COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut 06520

COWLES FOUNDATION DISCUSSION PAPER NO. 799

Limiting Distributions of the Number of Pure Strategy Nash Equilibria in n-Person Games

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August 1986

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BY

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ABSTRACT

In this paper, we study the number of pure strategy Nash equilibria in a "random" n-person non-cooperative game in which all players have a countable number of strategies. We provide explicit expressions for the expected number of pure strategy N.E., and show that the distribution of the number of pure strategy N.E. approaches the Poisson distribution with mean 1 as the numbers of strategies of two or more players go to infinity.

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*This paper forms part of the author's doctoral dissertation, but new proofs are given for several of the results. The printing and distribution of the paper is supported by the Department of the Navy Contract N00014-77-C-0518 issued by the Office of Naval Research under the Contract Authority NR 047-006. However, the content does not necessarily reflect the position or the policy of the Department of the Navy or the Government, and no official endorsement should be inferred.

+The author wishes to thank her thesis adviser, Martin Shubik, for suggesting this research, and Michael R. Powers, for his assistance in revising several of the proofs.

1. Introduction

The Nash equilibrium (N.E.) solution concept is often used to solve n-person non-cooperative games (games where there is no cooperation among players), because of the appealing notion of stability that it embodies. A N.E. solution specifies strategies for all players, such that each player gets his most preferred payoff, given the N.E. strategies adopted by the other players. Any n-person game in which each player has a countable number of strategies can be represented in matrix form, as in the following example, for $n = 2$:

Player 2

	$s_1 \backslash s_2$	1	2	3	4	
Player 1	1	(21,32)	(43,56)	(31,27)	(24,11)	
	2	(12,46)	(56,34)	(18,54)	(28,45)	payoffs in dollars
	3	(14,38)	(25,30)	(42,10)	(30,39)	

where s_k ($k = 1, 2$) denotes the strategy of Player k, and the ordered pairs denote componentwise the payoffs of Players 1 and 2. In our example, the pure strategy N.E. solution is $s_1 = 3$ and $s_2 = 4$.

In any matrix game, the number of pure strategy N.E. can be as small as zero or as large as the number of possible outcomes. Clearly, it can be a horrendous task to find the number of pure strategy N.E. in an n-person game where the payoffs are arranged in a seemingly unstructured manner, n is greater than 2, and the numbers of strategies of the players are very large. In this paper, we shall study the number of pure strategy N.E. in a "random" n-person game. We choose to study only pure strategy N.E., because it is often impractical to use mixed strategies and because many people are uncomfortable with using them. To the delight of many people, our results will prove that it is not so rare to have one or more pure strategy N.E. From now on, without exception, all the strategies referred to in this paper are pure strategies, and all the games are non-cooperative games in which each player has a countable number of strategies. Once we adhere only to pure strategies, all that is important is the ordinality of the payoffs, and since we can transform any game with cardinal payoffs to one with ordinal payoffs for our analysis, we need only study ordinal games in this paper.

In 1968, Goldberg, Goldman and Newman found that the probability that a 2-person cardinal game has at least one pure strategy N.E converges to $1-e^{-1}$ as the numbers of strategies of both players go to infinity. Dresher extended this result to n-person games in 1970 and showed that the probability that an n-person cardinal game has at least one pure strategy N.E also converges to $1-e^{-1}$ as the numbers of strategies of two or more players go to infinity. These researchers generated the payoffs of their cardinal games from a continuous distribution, so that for any player, the probability is 1 that all his payoffs are distinct. Their work thus in fact involves only strictly ordinal games, i.e., games where there are no ties among the payoffs of any player. They did not conceive of the need for research in weakly ordinal games, i.e., games where ties among the payoffs of a player are admitted. This restriction of their work sorely limits its ability to model real situations, since people quite often have only a weak preference ordering over all the possible outcomes of an event.

In this paper, we increase the scope of previous research. We first study the distribution of the number of N.E. in a "random" n-person strictly ordinal game as the numbers of strategies of some players go to infinity. We find that the probability distribution of the number of N.E. in an n-person strictly ordinal game approaches a binomial distribution as the number of strategies of one player goes to infinity, and that it approaches the Poisson distribution with mean 1 as the numbers of strategies of two or more players go to infinity. Then, we extend our research to weakly ordinal games and study the correspondent limiting distributions in a "random" n-person weakly ordinal game. As compared to the results for the n-person strictly ordinal game, we find that ties in the payoffs of some players in the n-person weakly ordinal game increase the expected number of N.E. when the number of strategies of only one player goes to infinity, but, surprisingly, the distribution of the number of N.E. in the n-person weakly ordinal game approaches the same Poisson distribution when the numbers of strategies of two or more players go to infinity.

2. General Model

We now describe how we model a "random" n-person strictly ordinal game. Consider an n-person strictly ordinal game with Players 1, 2, ..., n, where Player k has m_k strategies and every player has a strictly ordinal preference over all of the M (= $\Pi\, m_k$) possible outcomes. We shall identify the outcomes as $O_1, O_2, \ldots, O_M$. In our model, we assume that (a) for each player, the ordinal payoffs associated with $O_1, O_2, \ldots, O_M$ are the result of a random drawing of M numbers from 1, 2, ..., M without replacement, and (b) the ordinal payoffs to each player are independent. We use the convention that the higher the number associated with an outcome, the more the player prefers it. If we let s_k denote Player k's strategy and $p_k(s_1, s_2, \ldots, s_k, \ldots, s_n)$ denote his ordinal payoffs, then a set of strategies $(s_1', s_2', \ldots, s_k', \ldots, s_n')$ is a N.E. of the game if and only if $p_k(s_1', s_2', \ldots, s_k', \ldots, s_n') = \max_{s_k} p_k(s_1', s_2', \ldots, s_k, \ldots, s_n')$, for all k. We shall also consider an analogous model for a weakly ordinal game by varying assumption (a)--for each player, the ordinal payoffs associated with $O_1, O_2, \ldots, O_M$ are the result of a random drawing of M numbers from 1, 2, ..., M *with* replacement.

In this paper we study the distribution of X, the number of N.E., as the number of strategies of different players goes to infinity. We present results for both the strictly and weakly ordinal game. We begin our research with the simple case of the 2-person strictly ordinal game, and then generalize our results to the n-person strictly ordinal games. Finally, we study the corresponding problems for the n-person weakly ordinal games.

We note that Assumption (b) is appropriate for games in which the players' evaluations of the outcomes are independent of each other. Both Assumptions (a) and (b) are natural and simple suppositions that researchers make when asked to find the number of N.E. in an apparently unstructured n-person game. Using our model, we are able to obtain a number of results that are both mathematically interesting and of value to researchers seeking the number of N.E. in an n-person game, since our work permits them to know ahead of time approximately how many N.E. they should expect to find. We would also like to note that not all of our results require assumptions (a) and (b), and all of them still hold if assumption (a) is relaxed appropriately, as we discuss later. For the moment, we adhere to assumptions (a) and (b) for the sake of simplicity.

3. Two-Person Strictly Ordinal Games

Consider two players, Players 1 and 2, who have respectively m_1 and m_2 strategies, and who both have strictly ordinal preferences over the $m_1 \times m_2$ outcomes of the game. Without loss of generality, we can assume that $m_1 \leq m_2$. Let us label the strategies of Players 1 and 2 by i and j respectively, where $1 \leq i \leq m_1$, and $1 \leq j \leq m_2$, and let $(s_1, s_2) = (i,j)$ be componentwise the strategies chosen by Players 1 and 2. Let us denote the ordinal payoffs of (i,j) to Players 1 and 2 by A_{ij} and B_{ij} respectively, and assume that A_{ij} and B_{ij} are each independently obtained by a random drawing of $m_1 \times m_2$ numbers from 1 to $m_1 \times m_2$ without replacement. We further define the following indicator functions:

$$I[A_{i'j}] = 1 \quad \text{if } A_{i'j} = \max_i \{A_{ij}\}$$
$$= 0 \quad \text{otherwise}$$
$$I[B_{ij'}] = 1 \quad \text{if } B_{ij'} = \max_j \{B_{ij}\}$$
$$= 0 \quad \text{otherwise}$$

Let $I_{ij} = I[A_{ij}] \times I[B_{ij}]$. Then

$$I_{ij} = 1 \quad \text{if } (i,j) \text{ is a pair of N.E strategies}$$
$$= 0 \quad \text{otherwise}$$

and $$I_j = \sum_i I_{ij} = 1 \quad \text{if there is a N.E. associated with strategy } j$$
$$= 0 \quad \text{otherwise}$$

With the aid of the above notation, we are ready to present our results.

Proposition 1: The maximum number of possible N.E. is Min $\{m_1, m_2\}$.

Proof: Consider $1 \leq i' \leq m_1, 1 \leq j' \leq m_2$, $1 \leq i \leq m_1$ and $1 \leq j \leq m_2$, where $i \neq i'$ and $j \neq j'$. If (i',j') is a pair of N.E. strategies, then (i',j) cannot be a pair of N.E. strategies, since Player 2 has strictly ordinal preferences over all of the outcomes. Similarly, (i,j') cannot be a pair of N.E. strategies. Hence, if we fix the strategy of one player, there is at most one N.E. associated with this fixed strategy, and so an upper bound for the maximum number of N.E. is Min $\{m_1, m_2\}$. The following example (with its N.E. highlighted in boldface) illustrates how this upper bound can easily be achieved:

$s_1 \backslash s_2$	1	2	3
1	**(6,6)**	(4,4)	(3,3)
2	(2,2)	**(5,5)**	(1,1)

Q.E.D.

