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**EQUILIBRIA IN EXCHANGE ECONOMIES WITH A
COUNTABLE NUMBER OF AGENTS**

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EQUILIBRIA IN EXCHANGE ECONOMIES WITH A COUNTABLE NUMBER OF AGENTS

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The existence of equilibria is established in an overlapping generations exchange economy, where each generation lives for two periods and the commodity space is the positive cone of an infinite dimensional Riesz space. In particular, we establish the existence of equilibria in the stochastic overlapping generations model, i.e., we establish the existence of equilibria when the commodity space in each period is L_∞ equipped with the Mackey topology $\tau(L_\infty, L_1)$.

1. INTRODUCTION

P. Samuelson's consumption loan model [41] and its various extensions, i.e., overlapping generations models [7, 8, 22], constitute one of the two major paradigms in general equilibrium analysis; the other is, of course, the Arrow-Debreu model [6, 15]. For an insightful comparison of these two models see the work of J. Geanakoplos [19].

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An essential feature of overlapping generations models is that each generation's commodity space is a subset of a finite dimensional vector space. Consequently, uncertainty can only be included in these models if we posit a finite dimensional state space. This restrictive assumption precludes an overlapping generations analysis of financial markets modeled on L_2 -Hilbert spaces as in the recent work of J. H. Harrison and D. M. Kreps [24] or in the work of D. Duffie and C. F. Huang [17] or consideration of a stochastic overlapping generations model with commodity space L_∞ .

In this paper, we shall prove the existence of equilibria in overlapping generations models where each agent's consumption set lies in the positive cone of an infinite dimensional Riesz space. As such, our overlapping generations models are instances of exchange economies with a countable number of agents and an infinite dimensional commodity space. The first explicit treatment of overlapping generations models as special cases of economies with a "double infinity" of agents and commodities (or large square economies as they are called by J. Ostroy [33]) is due to C. A. Wilson [44]. Although Wilson's model is a special case of our more general analysis, his work has been the seminal influence on the research reported in this paper.

There has been a recent renaissance in the general equilibrium analysis of economies with infinite dimensional commodity spaces, since T. F. Bewley's path breaking work [10] in 1971. Most of this work on existence of equilibria has assumed either a finite number of agents as in [2, 28, 31, 34, 37, 45, 46] or a continuum of traders as in [48] or a measure space of agents as in [25, 30] or a nonstandard number of agents as in [12]. An exception is the recent paper by S. F. Richard and S. Srivastava [38], where they consider economies with a countable number of agents.

Similarly, the recent explosion of papers on overlapping generations models has been, for the most part, concerned with the issues of Pareto optimality [13, 14, 21] and indeterminacy [18, 20, 27]. In contrast, this paper is concerned with the existence of equilibria in exchange economies with a countable number of agents and an infinite dimensional Riesz space of commodities, with particular attention to the overlapping generations model.

Our method of proof derives from the arguments of T. F. Bewley [10], in that we consider a countable family of subeconomies where the n^{th} subeconomy consists of the first n agents. Using A. Mas-Colell's recent equilibrium existence theorem for finite exchange economies with a Riesz space of commodities [31], we construct a sequence of equilibria, one for each subeconomy. We must now take the "limit" of these allocations and prices. At this point, we have a choice of price normalizations. We can normalize prices such that the social endowment has unit value (in which case, the limit price may be a non-zero singular price that assigns zero value to each agent's initial endowment). Or we can normalize prices according to C. A. Wilson [44] so that the first agent's initial endowment has unit value (in which case, the limit price may assign infinite value to the social endowment).

For the general case, we adopt the first approach and prove the existence of a very weak equilibrium notion which we call a weak quasiequilibrium. In a weak quasiequilibrium, the limit price supports the limit allocation but budget equality may not hold for any agent.

For the special case of overlapping generations models, we adopt the second normalization. Our major innovation here is the explicit construction of the commodity space for the overlapping generations models as the inductive limit of subspaces which contain the consumption sets of each generation. This new space *does not* contain the social endowment. Moreover, the dual of this Riesz space is the space of prices which contains our limit price. Our construction of the commodity and price spaces for the overlapping generations models explicates the fundamental difference between these models and Arrow-Debreu models, i.e., the failure of Walras' law in overlapping generations models. In fact, since the social endowment does not lie in the commodity space of the agents, Walras' law is not even defined in our overlapping generations models. From this perspective, the surprising fact about overlapping generations models is not the suboptimality or indeterminacy of equilibria but rather the existence of equilibria. The construction of the commodity and price spaces as inductive and projective limits respectively, is basic to our analysis and its mathematical foundation is discussed in detail in section 3.

The intended interpretation of Samuelson's original Consumption Loan Model was an infinite horizon version of Irving Fisher's general equilibrium model of intertemporal exchange as explicated in Fisher's

classic work "The Theory of Interest". It is fitting that impatience or myopia which played such an important role in Fisher's analysis of capital markets should also be central to our study of intertemporal exchange. In section 4 of this paper, we introduce a notion of myopic preferences which is a proper generalization of the notions of impatience introduced by D. J. Brown and L. M. Lewis [11] and subsequent authors [32, 35].

The major result of this paper is Theorem 8.1. It simply asserts that if in an overlapping generations model each generation's consumption set is the positive cone of a Riesz space E so that E together with its dual E' constitute a symmetric Riesz-dual system, then equilibria always exist. Symmetric-Riesz-dual systems as models of the commodity-price duality were introduced into general equilibrium analysis by C. D. Aliprantis and D. J. Brown [1]. Special cases of symmetric Riesz dual systems are (1) L_∞ paired with L_1 and (2) L_2 paired with L_2 . Hence, our existence theorem demonstrates the existence of equilibria in stochastic overlapping generations models and in overlapping generations models with financial markets.

2. MATHEMATICAL PRELIMINARIES

This work will utilize the theory of Riesz spaces. For details and extensive treatments of the theory of Riesz spaces we refer the reader to the books [4, 5, 29, 43, 47]. We review briefly below the basic concepts needed for our study.

A Riesz space E is a partially ordered (real) vector space for which every finite set has a least upper bound or supremum (and also a greatest lower bound or infimum). The supremum and infimum of the set $\{x, y\}$ are denoted by $x \vee y$ and $x \wedge y$, respectively; i.e., $x \vee y = \sup\{x, y\}$ and $x \wedge y = \inf\{x, y\}$. If x is an element in a Riesz space, then the elements

$$x^+ := x \vee 0, \quad x^- := (-x) \vee 0 \quad \text{and} \quad |x| := x \vee (-x)$$

are called the positive part, the negative part and the absolute value of x , respectively. We have

$$x = x^+ - x^- \quad \text{and} \quad |x| = x^+ + x^-.$$

The cone of positive elements E^+ consists of all elements $x \in E$ with $x \geq 0$, i.e.,

$$E^+ = \{x \in E: x \geq 0\}.$$

The symbol $x > 0$ means $x \geq 0$ and $x \neq 0$.

For the rest of this section the letter E will denote a Riesz space. A subset A of E is said to be a *solid set* whenever $|x| \leq |y|$ and $y \in A$ imply $x \in A$. The *principal ideal* A_x generated by an element $x \in E$ is the smallest ideal containing x , and is precisely the set

$$A_x = \{y \in E: \exists \lambda > 0 \text{ with } |y| \leq \lambda|x|\}.$$

A net $\{x_\alpha\}$ of E is said to be *non-decreasing* (in symbols, $x_\alpha \uparrow$) whenever $\alpha \geq \beta$ implies $x_\alpha \geq x_\beta$ in E . The symbol $x_\alpha \uparrow x$ means that $x_\alpha \uparrow$ and $x = \sup\{x_\alpha\}$ both hold. The meanings of $x_\alpha \downarrow$ and $x_\alpha \downarrow x$ are similar. A net $\{x_\alpha\}$ is said to be *order convergent* to x (in symbols, $x_\alpha \xrightarrow{o} x$) whenever there exists another net $\{y_\alpha\}$ with the same indexed set satisfying $y_\alpha \downarrow 0$ and $|x_\alpha - x| \leq y_\alpha$ for all α . A subset A of E is said to be *order closed* whenever $\{x_\alpha\} \subseteq A$ and $x_\alpha \xrightarrow{o} x$ imply $x \in A$. An order closed ideal is

referred to as a *band*. A Riesz space is said to be *Dedekind complete* whenever $x_\alpha \uparrow x$ implies that the supremum of the set $\{x_\alpha\}$ exists in E .

Two elements x and y of E are said to be *disjoint* (in symbols, $x \perp y$) whenever $|x| \wedge |y| = 0$ holds. The disjoint complement of a non-empty subset D of E is the set

$$D^d = \{x \in E: x \perp y \forall y \in D\} = \{x \in E: |x| \wedge |y| = 0 \forall y \in D\}.$$

The disjoint complement D^d is always a band of E .

If $x \leq y$, then the set

$$[x, y] := \{z \in E: x \leq z \leq y\}$$

is known as an *order interval* of E . The subsets of the order intervals are referred to as *order bounded sets*. A linear functional $f: E \rightarrow \mathcal{R}$ is said to be *order bounded* whenever it carries order bounded sets onto order bounded subsets of \mathcal{R} . If f is *positive* (i.e., if $x \geq 0$ implies $f(x) \geq 0$), then f is order bounded. The vector space of all order bounded linear functionals on E is called the *order dual* of E , and is denoted by E' . Remarkably, under the ordering $f \geq g$ whenever $f(x) \geq g(x)$ for all $x \in E^+$, the order dual E' is a Dedekind complete Riesz space. Its lattice operations are given by

$$\begin{aligned} f \vee g(x) &= \sup\{f(y) + g(z): y, z \in E^+ \text{ and } y + z = x\}, \text{ and} \\ f \wedge g(x) &= \inf\{f(y) + g(z): y, z \in E^+ \text{ and } y + z = x\} \end{aligned}$$

for all $x \in E^+$.

An order bounded linear functional $f: E \rightarrow \mathcal{R}$ is said to be *order continuous* (or a *normal integral*) whenever $x_\alpha \xrightarrow{0} 0$ in E implies $f(x_\alpha) \rightarrow 0$ in \mathcal{R} . The vector subspace of all order continuous linear functionals is a band of E' , and is denoted by E_n^- . It is important to keep in mind that

$$E' = E_n^- \oplus (E_n^-)^d$$

holds, i.e., each $\phi \in E'$ has a unique decomposition $\phi = \phi_n + \phi_s$, where ϕ_n is order continuous and $\phi_s \in (E_n^-)^d$. The linear functional ϕ_n is called the *order continuous* (or the *normal*) *component* of ϕ , and ϕ_s is called the *singular component* of ϕ . If $\phi_n = 0$ (i.e., if $\phi \in (E_n^-)^d$), then ϕ is called a *singular functional* otherwise it is called *non-singular*. Riesz spaces with an abundance of normal integrals will play a crucial role in our study and for this reason we give them a name.

Definition 2.1. A Riesz space E is said to be a *normal Riesz space*, whenever

1. E is Dedekind complete; and
2. E_n^- separates the points of E , i.e., if $x \neq 0$, then there exists some $\phi \in E_n^-$ with $\phi(x) \neq 0$.

It should be noted that every ideal of a normal Riesz space is a normal Riesz space in its own right. An ideal A of E is said to be *order dense* in E whenever for each $x \in E^+$ there exists a net $\{x_\alpha\}$ of A with $0 \leq x_\alpha \uparrow x$ in E . An ideal A is order dense if and only if $A^d = \{0\}$.

The null ideal of an order bounded linear functional ϕ is defined by

$$N_\phi := \{x \in E: |\phi|(|x|) = 0\},$$

and its carrier is the band $C_\phi := (N_\phi)^d$. Clearly, ϕ is strictly positive on C_ϕ , i.e., $0 < x \in C_\phi$ implies $\phi(x) > 0$. By the above the null ideal N_ϕ is order dense in E if and only if $C_\phi = \{0\}$.

It is important to note that on a normal Riesz space every singular functional has an order dense null ideal. The details follow.

Theorem 2.2. If E is a normal Riesz space and ϕ is a singular linear functional on E (i.e., $\phi \in (E_n^-)^d$), then its null ideal N_ϕ is order dense in E .

Proof. Assume $\phi \in (E_n^-)^\delta$, and let $x \in C_\phi$. If $\psi \in E_n^-$, then $\psi \perp \phi$, and so by [4, Theorem 3.9, p. 23] we have $C_\phi \subseteq N_\psi$. Thus, $|\psi|(|x|) = 0$, and in view of $|\psi(x)| \leq |\psi|(|x|)$, we see that $\psi(x) = 0$ for all $\psi \in E_n^-$. Therefore, $(N_\phi)^\delta = C_\phi = \{0\}$, and hence N_ϕ is order dense in E . ■

Now we turn our attention to topological properties of Riesz spaces. All topologies will be assumed to be Hausdorff. A linear topology τ on a Riesz space E is said to be *locally convex-solid* (and (E, τ) is called a *locally convex-solid Riesz space*) whenever it has a basis at zero consisting of convex and solid neighborhoods. Every locally convex-solid topology on E is generated by a family of lattice seminorms; a seminorm q on E is said to be a *lattice seminorm* whenever $|x| \leq |y|$ implies $q(x) \leq q(y)$. The topological dual E' of a locally convex-solid Riesz space E is always an ideal of E' .

A *Riesz dual system* (E, E') is a Riesz space E together with an ideal E' of E' that separates the points of E such that the duality is the natural one, i.e., $(x, x') = x'(x)$ holds for all $x \in E$ and all $x' \in E'$. If (E, E') is a Riesz dual system, then the absolute weak topology $|\sigma|(E, E')$ is the smallest locally convex-solid topology on E consistent with (E, E') . It is generated by the family of lattice seminorms $\{p_{x'}: x' \in E'\}$, where $p_{x'}(x) = |x'|(|x|)$ for all $x \in E$ and $x' \in E'$.

A Riesz dual system (E, E') is called *symmetric* whenever E (considered embedded naturally in $(E')^\sim$) is an ideal of $(E')^\sim$, i.e., whenever (E', E) is also a Riesz dual system. A Riesz dual system (E, E') is symmetric if and only if every order interval of E is $\sigma(E, E')$ -compact; see [5, Theorem 11.13, p. 170]. For simplicity, the topology $\sigma(E, E')$ will be denoted by w and it will be called the *weak topology*. If E is a normal Riesz space, then (E, E_n^-) is a symmetric Riesz dual system and so its order intervals are $\sigma(E, E_n^-)$ -compact. In addition, for a normal Riesz space E the Mackey topology $\tau(E, E_n^-)$ is locally convex-solid, i.e., $\tau(E, E_n^-) = |\tau|(E, E_n^-)$ holds; see [4, Corollary 20.12, p. 140]. The Mackey topology $\tau(E, E_n^-)$ was used by S. F. Richard and S. Srivastava in [38].

Special and important examples of locally convex-solid Riesz spaces are provided by the Banach lattices. Recall that a Riesz space equipped with a lattice norm is called a *normed Riesz space*. A complete normed Riesz space is known as a *Banach lattice*. A Banach lattice is said to be an *AM-space* whenever for all $x, y \geq 0$ we have $\|x \vee y\| = \max\{\|x\|, \|y\|\}$. An AM-space is said to have a *unit* $e > 0$ whenever

$$\|x\| = \inf\{\lambda > 0: |x| \leq \lambda e\}$$

holds for all x . Every AM-space with unit is lattice isometric to some $C(\Omega)$ for a unique Hausdorff compact topological space Ω , where the unit corresponds to the constant function one on Ω ; see [5, Theorem 12.28, p. 194].

It is important to keep in mind that if E is a Dedekind complete Riesz space and $x \neq 0$, then the principal ideal A_x under the lattice norm

$$\|y\|_\infty = \inf\{\lambda > 0: |y| \leq \lambda|x|\}, \quad y \in A_x,$$

is an AM-space having $|x|$ as a unit; see [5, Theorem 12.20, p. 187]. In addition, the locally convex-solid topology generated by $\|\cdot\|_\infty$ (which will be denoted by τ_∞) is the finest locally convex-solid topology that A_x admits.

Finally, we mention the symmetric Riesz dual systems associated with order continuous Fréchet lattices. Recall that a linear topology τ on a Riesz space is called *order continuous* whenever $x_\alpha \downarrow 0$ implies $x_\alpha \xrightarrow{\tau} 0$. Keep in mind that for a Riesz dual system (E, E') the weak topology on E is order continuous if and only if every locally convex-solid topology on E consistent with (E, E') is order continuous; see [5, Theorem 11.10, p. 168]. A Banach lattice whose norm induces an order continuous topology is called a *Banach lattice with order continuous norm*. The latter is, of course, equivalent to saying that $x_\alpha \downarrow 0$ implies $\|x_\alpha\| \downarrow 0$. Unless A_x is finite dimensional, the topology τ_∞ on A_x is never order continuous.

A *Fréchet lattice* is a complete metrizable locally convex-solid Riesz space. In a Fréchet lattice the classical Eberlein-Šmulian theorem (see [23, pp. 206–211]) guarantees that a subset is weakly compact if and only if it is weakly sequentially compact. An *order continuous Fréchet lattice* is a Fréchet lattice whose locally convex-solid topology is order continuous. If E is an order continuous Fréchet lattice, then E is Dedekind complete, its topological dual E' coincides with E_n^- (i.e., $E' = E_n^-$), and its order intervals are weakly compact. Thus, for an order continuous Fréchet lattice E , the dual system (E, E') is a symmetric Riesz dual system and the order intervals of E are weakly sequentially compact.

3. THE IDEAL GENERATED BY A COUNTABLE SET

As we have mentioned in the introduction, the main purpose of this work is to study economies with a countable number of agents. This naturally leads us to the study of ideals generated by a countable number of elements. As far as we know, there is not a comprehensive study of these ideals available in the mathematical literature. For this paper and for future reference, we gather below some of the remarkable algebraic and topological properties of the ideals generated by a countable number of elements.

