedly solves linear complementarity problems (LCPs). These are defined as:

Given $q \in \Re^N$, $M \in \Re^{NxN}$

Find $z \in \Re^N$, $z \geq 0$ such that

$$q + M z \geq 0 \qquad z' (q + M z) = 0$$

The LCP is a fundamental problem of mathematical prgramming. It contains linear and quadratic programs as special cases. (See Cottle and Dantzig [1968].)

The data for SLCP subproblems are determined by taking a first-order Taylor expansion of the nonlinear function, F. This linearization is given by:

$$q(\bar{z}) = F(\bar{z}) - \nabla F(\bar{z})\,\bar{z} \qquad M(\bar{z}) = \nabla F(\bar{z})$$

Homogeneity of excess demand implies that near the equilibrium, the basis associated with nonzero prices and activity levels is rank deficient. There are alternative methods for dealing with this difficulty. One could, for example, add a constraint that the prices lie in the unit simplex. In our implementation, we fix a numeraire price and drop the associated market clearance constraint. At equilibrium, the omitted market clears through Walras' law. Subproblem data are denoted $(q_i(\bar{z}), M_i(\bar{z}))$, indicating both the linearization point, $\bar{z}$, and the numeraire index, i.

The SLCP algorithm may be viewed as an extension of Newton's classical method for solving simultaneous nonlinear equations. The extension accomodates both linear and nonlinear inequalities. Applied to general equilibrium models, SLCP involves the following steps:

1. Select a starting point, z_0, and set the iteration counter k to zero.
2. Stipulate a numeraire commodity, index i.
3. Start iteration k+1, with $k \leftarrow k+1$.

4. Construct $q_i(z_{k-1})$ and $M_i(z_{k-1})$.
5. Apply Lemke's algorithm to solve the subproblem. If no solution exists or the problem fails to solve, select another numeraire index and repeat step 4.
6. Given the LCP solution, z, conduct a line search to determine the step length α_k and the next iterate:

   $$z_k = \alpha_k z + (1-\alpha_k) z_{k-1}.$$

   In the process of determining α_k, assess the current deviation from equilibrium.
7. If the deviation exceeds a prespecified convergence tolerance, ε, repeat steps 3-7.

The same algorithm may be used for partial equilibrium models. In those cases, supply and demand functions are expressed in nominal terms and Walras' law does not apply.

The convergence theory for this algorithm applied to the general equilibrium model is not well developed. For the partial equilibrium model, where ∇F is positive semidefinite, it has been proven that the algorithm converges from an arbitrary starting point and that the rate of convergence is quadratic. (See Pang and Chan [1983].) Due to income effects, the general equilibrium model fails to satisfy the conditions under which convergence can be guaranteed. We have, however, gathered substantial empirical evidence of the algorithm's efficiency and robustness. (See Mathiesen and Rutherford [1983].) For many models which have been formulated with MPS/GE, there is no comparable solution method from the standpoint of efficiency. Given our commitment to modeling on the microcomputer, we feel that SLCP is the best and perhaps only alternative for general purpose applications.

The implementation of SLCP developed for MPS/GE relies on LUSOL, a library of in-core basis factorization and update routines originally developed for the nonlinear programming code MINOS. (See Gill, Murray, Wright and Saunders [1986].) LUSOL is an integral component of the LCP solver in MPS/GE. The LCP routine, developed explicitly for solving subproblems, accepts an advanced basis so that in later iterations when the active set stabilizes, Lemke's algorithm is bypassed and the sequence of iterates is equivalent to Newton steps on the binding inequalities.

An industrious model builder can access the SLCP code directly by providing subroutines which evaluate $F(z)$ and $\nabla F(z)$. These subroutines, however, are tedious to write and debug. MPS/GE automatically generates functions and analytic gradients, reducing the work needed for model implementation and assuring optimum performance from the solution algorithm.

5.2. Performance

Figure 11 displays solution statistics for a variety of models which have been implemented with MPS/GE. From the standpoint of computational cost, several characteristics seem to be relevent, in particular:

• the number of commodities, activities, consumers and institutional constraints;

• the complexity of supply and demand functions, as evidenced by the density of the linearized system;

• the model structure, determined by the types of constraints and the pattern of nonzeros in the linearization.