Proposition 2: The expected number of N.E. is 1.

Proof: Let $P(i,j)$ be the probability that (i,j) is a pair of N.E. strategies. Then $E[\sum_{i,j} I_{ij}] = \sum_{i,j} E[I_{ij}] = \sum_{i,j} P(i,j)$. We know that $P(i,j)$ = P(Player 1's ordinal payoff is greatest when he chooses i | Player 2 chooses j) x P(Player 2's ordinal payoff is greatest when he chooses j | Player 1 chooses i) = $1/m_1 \times 1/m_2$. Hence, the expected number of N.E. is $\sum_{i,j} P(i,j) = m_1 m_2 \times 1/(m_1 m_2) = 1$.

Q.E.D.

Proposition 3: The probability distribution of the number of N.E. approaches the binomial $(m_1, 1/m_1)$ distribution as the number of strategies of Player 2 approaches infinity, and the number of strategies of Player 1 remains finite.

Proof: Let $\sum_i I[B_{ij}] = \lambda_j$, for each j , so that $\Sigma \lambda_j = m_1$. Then $P(I_j = 1 \mid \boldsymbol{\lambda}) = \lambda_j / m_1$ and $P(I_j = 0 \mid \boldsymbol{\lambda}) = 1 - \lambda_j / m_1$, where $\boldsymbol{\lambda} = (\lambda_j)$. Let X be the number of N.E. and $\varphi_X(t)$ be its moment generating function. Then

$$\varphi_X(t)$$
$$= \sum_{\boldsymbol{\lambda}} \varphi_{X|\boldsymbol{\lambda}}(t) \cdot P(\boldsymbol{\lambda})$$
$$= \Sigma\, \varphi_{\Sigma I_j|\boldsymbol{\lambda}}(t) \cdot P(\boldsymbol{\lambda})$$

Since I_j and $I_{j'}$ $(j \neq j')$ are conditionally independent given a particular $\boldsymbol{\lambda}$, we see that

$$\varphi_X(t)$$
$$= \Sigma \left[\, \Pi\, \varphi_{I_j|\boldsymbol{\lambda}}(t) \,\right] \cdot P(\boldsymbol{\lambda})$$
$$= \Sigma \left[\, \Pi\, (e^t \lambda_j / m_1 - \lambda_j / m_1 + 1) \,\right] \cdot P(\boldsymbol{\lambda}).$$

Let E be the event that $\lambda_j \leq 1$ for all j . Then

$P(E) = m_2(m_2-1)(m_2-2) \cdots (m_2-m_1+1)/m_2^{m_1}$, and

as $m_2 \longrightarrow \infty$, $P(E) \longrightarrow 1$, which means $\lambda_j = 0$ or 1 . Together with $\Sigma \lambda_j = m_1$, it follows that

$$\lim_{m_2 \to \infty} \varphi_X(t)$$
$$= \Sigma \left[\, (e^t/m_1 - 1/m_1 + 1)^{m_1} \,\right] \cdot P(\boldsymbol{\lambda})$$
$$= (e^t/m_1 - 1/m_1 + 1)^{m_1} ,$$

which is the moment generating function of the binomial distribution with parameters m_1 and $1/m_1$.

Q.E.D.

<u>Proposition</u> 4: The probability distribution of the number of N.E. approaches the Poisson distribution with mean 1 as the numbers of strategies of both players approach infinity.

Proof: We know from the proof of proposition 3 that

$$\varphi_{X|\boldsymbol{\lambda}}(t)$$

$$= \varphi_{\Sigma I_j|\boldsymbol{\lambda}}(t)$$

$$= \Pi\,(e^t\lambda_j/m_1 - \lambda_j/m_1 + 1)\ .$$

$$= \exp\{\log[\,\Pi\,(e^t\lambda_j/m_1 - \lambda_j/m_1 + 1)]\}$$

$$= \exp\{\,\Sigma\,[\,\log(\,(e^t-1)\,\lambda_j/m_1 + 1)]\}$$

$$= \exp\{\Sigma\,\{[(e^t-1)\,\lambda_j/m_1] - (1/2)[(e^t-1)\,\lambda_j/m_1]^2 + (1/3)[(e^t-1)\,\lambda_j/m_1]^3$$

$$- \cdots\}\}\ ,\ \text{for } -\infty \le t \le \log 2\ .$$

We note that

$$\varphi_{X|\boldsymbol{\lambda}}(t)$$

$$\le \exp\{\Sigma\,[(e^t-1)\,\lambda_j/m_1]\}$$

$$= \exp(e^t-1)\ , \qquad (1)$$

and $\varphi_{X|\boldsymbol{\lambda}}(t)$

$$\geq \exp\{\Sigma[(e^t-1)\lambda_j/m_1] - 1/2[(e^t-1)\lambda_j/m_1]^2\}\ . \qquad (2)$$

Let A be the event that $\lambda_j < cm_1^{1/3}$ for all j, and A' be the event that $\lambda_j \geq cm_1^{1/3}$ for at least one j, where c is a constant, then

$\varphi_X(t)$

$= E_{\boldsymbol\lambda}[\varphi_{X|\boldsymbol\lambda}(t)]$

$= E_{\boldsymbol\lambda\epsilon A}[\varphi_{X|\boldsymbol\lambda}(t)]\cdot P(A) + E_{\boldsymbol\lambda\epsilon A'}[\varphi_{X|\boldsymbol\lambda}(t)]\cdot P(A')$.

Note that P(A')

$$\leq m_2 \sum_{q=[cm_1^{1/3}]}^{m_1} \binom{m_1}{q} (1/m_2)^q (1-1/m_2)^{m_1-q}\ ,$$

and let $m_1 = m_2 = \eta$. Then P(A')

$$\leq \eta \sum_{q=[c\eta^{1/3}]}^{\eta} \binom{\eta}{q} (1/\eta)^q (1-1/\eta)^{\eta-q}$$

$$\leq \eta(\eta - c\eta^{1/3})[1/[c\eta^{1/3}]!]$$

$$\leq \eta^2/[c\eta^{1/3}]!\ .$$

(1) and (2) $\Longrightarrow 0 \leq E_{\boldsymbol\lambda\epsilon A'}[\varphi_{X|\boldsymbol\lambda}(t)] \leq \exp(e^t-1)$. Therefore as $\eta \longrightarrow \infty$,

$E_{\boldsymbol\lambda\epsilon A'}[\varphi_{X|\boldsymbol\lambda}(t)]\cdot P(A') = 0$.

From (1) and (2), we also know that $E_{\boldsymbol\lambda\epsilon A}[\varphi_{X|\boldsymbol\lambda}(t)]\cdot P(A)$ is bounded below by $E_{\boldsymbol\lambda\epsilon A}\{\exp\{\Sigma[(e^t-1)\lambda_j/\eta] - 1/2[(e^t-1)\lambda_j/\eta]^2\}\}\cdot P(A)$ and above by $\exp(e^t-1)\cdot P(A)$. Since these bounds both converge to $\exp(e^t-1)$ as $\eta \longrightarrow$

∞, it follows that $E_{\lambda\epsilon A}[\varphi_{X|\lambda}(t)]\cdot P(A) \longrightarrow \exp(e^t-1)$ as $\eta \longrightarrow \infty$. Thus,

$$\lim_{m_1, m_2 \to \infty} \varphi_X(t)$$

$$= \exp(e^t-1) ,$$

which is the moment generating function of the Poisson distribution with mean 1.

Q.E.D.

At this point, it is helpful to cultivate an intuition for why the above results hold. We shall first pinpoint the elements that were crucial in obtaining the results for the 2-person strictly ordinal game and then try to generalize them for the n-person strictly ordinal game. We shall quickly see that the insights used to solve the simpler problems can be transferred to the more complex ones in a straightforward manner.

In the 2-person strictly ordinal game, we solve for the maximum number of possible N.E. by observing that if we fix the strategy of one player, then there is at most one N.E. associated with that fixed strategy. A similar observation is helpful in figuring out the maximum number of possible N.E. for any n-person strictly ordinal game. We note that if we fix the strategies of n-1 players, then there is at most one N.E. associated with that combination of fixed strategies, since all the players have strictly ordinal preferences. Hence, an upper bound for the maximum number of possible N.E. is Min $\{\prod_{k\neq 1} m_k, \prod_{k\neq 2} m_k, \cdots, \prod_{k\neq n} m_k\}$. Unfortunately, it is not so easy to show that this upper bound can be achieved, but we will later do so via a constructive proof. We note that this result concerning the maximum number of possible N.E. is true for any n-person strictly ordinal game, because the observations that lead to the conclusion are not based on any assumptions on the way that the ordinal payoffs are generated.

In the 2-person strictly ordinal game, we calculate the expected number of N.E. under the two assumptions stated in our model by making use of the observation that E[number of N.E.] = total number of outcomes x P(an outcome is a N.E.). The same argument remains valid for the n-person strictly ordinal game, and when we apply it, we find that the expected number of N.E. is still 1.