For the discussion in this section E will denote a fixed Dedekind complete Riesz space. Also, we shall fix a sequence $\{\theta_n\}$ of E and we shall let A denote the ideal generated by the sequence $\{\theta_n\}$. Clearly, A is the same as the ideal generated by the sequence $\{\hat{\theta}_n\}$, where $\hat{\theta}_n = \sum_{i=1}^n |\theta_i|$, $n = 1, 2, \dots$. Thus, replacing each θ_n by $\hat{\theta}_n$, we can assume without loss of generality that $0 \leq \theta_n \uparrow$ holds in E . Note that

$$A = \{x \in E : \exists \lambda > 0 \text{ and } n \in \mathcal{N} \text{ with } |x| \leq \lambda \theta_n\}.$$

For each n we shall denote by A_n the principal ideal generated by θ_n , i.e.,

$$A_n = \{x \in E : \exists \lambda > 0 \text{ with } |x| \leq \lambda \theta_n\}.$$

The ideal A_n equipped with the lattice norm

$$\|x\|_n = \inf\{\lambda > 0 : |x| \leq \lambda \theta_n\}, \quad x \in A_n,$$

is a Banach lattice. In fact, A_n under $\|\cdot\|_n$ is an AM-space having θ_n as a unit. From $0 \leq \theta_n \uparrow$ in E , we see that $A_n \subseteq A_{n+1}$ holds for all n and $A = \bigcup_{n=1}^{\infty} A_n$. We shall denote by ξ_n the norm topology induced on A_n by $\|\cdot\|_n$. Since $\|x\|_{n+1} \leq \|x\|_n$ holds for all $x \in A_n$, it follows that $\xi_{n+1} \subseteq \xi_n$ holds on A_n . To avoid trivialities we shall also assume that the inclusion $A_n \subseteq A_{n+1}$ is proper for all n .

Definition 3.1. *The inductive limit topology ξ is the finest locally convex topology on A for which all the embeddings $i_n: (A_n, \xi_n) \hookrightarrow (A, \xi)$ are continuous.*

The topology ξ is uniquely determined in the following sense. If $\{x_n\}$ is another sequence that generates A (we can suppose $0 \leq x_n \uparrow$ in E) and B_n denotes the principal ideal generated by x_n , then the inductive limit topology of the sequence $\{B_n\}$ on A is precisely ξ .

To see this, let B_n be equipped with the lattice norm

$$\|x\|_n = \inf\{\lambda > 0 : |x| \leq \lambda x_n\}, \quad x \in B_n,$$

and let η_n denote the topology induced by $\|\cdot\|_n$ on B_n . Also, denote by η the finest locally convex topology on A such that all embeddings $j_n: (B_n, \eta_n) \hookrightarrow (A, \eta)$ are continuous. Now if k is fixed, then there exists some n and some $M > 0$ satisfying $\theta_k \leq M x_k$. This implies $A_k \subseteq B_n$ and $\|x\|_n \leq M \|x\|_k$ for all $x \in A_k$. Thus, the embedding $i_{nk}: (A_k, \xi_k) \hookrightarrow (B_n, \eta_n)$ is continuous and since all $j_n: (B_n, \eta_n) \hookrightarrow (A, \eta)$ are continuous we see that each $i_k = j_n \circ i_{nk}: (A_k, \xi_k) \hookrightarrow (A, \eta)$ is continuous. Therefore, $\eta \subseteq \xi$ must hold. By the symmetry of the situation, we infer that $\xi \subseteq \eta$, and hence $\eta = \xi$.

A basis at zero for the topology ξ consists of the sets of the form

$$V = \text{co}\left(\bigcup_{n=1}^{\infty} V_n\right),$$

where $V_n \subseteq A_n$ is a ξ_n -neighborhood of zero (and, of course, $\text{co } X$ denotes the convex hull of the set X in A). An immediate consequence of the above observation is that a linear functional $f: A \rightarrow \mathcal{R}$ is ξ -continuous if and only if f restricted to each A_n is ξ_n -continuous. The reader can find the general theory

of inductive limits in the books [23, 40, 42]. Next, we shall list the basic properties of the inductive limit topology ξ .

Theorem 3.2. *The locally convex space (A, ξ) is a locally convex-solid Riesz space whose topological dual coincides with the order dual of A , i.e., $(A, \xi)' = A^-$ holds.*

In particular, ξ is a Hausdorff topology if and only if the order dual A^- separates the points of A (and hence, in this case, (A, A') is a Riesz dual system).

Proof. If V_n is a solid ξ_n -neighborhood of A_n , then $\bigcup_{n=1}^{\infty} V_n$ is a solid subset of A , and hence $\text{co}(\bigcup_{n=1}^{\infty} V_n)$ is a solid ξ -neighborhood of A ; see [4, Theorem 1.3, p. 4]. This implies that ξ is also a locally solid topology.

Next, note that a linear functional $f: A \rightarrow \mathcal{R}$ is order bounded (i.e., $f \in A^-$) if and only if f -restricted to each A_n is order bounded. Since A_n is a Banach lattice, its order dual coincides with its norm dual [5, Corollary 12.5, p. 176], and so a linear functional $f: A \rightarrow \mathcal{R}$ is order bounded if and only if f restricted to each A_n is ξ_n -continuous (i.e., if and only if $f \in (A, \xi)'$). Thus, $(A, \xi)' = A^-$ holds. ■

It should be kept in mind that if E^- separates the points of E , then A^- also separates the points of A , and hence the inductive limit topology ξ is always a Hausdorff locally convex-solid topology on A .

For the rest of the discussion in this section we shall assume that A^- separates the points of A so that the inductive limit topology ξ is a Hausdorff locally convex-solid topology. More properties of the topology ξ are included in the next theorem.

Theorem 3.3. *The inductive limit topology ξ on A is barrelled, Mackey (i.e., $\xi = \tau(A, A^-)$) and bornological.*

Proof. See [40, pp. 81–82]. ■

Remarkably, the strong order dual A^- of A is a Fréchet lattice.

Theorem 3.4. *The order dual A^- with the strong topology $\beta(A^-, A)$ is a Fréchet lattice.*

Proof. Combine [23, Proposition 5, p. 171] with [23, Corollary 4, p. 166]. ■

Next, we shall discuss the case when ξ is a strict inductive limit. As we mentioned before, the inductive limit topology ξ on A is independent of the generating sequence $\{\theta_n\}$.

Now assume that there exists a disjoint sequence $\{\omega_n\}$ of E^+ (i.e., $\omega_n \wedge \omega_m = 0$ for $n \neq m$) that generates the ideal A ; we can assume that $\omega_n > 0$ holds for all n . Put $\tilde{\omega}_n = \sum_{i=1}^n \omega_i$ and let C_n denote the principal ideal generated by $\tilde{\omega}_n$. Note that $\tilde{\omega}_n \wedge \omega_{n+1} = 0$ holds for all n . Let $x \in C_n$. If $\lambda > 0$ satisfies $|x| \leq \lambda \tilde{\omega}_n$, then clearly $|x| \leq \lambda \tilde{\omega}_{n+1}$ holds. On the other hand, if $\lambda > 0$ satisfies $|x| \leq \lambda \tilde{\omega}_{n+1}$, then we have

$$\begin{aligned} |x| &= |x| \wedge \lambda \tilde{\omega}_{n+1} = |x| \wedge \lambda \tilde{\omega}_n + |x| \wedge \lambda \omega_{n+1} \\ &= |x| \wedge \lambda \tilde{\omega}_n \leq \lambda \tilde{\omega}_n. \end{aligned}$$

Thus, a constant $\lambda > 0$ satisfies $|x| \leq \lambda \tilde{\omega}_n$ if and only if $|x| \leq \lambda \tilde{\omega}_{n+1}$. This shows that $\|x\|_n = \|x\|_{n+1}$ holds for all $x \in C_n$, i.e., $\|\cdot\|_{n+1}$ restricted to C_n is precisely $\|\cdot\|_n$. It is, therefore, immediate that C_n is ξ_{n+1} -closed in C_{n+1} . In this case, ξ is the strict inductive limit of the sequence $\{C_n\}$ of AM-spaces. I. Kawai [26, Theorem 6.6, p. 311] has also proven the converse.

Theorem 3.5. (Kawai) *The inductive limit topology ξ on A is a strict inductive limit topology if and only if A is generated by a disjoint sequence of non-zero positive elements.*

It is interesting to note that when ξ is the strict inductive limit, the ideal A has a nice representation; see [26, Theorem 6.6, p. 311] for details.

Theorem 3.6. (Kawai) *If ξ is a strict inductive limit, then there exists a locally compact and σ -compact Hausdorff topological space Ω such that A is lattice isomorphic to $C_c(\Omega)$ (the Riesz space of*

all continuous real-valued functions on Ω with compact support).

In addition, if $H = \{h \in C(\Omega): h(\omega) > 0 \forall \omega \in \Omega\}$, then the sets

$$V_h = \{f \in C_c(\Omega): |f(\omega)| \leq h(\omega) \forall \omega \in \Omega\}, h \in H,$$

form a basis at zero for the ξ -neighborhoods.

When ξ is the strict inductive limit, then it also has a number of extra properties.

Theorem 3.7. *If ξ is the strict inductive limit, then:*

1. *The locally convex-solid Riesz space (A, ξ) is topologically complete and non-metrizable.*
2. *The topology ξ induces ξ_n on each A_n and each A_n is ξ -closed in A .*
3. *A subset of A is ξ -bounded if and only if it is contained in some A_n and is ξ_n -bounded there.*

Recall that a topological vector space (X, τ) is said to have the *Dunford-Pettis property* whenever $x_n \xrightarrow{w} x$ in X and $f_n \xrightarrow{w} f$ in X' (the topological dual of (X, τ)) imply $f_n(x_n) \rightarrow f(x)$. The reader will notice here that the Dunford-Pettis property is nothing else but a joint sequential continuity of the evaluation map $(x, p) \mapsto p \cdot x$. The lack of joint continuity of the evaluation map is one of the major differences between economies with finite and infinite dimensional commodity spaces. For more about the Dunford-Pettis property see [5, Section 19].

Theorem 3.8. *If ξ is the strict inductive limit, then (A, ξ) has the Dunford-Pettis property.*

Proof. Assume $x_n \xrightarrow{w} x$ in A and $p_n \xrightarrow{w} p$ in A' . Then the set $\{x, x_1, x_2, \dots\}$ is weakly bounded, and hence ξ -bounded. By Theorem 3.7(3) there exists some k such that $\{x, x_1, x_2, \dots\} \subseteq A_k$.

Now consider each p_n restricted to A_k . Clearly, $p_n \in A'_k$ for each n , and moreover, $p_n \xrightarrow{w} p$ in A' implies $p_n \xrightarrow{w} p$ in A'_k . By a theorem of A. Grothendieck [5, Theorem 13.13, p. 211], we see that $p_n \xrightarrow{w} p$ also holds in A'_k . Since A_k has the Dunford-Pettis property [5, Theorem 19.6, p. 336], we infer that $p_n \cdot x_n \rightarrow p \cdot x$, as desired. ■

By Theorem 3.4 we know that the strong dual of A is a Fréchet lattice. When ξ is the strict inductive limit, then the strong dual of A is, in fact, an order continuous Fréchet lattice.

Theorem 3.9. *If ξ is the strict inductive limit, then A^- with the strong topology $\beta(A^-, A)$ is an order continuous Fréchet lattice*.*

Proof. Assume that ξ is the strict inductive limit. Let $f_\alpha \downarrow 0$ hold in A^- and let B be a ξ -bounded subset of A . We have to show that $\{f_\alpha\}$ converges to zero uniformly on B .

By Theorem 3.7(3) there exists some n such that $B \subseteq A_n$. If we consider each f_α restricted to A_n , then $\{f_\alpha\}$ as a net of A'_n satisfies $f_\alpha \downarrow 0$. Since A_n is an AM-space, its norm dual A'_n is an AL-space and so A'_n has order continuous norm. Therefore, $\|f_\alpha\| \downarrow 0$ holds, and from this we see that $\{f_\alpha\}$ converges to zero uniformly on B . ■

Now assume that A is generated by a disjoint sequence $\{\omega_n\}$ of non-zero positive elements, so that ξ is the strict inductive limit topology. We shall also assume one extra condition; namely that $\omega = \sup\{\omega_n: n = 1, 2, \dots\}$ exists in E , i.e., we shall assume that

$$\hat{\omega}_n = \sum_{i=1}^n \omega_i \uparrow \omega$$

holds in E .

* In this case, it turns out that $(A^-, \beta(A^-, A))$ is also the projective limit of the sequence $\{A'_n\}$. For details see [40, Proposition 15, p. 85].

If A_ω denotes the principal ideal generated by ω in E , then we have the following ideal inclusions

$$A \subseteq A_\omega \subseteq E,$$

where the ideal A is order dense in A_ω . We shall denote by τ_∞ the locally convex-solid topology on A_ω generated by the lattice norm

$$\|x\|_\infty = \inf\{\lambda > 0: |x| \leq \lambda\omega\}, \quad x \in A_\omega.$$

Notice that the lattice norm $\|\cdot\|_\infty$ restricted to each A_n (the principal ideal generated by ω_n): satisfies $\|x\|_\infty \leq \|x\|_n$ for all $x \in A_n$, and so the inclusions $i_n: (A_n, \xi_n) \hookrightarrow (A, \tau_\infty)$ are all continuous. This implies that on A we have $\tau_\infty \subseteq \xi$, where the inclusion is proper by Theorem 3.7(1). Since any locally convex-solid topology τ on A_ω satisfies $\tau \subseteq \tau_\infty$ [4, Theorem 16.7, p. 112], we see that

$$\tau \subseteq \tau_\infty \subset \xi$$

holds on A . In addition, it should be noted that ξ cannot be extended to a locally convex-solid topology on A_ω . (Indeed, if ξ extends to a locally convex-solid topology on A_ω , say $\hat{\xi}$, then $\hat{\xi} \subseteq \tau_\infty$ must hold on A_ω . Therefore, $\xi = \tau_\infty$ on A , which means that ξ is metrizable on A , contrary to Theorem 3.7(1).)

The following example illustrates all the preceding spaces and topologies.

Example 3.10. Let $E = \mathcal{R}^\infty$, the vector space of all real sequences, and denote by τ the locally convex-solid topology of pointwise convergence. For each n let $\omega_n = (0, \dots, 0, 1, 0, 0, \dots)$, where the 1 occupies the n^{th} position. Clearly, $\{\omega_n\}$ is a disjoint sequence of positive elements, such that

$$\omega = \sup\{\omega_n: n = 1, 2, \dots\} = (1, 1, 1, \dots).$$

It is easily seen that

1. $A_\omega = \ell_\infty$;
2. $A = \phi = \{(x_1, x_2, \dots) \in \mathcal{R}^\infty: \exists k \text{ with } x_n = 0 \forall n \geq k\}$;
3. $A' = A^- = \mathcal{R}^\infty$; and
4. the topology ξ is the locally convex-solid topology on ϕ having a basis at zero consisting of the sets of the form

$$V = \{(x_1, x_2, \dots) \in \phi: |x_i| \leq y_i \forall i \in \mathcal{N}\},$$

where $(y_1, y_2, \dots) \in \mathcal{R}^\infty$ satisfies $y_i > 0$ for each i . ■

Finally, we close the section with the following remark. If $\{\omega_n\}$ is a disjoint sequence, then

$$\omega = \sup\{\omega_n: n = 1, 2, \dots\}$$

exists in the universal completion E^u of E . This means that A_ω can be defined as the ideal generated by ω in E^u , and so the $\|\cdot\|_\infty$ norm on A_ω always induces τ_∞ on A . For the concept of the "universal completion" of a Riesz space see [29] and [4].

4. PREFERENCES AND UTILITY FUNCTIONS

In this section E will denote an Archimedean Riesz space and τ a linear topology on E . For this paper a preference is a binary relation on E^+ which is complete, reflexive and transitive. A preference \succeq is said to be:

1. monotone, whenever $x > y \geq 0$ implies $x \succeq y$;
2. strictly monotone, whenever $x > y \geq 0$ implies $x \succ y$;
3. convex, whenever the set $\{y \in E^+ : y \succeq x\}$ is convex for each $x \in E^+$; and
4. τ -continuous, whenever for each $x \in E^+$ the sets $\{y \in E^+ : y \succeq x\}$ and $\{z \in E^+ : x \succeq z\}$ are both τ -closed in E^+ .

A commodity bundle v is said to be strongly desirable for a preference \succeq whenever for each $x \in E^+$ and each $\alpha > 0$ we have $x + \alpha v \succ x$.

A. Mas-Colell [31] introduced the notion of uniform properness for preferences as follows.

Definition 4.1. (Mas-Colell) Let E be a Riesz space, τ a linear topology on E and \succeq a preference on E^+ . Then \succeq is said to be uniformly τ -proper whenever there exists some $v > 0$ and some τ -neighborhood V of zero such that $x - \alpha v + z \succeq x$ in E^+ with $\alpha > 0$ implies $z \notin \alpha V$.

Any vector v that satisfies the preceding property will be referred to as a vector of uniform properness for \succeq .

Any vector v of uniform properness for a preference \succeq is automatically a strongly desirable bundle. Indeed, if $x \succeq x + \alpha v$ holds for some $x \in E^+$ and some $\alpha > 0$, then from

$$x = (x + \alpha v) - \alpha v + 0 \succeq x + \alpha v$$

and the uniform properness, it follows that $0 \notin \alpha V$, which is impossible. Hence, $x + \alpha v \succ x$ holds for all $\alpha > 0$ and all $x \in E^+$.

If \succeq is a preference and $x \in E^+$, then (as usual) the set $\{y \in E^+ : y \succeq x\}$ will be denoted by $P(x)$, that is,

$$P(x) = \{y \in E^+ : y \succeq x\}.$$

Theorem 4.2. (Mas-Colell) Let τ be a locally convex topology on a Riesz space E and let \succeq be a preference on E^+ . Then \succeq is uniformly τ -proper if and only if there exists a non-empty τ -open convex cone Γ such that

- a) $\Gamma \cap (-E^+) \neq \emptyset$; and
- b) $(x + \Gamma) \cap P(x) = \emptyset$ for all $x \in E^+$.

Proof. Assume that \succeq is uniformly τ -proper, and let $v > 0$ be a vector of uniform properness corresponding to some open convex and τ -neighborhood V of zero. Consider the non-empty τ -open convex cone

$$\Gamma = \{w \in E : \exists \alpha > 0 \text{ and } y \in V \text{ with } w = \alpha(y - v)\}.$$

From $-v \in \Gamma$, we see that $\Gamma \cap (-E^+) \neq \emptyset$. Now let $x \in E^+$. If $z \in (x + \Gamma) \cap P(x)$, then pick $\alpha > 0$ and $y \in V$ with

$$z = x + \alpha(y - v) = x - \alpha v + \alpha y \succeq x,$$

and so by the uniform τ -properness we have $\alpha y \notin \alpha V$, i.e., $y \notin V$, which is impossible. Consequently, $(x + \Gamma) \cap P(x) = \emptyset$ for all $x \in E^+$.