The first two models in Figure 11 are the standard test problems presented in Scarf [1973]. They are considerably smaller than the empirical models, and this suggests that they should not be used as the basis for demonstrating an algorithm's effectiveness.

The third problem is a highly-stylized spatial equilibrium model due to von Thunen, a 19th century economist. His model was generalized in a partial equilibrium format by MacKinnon [1976]. Rowse [1981] and Mathiesen [1985] also report numerical results with MacKinnon's model. The version implemented with MPS/GE is an extension of MacKinnon's test problem in which income flows are closed. (See Rutherford [1986] for details.)

Figure 11: Solution Statistics for Selected MPS/GE Models

Model	S	M	C	A	NAEL	Iters.	Time[14]
SCARF	8	6	5	0	126	3	:28
HANSEN	26	14	4	0	393	3	:49
VON THUNEN	72	24	3	0	729	6	5:48
VEMOD	81	86	6	0	1501	4	****
LTM-3	63	83	3	6	2032	13	30:14
LTM-4	72	98	4	0	1449	12	18:23
MISMOD	68	103	13	0	2303	10	*****
GEMTAP	62	65	14	0	2546	8	23:11
CAMEROON	49	73	4	0	1026	5	6:49

Key:

S	Production sectors.	NAEL	Nonzeros in ∇F.
M	Commodity markets.	Iters	SLCP iterations.
C	Consumers.	Time	Min:sec on 4.77 MHz
A	Auxiliary constraints.		IBM PC with 8087.

14 The time and number of iterations required to solve a particular scenario depends to a large degree on the availability of a good starting point. The values shown here for LTM-3, LTM-4 and GEMTAP are for "cold" starts. They are 13:30, 7:37 and 6:49, respectively, when started from related solutions.

Times are not reported for VEMOD and MISMOD because they use an earlier version of the program for which timing calls were not included.

VEMOD is a recursively dynamic model of international trade. It was a central focus of a project undertaken at the Norwegian School of Economics' Center for Applied Research. (See Haaland, Norman, Rutherford and Wergeland [1986].) The model is designed to expore the theory of comparative advantage in a large-scale Heckscher-Ohlin model. The data set referenced in Figure 11 identifies 6 regions, 3 primary factors and 5 traded goods. The difficulties encountered in developing this model inspired development of MPS/GE.

LTM is an intertemporal model of oil markets, economic growth and balance of payment constraints. (See Manne and Rutherford [1985] and Rutherford [1986].) In this model an equilibrium is obtained only when consumers and producers are consistent in their expectations of future prices and quantities. The data sets reported here include 5 time periods, representing 1990 through 2020 in 10 year intervals. LTM is intended for studying the interrelated issues of debt service and oil depletion. The three region version of the model studies a world economy in which the U.S., Mexico and an aggregate "rest-of-world" are distinguished. The four region version aggregates the market economies into U.S., other OECD, OPEC and non-OPEC developing countries. The three region version of the model, designed to explore policy alternatives for Mexican economic growth, includes period-by-period trade balance constraints. These constraints, supported by endogenous import premia, add considerable complexity to the model and increase density of the linearized system.

MISMOD is a general equilibrium model of the Norwegian economy which was developed for the Norwegian Industry Department. (See Mathiesen [1986].) The model provides a detailed representation of Norwegian industry, trade, taxes and demographic structure. It includes endogenous (wage-responsive) labor migration between regions of the country and labor categories. This is the largest and most detailed model yet constructed with MPS/GE.

GEMTAP is a model of the U.S. economy and tax system which has been used in a number of policy studies. (See Shoven and Whalley [1972] and Ballard et al. [1985].) The model contains a detailed representation of industrial structure and tax distortions. It includes ad-valorem taxes on production and consumption, linear income taxes and taxes on factor payments.

CAMEROON is a model of the Cameroon economy developed at the World Bank. (See Condon, Dahl and Devarajan [1986].) It is formulated in the style of Dervis, de Melo and Robinson [1982], including a number of income groups, taxes and wage distortions with alternative closure rules for foreign exchange.