In the 2-person strictly ordinal game, when we match every strategy, i, of Player 1, with the best response, s(i), of Player 2 (i.e., the strategy that gives Player 2 his most preferred outcome), we may find that several strategies of Player 1 are associated with the same strategy of Player 2. Consider performing this matching when Player 1 has a finite number of strategies, but Player 2 has infinitely many strategies, which are all equally likely to be the best response for any strategy of Player 1. Now, intuitively, the probability is 1 that each strategy of Player 1 is associated with a distinct strategy of Player 2, since it is impossible that among an infinite number of strategies, the finite number of best responses do not come from distinct strategies. We then observe that the N.E. of the game must come from one of the (i,s(i)). The probability that any of them is a N.E. is the same as P(Player 1's ordinal payoff is greatest when he chooses i | Player 2 chooses s(i)), which by the symmetry of the way the ordinal payoffs are generated, equals $1/m_1$. If we let X_i be the event that (i,s(i)) is a N.E., then we can identify X_i as a Bernoulli random variable with mean $1/m_1$, and $\sum_i X_i$ as a binomial random variable with parameters m_1 and $1/m_1$. The above insight provides us with the distribution of the number of N.E. when the number of strategies of only one player goes to infinity.

To generalize proposition 3 for the n-person strictly ordinal game, all that we have to do is to match every combination of strategies of Players 1,2,..., n-1 with the best response of Player n. Without loss of generality, we can assume that the number of strategies of Player n goes to infinity and the numbers of strategies of the rest of the players remain finite. If we mimic the argument for the 2-person strictly ordinal game carefully, we can show that the distribution of the number of N.E. in the n-person strictly ordinal game is binomial with parameters $\prod_{k\neq n} m_k$ and $1/\prod_{k\neq n} m_k$. Notice that our proof of proposition 3 is not as direct as the one we just suggested, but a proof based on our discussion is included in the appendix (p. 41). In addition to employing the properties that we just discussed, our proof also utilizes properties of moment generating functions. The advantage of this approach is that it is also applicable in proving proposition 4.

Finally, we would also like to know the distribution of the number of N.E. as the numbers of strategies of more than one player go to infinity. In the 2-person strictly ordinal game, it would be easy to find the limiting distribution when both m_1 and m_2 go to infinity if we knew that the limit exists. Then all that we would have to consider is what happens to the binomial $(m_1, 1/m_1)$ distribution when m_1 goes to infinity, since $\lim_{m_1, m_2 \to \infty} P(X_{m_1, m_2} = x) = \lim_{m_1 \to \infty} \lim_{m_2 \to \infty} P(X_{m_1, m_2} = x)$, if $\lim_{m_1, m_2 \to \infty} P(X_{m_1, m_2} = x)$ exists (see, for example, Gelbaum and Olmsted, p. 117). We know that $\lim_{m_1\to\infty}\lim_{m_2\to\infty} P(X_{m_1,m_2}=x) = e^{-1}/x!$ (by the DeMoivre -Laplace limit theorem), and so if the limit exists, then the distribution of the number of N.E. approaches the Poisson distribution with mean 1 as the numbers of strategies of both players go to infinity. Unfortunately, we do not know that the limit exists, and so we cannot use the result of proposition 3. However, we can solve for the limiting distribution by formulating the problem in the same way as we did in the proof of proposition 3, and then identifying the moment generating function in the limit as that of a Poisson distribution with mean 1.

The proof for proposition 4 can easily be generalized for the n-person strictly ordinal game. The notation for the n-person case is quite cumbersome, but the way the techniques works can be sufficiently well demonstrated by illustrating the proof for $n = 3$.

Alternate proofs for propositions 3 and 4 by way of considering the distribution of the number of N.E. when the numbers of strategies of the players are finite can be found in Powers (1986), pp. 8-10 and 21-24 .

4. n-Person Strictly Ordinal Games

Consider n players, Players 1, 2, ..., n, where Player k has m_k strategies, and each player has strictly ordinal preferences over the $m_1 \times m_2 \times \cdots \times m_n$ outcomes of the game. Without loss of generality, we can assume that $m_1 \le m_2 \le \cdots \le m_n$. Let $h = (s_1, s_2, \ldots, s_{n-2})$, $i = s_{n-1}$ and $j = s_n$, so that (h,i,j) specifies a particular combination of strategies of the first n-2 players with the strategies of Players n−1 and n, and let h be called a "compound strategy." Furthermore, let $A_{hij} = ((A_{hij})_1, (A_{hij})_2, \ldots, (A_{hij})_{n-2})$ denote componentwise the ordinal payoffs of (h,i,j) to Players 1, 2, ..., n−2, and B_{hij}, C_{hij} denote the ordinal payoffs of (h,i,j) to Players n−1 and n, respectively. As in the case of the 2-person strictly ordinal game, we assume that $(A_{hij})_\alpha$ ($\alpha = 1, 2, \ldots, n-2$), B_{hij} and C_{hij} are all independently obtained by a random drawing of $\Pi\, m_k$ numbers from 1 to $\Pi\, m_k$ without replacement. In an analogous fashion, we define the following indicator functions:

$$I[(A(s_1, s_2, \ldots, s_\alpha', \ldots, s_{n-2}, i, j))_\alpha] = 1,$$
$$\text{if } (A(s_1, s_2, \ldots, s_\alpha', \ldots, s_{n-2}, i, j))_\alpha = \operatorname*{Max}_{s_\alpha}\{ (A(s_1, s_2, \ldots, s_\alpha, \ldots, s_{n-2}, i, j))_\alpha\}$$
$$I[(A(s_1, s_2, \ldots, s_\alpha', \ldots, s_{n-2}, i, j))_\alpha] = 0 \quad \text{otherwise}$$

$$I[B_{hi'j}] = 1 \quad \text{if } [B_{hi'j}] = \operatorname*{Max}_{i}\{B_{hij}\}$$
$$= 0 \quad \text{otherwise}$$

$$I[C_{hij'}] = 1 \quad \text{if } [C_{hij'}] = \operatorname*{Max}_{j}\{C_{hij}\}$$
$$= 0 \quad \text{otherwise}$$

Let $I[A_{hij}] = \prod_\alpha I[(A(s_1, s_2, \ldots, s_\alpha, \ldots, s_{n-2}, i, j))_\alpha]$, and

$$I_{hij} = I[A_{hij}] \times I[B_{hij}] \times I[C_{hij}].$$

Then $I_{hij} = 1$ if (h,i,j) is a N.E.

= 0 otherwise

and $I_{hj} = \sum_i I_{hij} = 1$ if there is a N.E. associated with strategy (h,j)

= 0 otherwise

With this notation, we are ready to present the generalized results concerning the number of N.E. in the n-person strictly ordinal game.

We have argued in section 3 that an upper bound for the maximum number of possible N.E. is min $\{\prod_{k\neq 1} m_k, \prod_{k\neq 2} m_k, \ldots, \prod_{k\neq n} m_k\}$. We shall soon prove that this upper bound can be achieved, but before we proceed to the rather laborious proof, we would like to indicate the spirit of the proof by providing two examples of ordinal games (with their N.E. highlighted in boldface) that achieve the upper bound for the maximum number of possible N.E. Consider first the following 3x4 (2-person) strictly ordinal game with 3 (= min {3,4}) N.E.:

$s_1 \backslash s_2$	1	2	3	4
1	**(12,12)**	(9,9)	(8,8)	(7,7)
2	(6,6)	**(11,11)**	(5,5)	(4,4)
3	(3,3)	(2,2)	**(10,10)**	(1,1)

We can construct a 3x4x5 (3-person) strictly ordinal game with 12 (= min {4x5, 3x5, 3x4}) N.E. and position them in the following manner :

$s_1 \backslash s_2$	1	2	3	4	
1	**(60,60,60)**	(48,48,48)	(47,47,47)	(46,46,46)	
2	(45,45,45)	**(59,59,59)**	(44,44,44)	(43,43,43)	$s_3 = 1$
3	(42,42,42)	(41,41,41)	**(58,58,58)**	(40,40,40)	

$s_1 \backslash s_2$	1	2	3	4
1	(39,39,39)	**(57,57,57)**	(38,38,38)	(37,37,37)
2	(36,36,36)	(35,35,35)	**(56,56,56)**	(34,34,34)
3	(33,33,33)	(32,32,32)	(31,31,31)	**(55,55,55)**

$s_3 = 2$

$s_1 \backslash s_2$	1	2	3	4
1	(30,30,30)	(29,29,29)	**(54,54,54)**	(28,28,28)
2	(27,27,27)	(26,26,26)	(25,25,25)	**(53,53,53)**
3	**(52,52,52)**	(24,24,24)	(23,23,23)	(22,22,22)

$s_3 = 3$

$s_1 \backslash s_2$	1	2	3	4
1	(21,21,21)	(20,20,20)	(19,19,19)	**(51,51,51)**
2	**(50,50,50)**	(18,18,18)	(17,17,17)	(16,16,16)
3	(15,15,15)	**(49,49,49)**	(14,14,14)	(13,13,13)

$s_3 = 4$

$s_1 \backslash s_2$	1	2	3	4
1	(12,12,12)	(11,11,11)	(10,10,10)	(9,9,9)
2	(8,8,8)	(7,7,7)	(6,6,6)	(5,5,5)
3	(4,4,4)	(3,3,3)	(2,2,2)	(1,1,1)

$s_3 = 5$

Proposition 5: Let $f(n)$ be the maximum number of possible N.E. in an n-person strictly ordinal game. Then for all $n \geq 2$, $f(n) = \min\{ \prod_{k \neq 1} m_k, \prod_{k \neq 2} m_k, \ldots, \prod_{k \neq n} m_k \}$.