For the converse assume that there exists a non-empty τ -open cone Γ satisfying (a) and (b). Pick some $w \in \Gamma \cap (-E^+)$ and some τ -open convex neighborhood V of zero with $w + V \subseteq \Gamma$. Put $v = -w > 0$, and let $x - \alpha v + z \succeq x$ in E^+ with $\alpha > 0$. If $z \in \alpha V$, then $z = \alpha y$ for some $y \in V$, and so

$$x - \alpha v + z = x + \alpha(y - v) \in (x + \Gamma) \cap P(x) = \emptyset,$$

which is a contradiction. Thus, $x - \alpha v + z \succeq x$ in E^+ with $\alpha > 0$ implies $z \notin \alpha V$. ■

Recently, S. F. Richard [36] has shown that a uniformly proper preference can be extended to a preference on a closed convex set with a non-empty interior containing E^+ ; see also [39]. On an AM-space with unit a monotone preference with a strongly desirable commodity is automatically uniformly norm proper. This was pointed out by A. Mas-Colell [31].

Theorem 4.3. (Mas-Colell) *If a monotone preference on the positive cone of an AM-space with unit has a strongly desirable commodity, then it is uniformly norm proper.*

Proof. Let E be an AM-space with unit and let \succeq be a monotone preference on E^+ having a strongly desirable commodity $v > 0$. Observe that if $w \in \text{Int}(E^+)$, then $x + w \succ x$ holds for all $x \in E^+$. Indeed, if $w \in \text{Int}(E^+)$, then pick some $\alpha > 0$ with $w - \alpha v \in E^+$ and note that for $x \in E^+$ we have

$$x + w = x + \alpha v + (w - \alpha v) \succeq x + \alpha v \succ x.$$

Now consider the non-empty open convex cone $\Gamma = -\text{Int}(E^+)$. Clearly, $\Gamma \cap (-E^+) \neq \emptyset$. On the other hand, if $x \in E^+$, then we claim that $(x + \Gamma) \cap \{y \in E^+ : y \succeq x\} = \emptyset$. Indeed, if this is not the case, then there exists some $w \in \text{Int}(E^+)$ with $x - w \geq 0$ and $x - w \succeq x$, and so we must have

$$x = (x - w) + w \succ x - w \succeq x,$$

which is impossible. Therefore,

$$(x + \Gamma) \cap \{y \in E^+ : y \succeq x\} = \emptyset$$

holds for all $x \in E^+$, and hence by Theorem 4.2 the preference \succeq is uniformly norm proper. ■

We now turn our attention to utility functions. Recall that every function $u: E^+ \rightarrow \mathcal{R}$ defines a preference by saying that $x \succeq y$ whenever $u(x) \geq u(y)$. If \succeq is a preference, then a function $u: E^+ \rightarrow \mathcal{R}$ that satisfies $x \succeq y$ if and only if $u(x) \geq u(y)$ is called a *utility function representing \succeq* .

The order continuous utility functions will play a crucial role in our study and for this reason we give them a name.

Definition 4.4. *An order continuous utility function $u: E^+ \rightarrow \mathcal{R}$ will be referred to as a myopic utility function, i.e., a function $u: E^+ \rightarrow \mathcal{R}$ is said to be myopic whenever $x_\alpha \xrightarrow{o} x$ in E^+ implies $u(x_\alpha) \rightarrow u(x)$ in \mathcal{R} .*

Myopia (i.e. order continuity) should be interpreted as a mathematical notion that captures the economic intuition of impatience; see [11]. A different notion of impatience was also introduced recently by R. A. Becker, J. H. Boyd and C. Foias in [9].

Note that if a utility function $u: E^+ \rightarrow \mathcal{R}$ is continuous for an order continuous locally solid topology τ (i.e., $x_\alpha \xrightarrow{o} x$ implies $x_\alpha \xrightarrow{\tau} x$), then u is automatically myopic. Also, if E is a Fréchet lattice, then every myopic utility function $u: E^+ \rightarrow \mathcal{R}$ is continuous. This follows immediately from the fact that in a Fréchet lattice every topologically convergent sequence to some point x has an order convergent subsequence to x ; see [4, Exercise 8, p. 123].

A myopic utility function is not necessarily topologically continuous and a topologically continuous utility function need not be myopic. The next two examples clarify the situation.

Example 4.5. (A myopic utility function which is not topologically continuous) Let $E = \ell_1$ and let τ be the order continuous locally convex-solid topology induced on E by the ℓ_2 -norm. Now consider the utility function $u: E^+ \rightarrow \mathcal{R}$ defined by

$$u(x) = \sum_{i=1}^{\infty} x_i, \quad x = (x_1, x_2, \dots) \in E^+.$$

Clearly, u is strictly monotone and concave, and moreover, we claim that it is also myopic. To see the latter, let $x_\alpha \xrightarrow{o} x$ in E^+ , where $x_\alpha = (x_1^\alpha, x_2^\alpha, \dots)$ and $x = (x_1, x_2, \dots)$. Pick a net $\{y_\alpha\}$ of E^+ such that $|x_\alpha - x| \leq y_\alpha$ for each α and $y_\alpha \downarrow 0$. From

$$|u(x_\alpha) - u(x)| \leq \sum_{i=1}^{\infty} |x_i^\alpha - x_i| \leq \sum_{i=1}^{\infty} y_i^\alpha = \|y_\alpha\|_1$$

and $\|y_n\|_1 \downarrow 0$, we see that $u(x_n) \rightarrow u(x)$, and so u is order continuous.

Now we claim that the utility function is not r -continuous. To see this, for each n pick some $k_n > n$ with $\sum_{i=n}^{k_n} \frac{1}{i} > 1$ and let

$$x_n = \left(\frac{1}{n}, \frac{1}{n+1}, \dots, \frac{1}{k_n}, 0, 0, \dots\right), \quad n = 1, 2, \dots$$

Then $\{x_n\}$ is a sequence of E^+ satisfying $\lim_{n \rightarrow \infty} \|x_n\|_2 = 0$ (i.e., $x_n \xrightarrow{r} 0$). On the other hand, the inequalities

$$u(x_n) = \sum_{i=n}^{k_n} \frac{1}{i} > 1 > 0 = u(0),$$

show that $u(x_n) \not\rightarrow 0$, and hence u is not r -continuous. ■

Example 4.6. (A topologically continuous utility function which is not myopic) Consider the Riesz space $C[0,1]$ and define $u: (C[0,1])^+ \rightarrow \mathcal{R}$ by

$$u(x) = \int_0^1 \sqrt{x(t)} dt.$$

Then u is $\|\cdot\|_\infty$ -continuous, strictly monotone, strictly concave and it fails to be order continuous; see [5, Exercise 15, p. 199]. ■

The myopic utility functions have the following interesting continuity property.

Theorem 4.7. If $u: E^+ \rightarrow \mathcal{R}$ is a myopic utility function, then on every principal ideal of E the utility function u is $\|\cdot\|_\infty$ -continuous.

Proof. Let $x \in E^+$, and let $\{y_n\}$ be a sequence of A_x such that $\|y - y_n\|_\infty \rightarrow 0$. Put

$$\epsilon_n = \sup\{\|y_i - y\|_\infty : i \geq n\}$$

and note that $\epsilon_n \downarrow 0$ and that $|y_n - y| \leq \epsilon_n x$ for all n . Since $\epsilon_n x \downarrow 0$ holds in E , it follows that $y_n \xrightarrow{o} y$ in E , and so by the order continuity of u , we see that $u(y_n) \rightarrow u(y)$. ■

Recall that a utility function $u: E^+ \rightarrow \mathcal{R}$ is said to be quasi-concave whenever

$$u(\alpha x + (1 - \alpha)y) \geq \min\{u(x), u(y)\}$$

holds for all $x, y \in E^+$ and all $0 < \alpha < 1$. It is well known that a utility function is quasi-concave if and only if it represents a convex preference.

Our next result presents a useful continuity property of the myopic quasi-concave utility functions.

Theorem 4.8. Let E be a normal Riesz space, let $a \in E^+$ and let $\{x_n\}$ be a sequence of $[0, a]$. If x is a $\sigma(E, E_n^-)$ -accumulation point of $\{x_n\}$ and a utility function $u: E^+ \rightarrow \mathcal{R}$ is monotone, quasi-concave and myopic, then

$$u(x) \geq \liminf u(x_n).$$

Proof. Assume that E , $\{x_n\}$, x and $u: E^+ \rightarrow \mathcal{R}$ satisfy the hypotheses of the theorem. Fix $\epsilon > 0$.

Next consider the ideal

$$C = \bigcup_{\phi \in E^+} C_\phi,$$

and note that C is order dense in E . To see this, let $0 \leq z \in C^d$. Then $z \perp C_\phi$ holds and so $z \in C_\phi^d = N_\phi$ for all ϕ . Thus, $\phi(z) = 0$ for all $\phi \in E_n^-$ and since E_n^- separates the points of E , we see that $z = 0$. Therefore, C is order dense in E .

Now by the order continuity of u and an easy inductive argument, it follows that there exist sequences $\{y_n\}$ of E^+ and $\{\phi_n\}$ of $(E_n^-)^+$ such that

- a) $y_n \in C_{\phi_n}$ and $0 \leq y_n \leq x_n$ for all n ;
- b) $\phi_k(x_n - y_n) < 2^{-n}$ for $1 \leq k \leq n$; and
- c) $u(y_n) > u(x_n) - \epsilon$.

Since x is a $\sigma(E, E_n^-)$ -accumulation point of the convex hull of the set $\{x_k: k \geq n\}$, it is also a $|\sigma|(E, E_n^-)$ -accumulation point of the convex hull of $\{x_k: k \geq n\}$. Thus, for each n there exists some $\zeta_n \in \text{co}\{x_k: k \geq n\}$ satisfying

$$\phi_k(|x - \zeta_n|) < 2^{-n} \text{ for } 1 \leq k \leq n.$$

Write ζ_n as a convex combination $\zeta_n = \sum_{i=1}^{m_n} \lambda_i^n x_{n_i}$, where $n_i \geq n$ for $1 \leq i \leq m_n$, and then put

$$z_n = \sum_{i=1}^{m_n} \lambda_i^n y_{n_i}.$$

From (b) and

$$|x - z_n| \leq |x - \zeta_n| + |\zeta_n - z_n| = |x - \zeta_n| + \sum_{i=1}^{m_n} \lambda_i^n (x_{n_i} - y_{n_i}),$$

we see that

$$\phi_k(|x - z_n|) < 2^{-n} + 2^{-n} = 2^{1-n} \text{ for } 1 \leq k \leq n. \quad (1)$$

Taking into account that u is quasi-concave, it follows from (c) that

$$\begin{aligned} u(z_n) &\geq \min\{u(y_{n_i}): 1 \leq i \leq m_n\} \\ &\geq \min\{u(x_{n_i}): 1 \leq i \leq m_n\} - \epsilon, \end{aligned}$$

and consequently

$$u(z_n) \geq \inf\{u(x_k): k \geq n\} - \epsilon \text{ for all } n. \quad (2)$$

Our next goal is to establish that the sequence $\{z_n\}$ is order convergent. For each n write $E = N_{\phi_n} \oplus C_{\phi_n}$, and then let h_n be the projection of x onto C_{ϕ_n} . Put $h = \sup\{h_n\} \leq x$, and we claim that $z_n \xrightarrow{o} h$.

To see this, note first that from

$$0 \leq \phi_k(x - h) \leq \phi_k(x - h_k) = 0,$$

we have $\phi_k(x - h) = 0$ for all k . Thus, from (1) and the inequality

$$|h - z_n| = |x - z_n - (x - h)| \leq |x - z_n| + x - h,$$

it follows that

$$\phi_k(|h - z_n|) < 2^{1-n} \text{ for } 1 \leq k \leq n. \quad (3)$$

Put $f_n = \sup\{|h - z_k|: k \geq n\}$, and note that $|h - z_n| \leq f_n$ holds for all n . Thus, in order to establish that $z_n \xrightarrow{o} h$ it suffices to show that $f_n \downarrow 0$. To this end, let $0 \leq f \leq f_n$ hold for all n . Then from (3), we have

$$\phi_m(f) \leq \phi_m(f_n) \leq \sum_{k=n}^{\infty} \phi_m(|h - z_k|) \leq \sum_{k=n}^{\infty} 2^{1-k} = 2^{2-n}$$

for all $n \geq m$, and so $\phi_m(f) = 0$ for all m , i.e., $f \in N_{\phi_m}$ for all m . Therefore, $f \perp C_{\phi_m}$ for all m . This implies $f \perp h$ and $f \perp y_n$ for all n , and hence $f \perp z_n$ for all n . In turn, the latter implies $f \perp |h - z_n|$ for all n , and so $f \perp f_1$. From $0 \leq f \leq f_1$, we infer that $f = 0$. Thus, $f_n \downarrow 0$, and hence $z_n \xrightarrow{o} h$ holds.

Now by the order continuity of u , we see that $u(h) = \lim u(z_n)$. A glance at (2) reveals that

$$u(h) = \lim u(z_n) \geq \liminf u(z_n) - \varepsilon.$$

In view of $0 \leq h \leq x$ and the monotonicity of u , we have $u(x) \geq u(h)$, and so

$$u(x) \geq \liminf u(z_n) - \varepsilon$$

holds. Since $\varepsilon > 0$ is arbitrary, the latter implies $u(x) \geq \liminf u(z_n)$, and the proof of the theorem is finished. ■

5. THE ECONOMIC MODEL

As the title of the paper indicates, we shall study equilibria for pure exchange economies. The following six basic properties will characterize the economic model of our study.

1. The commodity-price duality is described by a Riesz dual system $\langle E, E' \rangle$; E is the commodity space and E' is the price space. In accordance with the economic tradition, the value of the bundle $x \in E$ at prices $p \in E'$ will be denoted by $p \cdot x$, i.e., $p \cdot x = \langle x, p \rangle$.
2. There is an at most countable number of consumers indexed by i ; the set of consumers will be denoted by \mathcal{N} .
3. Each consumer has E^+ as his consumption set.
4. Each consumer i has an initial endowment $\omega_i > 0$. If \mathcal{F} denotes the set of all finite subsets of \mathcal{N} , then the total endowment ω is defined by

$$\omega = \sup \left\{ \sum_{i \in F} \omega_i : F \in \mathcal{F} \right\},$$

where the supremum is assumed to exist in E .

5. Each consumer i has a convex and monotone preference relation \succsim_i .
6. The total endowment ω is strongly desirable by each consumer i , i.e.,

$$x + \alpha \omega \succsim_i x$$

holds for each $i \in \mathcal{N}$, each $x \in E^+$ and each $\alpha > 0$.

Definition 5.1. A pure exchange economy (or simply an economy) \mathcal{E} is a triplet

$$\mathcal{E} = (\langle E, E' \rangle, \{\omega_i : i \in \mathcal{N}\}, \{\succsim_i : i \in \mathcal{N}\}),$$

where the components of \mathcal{E} satisfy properties (1) through (6) above.

From now on we shall assume that \mathcal{E} is a fixed economy. An *allocation* for the economy \mathcal{E} (or simply an *allocation*) is an assignment $\{x_i: i \in \mathcal{N}\}$ such that $x_i \geq 0$ for each i and

$$\sup\left\{\sum_{i \in \mathcal{F}} x_i: \mathcal{F} \in \mathcal{F}\right\} = \omega.$$

Note that if \mathcal{N} is finite, say $\mathcal{N} = \{1, \dots, n\}$, then an allocation is a vector (x_1, \dots, x_n) such that $x_i \in E^+$ for each i and

$$\sum_{i=1}^n x_i = \sum_{i=1}^n \omega_i = \omega.$$

If \mathcal{N} is countable, say $\mathcal{N} = \{1, 2, \dots\}$, then an allocation is a sequence (x_1, x_2, \dots) such that $x_i \in E^+$ for each i and

$$\sup\left\{\sum_{i=1}^n x_i: n = 1, 2, \dots\right\} = \omega.$$

We now come to the definitions of the various equilibria concepts for our economy.

Definition 5.2. An allocation $\{x_i: i \in \mathcal{N}\}$ is said to be:

1. A **Walrasian** (or a **competitive**) equilibrium; whenever there exists some non-zero price $p \in E'$ such that each x_i is a maximal element in the i^{th} consumer's budget set $B_i(p)$, where as usual $B_i(p) = \{x \in E^+: p \cdot x \leq p \cdot \omega_i\}$.
2. A **quasiequilibrium**; whenever there exists a non-zero price $p \in E'$ such that
 - a) $p \cdot x_i = p \cdot \omega_i$ for each $i \in \mathcal{N}$; and
 - b) $x \succeq_i z_i$ in E^+ implies $p \cdot x \geq p \cdot \omega_i$.
3. A **weak quasiequilibrium**; whenever there exists a non-zero price $p \in E'$ such that $x \succeq_i z_i$ in E^+ implies $p \cdot x \geq p \cdot \omega_i$.

Any non-zero price p for which $x \succeq_i z_i$ in E^+ implies $p \cdot x \geq p \cdot \omega_i$ is said to be a *price supporting the allocation* $\{x_i: i \in \mathcal{N}\}$. Supporting prices are necessarily positive prices. To see this, let a price p support an allocation $\{x_i: i \in \mathcal{N}\}$, and let $z \geq 0$. Then $x_1 + \epsilon^{-1}z \succeq_1 x_1$ holds for all $\epsilon > 0$, and so $p \cdot (x_1 + \epsilon^{-1}z) \geq p \cdot \omega_1$. That is, $p \cdot z \geq \epsilon p \cdot (\omega_1 - x_1)$ holds for all $\epsilon > 0$, from which it follows that $p \cdot z \geq 0$, i.e., p is a positive price.

Also, it should be clear that in our model the following implications hold:

$$\text{Walrasian Equilibrium} \implies \text{Quasiequilibrium} \implies \text{Weak Quasiequilibrium}.$$

If the economy has a finite number of consumers, then it should be obvious that the notions of weak quasiequilibrium and quasiequilibrium coincide. As a matter of fact, if \mathcal{N} is finite, then any price supporting an allocation as a weak quasiequilibrium it also supports it as a quasiequilibrium. However, when we have a countable number of consumers the situation is quite different. It may very well happen that some non-zero price satisfies $p \cdot \omega_i = 0$ for all i (i.e., every consumer has zero wealth) and $p \cdot \omega > 0$. In this case, of course, the price p is not order continuous, and every allocation is a weak quasiequilibrium, and, in particular, $(\omega_1, \omega_2, \dots)$ is itself a quasiequilibrium.