6. CONCLUSION

We have described a modeling format for applied general equilibrium analysis of large- and small-scale models on microcomputers. The basis for the system is Mathiesen's SLCP algorithm which efficiently solves for equilibrium prices and quantities. The system provides a model builder with access to the algorithm without writing model-specific code. By eliminating the time consuming and tedious tasks involved in representing and altering functional forms, MPS/GE enhances the usefulness of the SLCP algorithm. The development of a general purpose representation for AGE models has also provided a synthesis of the economic features which play a role in empirical AGE models.

References

Adelman, I. and S. Robinson (1978), *Income Distribution Policies in Developing Countries*, Stanford University Press.

Ballard, C., D. Fullerton, J.B. Shoven and J. Whalley (1985), *A General Equilibrium Model for Tax Policy Evaluation*, National Bureau of Economic Research Monograph, The University of Chicago Press.

Broadie, M. N. (1983a), "OCTASOLV user's guide," Systems Optimization Laboratory Technical Resport No. 83-9, Department of Operations Research, Stanford University.

Broadie, M. N. (1983b), "Subdivisions and antiprisms for PL homotopy algorithms," Ph.D. thesis, Department of Operations Research, Stanford University.

Cottle, R.W. and G.B. Dantzig (1968), "Complementary pivot theory of mathematical programming," *Linear Algebra and its Applications*, 1:103-125.

de Boer, P. M. C. (1982), *Price Effects in Input-Output Relations: A Theoretical and Empirical Study for the Netherlands 1949-1967*, Springer-Verlag.

Dervis, K., J. de Melo and S. Robinson (1982), *General Equilibrium Models of Development Policy*, Cabridge University Press.

Dixit A.K. and V. Norman (1980), *Theory of International Trade: A Dual, General Equilibrium Approach*, Cabridge University Press.

Drud, A., D. Kendrick and A. Meeraus (1986), "Hercules: A system for development of multisectoral economywide models," mimeo, World Bank Development Research Department.

Eaves, B.C. (1978), "A locally quadratically convergent algorithm for computing stationary points," Technical Report SOL 78-13, Department of Operations Research, Stanford University.

Feltenstein, A. (1984), "Money and bonds in a disaggregated open economy", in Scarf and Shoven (eds.) *Applied General Equilibrium Analysis*.

Gill, P., W. Murray, M. Saunders and M. Wright (1986), "Maintaining LU factors of a general sparse matrix," Technical Report SOL 86-8, Department of Operations Research, Stanford University.

Ginsburgh, V., and J. Waelbroeck (1980), *Activity Analysis and General Equilibrium Modeling*, North-Holland, Amsterdam.

Haaland, J., V. Norman, T. Rutherford and T. Wergeland (1986), "VEMOD: A Ricardo-Heckscher-Ohlin-Jones Model of World Trade", working paper MU-11/86, Center for Applied Research, Norwegian School of Economics, Bergen, Norway.

Harberger, A.C. (1964), "Taxation, resource allocation and welfare," in *The Role of Direct and Indirect Taxes in the Federal Reserve System*, Princeton University Press.

Harris, R. (1985), "Applied general equilibrium analysis of small open economies with scale economies and imperfect competition," *American Economic Review*, 74:1016-1032.

Hogan, W. W. (1975), "Energy policy models for project independence," *Computation and Operations Research*, 2:251-271.

Lemke, L. (1965), "Bimatrix equilibrium points and mathematical programming," *Management Science*, 11:681-689.

MacKinnon, J. (1976), "A technique for the solution of spatial equilibrium models," *Journal of Regional Science*, (16) 3:293-307.

Manne, A. and T. Rutherford, "LTM: A Long-Term Model of Mexico's Growth and Balance-of-Payment Constraints," Department of Operations Research, Stanford University.

Malinvaud, E. (1977), *The Theory of Unemployment Reconsidered*, Basil Blackewll.