Proof: Since we have assumed, without loss of generality, that $m_1 \leq m_2 \leq \cdots$

$\leq m_{n-1}$, it follows that $\min\{\prod_{k\neq 1} m_k, \prod_{k\neq 2} m_k, \ldots, \prod_{k\neq n} m_k\} = \prod_{k\neq n} m_k$. We now proceed to prove the proposition by induction.

(i) When $n = 2$, we know from proposition 1 of section 3 that $f(2) = \min\{m_1, m_2\}$, which is equal to $\min\{\prod_{k\neq 1} m_k, \prod_{k\neq 2} m_k\}$.

(ii) Assume $f(n-1) = \min\{\prod_{k\neq 1} m_k, \prod_{k\neq 2} m_k, \ldots, \prod_{k\neq n-1} m_k\} = \prod_{k\neq n-1} m_k$. This means the maximum number of possible N.E. in the (n-1)-person strictly ordinal game, where Player k has m_k strategies, is $f(n-1)$. If Player k labels his strategies with the numbers 1, 2, ..., m_k, then the preceding statement implies that we can find a collection of $f(n-1)$ ordered (n-1)-tuples, where the k^{th} entry of each element represents the strategy of the k^{th} player, and every element in this collection differs from every other in at least 2 entries (since every player has strictly ordinal preferences over all of the outcomes). Let us denote such a collection by Q_{n-1}. We shall now show that from Q_{n-1} we can construct a collection of $f(n)$ ordered n-tuples, such that the additional entry is a strategy of the n^{th} player and every element in this collection differs from every other in at least 2 entries.

First of all, let $Q_{n-1} = \{a^{(1)}, a^{(2)}, \ldots, a^{f(n-1)}\}$, where each element in Q_{n-1} is an ordered (n-1)-tuple. We shall construct Q_n in the following manner (addition is done in modulus m_{n-1}):

$a_1^{(1)}$	$a_2^{(1)}$	$a_3^{(1)}$	...	$a_{n-1}^{(1)}$	1
$a_1^{(2)}$	$a_2^{(2)}$	$a_3^{(2)}$	...	$a_{n-1}^{(2)}$	1

$a_1(3)$	$a_2(3)$	$a_3(3)$	$\cdots$	$a_{n-1}(3)$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$a_1(f(n-1))$	$a_2(f(n-1))$	$a_3(f(n-1))$	$\cdots$	$a_{n-1}(f(n-1))$	1
$a_1(1)$	$a_2(1)$	$a_3(1)$	$\cdots$	$a_{n-1}(1)+1$	2
$a_1(2)$	$a_2(2)$	$a_3(2)$	$\cdots$	$a_{n-1}(2)+1$	2
$a_1(3)$	$a_2(3)$	$a_3(3)$	$\cdots$	$a_{n-1}(3)+1$	2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$a_1(f(n-1))$	$a_2(f(n-1))$	$a_3(f(n-1))$	$\cdots$	$a_{n-1}(f(n-1))+1$	2
$a_1(1)$	$a_2(1)$	$a_3(1)$	$\cdots$	$a_{n-1}(1)+2$	3
$a_1(2)$	$a_2(2)$	$a_3(2)$	$\cdots$	$a_{n-1}(2)+2$	3
$a_1(3)$	$a_2(3)$	$a_3(3)$	$\cdots$	$a_{n-1}(3)+2$	3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$a_1(f(n-1))$	$a_2(f(n-1))$	$a_3(f(n-1))$	$\cdots$	$a_{n-1}(f(n-1))+2$	3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$a_1(1)$	$a_2(1)$	$a_3(1)$	$\cdots$	$a_{n-1}(1) + m_{n-1} - 1$	m_{n-1}
$a_1(2)$	$a_2(2)$	$a_3(2)$	$\cdots$	$a_{n-1}(2) + m_{n-1} - 1$	m_{n-1}
$a_1(3)$	$a_2(3)$	$a_3(3)$	$\cdots$	$a_{n-1}(3) + m_{n-1} - 1$	m_{n-1}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$a_1(f(n-1))$	$a_2(f(n-1))$	$a_3(f(n-1))$	$\cdots$	$a_{n-1}(f(n-1)) + m_{n-1} - 1$	m_{n-1}

In order to facilitate our discussion, we organize the elements in Q_n by identifying all elements having the same last entry as one group, and within each group, we denote an element by its row number in the group. We first note that Q_n has $f(n-1) \times m_{n-1} = \prod_{k \neq n} m_k = f(n)$ elements, since $m_1 \leq m_2 \leq \cdots \leq m_{n-1} \leq m_n$. For any two elements in this collection we can make the following observations:

(1) If they are from different rows within the same group, then they differ from each other in at least two entries, since every element in Q_{n-1} differs from every other in at least two entries.

(2) If they are from corresponding rows in two different groups, then they differ from each other in the $(n-1)^{th}$ and n^{th} entries.

(3) If they have different row numbers and are from two different groups, then we note first that they differ from each other in the n^{th} entry. We also know that any two elements from the same group differ from each other in at least two entries (see remark (1)), and so we can claim that two elements with different row numbers from different groups must differ in at least one more entry besides the n^{th}.

We conclude that Q_n is a collection of $f(n)$ elements, where the entries of each element represent componentwise the strategies of Players 1, 2, ..., n, and every element in this collection differs from every other element in at least two entries. We note that the $f(n)$ elements give permissible coordinates for N.E. in an n-person strictly ordinal game. We now argue that in an n-person strictly ordinal game it is possible to have $f(n)$ N.E. to fill all of these $f(n)$ positions. This can be realized in at least one way, if all players prefer the outcomes of strategies represented by the elements in Q_n to the outcomes of strategies not represented by elements in Q_n. This proves that f(n) is a lower bound for the maximum number of N.E. in an n-person strictly ordinal game, and since we have shown in section 3 that f(n) is also an upper bound, we are done.

Q.E.D.

We now establish that proposition 3 in section 3 holds in a more general setting, namely, in the n-person strictly ordinal game.

Theorem 1: The probability distribution of the number of N.E. approaches the binomial $(\prod_{k \neq n} m_k,\ 1/\prod_{k \neq n} m_k)$ distribution as the number of strategies of Player n approaches infinity and the numbers of strategies of the other players remain finite.

Proof: We first prove the case where $n = 3$, and then indicate how the proof can be generalized for $n > 3$ by correctly identifying the corresponding variables.

Let $\sum_i I[C_{hij}] = \lambda_{hj}$, for all (h,j), so that $\sum_{h,j} \lambda_{hj} = m_1 m_2$. Then

$P(I_{hj} = 1 \mid \boldsymbol{\lambda}) = \lambda_{hj}/(m_1 m_2)$ and $P(I_{hj} = 0 \mid \boldsymbol{\lambda}) = 1 - \lambda_{hj}/(m_1 m_2)$,

where $\boldsymbol{\lambda} = (\lambda_{hj})$.

Let X be the number of N.E. and $\varphi_X(t)$ be its moment generating function.

Then

$$\varphi_X(t)$$
$$= \sum_{\boldsymbol{\lambda}} \varphi_{X|\boldsymbol{\lambda}}(t) \cdot P(\boldsymbol{\lambda})$$
$$= \sum \varphi_{\Sigma I_{hj}|\boldsymbol{\lambda}}(t) \cdot P(\boldsymbol{\lambda}) .$$

Since I_{hj} and $I_{h'j'}$ ($h \neq h'$ and $j \neq j'$) are conditionally independent given a particular $\boldsymbol{\lambda}$, we see that

$\varphi_X(t)$

$= \Sigma [\Pi\, \varphi_{I_{hj}|\boldsymbol{\lambda}}(t)] \cdot P(\boldsymbol{\lambda})$

$= \Sigma [\Pi\, (e^t \lambda_{hj}/(m_1 m_2) - \lambda_{hj}/(m_1 m_2) + 1)] \cdot P(\boldsymbol{\lambda})$.

Let E be the event that $\Sigma\, \lambda_{hj} \leq 1$ for all j . Then

$P(E) = m_3(m_3-1)(m_3-2) \cdot \cdot \cdot (m_3 - m_1 m_2 + 1)/m_3^{(m_1 m_2)}$, and

as $m_3 \longrightarrow \infty$, $P(E) \longrightarrow 1$, which means that $\lambda_{hj} = 0$ or 1 . Together with

$\Sigma\, \lambda_{hj} = m_1 m_2$, it follows that

$\lim_{m_3 \to \infty} \varphi_X(t)$

$= \Sigma [(e^t/(m_1 m_2) - 1/(m_1 m_2) + 1)^{(m_1 m_2)}] \cdot P(\boldsymbol{\lambda})$

$= (e^t/(m_1 m_2) - 1/(m_1 m_2) + 1)^{(m_1 m_2)}$,

which is the moment generating function of the binomial distribution with parameters $m_1 m_2$ and $1/(m_1 m_2)$.