The following result gives some connections between weak quasiequilibria and quasiequilibria. We shall discuss these connections in more detail in the next section.

Theorem 5.3. For an allocation (x_1, x_2, \dots) in an economy with a countable number of consumers the following statements hold.

1. If the allocation is a weak quasiequilibrium supported by an order continuous price, then the allocation is a quasiequilibrium.

2. If a price p supports the allocation as a quasiequilibrium and for some i we have $p \cdot \omega_i > 0$ and the preference \succeq_i is either myopic or continuous for some linear topology on E , then x_i is a maximal element in the budget set $B_i(p)$ of the i^{th} consumer.

Proof. (1) Assume that (x_1, x_2, \dots) is a weak quasiequilibrium which is supported by an order continuous price p . From $x_i \succeq_i x_i$, we see that $p \cdot x_i \geq p \cdot \omega_i$ holds for all i . On the other hand, the order continuity of p implies

$$\sum_{i=1}^{\infty} p \cdot x_i = \lim_{n \rightarrow \infty} p \cdot \sum_{i=1}^n x_i \leq p \cdot \omega = \sum_{i=1}^{\infty} p \cdot \omega_i,$$

and so the inequality $p \cdot x_i > p \cdot \omega_i$ for some i is impossible. Thus, $p \cdot x_i = p \cdot \omega_i$ must hold for all i . In other words, p supports (x_1, x_2, \dots) as a quasiequilibrium.

(2) To see this, assume by way of contradiction that there exists some $x \in B_i(p)$ satisfying $x \succ_i x_i$. Then $p \cdot x \geq p \cdot \omega_i$ must also hold, and in view of $x \in B_i(p)$, we have $p \cdot x = p \cdot \omega_i$. Since \succeq_i is either myopic or continuous for some linear topology on E , there exists some $0 < \delta < 1$ such that $\delta x \succ_i x_i$. In view of $p \cdot \omega_i > 0$, the latter implies

$$p \cdot \omega_i \leq p \cdot (\delta x) = \delta p \cdot x = \delta p \cdot \omega_i < p \cdot \omega_i,$$

which is impossible. Hence, x_i is a maximal element in $B_i(p)$, and the proof of the theorem is finished. ■

For the rest of this section we shall also consider economies with a finite number of consumers. For simplicity we shall refer to these economies as *finite economies*.

According to A. Mas-Colell [31] a finite economy with set of consumers $\mathcal{N} = \{1, \dots, m\}$ is said to satisfy the closedness condition whenever for every sequence of allocations $\{(x_{n,1}, \dots, x_{n,m})\}$ which satisfies $x_{n+1,i} \succeq_i x_{n,i}$ for all n and all $i = 1, \dots, m$ there exists another allocation (x_1, \dots, x_m) satisfying $x_i \succeq_i x_{n,i}$ for all n and all $i = 1, \dots, m$.

It is interesting to know that when the preferences are represented by myopic utility functions, the closedness condition is always satisfied.

Lemma 5.4. *If the commodity space E is a normal Riesz space, then every finite economy whose preferences are represented by myopic utility functions satisfies the closedness condition.*

Proof. Assume that E is a normal Riesz space, that the set of consumers is $\mathcal{N} = \{1, \dots, m\}$ and that the preference of each consumer is represented by a myopic utility function u_i . Let $\{(x_{n,1}, \dots, x_{n,m})\}$ be a sequence of allocations such that $x_{n+1,i} \succeq_i x_{n,i}$ holds for all n and all i . We have to show that there exists an allocation (x_1, \dots, x_m) satisfying $x_i \succeq_i x_{n,i}$ for all n and all i .

To this end, consider the order interval $[0, \omega]$ equipped with the topology $\sigma(E, E_n^-)$ and let τ denote the product topology on $[0, \omega]^m$. Since $[0, \omega]$ is $\sigma(E, E_n^-)$ -compact, it follows that $[0, \omega]^m$ is τ -compact. Now the sequence $\{(x_{n,1}, \dots, x_{n,m})\}$ is a sequence of $[0, \omega]^m$, and so it has a τ -accumulation point, say (x_1, \dots, x_m) . Clearly, (x_1, \dots, x_m) is an allocation, and each x_i is a $\sigma(E, E_n^-)$ -accumulation point of the sequence $\{x_{n,i}\}$. By Theorem 4.8 we have

$$u_i(x_i) \geq \liminf_{n \rightarrow \infty} u_i(x_{n,i}) = \sup\{u_i(x_{n,i}) : n = 1, 2, \dots\},$$

and so $x_i \succeq_i x_{n,i}$ holds for all n and all i , as desired. ■

For finite economies A. Mas-Colell [31] established the following basic existence theorem of quasiequilibria.

Theorem 5.5. (Mas-Colell) *Assume that in a finite economy each consumer has a convex, monotone, τ -continuous and uniformly τ -proper preference with respect to a consistent locally convex-solid topology τ on E . If the economy satisfies the closedness condition and the total endowment ω is strongly desirable by each consumer, then the economy has quasiequilibria.*

And now we come to a remarkable application of the preceding theorem to finite economies whose consumers' preferences are represented by myopic utility functions. Recall that for each $a \geq 0$ in a Dedekind complete Riesz space the principal ideal A_a under the lattice norm

$$\|x\|_\infty = \inf\{\lambda > 0: |x| \leq \lambda a\}, \quad x \in A_a$$

is an AM-space with unit a . The norm dual of $(A_a, \|\cdot\|_\infty)$ will be denoted by A'_a , and so (A_a, A'_a) is a Riesz dual system.

Theorem 5.6. *Assume that the commodity space of a finite economy is a normal Riesz space and that consumers' preferences are represented by myopic utility functions. If $a \geq \omega$ and the total endowment ω is strongly desirable by each consumer on A_a , then the finite economy has a quasiequilibrium with respect to the Riesz dual system (A_a, A'_a) .*

Proof. Let $a \geq \omega$ be fixed and consider the finite economy with respect to the Riesz dual system (A_a, A'_a) and with the original agents' characteristics restricted to A_a . By Theorem 4.7 we know that every utility function is $\|\cdot\|_\infty$ -continuous on A_a and Theorem 4.3 guarantees that all preferences are uniformly $\|\cdot\|_\infty$ -proper on A_a . In addition, by Lemma 5.4 the finite economy satisfies the closedness condition with respect to the Riesz dual system (A_a, A'_a) , and our conclusion follows from Theorem 5.5.

It should be noted that the supporting prices can be normalized with respect to a , i.e., if $p \in A'_a$ supports an allocation with respect to (A_a, A'_a) , then we can choose p to satisfy $p \cdot a = 1$. ■

The preceding result will play a crucial role in our existence proofs of equilibria in the next sections.

6. EQUILIBRIA IN INFINITE ECONOMIES

Here we shall consider economies with a countable number of consumers $\mathcal{N} = \{1, 2, \dots\}$. For brevity, we shall call these economies *infinite economies*. Two extra hypotheses will be assumed throughout this section.

1. *The commodity space E is a normal Riesz space; and*
2. *Each consumer's preference is represented by a monotone, quasi-concave and myopic utility function.*

We shall say that *the income distribution is positive* (resp. *strictly positive*) at prices $0 < p \in E'$ whenever $p \cdot \omega_i > 0$ holds for at least one i (resp. $p \cdot \omega_i > 0$ holds for all i). The *income distribution is zero* at prices $p \geq 0$ whenever $p \cdot \omega_i = 0$ for all i (in which case, of course, every allocation is a weak quasiequilibrium supported by p).

Note that (by Theorem 5.3(2)) if a quasiequilibrium is supported by a price whose income distribution is strictly positive, then the quasiequilibrium is, in fact, a Walrasian equilibrium.

The next example illustrates the above definitions.

Example 6.1. Consider the Riesz dual dual system $(\ell_\infty, \ell'_\infty)$, where ℓ'_∞ is the norm dual of ℓ_∞ . Clearly, ℓ_∞ is a normal Riesz space.

Consider now a countable number of consumers with initial endowments

$$\omega_i = (0, \dots, 0, 1, 0, 0, \dots), \quad i = 1, 2, \dots,$$

where the 1 occupies the i^{th} position, and utility functions given by the formulas

$$u_i(x_1, x_2, \dots) = x_i, \quad i = 1, 2, \dots$$

Note that $\omega = (1, 1, 1, \dots)$, and that this infinite economy satisfies our hypotheses.

Now take any Banach limit $0 < p \in \mathcal{L}_\infty$ (see [5, Theorem 14.18, p. 233]) and note that $p \cdot \omega_i = 0$ holds for all i , i.e., the income distribution is zero at prices p .

On the other hand, if we change the initial endowments to

$$\omega_1^* = (1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots),$$

and

$$\omega_i^* = (0, \dots, 0, \frac{1}{2}, 0, 0, \dots) \text{ for } i = 2, 3, \dots,$$

then it is not difficult to see that the income distribution is positive for any price $0 < p \in \mathcal{L}_\infty$. ■

The importance of supportability by prices whose income distribution is positive is demonstrated in the next result, which for the $(L_\infty(\mu), L_1(\mu))$ case was also proven by S. F. Richard and S. Srivastava in [38].

Theorem 6.2. *If an allocation (x_1, x_2, \dots) is a weak quasiequilibrium supported by a price $p \in E'$ whose income distribution is positive, then the allocation is a quasiequilibrium supported by the normal component of p .*

Proof. Let (x_1, x_2, \dots) be an allocation supported by a price $0 < p \in E'$ whose income distribution is positive. Fix some k with $p \cdot \omega_k > 0$. Also, write $p = q + \phi$ with q order continuous and ϕ a singular price. The proof will be completed in three steps.

1. *The order continuous component q of p is non-zero.*

Since ϕ is singular and E'_n separates the points of E , it follows from Theorem 2.2 that the null ideal of ϕ ,

$$N = \{x \in E: \phi \cdot |x| = 0\},$$

is order dense in E . Thus there exists a net $\{z_\alpha\}$ of N with $0 \leq z_\alpha \uparrow x_k + \omega$. In view of $x_k + \omega \succ_k x_k$, it follows from the order continuity of the utility functions that there exists some β such that $z_\alpha \succ_k x_k$ for all $\alpha \geq \beta$. Since p supports (x_1, x_2, \dots) , we see that

$$q \cdot z_\alpha = q \cdot z_\alpha + \phi \cdot z_\alpha = p \cdot z_\alpha \geq p \cdot \omega_k > 0$$

for all $\alpha \geq \beta$, and hence $q > 0$ holds.

2. *If $x \succeq_i x_i$, then $q \cdot x \geq q \cdot \omega_i$ holds.*

Let $x \succ_i x_i$ for some i , and assume by way of contradiction that $q \cdot x < q \cdot \omega_i$. Since N is order dense in E , there exists a net $\{y_\alpha\}$ of N with $0 \leq y_\alpha \uparrow x$. Clearly,

$$p \cdot y_\alpha = q \cdot y_\alpha + \phi \cdot y_\alpha = q \cdot y_\alpha \leq q \cdot x < q \cdot \omega_i \leq p \cdot \omega_i \tag{*}$$

holds for all α . On the other hand, the order continuity of the utility functions implies $y_\alpha \succ_i x_i$ for all sufficiently large α . In view of the supportability of (x_1, x_2, \dots) by p , the latter implies $p \cdot y_\alpha \geq p \cdot \omega_i$ for all sufficiently large α , contrary to (*). Hence, $x \succ_i x_i$ implies $q \cdot x \geq q \cdot \omega_i$.

Now let $x \succeq_i x_i$. Then $x + \varepsilon \omega \succ_i x_i$ holds for all $\varepsilon > 0$, and so by the above $q \cdot x + \varepsilon q \cdot \omega \geq q \cdot \omega_i$. Since $\varepsilon > 0$ is arbitrary, we infer that $q \cdot x \geq q \cdot \omega_i$ holds.

3. *$q \cdot x_i = q \cdot \omega_i$ holds for all i .*

From the order continuity of q , we infer that

$$\sum_{i=1}^{\infty} q \cdot x_i = \sum_{i=1}^{\infty} q \cdot \omega_i. \tag{**}$$

On the other hand $x_i \geq \sum_i x_i$ implies $q \cdot x_i \geq q \cdot \omega_i$ for all i . Now from (**), it easily follows that $q \cdot x_i = q \cdot \omega_i$ must hold for each i . The proof of the theorem is now complete. ■

Recall that an allocation (x_1, x_2, \dots) is said to be *weakly Pareto optimal* whenever there is no other allocation (y_1, y_2, \dots) such that $y_i \succ_i x_i$ holds for all i .

The quasiequilibria exhibit the following interesting property, which for the $(L_\infty(\mu), L_1(\mu))$ case was also established in [38].

Theorem 6.3. *Every quasiequilibrium which is supported by a price with positive income distribution is weakly Pareto optimal.*

Proof. Let (x_1, x_2, \dots) be an allocation supported by a price $p \geq 0$ whose income distribution is positive. By Theorem 6.2, we can suppose without loss of generality that p is an order continuous price. Also, we can suppose that $p \cdot \omega = 1$.

Now assume by way of contradiction that there exists an allocation (y_1, y_2, \dots) such that $y_i \succ_i x_i$ holds for each i . Since p is order continuous, it follows that

$$\sum_{i=1}^{\infty} p \cdot y_i = \sum_{i=1}^{\infty} p \cdot \omega_i = p \cdot \omega = 1. \quad (***)$$

In view of the supportability of (x_1, x_2, \dots) by p , we see that $p \cdot y_i \geq p \cdot \omega_i$ for all i , and from (***) it follows that in fact $p \cdot y_i = p \cdot \omega_i$ holds for all i .

Next, fix some k with $p \cdot \omega_k > 0$. From $(1 - \frac{1}{n})y_k \uparrow_n y_k$ and the order continuity of u_k , we see that $(1 - \frac{1}{n})y_k \succ_k x_k$ holds for all sufficiently large n , and so

$$p \cdot \omega_k = p \cdot y_k > (1 - \frac{1}{n})p \cdot y_k = p \cdot [(1 - \frac{1}{n})y_k] \geq p \cdot \omega_k$$

for all sufficiently large n , which is a contradiction. Therefore, (x_1, x_2, \dots) is a weak Pareto optimal allocation, as claimed. ■

We now come to the major result of this section. Namely, we shall establish next that every infinite economy has always a weak quasiequilibrium with respect to the Riesz dual system (A_ω, A'_ω) .

Theorem 6.4. *Every infinite economy has a weak quasiequilibrium with respect to the Riesz dual system (A_a, A'_a) for each $a \geq \omega$.*

Proof. Fix $a \geq \omega$. Let \mathcal{E}_n denote the finite economy with Riesz dual system (A_a, A'_a) and $\{1, \dots, n\}$ consumers (and, of course, with the original consumers' characteristics). Then, according to Theorem 5.6, each economy \mathcal{E}_n has a quasiequilibrium with respect to (A_a, A'_a) . Let (x_1^n, \dots, x_n^n) be a quasiequilibrium for \mathcal{E}_n supported by a price $0 < p_n \in A'_a$. We can assume that $p_n \cdot a = 1$.

The set $B = \{p \in A'_a: p \geq 0 \text{ and } p \cdot a = 1\}$ equipped with the topology $\sigma(A'_a, A_a)$ is a compact topological space. Also, the order interval $[0, a]$ equipped with the topology $\sigma(E, E_n^-)$ is compact. Thus, by Tychonoff's classical compactness theorem, the product topological space $X = B \times [0, a]^N$ (with the product topology) is a compact space. Now for each n let

$$x_n = (p_n, x_1^n, \dots, x_n^n, 0, 0, \dots) \in X.$$

Since X is compact, the sequence $\{x_n\}$ has an accumulation point, say (p, x_1, x_2, \dots) . Clearly, each x_i is a $\sigma(E, E_n^-)$ -accumulation point of the sequence $(x_i^1, x_i^2, x_i^3, \dots)$. The existence of the weak quasiequilibrium will be established by steps.

1. We have $p \cdot a = 1$, and hence $p > 0$.

This follows immediately from $p_n \cdot a = 1$ and the fact that p is a $\sigma(A'_a, A_a)$ -accumulation point of $\{p_n\}$.

2. If $0 \leq x \in A_a$ satisfies $x \succeq_i x_i$, then $p \cdot x \geq p \cdot \omega_i$.

Let $0 \leq x \in A_a$ satisfy $x \succ_i x_i$, i.e., $u_i(x) > u_i(x_i)$. Since x_i is a $\sigma(E, E_n^-)$ -accumulation point of $\{x_i^n\}$, it follows from Theorem 4.8 that

$$u_i(x) \geq \liminf_{n \rightarrow \infty} u_i(x_i^n).$$

Thus, for each $m > i$ there exists some $k > m$ with $u_i(x) > u_i(x_i^k)$ (i.e., $x \succ_i x_i^k$), and so (since p_k supports (x_1^k, \dots, x_k^k)) we see that $p_k \cdot x \geq p_k \cdot \omega_i$. The latter property easily implies $p \cdot x \geq p \cdot \omega_i$.

Now let $0 \leq x \in A_a$ satisfy $x \succeq_i x_i$. Then $x + \varepsilon \omega \succ_i x_i$ holds for all $\varepsilon > 0$, and so by the above $p \cdot x + \varepsilon p \cdot \omega \geq p \cdot \omega_i$. Since $\varepsilon > 0$ is arbitrary, we infer that $p \cdot x \geq p \cdot \omega_i$.

3. For each k we have $\sum_{i=1}^k x_i \leq \omega$.

If $n \geq k$, then

$$\sum_{i=1}^k x_i^n \leq \sum_{i=1}^n x_i^n = \sum_{i=1}^n \omega_i \leq \omega,$$

and so $\sum_{i=1}^k x_i \leq \omega$ must hold for each k .

Now let $x = \sup\{\sum_{i=1}^n x_i : n = 1, 2, \dots\} \leq \omega$. Consider (y_1, y_2, \dots) , where $y_1 = x_1 + \omega - x$ and $y_i = x_i$ for $i \geq 2$, and note that (y_1, y_2, \dots) is an allocation.

4. If $0 \leq x \in A_a$ satisfies $x \succeq_i y_i$, then $p \cdot x \geq p \cdot \omega_i$ holds.

Since $y_i \geq x_i$ holds for all i , it follows from the monotonicity of preferences that $y_i \succeq_i x_i$ holds for all i , and this easily implies that (y_1, y_2, \dots) is a weak quasiequilibrium. ■

We now turn our attention to replication properties of our infinite economy \mathcal{E} . By the r -replica \mathcal{E}^r of \mathcal{E} we shall mean a new economy having the following characteristics.