Mathiesen, L. (1986), "A Model for the Analysis of the Economic Effects on Sectoral Income of Changes in Norwegian Taxes, Support and Regulations" (in Norwegian), Center for Applied Research Report No. 4'86, Norwegian School of Economics.

Mathiesen, L. (1985a), "Computational experience in solving equilibrium models by a sequence of linear complementarity problems," *Operations Research*, (33) 6:1225-1250.

Mathiesen, L. (1985b), "Computation of economic equilibria by a sequence of linear complementarity problems," *Mathematical Programming Study*, 23, North-Holland.

Mathiesen, L., and T. Rutherford (1983), "Testing the robustness of an iterative LCP algorithm for solving Walrasian equilibrium models," Discussion paper 0883, Norwegian School of Economics and Business Administration, Bergen, Norway.

Meeraus, A. (1983), "An algebraic approach to modeling," *Journal of Economic Dynamics and Control*, Volume 5, No. 1, February.

Merrill, O. H. (1972), "Applications and extensions of an algorithm that computes fixed points to certain upper semi-comtinuous point-to-set mappins, Ph.D. dissertation, Department of Industrial Engineering, University of Michigan.

Pang, J. S. and D. Chan (1982), "Iterative methods for variational and complementarity problems," *Mathematical Programming*, 24 : 284-313.

Pearson, K. R. (1986), "Automating the computation of solutions of large economic models," IMPACT Research Centre, Preliminary Working Paper No. IP-27, Melbourne, Australia; March.

Petri, P. (1984), *Modeling Japanese-American Trade, A Study of Asymmetric Interdependence*, Harvard University Press.

Pigott, J. and J. Whalley (1985), *New Developments in General Equilibrium Analysis*, Cambridge University Press.

Powell, A.A. and F.H.G. Gruen (1968), "The constant elasticity of transformation production frontier and linear supply system," *International Economic Review* 9:315-328.

Preckel, P. (1983), "Intertemporal Equilibrium Models: Developments and Results," Ph.D. dissertation, Department of Operations Research, Stanford University.

Robinson, S. M. (1975), "A quadratically-convergent algorithm for general nonlinear programming problems," *Mathematical Programming Study*, 3, North-Holland.

Rowse, J. (1982), "On the Solution of Spatial Equilibrium Models," *Decision Science* 13:619-637.

Rutherford, T.F. (1987), "User's Guide to MPS/GE," Office of Technology Licensing, Stanford University.

Rutherford, T.F. (1986), "Applied General Equilibrium Modeling", Ph.D. dissertation, Department of Operations Research, Stanford University.

Rutherford, T.F. (1982), "Computing general equilibria in a complementarity format. NORGE: a general equilibrium model of the Norwegian economy," Engineer's Degree Thesis, Department of Operations Research, Stanford University.

Scarf, H. E., with collaboration of Terje Hansen, (1973) *The Computation of Economic Equilibria*, Yale University Press.

Scarf, H. E. and J. B. Shoven (eds.), (1984) *Applied General Equilibrium Analysis*, Cambridge University Press.

Shoven, J.B., and J. Whalley (1972), "A general equilibrium calculation of the effects of differential taxation of income from capital in the U.S.," *Journal of Public Economics*, 1:281-321.

Srinivasan, T. N. and J. Whalley (eds.), (1986), *General Equilibrium Trade Policy Modeling*, The MIT Press.

Todd, M. J. (1984), "Efficient methods of computing economic equilibria," in *Applied General Equilibrium Analysis*, H. E. Scarf and J. B. Shoven, eds., Cambridge University Press.

Todd, M.J. (1980), "PLAGO: a FORTRAN implementation of a piecewise-linear homotopy algorithm for solving systems of nonlinear equations," Technical Report No. 454, School of Operations Research and Industrial Engineering, Cornell University, Ithaca, New York (revised 1983, 1985).

van der Laan, G. and A. J. Talman, (1979), "A restart algorithm for computing fixed points without an extra dimension," *Mathematical Programming Study*, 17, North-Holland.

Whalley, J. (1986), "What have we learned from general equilibrium tax policy models?", working paper no. 8625c, Department of Economics, The University of Western Ontario.