For the proof of the theorem when $n > 3$, simply replace m_1 with $m_1 \times m_2 \times \cdot \cdot \cdot \times m_{n-2}$, m_2 with m_{n-1} and m_3 with m_n .

Q.E.D.

We now present our most interesting result so far for the distribution of the number of N.E.:

<u>Theorem</u> 2: The probability distribution of the number of N.E. approaches the Poisson distribution with mean 1 as the numbers of strategies of two or more players approach infinity.

Proof: As in the proof of theorem 1, we begin by demonstrating the proof for $n = 3$. We know from the proof of theorem 1 that

$$\varphi_{X|\boldsymbol{\lambda}}(t)$$

$$= \varphi_{\Sigma I_{hj}|\boldsymbol{\lambda}}(t)$$

$$= \Pi\ (e^t\lambda_{hj}/(m_1m_2) - \lambda_{hj}/(m_1m_2) + 1)\ .$$

$$= \exp\ \{\log\ [\ \Pi\ (e^t\lambda_{hj}/(m_1m_2) - \lambda_{hj}/(m_1m_2) + 1)]\}$$

$$= \exp\ \{\Sigma\ [\ \log\ (\ (e^t - 1)\ \lambda_{hj}/(m_1m_2) + 1)\ ]\}$$

$$= \exp\ \{\Sigma\ \{[(e^t - 1)\ \lambda_{hj}/(m_1m_2)] - (1/2)[(e^t - 1)\ \lambda_{hj}/(m_1m_2)]^2 +$$

$$(1/3)[(e^t - 1)\lambda_{hj}/(m_1m_2)]^3 - \cdots\}\ ,\ \text{for } -\infty \le t \le \log 2\ .$$

We note that

$$\varphi_{X|\boldsymbol{\lambda}}(t)$$

$$\le \exp\ \{\Sigma\ [(e^t - 1)\ \lambda_{hj}/(m_1m_2)]\}$$

$$= \exp(e^t - 1)\ , \qquad (1)$$

and $\varphi_{X|\boldsymbol{\lambda}}(t)$

$$\ge \exp\ \{\Sigma\ [(e^t - 1)\ \lambda_{hj}/(m_1m_2)] - 1/2[(e^t - 1)\ \lambda_{hj}/(m_1m_2)]^2\}\ . \qquad (2)$$

Let A be the event that $\lambda_{hj} < cm_2^{1/3}$ for all (h,j), and A' be the event that $\lambda_{hj} \ge cm_2^{1/3}$ for at least one (h,j), where c is a constant. Then

$\varphi_X(t)$

$= E_{\boldsymbol{\lambda}}[\varphi_{X|\boldsymbol{\lambda}}(t)]$

$= E_{\boldsymbol{\lambda}\epsilon A}[\varphi_{X|\boldsymbol{\lambda}}(t)]\cdot P(A) + E_{\boldsymbol{\lambda}\epsilon A'}[\varphi_{X|\boldsymbol{\lambda}}(t)]\cdot P(A')$.

Note that $P(A')$

$\le m_1 m_3 \sum_{q=[cm_2^{1/3}]}^{m_2} \binom{m_2}{q} (1/m_3)^q (1-1/m_3)^{m_2-q}$,

and let $m_2 = m_3 = \eta$. Then $P(A')$

$\le m_1\eta \sum_{q=[c\eta^{1/3}]}^{\eta} \binom{\eta}{q} (1/\eta)^q (1-1/\eta)^{\eta-q}$

$\le m_1\eta(\eta - c\eta^{1/3})[1/[c\eta^{1/3}]!]$

$\le m_1\eta^2/[c\eta^{1/3}]!$

(1) and (2) $\Longrightarrow 0 \le E_{\boldsymbol{\lambda}\epsilon A'}[\varphi_{X|\boldsymbol{\lambda}}(t)] \le \exp(e^t-1)$. Therefore, as $\eta \longrightarrow \infty$,

$E_{\boldsymbol{\lambda}\epsilon A'}[\varphi_{X|\boldsymbol{\lambda}}(t)]\cdot P(A') = 0$.

From (1) and (2), we also know that $E_{\boldsymbol{\lambda}\epsilon A}[\varphi_{X|\boldsymbol{\lambda}}(t)]\cdot P(A)$ is bounded below

by $E_{\boldsymbol{\lambda}\epsilon A}\{\exp\{\Sigma[(e^t-1)\lambda_{hj}/(m_1\eta)] - 1/2[(e^t-1)\lambda_{hj}/(m_1\eta)]^2\}\}\cdot P(A)$ and

above by $\exp(e^t-1)\cdot P(A)$. Since these bounds both converge to $\exp(e^t-1)$ as

$\eta \longrightarrow \infty$, it follows that $E_{\boldsymbol{\lambda}\epsilon A}[\varphi_{X/\boldsymbol{\lambda}}(t)]\cdot P(A) = \exp(e^t-1)$ as $\eta \longrightarrow \infty$.

Thus, $\lim_{m_2, m_3 \to \infty} \varphi_X(t)$

$= \exp(e^t-1)$,

which is the moment generating function of the Poisson distribution with mean

1. Since the result holds for all m_1, it holds as m_1 goes to infinity, and thus we have proved the theorem for $n = 3$.

For the proof of the theorem when $n > 3$, simply replace m_1 with $m_1 \times m_2 \times \cdots \times m_{n-2}$, m_2 with m_{n-1} and m_3 with m_n.

Q.E.D.

We would like to make two remarks about Theorem 2:

(1) For the Poisson distribution with mean 1, $P(X^{(n)} = 0) = 0.3679$, $P(X^{(n)} = 1) = 0.3679$, $P(X^{(n)} = 2) = 0.1839$, $P(X^{(n)} = 3) = 0.0613$, ... Thus, we can see that $P(X^{(n)} > 3) = 0.0190$, so that even though there are infinitely many possible outcomes when the numbers of strategies of two or more players go to infinity, the chance is very small that there are more than three N.E. We also note that it is not so rare to have a N.E.

(2) In section 3, we gave an intuitive argument why the result of the theorem holds when the numbers of strategies of two players go to infinity. Actually, a stronger result than the one that we anticipated holds, since the result is also valid when the numbers of strategies of more than two players go to infinity.

Now we turn to weakly ordinal games. Instead of stating the results for the 2-person weakly ordinal game first, we proceed directly to the n-person weakly ordinal game. As in the case of strictly ordinal games, we present the results first and then provide the intuition behind them.

5. n-Person Weakly Ordinal Games

We use the same notation as that in the n-person strictly ordinal game, but this time we consider n players who may have weakly ordinal preferences over the possible outcomes. Again, without loss of generality, we assume that $m_1 \leq m_2 \leq \cdots \leq m_n$, and let $(s_1, s_2, \ldots, s_n)$ be componentwise the strategies chosen by Players 1, 2, ..., n. Let us denote the ordinal payoffs of Player k by $p_k(s_1, s_2, \ldots, s_n)$, and assume that independent of the payoffs of the other players, these $p_k(s_1, s_2, \ldots, s_n)$ are obtained by a random drawing of M $(= \prod m_k)$ numbers from 1 to M with replacement.

This model of a "random" n-person weakly ordinal game is a natural extension of our model of a "random" n-person strctly ordinal game. The drawing with replacement simulates having ties among payoffs for a player. Let $Z^{(n)}$ be the number of N.E. in this weakly ordinal game. We know that given the strategies of the rest of the players, there are possibly more than one maximal outcome for each player, where a maximal outcome is an outcome that yields the most preferred ordinal payoff given the strategies chosen by the rest of the players. Hence, we anticipate a greater number of N.E. in this game than that in our strictly ordinal game, and this belief is confirmed by our next lemma.

<u>Lemma</u>: $E[Z^{(n)}] \geq 1$.

Proof: Fix the strategies of all but Player k, and consider the one or more maximal outcomes for Player k. Identify one of these outcomes as the "designated" maximal outcome and the rest, if any, as "undesignated" maximal outcomes. Then define

$$I_k(s_1, s_2, \ldots, s_n) = \begin{cases} 1 & \text{if } (s_1, s_2, \ldots, s_n) \text{ gives Player k a designated maximal outcome} \\ 0 & \text{otherwise} \end{cases}$$

$$J_k(s_1, s_2, \ldots, s_n) = \begin{cases} 1 & \text{if } (s_1, s_2, \ldots, s_n) \text{ gives Player k an undesignated maximal outcome} \\ 0 & \text{otherwise} \end{cases}$$

and let

$$I(s_1, s_2, \ldots, s_n) = \prod_k I_k(s_1, s_2, \ldots, s_n)$$

$$J(s_1, s_2, \ldots, s_n) = \prod_k [I_k(s_1, s_2, \ldots, s_n) + J_k(s_1, s_2, \ldots, s_n)] - \prod_k [I_k(s_1, s_2, \ldots, s_n)] .$$

We recognize that

$\sum_{(s_1, s_2, \ldots, s_n)} I(s_1, s_2, \ldots, s_n) = X^{(n)}$, the number of N.E. in the strictly ordinal game, and we know that $E[X^{(n)}] = 1$ (section 3). Let us also define

$\sum_{(s_1, s_2, \ldots, s_n)} J(s_1, s_2, \ldots, s_n) = Y^{(n)}$. Since $Y^{(n)}$ is a non-negative random variable, it follows that $E[Y^{(n)}] \geq 0$. We can see that $Z^{(n)} = X^{(n)} + Y^{(n)}$, and so $E[Z^{(n)}] = E[X^{(n)}] + E[Y^{(n)}] \geq 1$, for all $m_1, m_2, \ldots, m_n$.