1. Its Riesz dual system is (E, E^r) .
2. There are $\mathcal{N} \times \{1, \dots, r\}$ consumers indexed by (i, j) .
3. Each consumer (i, j) has an initial endowment ω_{ij} , i.e., $\omega_{ij} = \omega_i$ for $i = 1, 2, \dots$ and $j = 1, \dots, r$.
4. Each consumer (i, j) has u_i as his utility function, i.e., $u_{ij} = u_i$.

The consumers (i, j) ($j = 1, \dots, r$) are referred to as *consumers of type i* . Clearly, the total endowment of \mathcal{E}^r is $r\omega$. Every allocation (x_1, x_2, \dots) of \mathcal{E} defines an allocation on each \mathcal{E}^r by assigning $x_{ij} = x_i$ for each $(i, j) \in \mathcal{N} \times \{1, \dots, r\}$. Such an allocation of \mathcal{E}^r is called an *equal treatment allocation*. Thus, every allocation of \mathcal{E} can be considered as an (equal treatment) allocation of every replica \mathcal{E}^r .

An allocation (x_1, x_2, \dots) is said to be *blocked* by a finite coalition F (i.e., by a finite non-empty subset F of \mathcal{N}) whenever there exists another allocation (y_1, y_2, \dots) satisfying

$$\sum_{i \in F} y_i = \sum_{i \in F} \omega_i \text{ and } y_i \succ_i x_i \text{ for each } i \in F.$$

The finite core (or simply the **f-core**) of \mathcal{E} consists of all allocations that cannot be blocked by any finite coalition of \mathcal{N} .

Definition 6.5. An allocation (x_1, x_2, \dots) of \mathcal{E} is said to be an **Edgeworth equilibrium** whenever it belongs to the **f-core** of every replica of the economy \mathcal{E} .

The properties of Edgeworth equilibria for finite economies were studied extensively by the authors in [2, 3]. The next theorem presents a relation between Edgeworth equilibria and weak quasiequilibria for infinite economies.

Theorem 6.6. Every Edgeworth equilibrium is a weak quasiequilibrium with respect to the Riesz dual system (A_ω, A'_ω) .

Proof. Let (x_1, x_2, \dots) be an Edgeworth equilibrium. For each consumer i , let $F_i = \{y \in A_i^+ : y \succ_i x_i\}$ and $G_i = F_i - \omega_i$. In view of $x_i + \omega \succ_i x_i$, we see that both F_i and G_i are non-empty convex sets for each i . Since (by Theorem 4.7) the utility functions are $\|\cdot\|_\infty$ -continuous on A_ω , it follows that the F_i and G_i are also $\|\cdot\|_\infty$ -open subsets of A_ω . Now put

$$G = \text{co}\left(\bigcup_{i=1}^{\infty} G_i\right),$$

and note that G as a subset of A_ω has $\|\cdot\|_\infty$ -interior points. We claim that $0 \notin G$.

To see this, assume by way of contradiction that $0 \in G$. Then there exists some k , $y_i \in F$ (i.e., $y_i \succ_i x_i$) and constants $0 \leq \lambda_i \leq 1$ ($1 \leq i \leq k$) such that $\sum_{i=1}^k \lambda_i = 1$ and $\sum_{i=1}^k \lambda_i (y_i - \omega_i) = 0$. Let $S = \{i : \lambda_i > 0\}$. Clearly, $S \neq \emptyset$ and

$$\sum_{i \in S} \lambda_i y_i = \sum_{i \in S} \lambda_i \omega_i. \quad (*)$$

Now if $i \in S$, then let n_i denote the smallest integer greater or equal than $n\lambda_i$ (i.e., $0 \leq n_i - n\lambda_i \leq 1$). Since $\lim_{n \rightarrow \infty} \frac{n\lambda_i}{n_i} = 1$, we see that $\frac{n\lambda_i}{n_i} y_i \xrightarrow{o} y_i$, and so by the order continuity of the utility functions we can choose n large enough so that

$$z_i = \frac{n\lambda_i}{n_i} y_i \succ_i x_i \text{ for all } i \in S. \quad (**)$$

From (*) we get

$$\sum_{i \in S} n_i z_i = \sum_{i \in S} n\lambda_i y_i = \sum_{i \in S} n\lambda_i \omega_i \leq \sum_{i \in S} n_i \omega_i.$$

The preceding inequality, coupled with (**), shows that (x_1, x_2, \dots) can be blocked by an allocation in some replication of the economy, which is impossible. Hence, $0 \notin G$ must hold, as claimed. (The preceding argument is essentially due to G. Debreu and H. Scarf [16].)

The proof can be completed now by a separation argument. Since G has $\|\cdot\|_\infty$ -interior points, there exists some non-zero price $p \in A'_\omega$ such that $g \in G$ implies $p \cdot g \geq 0$. It follows that $x \succ_i x_i$ in A_i^+ implies $p \cdot x \geq p \cdot \omega_i$, and from this we see that p is a price supporting (x_1, x_2, \dots) on A_ω . ■

In the next section we shall use the preceding theorem to show that every equilibrium in our overlapping generations model is a weak quasiequilibrium with respect to the Riesz dual system (A_ω, A'_ω) .

7. THE OVERLAPPING GENERATIONS MODEL

In our overlapping generations model the index t will denote the time period. The commodity-price duality at period t will be represented by a symmetric Riesz dual system (E_t, E'_t) (and so E_t is a normal Riesz space). Consequently, we have a sequence $((E_1, E'_1), (E_2, E'_2), \dots)$ of symmetric Riesz dual systems each member of which designates the commodity-price duality at the corresponding time period. Note that we allow a (possibly) different commodity space at each time period in order to capture the economic intuition, according to which, as we progress into the future new commodities enter the market and old ones cease to exist. We shall write

$$E = E_1 \times E_2 \times \dots \quad \text{and} \quad E' = E'_1 \times E'_2 \times \dots$$

Here are some examples of symmetric Riesz dual systems

$$(L_p(\mu), L_q(\mu)) \quad (1 < p < \infty, 1 < q < \infty, \frac{1}{p} + \frac{1}{q} = 1),$$

$$(\mathcal{R}^n, \mathcal{R}^n), \quad (ca(\Omega), ca'(\Omega)) \quad (\mathcal{R}^\infty, \phi) \quad (\phi, \mathcal{R}^\infty), \quad \text{and} \quad (L_\infty, l_1).$$

If the measure μ is σ -finite, then

$$(L_1(\mu), L_\infty(\mu)) \quad \text{and} \quad (L_\infty(\mu), L_1(\mu))$$

are also symmetric Riesz dual systems.

We shall assume that each consumer has a two-period lifetime. Thus, consumer t is born at period t and lives all his life in periods t and $t+1$. Each consumer trades and has tastes for commodities only during his life-time period. We suppose that consumer t gets an initial endowment $0 < \omega_t^i \in E_t$ at period t and $0 < \omega_t^{i+1} \in E_{t+1}$ at period $t+1$ (and, of course, nothing else in any other periods). Consequently, his initial endowment ω_t can be represented by the vector

$$\omega_t = (0, \dots, 0, \omega_t^i, \omega_t^{i+1}, 0, 0, \dots) \in E,$$

where ω_t^i and ω_t^{i+1} occupy the positions t and $t+1$, respectively. Also, we shall assume that the "father" of consumer 1 (i.e., consumer 0) is present in the model at period 1. He will be designated as consumer 0 and his endowment will be taken to be of the form

$$\omega_0 = (\omega_0^1, 0, 0, \dots)$$

with $0 < \omega_0^1 \in E_1$. Thus, the total endowment is represented by the vector

$$\omega = \sum_{t=0}^{\infty} \omega_t = (\omega_0^1 + \omega_1^1, \omega_1^2 + \omega_2^2, \omega_2^3 + \omega_3^3, \dots) \in E.$$

The vectors of the form

$$x_t = (0, \dots, 0, x_t^i, x_t^{i+1}, 0, 0, \dots),$$

where $x_t^i \in E_t^+$ and $x_t^{i+1} \in E_{t+1}^+$ represent the commodity bundles for consumer t during his life time. Each consumer t maximizes a utility function u_t defined on his commodity space, i.e., u_t is a function from $E_t^+ \times E_{t+1}^+$ into \mathcal{R} . The value of u_t at the commodity bundle $x_t = (0, \dots, 0, x_t^i, x_t^{i+1}, 0, 0, \dots)$ will be denoted by $u_t(x_t^i, x_t^{i+1})$. When needed to simplify notation, we shall consider u_t defined everywhere on $E_1^+ \times E_2^+ \times \dots$ by the formula

$$u_t(x^1, x^2, \dots) = u_t(x^i, x^{i+1}).$$

The utility functions will be assumed to satisfy the following properties.

1. Each u_t is quasi-concave;
2. Each u_t is strictly monotone on $E_t^+ \times E_{t+1}^+$, that is, $(x, y) > (x_1, y_1)$ in $E_t^+ \times E_{t+1}^+$ implies $u_t(x, y) > u_t(x_1, y_1)$; and
3. Each u_t is weakly continuous on the order bounded subsets of $E_t^+ \times E_{t+1}^+$, i.e., if an order bounded net $\{(x_\alpha^i, x_\alpha^{i+1})\}$ satisfies $(x_\alpha^i, x_\alpha^{i+1}) \xrightarrow{w} (x, y)$ in $E_t^+ \times E_{t+1}^+$, then $\lim_\alpha u_t(x_\alpha^i, x_\alpha^{i+1}) = u_t(x, y)$.

The case $t = 0$ is a special case. The utility function u_0 is a function of one variable defined on E_1^+ . It is also assumed to satisfy properties (1), (2) and (3) above.

Remark. Since the locally convex-solid topology $|\sigma|(E_t, E_t')$ on each E_t is order continuous, it is easy to see that order convergence in $E_t \times E_{t+1}$ implies $|\sigma|(E_t, E_t')$ -convergence (and hence, weak convergence). Therefore, since each order convergent net is eventually order bounded, it follows from property (3) that the utility functions for our overlapping generations model are all myopic.

Next, we present examples of utility functions that satisfy properties (1), (2) and (3) above.

Example 7.1. Here are two utility functions that satisfy our properties (whose straightforward verifications are left for the reader).

1. Consider two symmetric Riesz dual systems of the form (ℓ_p, ℓ_q) and (ℓ_r, ℓ_s) , where $1 \leq p, r \leq \infty$ and $1 \leq q, s \leq \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{1}{s} = 1$. Then the utility function $u: \ell_p^+ \times \ell_r^+ \rightarrow \mathcal{R}$ defined by

$$u(x, y) = \sum_{n=1}^{\infty} \frac{\sqrt{x_n + y_n}}{n^2}, \quad x = (x_1, x_2, \dots) \in \ell_p^+, \quad y = (y_1, y_2, \dots) \in \ell_r^+,$$

satisfies our properties.

2. Consider the symmetric Riesz dual systems (ℓ_p, ℓ_q) and $(L_r[0, 1], L_s[0, 1])$, where $1 \leq p, r \leq \infty$ and $1 \leq q, s \leq \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{1}{s} = 1$. Fix some strictly positive function $h \in L_s[0, 1]$ (for instance let $h(x) = x^2$). Then the utility function $u: \ell_p^+ \times L_r^+[0, 1] \rightarrow \mathcal{R}$ defined by

$$u(x, f) = \int_0^1 f(x)h(x) dx + \sum_{n=1}^{\infty} \frac{\sqrt{x_n}}{n^2}, \quad x = (x_1, x_2, \dots) \in \ell_p^+, \quad f \in L_r^+[0, 1],$$

satisfies our properties. ■

For our discussion, we shall employ the notation

$$\theta_t = \omega_{t-1}^t + \omega_t^t, \quad t = 1, 2, \dots$$

Clearly, the bundle $\theta_t \in E_t^+$ represents the total endowment present at period t . In particular, we have $\omega = (\theta_1, \theta_2, \dots)$.

The ideal generated by θ_t in E_t will be denoted by Θ_t . That is,

$$\Theta_t = \{x \in E_t: \exists \lambda > 0 \text{ with } |x| \leq \lambda \theta_t\}.$$

The ideal Θ_t under the norm $\|x\|_{\infty} = \inf\{\lambda > 0: |x| \leq \lambda \theta_t\}$ is an AM-space with unit. As usual, the norm dual of Θ_t will be denoted by Θ_t' .

We shall denote the ideal generated by $\{\omega_t: t = 0, 1, \dots, n\}$ in E by A_n . Clearly,

$$A_n = \Theta_1 \times \Theta_2 \times \dots \times \Theta_n \times \Omega_n \times 0 \times 0 \times \dots,$$

where $0 = \{0\}$ and Ω_n denotes the ideal generated by ω_n^{n+1} in E_{n+1} . Obviously, each A_n is an AM-space having $(\theta_1, \dots, \theta_n, \omega_n^{n+1}, 0, 0, \dots)$ as unit. Note that for each n we have the proper inclusion $A_n \subseteq A_{n+1}$. Finally, we shall denote by A the ideal generated in E by the sequence $\{\omega_n: n = 0, 1, 2, \dots\}$. It follows that

$$A = \bigcup_{n=1}^{\infty} A_n.$$

Let ξ denote the inductive limit topology generated by the sequence $\{A_n\}$ on A . Since the ideal A is also generated by the disjoint sequence $\{(0, \dots, 0, \theta_n, 0, 0, \dots): n = 1, 2, \dots\}$, we see that ξ is also the strict inductive limit topology.

Theorem 7.2. *The topological dual of (A, ξ) is*

$$A' = \Theta_1' \times \Theta_2' \times \dots,$$

where the duality between A and A' is given by

$$p \cdot x = \sum_{t=1}^{\infty} p_t \cdot x_t,$$

for all $\mathbf{x} = (x_1, x_2, \dots)$ and $\mathbf{p} = (p_1, p_2, \dots)$.

Proof. By Theorem 3.2 we know that $\mathbf{A}' = \mathbf{A}^-$. So, it suffices to show that $\mathbf{A}^- = \Theta'_1 \times \Theta'_2 \times \dots$. Note first that if $\mathbf{p} = (p_1, p_2, \dots) \in \Theta'_1 \times \Theta'_2 \times \dots$, then the formula

$$\mathbf{p} \cdot \mathbf{x} = \sum_{t=1}^{\infty} p_t \cdot x_t, \quad \mathbf{x} = (x_1, x_2, \dots) \in \mathbf{A},$$

clearly defines an order bounded linear functional on \mathbf{A} .

Now let $\mathbf{p} \in \mathbf{A}^-$. For each t define $p_t \in \Theta'_t$ by

$$p_t \cdot \mathbf{x} = \mathbf{p} \cdot (0, \dots, 0, x_t, 0, 0, \dots), \quad x_t \in \Theta_t,$$

where x occupies position t . If $\mathbf{x} = (x_1, x_2, \dots) \in \mathbf{A}$, then $x_t \in \Theta_t$ for each t and $x_t = 0$ for all but a finite number of t . Thus,

$$\mathbf{p} \cdot \mathbf{x} = \sum_{t=1}^{\infty} \mathbf{p} \cdot (0, \dots, 0, x_t, 0, 0, \dots) = \sum_{t=1}^{\infty} p_t \cdot x_t,$$

and so \mathbf{p} can be identified with the sequence (p_1, p_2, \dots) , and our conclusion follows. ■

Unless otherwise stated to the contrary, by a price for the overlapping generations model we shall simply mean a positive element of \mathbf{A}' . Thus, according to Theorem 7.2, a price is any sequence of the form

$$\mathbf{p} = (p_1, p_2, \dots),$$

where $0 \leq p_t \in \Theta'_t$ for each t .

A sequence $\mathbf{x} = (x_0, x_1, x_2, \dots)$, where

$$0 \leq x_0 = (x_0^1, 0, 0, \dots) \in \mathbf{A}$$

and

$$0 \leq x_t = (0, \dots, 0, x_t^i, x_t^{i+1}, 0, 0, \dots) \in \mathbf{A} \quad \text{for } t \geq 1,$$

is said to be an allocation whenever $\sum_{i=0}^{\infty} x_i = \omega$ (or equivalently, whenever $x_{i-1}^i + x_i^i = \theta_i$ holds for all $t = 1, 2, \dots$). A non-zero price $\mathbf{p} = (p_1, p_2, \dots)$ is said to support an allocation $\mathbf{x} = (x_0, x_1, x_2, \dots)$ whenever

- a) $x \succeq_0 x_0^1$ in E_1^+ implies $p_1 \cdot x \geq p_1 \cdot \omega_0^1$; and
- b) $(x, y) \succeq_t (x_t^i, x_t^{i+1})$ in $E_t^+ \times E_{t+1}^+$ implies

$$p_t \cdot x + p_{t+1} \cdot y \geq p_t \cdot \omega_t^i + p_{t+1} \cdot \omega_t^{i+1}$$

for all $t = 1, 2, \dots$.

It should be noted that if a price $\mathbf{p} = (p_1, p_2, \dots)$ supports an allocation (x_0, x_1, x_2, \dots) , then we have $p_1 \cdot x_0^1 \geq p_1 \cdot \omega_0^1$ and

$$p_t \cdot x_t^i + p_{t+1} \cdot x_t^{i+1} \geq p_t \cdot \omega_t^i + p_{t+1} \cdot \omega_t^{i+1}$$

for all $t = 1, 2, \dots$.

Definition 7.3. An allocation $\mathbf{x} = (x_0, x_1, x_2, \dots)$ is said to be an equilibrium for the overlapping generations model if it can be supported by a non-zero price $\mathbf{p} = (p_1, p_2, \dots)$ such that

- a) $p_1 \cdot x_0^1 = p_1 \cdot \omega_0^1$; and
- b) $p_t \cdot x_t^i + p_{t+1} \cdot x_t^{i+1} = p_t \cdot \omega_t^i + p_{t+1} \cdot \omega_t^{i+1}$ for $t \geq 1$.