Q.E.D.

<u>Proposition</u> 6: The expected number of N.E. approaches $(1/m)(1/(1-e^{-1/m}))$, where $m = m_1 \times m_2 \times \ldots \times m_{n-1}$, as the number of strategies of Player n goes to infinity and the numbers of strategies of the other players remain finite.

Proof: $E[Z^{(n)}]$

$= (\Pi m_k)\, \Pi\, \{(\Pi m_k)^{-m_k} [(\Pi m_k)^{m_k-1} + (\Pi m_k-1)^{m_k-1} + (\Pi m_k-2)^{m_k-1} + \cdots +$

$(\Pi m_k-(\Pi m_k-1))^{m_k-1}]\}$.

Let $\eta = m_n$ and let v be the index for Players 1, 2, ..., n−1, so that $\Pi m_v = m$ and $\Pi m_k = m\eta$. Then

$E[Z^{(n)}]$

$= m\, \Pi\{(m\eta)^{-m_v} [(m\eta)^{m_v-1} + (m\eta-1)^{m_v-1} + (m\eta-2)^{m_v-1} + \cdots +$

$(m\eta-(m\eta-1))^{m_v-1}]\}\cdot \eta\{(m\eta)^{-\eta}[(m\eta)^{\eta-1} + (m\eta-1)^{\eta-1} + (m\eta-2)^{\eta-1} + \cdots +$

$(m\eta-(m\eta-1))^{\eta-1}]\}$

$= m \prod_v [(m\eta)^{-1} \cdot \sum_{i=0}^{m\eta-1} (1-i/(m\eta))^{m_v-1}] \cdot [(1/m) \sum_{i=0}^{m\eta-1} (1-i/(m\eta))^{\eta-1}]$, and

$\lim_{\eta\to\infty} E[Z^{(n)}]$

$= m\, [\prod_v \int_0^1 (1-\theta)^{m_v-1}\, d\theta]\cdot [(1/m) \sum_{i=0}^{m\eta-1} (1-i/(m\eta))^{\eta-1}]$ (where θ is just a dummy variable)

$= m(1/m)\cdot[(1/m)\cdot \lim_{\eta\to\infty} \sum_{i=0}^{m\eta-1} (1-i/(m\eta))^{\eta-1}]$

$= (1/m)(1/(1-e^{-1/m}))$.

Q.E.D.

Since the solution to the last limit is not found in any table or textbook that we have consulted, we have provided a proof for the relevant identity in the appendix (p. 42).

We would like to point out that $(1/m)(1/(1-e^{-1/m}))$ is a decreasing function on $m \in [1,\infty)$, taking values between $e/(e-1)$ ($\approx$1.5820) and 1. This is consistent with the result in our Lemma. Our next step is to study the distribution of the number of N.E. when the number of strategies of Player n goes to infinity and the numbers of strategies of the other players remain finite.

Theorem 3: The moment generating function of the probability distribution of the number of N.E. approaches $\varphi_z(t) = (1/(1-e^{-\lambda}))\cdot(e^{\lambda^2(e^t-1)} - e^{-\lambda}))^{1/\lambda}$, where $\lambda = 1/m$ ($m = m_1 \times m_2 \times \ldots \times m_{n-1}$), as the number of strategies of Player n goes to infinity while the numbers of strategies of the other players remain finite.

Proof: Let W equal the number of maximal outcomes of Player n for a given compound strategy chosen by Players 1, 2, ..., n-1, let $M = \Pi m_k$, the total number of possible outcomes, and let $\eta = m_n$. Then

$P(W = w) = M/M^{\eta} = M^{-(\eta-1)}$, for $w = \eta$, and

$$P(W = w) = \binom{\eta}{w}((M-1)^{\eta-w} + (M-2)^{\eta-w} + \cdots + 1^{\eta-w})/M^{\eta}, \text{ for } 0 < w < \eta$$

$$= \binom{\eta}{w}(1/M)^{w}[(1-1/M)^{\eta-w} + (1-2/M)^{\eta-w} + \cdots + (1-(M-1)/M)^{\eta-w}].$$

For any $(s_1, s_2, \ldots, s_n)$, the probability that Player k's strategy gives him a maximal payoff given the strategies chosen by the rest of the players is $(M^{m_k-1} + (M-1)^{m_k-1} + (M-2)^{m_k-1} + \cdots + 1)/M^{m_k}$. Let R be the event that this is true for Players 1, 2, ..., n-1. Then $P(R) = \prod_{k=1}^{n-1}(M^{m_k-1} + (M-1)^{m_k-1} + (M-2)^{m_k-1} + \cdots + 1)/M^{m_k}$.

Now, let Z_i be the number of N.E. when the strategies of Players 1, 2, ..., n-1 are fixed. Then for $\zeta \geq 1$,

$$P(Z_i = \zeta) = P(W = \eta) \binom{\eta}{\zeta} (P(R))^{\zeta} (1-P(R))^{\eta-\zeta} + \sum_{w=\zeta}^{\eta-1} P(W = w) \binom{w}{\zeta} (P(R))^{\zeta} (1-P(R))^{w-\zeta}.$$

If we let $\lambda = 1/m$, then as $\eta \to \infty$,

$$\lim_{\eta\to\infty} P(Z_i = \zeta) = 0 + \sum_{w=\zeta}^{\infty} (1/w!)\, \lambda^{w} [e^{-\lambda}/(1-e^{-\lambda})] \binom{w}{\zeta} \lambda^{\zeta} (1-\lambda)^{w-\zeta}$$

$$= [\lambda^{2\zeta} e^{-\lambda}/(\zeta!(1-e^{-\lambda}))] \cdot [\Sigma\, (\lambda^{w-\zeta}/(w-\zeta)!) \cdot (1-\lambda)^{w-\zeta}]$$

$$= [\lambda^{2\zeta} e^{-\lambda}/(\zeta!(1-e^{-\lambda}))] \cdot e^{\lambda(1-\lambda)}$$

$$= \lambda^{2\zeta} e^{-\lambda^2}/(\zeta!(1-e^{-\lambda})), \text{ and}$$

$$\lim_{\eta\to\infty} P(Z_i = 0) = 1 - \sum_{\zeta=1}^{\infty} [\lambda^{2\zeta} e^{-\lambda^2}/(\zeta!(1-e^{-\lambda}))]$$

$$= 1 - (1/(1-e^{-\lambda})) \cdot [\sum_{\zeta=1}^{\infty} (\lambda^2)^{\zeta} e^{-\lambda^2}/\zeta!]$$

$$= 1 - (1/(1-e^{-\lambda})) \cdot (1-e^{-\lambda^2})$$

$$= (e^{-\lambda^2} - e^{-\lambda})/(1 - e^{-\lambda}).$$

Hence, the moment generating function of the limiting distribution of Z_i, as $\eta \to \infty$, is

$$E[e^{tZ_i}] = (e^{-\lambda^2} - e^{-\lambda})/(1-e^{-\lambda}) + \sum_{\zeta=1}^{\infty} (e^{t\zeta} \lambda^{2\zeta} e^{-\lambda^2})/(\zeta!(1-e^{-\lambda}))$$

$$= (e^{-\lambda^2} - e^{-\lambda})/(1-e^{-\lambda}) + (1/(1-e^{-\lambda}))(e^{\lambda^2(e^t-1)} - e^{-\lambda^2})$$

$$= (1/(1-e^{-\lambda}))(e^{\lambda^2(e^t-1)} - e^{-\lambda}).$$

We note that

$$\lim_{\eta\to\infty} E[W]$$

$$= \lim_{\eta\to\infty} \{\eta \bar{M}^{(\eta-1)} + \sum_{w=1}^{\eta-1} w \binom{\eta}{w} (1/M)^w [(1-1/M)^{\eta-w} + (1-2/M)^{\eta-w} + \cdots +$$

$$(1-(M-1)/M)^{\eta-w}]\}$$

$$= 0 + \sum_{w=1}^{\infty} (1/(w-1)!)\, \lambda^w \cdot (e^{-\lambda} / (1-e^{-\lambda}))$$

$$= \lambda/(1-e^{-\lambda}) < \infty .$$

Thus, for a given compound strategy chosen by Players 1, 2, ..., n-1, the number of maximal outcomes of Player n is finite in the limit as $\eta\to\infty$ with probability 1. Since the number of compound strategies is finite, it follows that in the limit as $\eta\to\infty$, the maximal outcomes of Player n for all compound strategies chosen by Players 1, 2, ..., n-1 are achieved with distinctly different strategies with probability 1. Therefore, the number of N.E. for any compound strategy i is independent of the number of N.E. for any other compound strategy, and so the moment generating function of $Z^{(n)} = \sum_{i=1}^{m} Z_i$, the total number of N.E., is $(E[e^{tZ_i}])^{1/\lambda} = \{[(1/(1-e^{-\lambda})]\cdot[e^{\lambda^2(e^t-1)} - e^{-\lambda}]\}^{1/\lambda}$.