The alert reader should recognize immediately that an equilibrium for our overlapping generations model is a quasiequilibrium for the infinite economy having Riesz dual system (A, A') and consumers' characteristics $\{(\omega_t, \Sigma_t): t = 0, 1, 2, \dots\}$. In fact, it is a Walrasian equilibrium since we claim that an equilibrium $x = (x_0, x_1, x_2, \dots)$ for the overlapping generations model satisfies $p \cdot \omega_t > 0$ for all t . To see this, pick a price $0 < p = (p_1, p_2, \dots) \in A'$ that supports the allocation and note that $p \cdot \omega_t > 0$ must hold for some t . Now if $p \cdot \omega_t = 0$ holds for at least one t , then there exist two consecutive non-negative integers r and s such that $p \cdot \omega_r > 0$ and $p \cdot \omega_s = 0$. Clearly, x_r is a maximal element in the budget set of consumer r . On the other hand, $x_r + \omega_s \succ_r x_r$ and $p \cdot (x_r + \omega_s) = p \cdot x_r = p \cdot \omega_r$ show that x_r is not a maximal element in the budget set of consumer r , a contradiction.

Therefore, we have the following connection between equilibria for the overlapping generations model and Walrasian equilibria.

Theorem 7.4. *An allocation is an equilibrium for the overlapping generations model if and only if it is a Walrasian equilibrium for the infinite economy with Riesz dual system (A, A') and consumers' characteristics $\{(\omega_t, \Sigma_t): t = 0, 1, 2, \dots\}$.*

Since a Walrasian equilibrium is an Edgeworth equilibrium, it follows from Theorem 6.6 that every equilibrium for our overlapping generations model is a weak quasiequilibrium with respect to the Riesz dual system (A_ω, A'_ω) .

8. EXISTENCE OF EQUILIBRIA IN THE OVERLAPPING GENERATIONS MODEL

The purpose of this section is to prove the main result of this work. It can be stated as follows.

Theorem 8.1. *Every overlapping generations model has an equilibrium with respect to the Riesz dual system (A, A') that can be supported by an order continuous price.*

The proof of Theorem 8.1 is quite involved and it will be accomplished by a series of lemmas. We start with a basic definition.

Definition 8.2. *For each n we shall denote by \mathcal{E}_n the finite economy having Riesz dual system (A_n, A'_n) and set of agents $\{0, 1, \dots, n\}$ with their original characteristics.*

Intuitively speaking our overlapping generations model must be in some sense the "limit" of the sequence $\{\mathcal{E}_n\}$ of finite economies. This intuitive idea (which is the byproduct of the pioneering classical work of T. F. Bewley [10]) is the driving force behind our mathematically delicate proof of Theorem 8.1. Before passing to a "limit" of the sequence $\{\mathcal{E}_n\}$ we have to study its properties.

Our first important result is that each finite economy \mathcal{E}_n has a Walrasian equilibrium.

Lemma 8.3. *Every finite economy \mathcal{E}_n has a Walrasian equilibrium (x_0, x_1, \dots, x_n) of the form*

$$x_0 = (x_0^1, 0, 0, \dots) \quad \text{and} \quad x_t = (0, \dots, 0, x_t^t, x_t^{t+1}, 0, 0, \dots), \quad 1 \leq t \leq n.$$

Moreover, every non-zero price p that supports (x_0, x_1, \dots, x_n) satisfies

$$p \cdot \omega_t > 0 \quad \text{for each} \quad 0 \leq t \leq n.$$

Proof. From our previous discussion, we know that A_n coincides with the ideal generated in A by $(\theta_1, \theta_2, \dots, \theta_n, \omega_n^{n+1}, 0, 0, \dots)$, and so A_n is an AM-space with unit. Since (by the remark before

Lemma 7.1) all utility functions are myopic, it follows from Theorem 5.6 that there is a quasiequilibrium (x_0, x_1, \dots, x_n) supported by a price $0 \leq p \in A'_n$ such that $p \cdot (\sum_{i=0}^n \omega_i) = 1$. By the special nature of the utility functions, we see that

$$x_t = (0, \dots, 0, x_t^i, x_t^{i+1}, 0, 0, \dots) \text{ for } 1 \leq t \leq n \text{ and } x_0 = (x_0^1, 0, 0, \dots).$$

Next, we claim that $p \cdot \omega_0 > 0$ holds. To see this, assume by way of contradiction that $p \cdot \omega_0 = 0$. Then $p \cdot \omega_1 = 0$ must also hold. Otherwise, $p \cdot \omega_1 > 0$ and $x_1 + \omega_0 \succ_1 x_1$ imply that x_1 is not a maximal element in the budget set of consumer 1. Repeating the argument, we see that $p \cdot \omega_t = 0$ for $0 \leq t \leq 1$, and so

$$0 = \sum_{t=0}^n p \cdot \omega_t = p \cdot (\sum_{t=0}^n \omega_t) = 1,$$

which is impossible. Hence, $p \cdot \omega_0 > 0$ holds, and so x_0 is a maximal element in the budget set of consumer 0.

Now we claim that $p \cdot \omega_1 > 0$ holds. Indeed, if $p \cdot \omega_1 = 0$, then $x_0 + \omega_1 \succ_0 x_0$ implies that x_0 is not a maximal bundle in the budget set of consumer 0, a contradiction. Hence, $p \cdot \omega_1 > 0$. Repeating the argument, we see that $p \cdot \omega_2 > 0, \dots, p \cdot \omega_n > 0$, and so (x_0, x_1, \dots, x_n) is indeed a Walrasian equilibrium.

Also, it should be noted that the price p above is of the form

$$p = (p_1, p_2, \dots, p_n, p_{n+1}, 0, 0, \dots),$$

where $0 \leq p_t \in \Theta'_t$ ($1 \leq t \leq n$) and $0 \leq p_{n+1} \in \Omega'_t$. ■

In the sequel a Walrasian equilibrium (x_0, x_1, \dots, x_n) for the finite economy \mathcal{E}_n supported by a price $0 < p \in A'_n$ will be denoted by $(x_0, x_1, \dots, x_n; p)$. Also, it should be noted that every Walrasian equilibrium $(x_0, x_1, \dots, x_n; p)$ for the economy \mathcal{E}_n is necessarily of the form

$$x_0 = (x_0^1, 0, 0, \dots) \text{ and } x_t = (0, \dots, 0, x_t^i, x_t^{i+1}, 0, 0, \dots) \text{ for } 1 \leq t \leq n.$$

It will be useful to know that supporting prices of Walrasian equilibria for the economies \mathcal{E}_n are order continuous linear functionals on A_n . The next lemma takes care of this property.

Lemma 8.4. If $(x_0, x_1, \dots, x_n; p)$ is a Walrasian equilibrium for the finite economy \mathcal{E}_n , then p is order continuous on A_n .

Proof. We can assume that $p \cdot (\sum_{i=0}^n \omega_i) = 1$. Let $y_\alpha \downarrow 0$ in A_n and let $\varepsilon > 0$ be fixed.

Without loss of generality we can suppose that $0 \leq y_\alpha \leq \sum_{i=0}^n x_i$ holds for all α . Thus, by the Riesz Decomposition Property, we can write $y_\alpha = \sum_{i=0}^n y_\alpha^i$ with $0 \leq y_\alpha^i \leq x_i$ for $t = 0, 1, 2, \dots, n$. From $x_t + \varepsilon \omega_t \succ_t x_t$, $y_\alpha^i \xrightarrow{\alpha} 0$ and the weak continuity of the utility functions on the order bounded sets, we see that there exists some β such that $x_t + \varepsilon \omega_t - y_\alpha^i \succ_t x_t$ for all $\alpha \geq \beta$ and all $t = 0, 1, \dots, n$. By the supportability of p , we infer that

$$p \cdot x_t + \varepsilon p \cdot \omega_t - p \cdot y_\alpha^i \geq p \cdot x_t = p \cdot x_t,$$

and so $0 \leq p \cdot y_\alpha^i \leq \varepsilon p \cdot \omega_t$ for all $\alpha \geq \beta$ and all $0 \leq t \leq n$. Thus, for $\alpha \geq \beta$ we have

$$0 \leq p \cdot y_\alpha = \sum_{i=0}^n p \cdot y_\alpha^i \leq \sum_{i=0}^n \varepsilon p \cdot \omega_i = \varepsilon,$$

as desired. ■

We continue our discussion with an important property of nets. Recall that a net $\{y_\lambda\}_{\lambda \in \Lambda}$ is said to be a subnet of another net $\{z_\alpha\}_{\alpha \in \Lambda}$ whenever there exists a function $\sigma: \Lambda \rightarrow A$ such that

- a) $y_\lambda = x_{\sigma_\lambda}$ holds for all $\lambda \in A$; and
b) given $\alpha_0 \in A$ there exists some $\lambda_0 \in A$ so that $\lambda \geq \lambda_0$ implies $\sigma_\lambda \geq \alpha_0$.

Now if $\{y_\alpha\}$ is a subnet of a sequence $\{x_n\}$, then for any α_0 , we see that the set $S = \{y_\alpha: \alpha \geq \alpha_0\}$ contains infinitely many terms of the sequence $\{x_n\}$ (i.e., $x_n \in S$ holds for infinitely many n), and so there exists a subsequence $\{y_n\}$ of $\{x_n\}$ satisfying $y_n \in S$ for each n . This observation will be employed quite often in our proofs.

The next lemma presents a "growth" estimate for a sequence of Walrasian equilibria for the finite economies \mathcal{E}_n . It is the analogue of C. A. Wilson's Lemma 3 in [44].

Lemma 8.5. *If $\{(x_0^n, x_1^n, \dots, x_r^n; p_n)\}$ is a sequence of Walrasian equilibria for the finite economies \mathcal{E}_n , then for each pair m, ℓ there exists a constant $M > 0$ (depending upon m and ℓ) such that*

$$0 < p_n \cdot \omega_\ell \leq M p_n \cdot \omega_m$$

holds for all $n \geq \max\{m, \ell\}$.

Proof. Fix m and suppose by way of contradiction that there exists some natural number ℓ satisfying $\liminf_{n \geq k} \frac{p_n \cdot \omega_m}{p_n \cdot \omega_\ell} = 0$, where $k = \max\{\ell, m\}$. Thus, the sets

$$\{i \in \mathcal{N}: \liminf_{n \rightarrow \infty} \frac{p_n \cdot \omega_i}{p_n \cdot \omega_\ell} > 0\} \text{ and } \{i \in \mathcal{N}: \liminf_{n \rightarrow \infty} \frac{p_n \cdot \omega_i}{p_n \cdot \omega_\ell} = 0\}$$

are both non-empty. It follows that there exist two consecutive integers r and s such that

$$\liminf_{n \rightarrow \infty} \frac{p_n \cdot \omega_r}{p_n \cdot \omega_\ell} > 0 \text{ and } \liminf_{n \rightarrow \infty} \frac{p_n \cdot \omega_s}{p_n \cdot \omega_\ell} = 0.$$

By passing to a subsequence, we can assume that

$$\liminf_{n \rightarrow \infty} \frac{p_n \cdot \omega_r}{p_n \cdot \omega_\ell} > 0 \text{ and } \lim_{n \rightarrow \infty} \frac{p_n \cdot \omega_s}{p_n \cdot \omega_\ell} = 0. \quad (1)$$

In view of $x_r^n \in [0, \theta_r] \times [0, \theta_{r+1}]$ ($n \geq r$) and the weak compactness of the order intervals, we see that the sequence $\{x_r^n: n = 1, 2, \dots\}$ has a weakly convergent subnet $\{y_\alpha\}$, say $y_\alpha \xrightarrow{w} y$ holds in $E_r^+ \times E_{r+1}^+$.

From $y + \omega_s \succ_r y$ and the weak continuity of the utility function u_r on the order bounded sets, there exists some $0 < \delta < 1$ such that $\delta(y + \omega_s) \succ_r y$. Using the weak continuity of u_r once again, we see that there exists some α_0 satisfying $\delta(y_\alpha + \omega_s) \succ_r y_\alpha$ for all $\alpha \geq \alpha_0$. Therefore, there exists a strictly increasing sequence k_n of natural numbers satisfying $\delta(x_r^{k_n} + \omega_s) \succ_r x_r^{k_n}$; see the discussion preceding the lemma. Since (1) remains true if we replace p_n by p_{k_n} , we can assume without loss of generality that $\delta(x_r^n + \omega_s) \succ_r x_r^n$ holds for all $n \geq r$. Therefore, by the suportability of p_n , we have $\delta p_n \cdot (\omega_r + \omega_s) = p_n \cdot [\delta(x_r^n + \omega_s)] \geq p_n \cdot \omega_r$ for all $n \geq r$. Now note that

$$\begin{aligned} \delta \liminf_{n \rightarrow \infty} \frac{p_n \cdot \omega_r}{p_n \cdot \omega_\ell} &= \delta \liminf_{n \rightarrow \infty} \left(\frac{p_n \cdot \omega_r}{p_n \cdot \omega_\ell} + \frac{p_n \cdot \omega_s}{p_n \cdot \omega_\ell} \right) \\ &= \liminf_{n \rightarrow \infty} \frac{\delta p_n \cdot (\omega_r + \omega_s)}{p_n \cdot \omega_\ell} \\ &\geq \liminf_{n \rightarrow \infty} \frac{p_n \cdot \omega_r}{p_n \cdot \omega_\ell} > 0, \end{aligned}$$

which is impossible, and our conclusion follows. ■

Now consider each order interval $[0, \theta_i]$ equipped with the weak topology and let

$$\mathcal{X}_0 = [0, \theta_1] \times 0 \times 0 \times \dots$$

and

$$\mathcal{X}_t = 0 \times \cdots \times 0 \times [0, \theta_t] \times [0, \theta_{t+1}] \times 0 \times 0 \times \cdots \quad \text{for each } t = 1, 2, \dots,$$

where $0 = \{0\}$ and the order intervals $[0, \theta_t]$ and $[0, \theta_{t+1}]$ occupy the t and $t+1$ factors. Clearly, each \mathcal{X}_t is a compact topological space, and so by Tychonoff's classical compactness theorem the product topological space

$$\mathcal{X} = \prod_{i=0}^{\infty} \mathcal{X}_i$$

is a compact space.

The topological space \mathcal{X} will play an important role in our proofs. At this point, let us illustrate briefly its role. By Lemma 8.3 we know that every finite economy \mathcal{E}_n has a Walrasian equilibrium. For each n , let $(x_0^n, x_1^n, \dots, x_n^n)$ be a Walrasian equilibrium for \mathcal{E}_n , where

$$x_0^n = (x_{0,n}^1, 0, 0, \dots) \quad \text{and} \quad x_t^n = (0, \dots, 0, x_{t,n}^t, x_{t,n}^{t+1}, 0, 0, \dots) \quad \text{for } t \geq 1.$$

If we let

$$f_n = (x_0^n, x_1^n, \dots, x_n^n, 0, 0, \dots),$$

then $\{f_n\}$ is a sequence of \mathcal{X} , and so since \mathcal{X} is compact, it has an accumulation point $x = (x_0, x_1, \dots)$. It will turn out that the accumulation point x is an equilibrium for our overlapping generations model. The objective of our next goal is the establishment of this claim.

Definition 8.6. A sequence $\{(x_0^n, x_1^n, \dots, x_n^n; p_n)\}$ of Walrasian equilibria for the finite economies \mathcal{E}_n is said to be a fundamental sequence for the overlapping generations model whenever $p_n \omega_0 = 1$ holds for all n .

The reader should notice that (by Lemma 8.3) every overlapping generations model admits a fundamental sequence. As mentioned before, our next objective is to show that an appropriate "limit" of a fundamental sequence yields an equilibrium for the overlapping generations model.

Lemma 8.7. If $\{(x_0^n, x_1^n, \dots, x_n^n; p_n)\}$ is a fundamental sequence and $y_m \downarrow 0$ holds in some A_k , then the sequence $\{y_m\}$ converges uniformly to zero on $\{p_n; n \geq k\}$.

Proof. Let $\{(x_0^n, x_1^n, \dots, x_n^n; p_n)\}$ be a fundamental sequence and let a sequence $\{y_m\} \subseteq A_k$ satisfy $y_m \downarrow 0$. We have to show that given $\epsilon > 0$ there exists some m_0 such that

$$0 \leq p_n \cdot y_m < \epsilon$$

holds for all $n \geq k$ and all $m \geq m_0$.

Suppose by way of contradiction that for some $0 < \epsilon < 1$ the above conclusion is false. Then using Lemma 8.4 and an easy inductive argument, we see that there exist two strictly increasing sequences $\{n_\ell\}$ and $\{m_\ell\}$ of natural numbers such that

$$p_{n_\ell} \cdot y_{m_\ell} > \epsilon$$

holds for all ℓ . Thus, by passing to an appropriate subsequence (and relabelling) we can assume that

$$p_n \cdot y_m > \epsilon \quad \text{for all } n \geq k \quad \text{and all } m. \quad (*)$$

Next, note that without loss of generality we can also suppose that $0 \leq y_m \leq \sum_{i=0}^k \omega_i = \hat{\omega}_k$ holds for each m . Lemma 8.5 applied with $m = 0$ and $\ell = i$ for $0 \leq i \leq k+1$ guarantees the existence of some $M \geq 1$ such that

$$0 \leq p_n \omega_i \leq M$$

holds for all $n \geq k+1$ and all $i = 0, 1, \dots, k$. Put $\eta = \frac{\epsilon}{(k+2)M}$ so that $\epsilon = \eta(k+2)M$.

Now put

$$f_n = (x_0^n, x_1^n, \dots, x_k^n, 0, 0, \dots), \quad n = 1, 2, \dots,$$

and note that $\{f_n\}$ is a sequence of the compact topological space \mathcal{X} . Thus, there exists a subnet $\{g_\alpha\}$, where $g_\alpha = (g_1^\alpha, g_2^\alpha, \dots) \in \mathcal{X}$, of the sequence $\{f_n\}$ such that

$$\lim_{\alpha} g_\alpha = x = (x_0, x_1, \dots).$$

Clearly, $g_i^\alpha \xrightarrow{w} x_i$ holds in $E_i \times E_{i+1}$ for each i .

From $x_i + \eta\omega_i \succ_i x_i$, we see that there exists some $\delta > 0$ satisfying

$$u_i(x_i + \eta\omega_i) > u_i(x_i) + 2\delta$$

for all $i = 0, 1, \dots, k+1$. Put $v_i = x_i + \eta\omega_i$ and note that since each u_i ($i = 0, 1, \dots, k+1$) is weakly continuous on $[0, 2\hat{\omega}_{k+2}]$ (where $\hat{\omega}_{k+2} = \sum_{i=0}^{k+2} \omega_i$), there exists a weakly convex neighborhood V of zero for $E_1 \times \dots \times E_{k+1} \times E_{k+2}$ such that:

$$x \in [0, 2\hat{\omega}_{k+2}] \text{ with } x - v_i \in V \text{ for some } 0 \leq i \leq k+1 \text{ implies } |u_i(x) - u_i(v_i)| < \delta.$$

Next, fix a solid neighborhood W of zero for $E_1 \times \dots \times E_{k+2}$ with $W \subseteq \frac{1}{2}V$. (Here we use the fact that each $|\sigma|(E_i, E_i)$ is an order continuous locally convex-solid topology.) Since $y_m \downarrow 0$, there exists some m_0 such that $y_m \in W$ holds for all $m \geq m_0$.

In view of $g_i^\alpha \xrightarrow{w} x_i$ ($i = 0, 1, \dots, k+1$), and the weak continuity of the utility functions on the order bounded sets, there exists some α_0 such that

$$u_i(x_i + \eta\omega_i) > u_i(g_i^\alpha) + \delta \quad \text{and} \quad x_i - g_i^\alpha \in \frac{1}{2}V$$

hold for all $\alpha \geq \alpha_0$ and all $i = 0, 1, \dots, k+1$. By the discussion preceding Lemma 8.5, we see that for each $i = 0, 1, \dots, k+1$ and infinitely many n we have

$$u_i(x_i + \eta\omega_i) > u_i(x_i^n) + \delta \quad \text{and} \quad x_i - x_i^n \in \frac{1}{2}V. \quad (**)$$

Now define $z_{i,m}^n \in [0, 2\hat{\omega}_{k+2}]$ by $z_{i,m}^n = x_i^n + \eta\omega_i - x_i^n \wedge y_m$ and note that for all $m \geq m_0$, all $i = 0, 1, \dots, k+1$ and infinitely many n we have

$$v_i - z_{i,m}^n = x_i - x_i^n + x_i^n \wedge y_m \in \frac{1}{2}V + W \subseteq \frac{1}{2}V + \frac{1}{2}V = V.$$

Thus, $|u_i(z_{i,m}^n) - u_i(v_i)| < \delta$ holds for all $m \geq m_0$, all $i = 0, 1, \dots, k+1$ and infinitely many n . The latter, combined with (**), yields

$$u_i(z_{i,n}^n) > u_i(v_i) - \delta > [u_i(x_i^n) + \delta] - \delta = u_i(x_i^n),$$

and so

$$z_{i,m}^n = x_i^n + \eta\omega_i - x_i^n \wedge y_m \succ_i x_i^n$$

holds for all $m \geq m_0$, all $i = 0, 1, \dots, k+1$ and infinitely many n . Invoking the supportability of p_n , we obtain

$$p_n \cdot x_i^n + \eta p_n \cdot \omega_i - p_n \cdot (x_i^n \wedge y_m) \geq p_n \cdot \omega_i = p_n \cdot x_i^n,$$

and so

$$0 \leq p_n \cdot (x_i^n \wedge y_m) \leq \eta p_n \cdot \omega_i < \eta M$$

holds for all $m \geq m_0$, all $0 \leq i \leq k+1$ and infinitely many n . Therefore, for infinitely many n and all $m \geq m_0$ we have

$$\begin{aligned} p_n \cdot y_m &= p_n \cdot \left[\left(\sum_{i=0}^{k+1} x_i^n \right) \wedge y_m \right] \\ &\leq \sum_{i=0}^{k+1} p_n \cdot (x_i^n \wedge y_m) \\ &\leq \sum_{i=0}^{k+1} \eta M = \eta(k+2)M = \varepsilon, \end{aligned}$$

which contradicts $(*)$, and the proof of the lemma is finished. ■

The next result is the heart of our arguments. It asserts that the sequence of prices in a fundamental sequence is sequentially w^* -compact on every ideal A_k .

Lemma 8.8. *Let $\{(x_0^n, x_1^n, \dots, x_n^n; p_n)\}$ be a fundamental sequence and let $\{r_n\}$ be a subsequence of $\{p_n\}$. Then for each k there exists a subsequence $\{q_n\}$ of $\{r_n\}$ which converges pointwise on A_k (i.e., $\lim q_n \cdot y$ exists in \mathcal{R} for each $y \in A_k$).*

Proof. Let k be fixed. We shall show that the set $\{p_n: n \geq k\}$ as a subset of A'_k is relatively weakly compact, and from this our conclusion will follow. To establish this, by Grothendieck's classical compactness theorem [5, Theorem 13.10, p. 208] it suffices to show that the set $\{p_n: n \geq k\}$ is norm bounded and that every disjoint sequence of $[0, \hat{\omega}_k]$ converges to zero uniformly on $\{p_n: n \geq k\}$ (where, as usual, $\hat{\omega}_k = \sum_{i=0}^k \omega_i$).

To see that $\{p_n: n \geq k\}$ is norm bounded, note first that $\|p_n\| = p_n \cdot \hat{\omega}_k$ holds for all $n \geq k$. If $\{p_n: n \geq k\}$ is not norm bounded, then there exists an increasing sequence of natural numbers $\{t_n\}$ satisfying $p_{t_n} \cdot \hat{\omega}_k \geq n^2$ for all n . Let $y_n = \frac{1}{n} \hat{\omega}_k$. Then $y_n \downarrow 0$ holds in A_k and from Lemma 8.7, we see that $\{y_n\}$ converges to zero uniformly on $\{p_n: n \geq k\}$, contrary to $p_{t_n} \cdot y_n \geq n$ for all n . Hence $\{p_n: n \geq k\}$ is a norm bounded subset of A'_k .

Now let $\{v_n\}$ be a disjoint sequence of $[0, \hat{\omega}_k]$. For each n put $z_n = \sum_{i=1}^n v_i = \sqrt[n]{\sum_{i=1}^n v_i} \leq \hat{\omega}_k$. Since each E_t is Dedekind complete, there exists some $z \in [0, \hat{\omega}_k]$ with $z_n \uparrow z$. Next note that

$$0 \leq v_n = z_n - z_{n-1} \leq z - z_{n-1} = \zeta_n \downarrow 0.$$

By Lemma 8.7 the sequence $\{\zeta_n\}$ converges to zero uniformly on $\{p_n: n \geq k\}$, and from the relation $0 \leq p_t \cdot v_n \leq p_t \cdot \zeta_n$ ($t \geq k$), we infer that $\{v_n\}$ likewise converges to zero uniformly on $\{p_n: n \geq k\}$. The proof of the lemma is now complete. ■

Observe that if $\{(x_0^n, x_1^n, \dots, x_n^n; p_n)\}$ is a fundamental sequence, then for each $y \in A$ the value $p_n \cdot y$ is well defined for all but a finite number of n . Thus, for example, $\liminf p_n \cdot y$ exists in \mathcal{R} for each $y \in A^+$.

The next lemma asserts that the sequence of prices in a fundamental sequence has always a weak* convergent subsequence on A .

Lemma 8.9. *If $\{(x_0^n, x_1^n, \dots, x_n^n; p_n)\}$ is a fundamental sequence, then there exists a subsequence $\{q_n\}$ of $\{p_n\}$ which is w^* -convergent to a non-zero price in A' , i.e., there exists some $0 < q \in A'$ satisfying*

$$q \cdot y = \lim_{n \rightarrow \infty} q_n \cdot y$$

for each $y \in A$.

Proof. The desired subsequence will be constructed by a diagonal process using induction. To do this, we shall construct subsequences $\{r_n^\ell\}$, $\ell = 0, 1, 2, \dots$, of $\{p_n\}$ such that:

- a) $r_n^0 = p_n$ for each n ;
 b) For each $\ell = 1, 2, \dots$ the sequence $\{r_n^\ell\}$ converges pointwise on $A_{k_{\ell-1}}$, where $k_{\ell-1}$ is chosen to satisfy $r_1^{\ell-1} = p_{k_{\ell-1}}$.

Start by letting $r_n^0 = p_n$ for each n and $k_0 = 1$. By Lemma 8.8 there exists a subsequence $\{r_n^1\}$ of $\{r_n^0\}$ such that $\lim r_n^1 \cdot y$ exists in \mathcal{R} for each $y \in A_{k_0}$. Now for the inductive step, assume that a subsequence $\{r_n^\ell\}$ of $\{r_n^{\ell-1}\}$ has been chosen such that $\lim r_n^\ell \cdot y$ exists in \mathcal{R} for each $y \in A_{k_{\ell-1}}$, where $r_1^{\ell-1} = p_{k_{\ell-1}}$. Pick k_ℓ so that $r_1^\ell = p_{k_\ell}$, and then use Lemma 8.8 to extract a subsequence $\{r_n^{\ell+1}\}$ of $\{r_n^\ell\}$ such that $\lim r_n^{\ell+1} \cdot y$ exists in \mathcal{R} for each $y \in A_{k_\ell}$. The induction is now complete.

Next, consider the subsequence $\{q_n\}$ of $\{r_n\}$, where $q_n = r_n^{\ell+1}$. An easy argument shows that $\lim q_n \cdot y$ exists in \mathcal{R} for each $y \in A$. Therefore, if for each $y \in A$ we put

$$q \cdot y = \lim_{n \rightarrow \infty} q_n \cdot y,$$

then q defines a positive linear functional on A . From $q_n \cdot \omega_0 = 1$ for all n , we infer that $q \cdot \omega_0 = 1$, and so $q > 0$. The proof of the lemma is now complete. ■

For our next discussion we shall employ the compact topological space \mathcal{X} that was introduced before Definition 8.6. If (x_0, x_1, \dots, x_n) is a Walrasian equilibrium for some economy \mathcal{E}_n , then by identifying (x_0, x_1, \dots, x_n) with $(x_0, x_1, \dots, x_n, 0, 0, \dots)$, we can consider (x_0, x_1, \dots, x_n) as an element of \mathcal{X} .

We now come to the concept of a "limit" for a fundamental sequence.

Definition 8.10. A pair (x, p) , where $x = (x_0, x_1, x_2, \dots)$ is an allocation for the overlapping generations model and $0 < p = (p_1, p_2, \dots) \in A'$, is said to be an asymptotic limit for a fundamental sequence $\{(x_0^n, x_1^n, \dots, x_n^n; p_n)\}$ whenever there exists a subsequence $\{(x_0^{k_n}, x_1^{k_n}, \dots, x_{k_n}^{k_n}; p_{k_n})\}$ such that

1. x is an accumulation point of the sequence $\{(x_0^{k_n}, x_1^{k_n}, \dots, x_{k_n}^{k_n})\}$ in \mathcal{X} ; and
2. $p \cdot y = \lim p_{k_n} \cdot y$ holds in \mathcal{R} for each $y \in A$.

Asymptotic limits always exist.

Lemma 8.11. Every fundamental sequence has an asymptotic limit.

Proof. Let $\{(x_0^n, x_1^n, \dots, x_n^n; p_n)\}$ be a fundamental sequence. By Lemma 8.9 there exists a subsequence $\{p_{k_n}\}$ of $\{p_n\}$ such that $\lim p_{k_n} \cdot y$ exists in \mathcal{R} for each $y \in A$.

Now consider the sequence $\{(x_0^{k_n}, x_1^{k_n}, \dots, x_{k_n}^{k_n})\}$ of the compact topological space \mathcal{X} and note that any accumulation point $x = (x_0, x_1, x_2, \dots)$ of $\{(x_0^{k_n}, x_1^{k_n}, \dots, x_{k_n}^{k_n})\}$ is an allocation for the overlapping generations model. Therefore, (x, p) is an asymptotic limit for $\{(x_0^n, x_1^n, \dots, x_n^n; p_n)\}$. ■

Prices associated with asymptotic limits are necessarily order continuous.

Lemma 8.12. If (x, p) is an asymptotic limit, then the price p is order continuous on A .

Proof. Let (x, p) be an asymptotic limit for a fundamental sequence $\{(x_0^n, x_1^n, \dots, x_n^n; p_n)\}$. It suffices to show that p is order continuous on each A_k .

By Lemma 8.4 we know that each p_n ($n \geq k$) is order continuous on A_k . Since p is a pointwise limit of a subsequence of $\{p_n\}$ on A_k , it follows from a classical theorem of H. Nakano that p is order continuous on A_k ; see [5, Theorem 20.23, p. 145] or [6, Corollary 13.15, p. 212 and Exercise 14, p. 214]. ■

The next lemma tells us that if (x, p) is an asymptotic limit, then the price p supports x .

Lemma 8.13. If (x, p) is an asymptotic limit for a fundamental sequence, then the price p supports the allocation x on A .

Proof. Let (\mathbf{x}, \mathbf{p}) be an asymptotic limit of a fundamental sequence $\{(\mathbf{x}_0^n, \mathbf{x}_1^n, \dots, \mathbf{x}_n^n; \mathbf{p}_n)\}$. By passing to a subsequence (and relabelling), we can assume without loss of generality that

- a) $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \dots)$ is an accumulation point of the sequence $\{(\mathbf{x}_0^n, \mathbf{x}_1^n, \dots, \mathbf{x}_n^n)\}$ in \mathcal{X} ; and
- b) $\mathbf{p} \cdot \mathbf{y} = \lim \mathbf{p}_n \cdot \mathbf{y}$ holds in \mathcal{R} for each $\mathbf{y} \in \mathbf{A}$.

To see that \mathbf{p} supports \mathbf{x} , let $\mathbf{y} \succeq_i \mathbf{x}_i$ hold in \mathbf{A}^+ . Fix $\varepsilon > 0$ and note that $\mathbf{y} + \varepsilon \omega_i \succ_i \mathbf{x}_i$. Since \mathbf{x}_i is a weak accumulation point of the sequence $\{\mathbf{x}_i^n : n = 1, 2, \dots\}$ in $E_i \times E_{i+1}$ and the utility function u_i is weakly continuous on the order bounded sets, it follows that

$$\mathbf{y} + \varepsilon \omega_i \succ_i \mathbf{x}_i^n$$

holds for infinitely many $n \geq i$; see the discussion preceding Lemma 8.5. Thus, by the supportability of \mathbf{p}_n , we see that

$$\mathbf{p}_n \cdot \mathbf{y} + \varepsilon \mathbf{p}_n \cdot \omega_i \geq \mathbf{p}_n \cdot \mathbf{x}_i^n = \mathbf{p}_n \cdot \omega_i$$

holds for infinitely many $n \geq i$. By (b) above, it easily follows that $\mathbf{p} \cdot \mathbf{y} + \varepsilon \mathbf{p} \cdot \omega_i \geq \mathbf{p} \cdot \omega_i$ holds for all $\varepsilon > 0$, and so

$$\mathbf{p} \cdot \mathbf{y} \geq \mathbf{p} \cdot \omega_i,$$

as desired. ■

We shall complete the proof of Theorem 8.1 by establishing that if (\mathbf{x}, \mathbf{p}) is an asymptotic limit of a fundamental sequence, then budget equality holds for each consumer, i.e., we shall show that for each $t = 0, 1, 2, \dots$ we have

$$\mathbf{p} \cdot \mathbf{x}_t = p_t \cdot \mathbf{x}_t^t + p_{t+1} \cdot \mathbf{x}_t^{t+1} = p_t \cdot \omega_t^t + p_{t+1} \cdot \omega_t^{t+1} = \mathbf{p} \cdot \omega_t.$$

To do this, we shall fix a fundamental sequence $\{(\mathbf{x}_0^n, \mathbf{x}_1^n, \dots, \mathbf{x}_n^n; \mathbf{p}_n)\}$, where

$$\mathbf{x}_0^n = (x_{0,n}^1, 0, 0, \dots), \quad \mathbf{x}_t^n = (0, \dots, 0, x_{t,n}^t, x_{t,n}^{t+1}, 0, 0, \dots) \quad \text{for } t = 1, \dots, n; \text{ and}$$

$$\mathbf{p}_n = (p_n^1, \dots, p_n^n, p_n^{n+1}, 0, 0, \dots).$$

As usual, for an asymptotic limit (\mathbf{x}, \mathbf{p}) of a fundamental sequence $\{(\mathbf{x}_0^n, \mathbf{x}_1^n, \dots, \mathbf{x}_n^n; \mathbf{p}_n)\}$, we shall write

$$\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \dots), \quad \mathbf{p} = (p_1, p_2, \dots), \quad \mathbf{x}_0 = (x_0^1, 0, 0, \dots), \quad \text{and} \quad \mathbf{x}_t = (0, \dots, 0, x_t^t, x_t^{t+1}, 0, 0, \dots).$$

We can assume (by passing to a subsequence if necessary) that

- a) $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \dots)$ is an accumulation point of the sequence $\{(\mathbf{x}_0^n, \mathbf{x}_1^n, \dots, \mathbf{x}_n^n; \mathbf{p}_n)\}$ in \mathcal{X} ; and
- b) $\mathbf{p} \cdot \mathbf{y} = \lim \mathbf{p}_n \cdot \mathbf{y}$ for each $\mathbf{y} \in \mathbf{A}$.

Lemma 8.14. *If $\{(\mathbf{x}_0^n, \mathbf{x}_1^n, \dots, \mathbf{x}_n^n; \mathbf{p}_n)\}$ is a fundamental sequence, then for each n we have*

1. $p_n^1 \cdot x_{0,n}^1 = p_n^1 \cdot \omega_0^1$; and
2. $p_n^t \cdot x_{t,n}^t = p_n^t \cdot \omega_t^t$ and $p_n^{t+1} \cdot x_{t,n}^{t+1} = p_n^{t+1} \cdot \omega_t^{t+1}$ for $t = 1, \dots, n$.

Proof. Since $(\mathbf{x}_0^n, \mathbf{x}_1^n, \dots, \mathbf{x}_n^n; \mathbf{p}_n)$ is a Walrasian equilibrium, for each $t = 0, 1, \dots, n$ we have

$$\mathbf{p}_n \cdot \mathbf{x}_t^n = p_n^t \cdot x_{t,n}^t + p_n^{t+1} \cdot x_{t,n}^{t+1} = p_n^t \cdot \omega_t^t + p_n^{t+1} \cdot \omega_t^{t+1} = \mathbf{p}_n \cdot \omega_t. \quad (*)$$

From $\sum_{t=0}^n \mathbf{x}_t^n = \sum_{t=0}^n \omega_t$, we see that $x_{n,n}^{n+1} = \omega_n^{n+1}$ holds, and so $p_n^{n+1} \cdot x_{n,n}^{n+1} = p_n^{n+1} \cdot \omega_n^{n+1}$. Using (*) and letting $t = n$, we infer that

$$p_n^n \cdot x_{n,n}^n = p_n^n \cdot \omega_n^n,$$

i.e., the identities are true for $t = n$.