Q.E.D.

It is regrettable that we cannot extract the exact probability distribution from this moment generating function. However, we do know its mean already, and we can find out its variance and higher moments by using the moment generating function.

<u>Proposition</u> 7: The expected number of N.E. approaches 1 as the numbers of strategies of two or more players go to infinity.

Proof: $E[Z^{(n)}]$

$= (\Pi m_k) \Pi \{(\Pi m_k)^{-m_k} [(\Pi m_k)^{m_k-1} + (\Pi m_k-1)^{m_k-1} + (\Pi m_k-2)^{m_k-1} + \cdots +$

$(\Pi m_k-(\Pi m_k-1))^{m_k-1}]\}$.

Let $\eta = m_{n-u} = m_{n-u+1} = m_{n-u+2} = \cdots = m_n$, let v be the index for Players 1, 2, ..., n-u-1, and let $\mu = \Pi m_v$, so that $\Pi m_k = \mu\eta^{u+1}$. Then

$E[Z^{(n)}]$

$= \mu\ \Pi\{(\mu\eta^{u+1})^{-m_v} [(\mu\eta^{u+1})^{m_v-1} + (\mu\eta^{u+1}-1)^{m_v-1} + (\mu\eta^{u+1}-2)^{m_v-1} +$

$\cdots + (\mu\eta^{u+1}-(\mu\eta^{u+1}-1))^{m_v-1}]\} \cdot \eta^{u+1}\{(\mu\eta^{u+1})^{-\eta} [(\mu\eta^{u+1})^{\eta-1} +$

$(\mu\eta^{u+1}-1)^{\eta-1} + (\mu\eta^{u+1}-2)^{\eta-1} + \cdots + (\mu\eta^{u+1}-(\mu\eta^{u+1}-1))^{\eta-1}]\}^{u+1}$

$= \mu \prod_v [(\mu\eta^{u+1})^{-1} \cdot \sum_{i=0}^{\mu\eta^{u+1}-1} (1-i/(\mu\eta^{u+1}))^{m_v-1}]\cdot$

$[(1/(\mu\eta^u)) \sum_{i=0}^{\mu\eta^{u+1}-1} (1-i/(\mu\eta^{u+1}))^{\eta-1}]^{u+1}$, and

$\lim_{\eta\to\infty} E[Z^{(n)}]$

$= \mu\ [\prod_v \int_0^1 (1-\theta)^{m_v-1}\, d\theta] \cdot \lim_{\eta\to\infty} [(1/(\mu\eta^u)) \sum_{i=0}^{\mu\eta^{u+1}-1} (1-i/(\mu\eta^{u+1}))^{\eta-1}]^{u+1}$

(where θ is just a dummy variable)

$= \mu(1/\mu) \cdot \lim_{\eta\to\infty} [(1/(\mu\eta^u)) \sum_{i=0}^{\mu\eta^{u+1}-1} (1-i/(\mu\eta^{u+1}))^{\eta-1}]^{u+1}$.

Now, $[(1/(\mu\eta^u)) \sum_{i=0}^{\mu\eta^{u+1}-1} (1-i/(\mu\eta^{u+1}))^{\eta-1}]^{u+1}$

$\le [(1/(\mu\eta^u)) \sum_{i=0}^{\mu\eta^{u+1}-1} (e^{-i/(\mu\eta^{u+1})})^{\eta-1}]^{u+1}$

$\leq [(1/(\mu\eta^{u})) \sum_{i=0}^{\infty} (e^{-i/(\mu\eta^{u+1})})^{\eta-1}]^{u+1}$

$= [(1/(\mu\eta^{u})) \cdot 1/(1-e^{-(\eta-1)/(\mu\eta^{u+1})})]^{u+1}$

$= [(1/\mu)\cdot(\eta^{-u})/(1-e^{-(\eta-1)/(\mu\eta^{u+1})})]^{u+1}$.

Using L'Hôpital's rule, we find that $\lim_{\eta\to\infty} (\eta^{-u})/(1-e^{-(\eta-1)/(\mu\eta^{u+1})}) = \mu$.

Hence, $\lim_{\eta\to\infty} E[Z^{(n)}] \leq 1$. Together with our lemma, it follows that

$\lim_{\eta\to\infty} E[Z^{(n)}] = 1$.

Q.E.D.

<u>Theorem</u> 4: The probability distribution of the number of N.E. approaches the Poisson distribution with mean 1 as the numbers of strategies of two or more players go to infinity.

Proof: Let $\eta = m_{n-u} = m_{n-u+1} = m_{n-u+2} = \cdots = m_n$. Then the following are true:

$\lim_{\eta\to\infty} E[X^{(n)}] = 1$ (from section 3)

$\lim_{\eta\to\infty} E[Z^{(n)}] = 1$ (from Proposition 7 in this section).

Since $E[Z^{(n)}] = E[X^{(n)}] + E[Y^{(n)}]$, we conclude that $\lim_{\eta\to\infty} E[Y^{(n)}] = 0$.

Using Markov's inequality and the fact that $Y^{(n)} \geq 0$, we see that for all $\epsilon > 0$,

$\lim_{\eta\to\infty} P(|Y^{(n)} - 0| \geq \epsilon) \leq \lim_{\eta\to\infty} E[Y^{(n)}]/\epsilon$

$\Longrightarrow \lim_{\eta\to\infty} P(|Y^{(n)} - 0| \geq \epsilon) = 0$

$\Longrightarrow \lim_{\eta\to\infty} P(|Y^{(n)} - 0| > \epsilon) = 0$

$\Longleftrightarrow Y^{(n)} \xrightarrow{P} 0$

$\Longrightarrow Y^{(n)} \xrightarrow{\text{dist.}} 0$

We know from Theorem 2 in section 4 that $X^{(n)} \xrightarrow{\text{dist.}} X$, where $X \sim$ Poisson (1). Since $Z^{(n)} = X^{(n)} + Y^{(n)}$, $X^{(n)} \xrightarrow{\text{dist.}} X$, and $Y^{(n)} \xrightarrow{\text{dist.}} 0$, we conclude by a standard convergence theorem (see, for example, Chung, p. 92) that $Z^{(n)} \xrightarrow{\text{dist.}} X$.

Q.E.D.

In our weakly ordinal game, each player's payoffs for all outcomes are obtained by a random drawing of M $(=\prod m_k)$ numbers from 1 to M with replacement. For the sake of completeness, we would like to mention that if the payoffs for Player k are obtained through a random drawing of M numbers from 1 to D_k, where D_k is a finite constant, then as the numbers of strategies of one or more players go to infinity, we would expect to see infinitely many N.E. Actually, we are able to prove an even stronger result: the probability that there are infinitely many N.E. is 1 as the numbers of strategies of one or more players go to infinity. In order not to distract the reader from our original model, we have put the proof of this result in the appendix (p. 43).

6. Intrepretation of the Results in n-Person Weakly Ordinal Games

It is very curious that the limiting distributions of the number of N.E. for the strictly and weakly ordinal n-person games are different when the number of strategies of only one player goes to infinity, but are the same when the numbers of strategies of two or more players go to infinity. In this section, we shall try to explain this apparent oddity.

In the n-person strictly ordinal game, the payoffs of different outcomes for a particular player are all distinct. Hence, there is always only one maximal outcome for a player when the strategies of the rest of the players are fixed. Now consider an n-person weakly ordinal game when the number of strategies of Player n goes to infinity while the numbers of strategies of the other players remain finite. Given any compound strategy chosen by Players 1, 2, ..., k-1, k+1, ..., n, the probability that all of the m_k ordinal payoffs are distinct for Player k (k = 1, 2, ..., n-1) goes to 1, since Player k's finite m_k ordinal payoffs are drawn with replacement from an infinite collection of numbers: 1, 2, ..., M (= $m_1 \times m_2 \times \cdots \times m_n$). Thus, in the limit, there is only one maximal outcome for Player k when the strategies of the rest of the players are fixed, and this is similar to the strictly ordinal n-person game. However, this is true only for Player k, where $k \neq n$. For Player n, there is, with positive probability, more than one maximal outcome in the limit when the strategies of the rest of the players are fixed, since in this case, an infinite number of ordinal payoffs, m_n, is drawn from M numbers, where M is just a finite constant times m_n. This crucial structural difference destroys any hope of having the same results in both the strictly and weakly ordinal games.

Next we consider the n-person weakly ordinal game when the numbers of strategies of two players go to infinity, and the numbers of strategies of the other (n-2) players remain finite. Consider Player k's m_k ordinal payoffs given a particular compound strategy chosen by the rest of the players. Since Player k's m_k ordinal payoffs are drawn from M (= $m_1 \times m_2 \times \cdots \times m_n$) numbers with replacement, and M approaches infinity at a faster rate than m_k, the probability that Player k's m_k ordinal payoffs are all distinct and that there is only one maximal outcome goes to 1. We note that this is true for all players, and that this structure is thus similar to the structure of the n-person strictly ordinal game. Hence, it is not surprising that the limiting distributions of the number of N.E. are the same for both the strictly and weakly ordinal games. We also observe that the foregoing discussion is equally relevant when the numbers of strategies of more than two players go to infinity.

Lest anyone think that for each player the ordinal payoffs of all outcomes become distinct as the numbers of strategies of two or more players go to infinity, we would like to remark that this is not true. In fact, even though the probability goes to one that there is only one maximal outcome for Player k (k = 1, 2, ..., n), for any given compound strategy chosen by the rest of the players, the probability does not go to one that there is only one maximal outcome for Player k for *all* given compound strategies chosen by the rest of the players. This may seem paradoxical at first, but it is plausible because the other players can choose from an infinite number of compound strategies. We know from the proof of Proposition 7 that the expected number of N.E. generated from these undesignated maxima goes to zero, and so we deduce that the undesignated maxima of the various players are never matched up to form additional N.E.

7. Relaxation of Assumptions

We recall that we have made two simplifying assumptions in our models. They are in essence: (a) the complete symmetry of the ordinal payoffs across outcomes for each player, and (b) the independence of different players' ordinal payoffs. We used assumption (b) to obtain almost all of our results (except Proposition 5, which holds without any conditions), but assumption (a) can be relaxed.

In our analysis of the number of N.E. in the n-person strictly ordinal game, we took advantage only of the symmetry across outcomes for each Player k, given any compound strategy i chosen by all of the other players. Therefore, the results of Theorem 1 and 2 still hold if the ordinal payoffs for Player k, given i, are drawn randomly without replacement from a sample of m_k or more distinct integers.

In the case of the n-person weakly ordinal game, if the ordinal payoffs for Player k, given any compound strategy i chosen by the other players, are drawn randomly with replacement from a sample of $f_{ki}d^{(1+\epsilon)}$ distinct integers, where $d = \text{Max}\{m_k\}$, $f_{ki} > 0$ and $\epsilon > 0$, then the results in Proposition 7 and Theorem 4 still hold. However, the results for Proposition 6 and Theorem 3 do not remain the same, as suggested by our discussion in the last section. Actually, the expected number of N.E. is 1 and the distribution of the number of N.E. approaches the binomial distribution in Theorem 2. (The proof is analogous to that of Theorem 4.)

Some readers may be interested in allowing the number of possible rankings of the outcomes by the players to be fM, a fraction of the total number of possible outcomes in the weakly ordinal game. This extension does not alter the results of Proposition 7 and Theorem 4. However, the expected number of N.E. in Proposition 6 becomes $(1/(fm))(1/(1-e^{-1/(fm)}))$, where $m = m_1 \times m_2 \times \cdots \times m_{n-1}$, and the result in Theorem 3 changes in a more complicated way.

In all cases, a relaxation of assumption (a) is possible in that we need for each player only symmetry among the ordinal payoffs of his outcomes for a given compound strategy chosen by the other players, and the number of possible rankings of the outcomes by the players to go to infinity at the right rate. Furthermore, we speculate that in both the strictly and weakly ordinal games, if the probabilities of each of Player k's ($k = 1, 2, \ldots, n$) strategies to bring about his most preferred outcome, given any compound strategy i chosen by the other players, goes to 0 as the numbers of strategies of more than one player go to infinity, then the distribution of the number of N.E. is still Poisson with mean 1.

8. Conclusion

In this paper, we have considered the number of N.E. in n-person non-cooperative games in matrix form. Since we have chosen to study only pure strategy N.E., in which only the ordinality or the ranking of the payoffs matters, and since all cardinal games can be transformed into ordinal games, we have performed our study in terms of ordinal games. Under two very simple assumptions, we obtained a number of interesting results concerning the number of N.E. in strictly and weakly ordinal games.

Specifically, we found limiting probability distributions of the numbers of N.E. as the numbers of strategies of some players go to infinity. Our results show that pure strategy N.E. are not uncommon in classes of n-person games that satisfy our assumptions.

Appendix

1. An alternate proof for proposition 3 in section 3

Proposition 3: The probability distribution of the number of N.E. approaches the binomial $(m_1, 1/m_1)$ distribution as the number of strategies of Player 2 approaches infinity and the number of strategies of Player 1 remains finite.

Proof:

$P(X_{m_1,m_2} = x)$

$= P(X_{m_1,m_2} = x \text{ and } \Sigma_i I[B_{ij}] \le 1 \ \forall_j) + P(X_{m_1,m_2} = x \text{ and } \Sigma_i I[B_{ij}] > 1$ for at least one j)

$= P(X_{m_1,m_2} = x \mid \Sigma_i I[B_{ij}] \le 1 \ \forall_j)\, P(\Sigma_i I[B_{ij}] \le 1 \ \forall_j) + P(X_{m_1,m_2} = x \mid \Sigma_i I[B_{ij}]$ > 1 for at least one j) $P(\Sigma_i I[B_{ij}] > 1$ for at least one j)

$= \binom{m_1}{x}(1/m_1)^x(1-1/m_1)^{m_1-x} m_2(m_2-1)(m_2-2)\cdots(m_2-m_1+1)/m_2^{m_1} + P(X_{m_1,m_2}$ $= x \mid \Sigma_i I[B_{ij}] > 1$ for at least one j$)(1-m_2(m_2-1)(m_2-2)\cdots(m_2-m_1+1)/m_2^{m_1})$.

Therefore, $\lim_{m_2\to\infty} P(X_{m_1,m_2} = x)$

$= \binom{m_1}{x}(1/m_1)^x(1-1/m_1)^{m_1-x}(1) + P(X_{m_1,m_2} = x / \Sigma_i I[B_{ij}] > 1$ for at least one j$)\cdot(0)$

$= \binom{m_1}{x}(1/m_1)^x(1-1/m_1)^{m_1-x}$

Q.E.D.

2. We would like to prove that

(i) $\lim_{v\to\infty} a_i(v) = a_i$ for every k = 0, 1, 2, . . . ,

(ii) $0 \le a_i(v) \le a_i$ for any v, i

and (iii) $\sum_{i=0}^{\infty} a_i < +\infty$

imply $\lim_{v\to\infty} \sum_{i=0}^{v-1} a_i(v) = \sum_{i=0}^{\infty} a_i$.

Proof: Let $S_v = \sum_{i=0}^{v-1} a_i(v)$.

Then $S_v \le \sum_{i=0}^{v-1} a_i(v) \le \sum_{i=0}^{\infty} a_i \implies \overline{\lim_{v\to\infty}}\, S_v \le \sum_{i=0}^{\infty} a_i$. (1)

Fix N, and note that for all $v \ge N$,

$S_v = \sum_{i=0}^{v-1} a_i(v) \ge \sum_{i=0}^{N-1} a_i(v)$, so that

$\underline{\lim}_{v\to\infty}\, S_v \ge \sum_{i=0}^{N-1} a_i \implies \underline{\lim}_{v\to\infty}\, S_v \ge \sum_{i=0}^{\infty} a_i$. (2)

(1) and (2) together imply that $\lim_{v\to\infty} \sum_{i=0}^{v-1} a_i(v) = \sum_{i=0}^{\infty} a_i$.

Q.E.D.

3. We would like to prove the following proposition:

If the ordinal payoffs of all outcomes for player k are drawn randomly with replacement from 1, 2, ..., D_k, where D_k is a constant, then the probability that there are infinitely many N.E. is 1 as the numbers of strategies of one or more players go to infinity.

Proof: Let $\eta = m_{n-u} = m_{n-u+1} = m_{n-u+2} = \cdots = m_n$, and $m_n \to \infty$.

Fix the strategies of the first n-u-1 players, and let E_h be the event that in outcome h, Players n-u, n-u+1, n-u+2, ... , n obtain their most preferred ordinal payoffs, D_{n-u}, D_{n-u+1}, D_{n-u+2}, ... , D_n.

Since the E_h are independent, and since $\sum_h P(E_h) = \eta^{u+1}[1/(D_{n-u} D_{n-u+1} \cdot D_{n-u+2} \cdots D_n)] = \infty$, it follows from the Borel-Cantelli Lemma that P(E_h appears infinitely often) = 1.

Let $Z^{(n)}{}_{n-u-1}$ = the number of N.E. given that the strategies of the first n-u-1 players are fixed, and let

$$\rho = \prod_{k=1}^{n-u-1} \{[D_k{}^{m_k-1} + (D_k-1)^{m_k-1} + (D_k-2)^{m_k-1} + \ldots + 1]/D_k{}^{m_k}\}.$$ Then,

for all finite z', $P(Z^{(n)}{}_{n-u-1} \leq z')$

$= P(Z^{(n)}{}_{n-u-1} \leq z' \mid E_h$ appears infinitely often)

$$= \lim_{\omega\to\infty} \sum_{q=0}^{z'} \binom{\omega}{q} \rho^q (1-\rho)^{\omega-q}$$

$= 0$.

Hence, $P(Z^{(n)}{}_{n-u-1} > z') = 1$

$\Longrightarrow P(Z^{(n)} > z') = 1$, for all finite z' .

Q.E.D.

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