From $x_{n-1,n}^n + x_{n,n}^n = \omega_{n-1}^n + \omega_n^n$, we see that $p_n^n \cdot x_{n-1,n}^n + p_n^n \cdot x_{n,n}^n = p_n^n \cdot \omega_{n-1}^n + p_n^n \cdot \omega_n^n$, and so

$$p_n^n \cdot x_{n-1,n}^n = p_n^n \cdot \omega_{n-1}^n.$$

Letting $t = n - 1$ in (*) and taking into account the above equality, we see that

$$p_n^{n-1} \cdot x_{n-1,n}^{n-1} = p_n^{n-1} \cdot \omega_{n-1}^{n-1},$$

and hence our identities are also true for $t = n - 1$.

Now the validity of the desired identities can be established by repeating the preceding arguments with $t = n - 2, n - 3, \dots, 1, 0$. ■

Our final objective is to "take" the limit as $n \rightarrow \infty$ in the identities of the preceding lemma. This will be done in the next lemma.

Lemma 8.15. *If (x, p) is an asymptotic limit of a fundamental sequence, then*

$$p_t \cdot \omega_t^i = p_t \cdot x_t^i \quad \text{and} \quad p_{t+1} \cdot \omega_t^{i+1} = p_{t+1} \cdot x_t^{i+1}$$

hold for all $t = 0, 1, 2, \dots$. In particular,

$$p \cdot x_t = p \cdot \omega_t > 0$$

holds for all $t = 0, 1, 2, \dots$.

Proof. Let (x, p) be an asymptotic limit for a fundamental sequence $\{(x_0^n, x_1^n, \dots, x_n^n; p_n)\}$. Fix $t \geq 1$ and let $\varepsilon > 0$. Lemma 8.5 applied with $m = 0$ and $\ell = t$ guarantees the existence of a constant $M > 0$ such that

$$0 < p_k \cdot \omega_t \leq M \tag{**}$$

holds for all $k \geq t$.

Now from

$$(x_t^i, x_t^{i+1}) + \varepsilon(\omega_t^i, \omega_t^{i+1}) \succ_t (x_t^i, x_t^{i+1}),$$

the fact that each x_i is a weak accumulation point of the sequence $\{x_i^n: n = 1, 2, \dots\}$ in $[0, \theta_i] \times [0, \theta_{i+1}]$ and the weak continuity of the utility functions on the order bounded sets, we see that for infinitely many $k \geq t$ we have

$$(x_t^i, x_{t,k}^{i+1}) + \varepsilon(\omega_t^i, \omega_t^{i+1}) \succ_t (x_{t,k}^i, x_{t,k}^{i+1}), \quad \text{and} \\ (x_{t,k}^i, x_{t,k}^{i+1}) + \varepsilon(\omega_t^i, \omega_t^{i+1}) \succ_t (x_{t,k}^i, x_{t,k}^{i+1})$$

Thus, in view of the supportability of p_k , we see that

$$p_k^i \cdot x_t^i + p_k^{i+1} \cdot x_{t,k}^{i+1} + \varepsilon p_k \cdot \omega_t \geq p_k^i \cdot x_{t,k}^i + p_k^{i+1} \cdot x_{t,k}^{i+1}, \quad \text{and} \\ p_k^i \cdot x_{t,k}^i + p_k^{i+1} \cdot x_{t,k}^{i+1} + \varepsilon p_k \cdot \omega_t \geq p_k^i \cdot x_{t,k}^i + p_k^{i+1} \cdot x_{t,k}^{i+1},$$

and so from (**) and Lemma 8.13, we infer that

$$p_k^i \cdot \omega_t^i = p_k^i \cdot x_{t,k}^i \leq p_k^i \cdot x_t^i + \varepsilon M \quad \text{and} \quad p_k^{i+1} \cdot \omega_t^{i+1} = p_k^{i+1} \cdot x_{t,k}^{i+1} \leq p_k^{i+1} \cdot x_t^{i+1} + \varepsilon M$$

hold for infinitely many $k \geq t$. In view of $p \cdot y = \lim p_n \cdot y$ for each $y \in A$, the latter inequalities imply

$$p_t \cdot \omega_t^i \leq p_t \cdot x_t^i + \varepsilon M \quad \text{and} \quad p_{t+1} \cdot \omega_t^{i+1} \leq p_{t+1} \cdot x_t^{i+1} + \varepsilon M,$$

for all $\varepsilon > 0$. Hence,

$$p_t \cdot \omega_t^i \leq p_t \cdot x_t^i \quad \text{and} \quad p_{t+1} \cdot \omega_t^{i+1} \leq p_{t+1} \cdot x_t^{i+1}$$

hold for all $t = 1, 2, \dots$. A similar argument also shows that $p_0 \cdot \omega_0^1 \leq p_0 \cdot x_0^1$.

Finally, from the relations

$$\begin{aligned} p_t \cdot \omega_{t-1}^i + p_t \cdot \omega_t^i &\leq p_t \cdot x_{t-1}^i + p_t \cdot x_t^i \\ &= p_t \cdot (x_{t-1}^i + x_t^i) \\ &= p_t \cdot (\omega_{t-1}^i + \omega_t^i) \leq p_t \cdot \omega_{t-1}^i + p_t \cdot \omega_t^i, \end{aligned}$$

it is easy to see that

$$p_t \cdot \omega_t^i = p_t \cdot x_t^i \quad \text{and} \quad p_t \cdot \omega_{t-1}^i = p_t \cdot x_{t-1}^i$$

hold for all $t = 1, 2, \dots$. Hence,

$$p \cdot x_t = p \cdot \omega_t$$

also holds for each $t = 0, 1, 2, \dots$, as desired. ■

Finally, we close the discussion in this section with an interesting observation. Let (x, p) be an asymptotic limit. Then

$$y_t = (0, \dots, 0, x_t^i, 0, 0, \dots) \quad \text{and} \quad z_t = (0, \dots, 0, x_t^{i+1}, 0, 0, \dots)$$

represent the allocations that consumer t is receiving at periods t and $t+1$, while

$$e_t = (0, \dots, 0, \omega_t^i, 0, 0, \dots) \quad \text{and} \quad e_t^* = (0, \dots, 0, \omega_t^{i+1}, 0, 0, \dots),$$

are his initial endowments at periods t and $t+1$, respectively. Lemma 8.14 tells us that

$$p \cdot y_t = p \cdot e_t \quad \text{and} \quad p \cdot z_t = p \cdot e_t^*,$$

which means that the values at prices p of the initial endowment and the allocation bundle are the same for a given consumer at each time period.

9. THE OVERLAPPING GENERATIONS MODEL AND PROPER PREFERENCES

We start our discussion by defining the Riesz space

$$\phi_E = \{(x_1, x_2, \dots) \in E: \exists k \text{ with } x_i = 0 \quad \forall i > k\}.$$

Under the duality

$$p \cdot x = \sum_{i=1}^{\infty} p_i \cdot x_i,$$

the dual system (ϕ_E, E') is clearly a symmetric Riesz dual system. The purpose of this section is to present a condition which guarantees the existence of equilibria for the overlapping generations model with respect to the Riesz dual system (ϕ_E, E') . We shall say that the overlapping generations model

has an equilibrium with respect to the Riesz dual system (ϕ_E, E') whenever there exists an allocation $x = (x_0, x_1, x_2, \dots)$ and some non-zero price $p = (p_1, p_2, \dots) \in E'$ such that

- i) $x \succeq_0 x_0^1$ in E_1^+ implies $p_1 \cdot x \geq p_1 \cdot x_0^1$;
- ii) $(x, y) \succeq_t (x_t^1, x_t^{1+1})$ in $E_t^+ \times E_{t+1}^+$ implies $p_t \cdot x + p_{t+1} \cdot y \geq p_t \cdot \omega_t^1 + p_{t+1} \cdot \omega_t^{1+1}$; and
- iii) $p \cdot x_t = p \cdot \omega_t$ holds for $t = 0, 1, 2, \dots$.

In our overlapping generations model each consumer $t \geq 1$ lives in periods t and $t+1$ and his utility function u_t is defined on $E_t^+ \times E_{t+1}^+$. Let us say that the preference \succeq_t induced by u_t is *uniformly proper* whenever there exist locally convex-solid topologies on E_t and E_{t+1} consistent with the dualities (E_t, E_t') and (E_{t+1}, E_{t+1}') such that each \succeq_t is uniformly proper with respect to the product topology on $E_t \times E_{t+1}$. Equivalently, u_t is uniformly proper if and only if it is uniformly proper for the Mackey topology $\tau(E_t \times E_{t+1}, E_t' \times E_{t+1}')$. The preference \succeq_0 is uniformly proper whenever it is uniformly proper on E_1 . Also, let us say that the overlapping generations model is *proper* whenever

- a) Each preference \succeq_t ($t = 0, 1, 2, \dots$) is uniformly proper; and
- b) Each $\theta_t = \omega_{t+1}^1 + \omega_t^1$ is a strictly positive element of E_t for each $t \geq 1$. (Recall that θ_t is strictly positive whenever $q \cdot \theta_t > 0$ holds for all $0 < q \in E_t'$.)

It should be clear that A is an ideal of ϕ_E which is also dense in ϕ_E with respect to the product of the Mackey topologies. In addition, it should be noted that our notion of equilibrium with respect to the Riesz dual system (ϕ_E, E') is an extension of the equilibrium notion of Definition 7.3.

The objective of this section is to prove the following result.

Theorem 9.1. *Every proper overlapping generations model has an equilibrium with respect to the symmetric Riesz dual system (ϕ_E, E') .*

The proof of this theorem will be accomplished with the help of a theorem which is of some independent interest in its own right. So far, we have seen that quite often an allocation can be supported by a price on an ideal of the original commodity space. It is, therefore, natural to ask whether or not such a supporting price can be extended to the whole commodity space. N. C. Yannelis and W. R. Zame [46] proved that if in a finite economy ω is strictly positive and all preferences are uniformly proper, then a supporting price on A_ω extends to a supporting price on E . Next, we shall state and present a different proof of this result.

Theorem 9.2. (Yannelis-Zame) *Consider a finite economy with set of consumers $\mathcal{N} = \{1, \dots, n\}$ and n positive vectors x_1, \dots, x_n . Let $a = \sum_{i=1}^n x_i$ and let p be a positive linear functional on E such that $x \succeq_i x_i$ in E^+ implies $p \cdot x \geq p \cdot x_i$.*

If for some locally convex-solid topology τ on E the preferences are uniformly τ -proper, then p is τ -continuous on the ideal A_a .

Proof. Assume that the price $0 \leq p \in E'$ and the positive vectors x_1, \dots, x_n satisfy the hypotheses of the theorem. For each i fix some $v_i > 0$ and some convex solid τ -neighborhood V_i of zero such that $x - \alpha v_i + z \succeq_i z$ in E^+ with $\alpha > 0$ implies $z \notin \alpha V_i$. Put $v = \sum_{i=1}^n v_i$ and $V = \bigcap_{i=1}^n V_i$.

Next, consider the Minkowski functional ρ of V , i.e.,

$$\rho(y) = \inf \{ \lambda > 0 : y \in \lambda V \}, \quad y \in E.$$

Clearly, ρ is a τ -continuous seminorm on E . Now let $0 \leq z \leq a = \sum_{i=1}^n x_i$. By the Riesz Decomposition Property we can write $z = \sum_{i=1}^n z_i$ with $0 \leq z_i \leq x_i$ for each i . Let $\alpha_i = \rho(z_i)$, and let $\epsilon > 0$ be fixed. Put $y_i = z_i + (\alpha_i + \epsilon)v_i - z_i \geq 0$, and that $z_i = y_i - (\alpha_i + \epsilon)v_i + z_i \geq 0$. If $y_i - (\alpha_i + \epsilon)v_i + z_i \succeq_i y_i$ holds, then by the uniform τ -properness of \succeq_i , we see that $z_i \notin (\alpha_i + \epsilon)V$, contrary to $\rho(z_i) = \alpha_i$. Therefore, $y_i \succ_i y_i - (\alpha_i + \epsilon)v_i + z_i = z_i$ holds, and so by the supportability of p , we obtain that

$$p \cdot y_i \geq p \cdot z_i = p \cdot [y_i - (\alpha_i + \epsilon)v_i + z_i] = p \cdot y_i - (\alpha_i + \epsilon)p \cdot v_i + p \cdot z_i.$$

Hence, $p \cdot z_i \leq (\alpha_i + \epsilon)p \cdot v_i$ holds for each i and all $\epsilon > 0$, and so

$$p \cdot z_i \leq \alpha_i p \cdot v_i = (p \cdot v_i) \rho(z_i) \leq (p \cdot v_i) \rho(z).$$

This implies

$$p \cdot z = \sum_{i=1}^n p \cdot z_i \leq \left(\sum_{i=1}^n p \cdot v_i \right) \rho(z) = (p \cdot v) \rho(z)$$

for all z with $0 \leq z \leq a$. An easy argument now shows that

$$|p \cdot z| \leq (p \cdot v) \rho(z)$$

holds for all $z \in A_a$, and so p is τ -continuous on A_a . ■

We continue our discussion with one more lemma dealing with proper preferences.

Lemma 9.3. Assume that

- a) (E, E') is a Riesz dual system;
- b) τ is a locally convex-solid topology on E consistent with (E, E') ;
- c) A is a τ -dense ideal of E ; and
- d) \succeq is a preference relation on E^+ .

Then the preference \succeq is uniformly τ -proper if and only if there exists a τ -neighborhood V of zero and some $0 < \alpha \in A$ such that

$$z - \alpha v + z \succeq x \text{ in } E^+ \text{ with } \alpha > 0 \text{ implies } z \notin \alpha V.$$

Proof. Assume that \succeq is uniformly τ -proper. Pick a τ -neighborhood W of zero and some $0 < u \in E$ such that

$$z - \alpha u + z \succeq x \text{ in } E^+ \text{ with } \alpha > 0 \text{ implies } z \notin \alpha W. \quad (*)$$

Choose a convex solid τ -neighborhood V of zero with $V + V \subseteq W$. Since A is τ -dense in E , there exists some $v \in A$ with $u - v \in V$. Replacing v by $u \wedge v^+$ (and taking into account the inequality $|u - u \wedge v^+| \leq |u - v|$), we can assume that $0 \leq v \leq u$ holds. Now we claim that

$$z - \alpha v + z \succeq x \text{ in } E^+ \text{ with } \alpha > 0 \text{ implies } z \notin \alpha V.$$

Indeed, if $z - \alpha v + z \succeq x$ holds in E^+ with $\alpha > 0$, then from $(*)$ and the relation

$$z - \alpha u + [z - \alpha(v - u)] = z - \alpha v + z \succeq x,$$

we see that $z - \alpha(v - u) \notin \alpha W$. On the other hand, if $z \in \alpha V$, then

$$z - \alpha(u - v) \in \alpha V + \alpha V = \alpha(V + V) \subseteq \alpha W,$$

which is impossible. Hence, $z \notin \alpha V$, and the proof of the lemma is finished. ■

Recall that an element $0 < x \in E$ is said to be *strictly positive* or a *quasi-interior point* (in symbols, $x \gg 0$) whenever $p \cdot x > 0$ holds for all $0 < p \in E'$. It is well known that an element $0 < x \in E$ is strictly positive if and only if its principal ideal A_x is weakly dense in E ; see [5, pp. 259-260].

Now let us complete the proof of Theorem 9.1. To this end, assume that the overlapping generations model is proper. By Theorem 8.1, we know that there exists an equilibrium (x, p) , where

$$x = (x_0, x_1, x_2, \dots) \text{ and } p = (p_1, p_2, \dots).$$

Let t be fixed. Observe that the price $p = (p_1, p_2, \dots) \in A'$ supports the vectors x_0, x_1, \dots, x_{t+1} on A . Since the preferences are uniformly proper on E with respect to the product of the Mackey topologies τ and A is τ -dense in E , it follows from Lemma 9.3 that the preferences are also uniformly τ -proper on

A. Thus, by Theorem 9.2, we infer that the price p is τ -continuous on A_α , where $\alpha = \sum_{i=0}^{t+1} x_i$. Since $\Theta_1 \times \Theta_2 \times \dots \times \Theta_t \times 0 \times 0 \dots$ is an ideal of A_α , we see that the individual price p_i is $\sigma(E_i, E_i')$ -continuous on Θ_i .

Since Θ_i is dense in E_i , the price $p_i: \Theta_i \rightarrow \mathcal{R}$ has a unique continuous extension p_i^* on E_i . We claim that the price $p^* = (p_1^*, p_2^*, \dots) \in E'$ supports x on $\phi_{\mathcal{R}}$. To see this, let $y \succeq_t x_t$ in $E_t^+ \times E_{t+1}^+$. Fix $\delta > 0$ and then pick a net $\{y_\alpha\} \subseteq \Theta_t^+ \times \Theta_{t+1}^+$ with $y_\alpha \xrightarrow{w} y + \delta\omega_t$. Replacing $\{y_\alpha\}$ by $\{y_\alpha \wedge (y + \delta\omega_t)\}$, we can assume that $\{y_\alpha\}$ is order bounded. In view of $y_\alpha \xrightarrow{w} y + \delta\omega_t$, $y + \delta\omega_t \succ_t x_t$ and the weak continuity of the utility functions on the order bounded sets, we can also assume that $y_\alpha \succ_t x_t$ holds for all α . Thus, by the supportability of p on A , we get $p \cdot y_\alpha \geq p \cdot \omega_t$ for all α , and by the weak continuity of p on $E_t \times E_{t+1}$, we see that $p \cdot y + \delta p \cdot \omega_t \geq p \cdot \omega_t$ for all $\delta > 0$. Thus, $y \succeq_t x_t$ in $E_t^+ \times E_{t+1}^+$ implies $p \cdot y \geq p \cdot \omega_t$, and the proof of Theorem 9.1 is complete.

Finally, we close the paper with a few remarks concerning the general overlapping generations model. That is, the overlapping generations model where we allow

- a) k persons to be born in each time period; and
- b) each person to live n periods.

It is not difficult to see that (with some appropriate modifications) the arguments up to Lemma 8.14 are valid in this case too. That is, it can be shown that if (x, p) is an asymptotic limit, then x is an allocation supported by the price p . However, the proofs about the budget equalities (Lemmas 8.14 and 8.15) cannot be replicated because they depend upon the two-time period assumption and the fact that each generation consists of a single individual. In special cases (for instance, when the symmetric Riesz dual systems are of the form $(\mathcal{R}^n, \mathcal{R}^n)$ or $(\mathcal{L}_n, \mathcal{L}_n)$) Theorem 8.1 is true. We conjecture that Theorem 8.1 is true for the general overlapping generations model.

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