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THEORY FOR AUTOREGRESSION

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0. ABSTRACT

This paper develops an asymptotic theory for a first order autoregression with a root near unity. Deviations from the unit root theory are measured through a noncentrality parameter c . When $c < 0$ we have a local alternative that is stationary and when $c > 0$ the local alternative is explosive. As c approaches the limits of its domain of definition ($\pm\infty$) it is shown that the asymptotic distributions known to apply under fixed stationary and explosive alternatives are obtained as special cases. Moreover, when $c = 0$ we have the standard unit root theory. Thus, the asymptotic theory that we present goes a long way towards unifying earlier theory for these individual special cases. The general theory is expressed in terms of functionals of the Wiener process.

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Some Key Words: Unit root; autoregression; near-integrated process; Brownian motion; noncentrality parameter.

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1. INTRODUCTION

There has recently been a growing interest in the asymptotic theory of autoregressive time series with roots on or near the unit circle. Dickey (1976), Fuller (1976) and Dickey and Fuller (1979) developed statistical tests for detecting the presence of a unit root in an AR(1). Subsequent papers by Solo (1984), by Said and Dickey (1984) and by the author (1985a) have extended these procedures to quite general integrated time series of the ARMA(p,1,q) type. The limiting distributions of the various test statistics proposed in these papers are known and are all non normal. However, these limiting distributions may usually be represented quite simply in terms of functionals of standard Brownian motion. Moreover, numerical tabulations of the relevant distributions have been obtained by Monte Carlo methods for the asymptotic case as well as for a range of finite sample sizes. These are reported in Fuller (1976), and in Dickey and Fuller (1979, 1981).

Autoregressive time series with roots that are near unity have also been studied. Evans and Savin (1981, 1984) found in extensive simulation experiments that the statistical properties of the coefficient estimator and associated t-test in a stationary AR(1) with a root near unity are close to those that apply when the model is a random walk even when the sample size is as large as $T = 100$. Similar results were found to apply when the AR(1) is mildly explosive.

Ahtola and Tiao (1984) recently studied the sampling behavior of the score function in an AR(1) as the autoregressive coefficient (α) approaches unity from below. These authors described such an AR(1) as nearly non-stationary; and their analysis helps to explain the progressive deterioration

as a $\dagger 1$ of the conventional normal asymptotic theory for the score function in this context.

The present paper deals with a closely related subject. We shall consider a time series $\{y_t\}_1^\infty$ that is generated by the model

$$(1) \quad y_t = ay_{t-1} + u_t ; \quad t = 1, 2, \dots$$

with

$$(2) \quad a = \exp(c/T)$$

where T is the sample size. When $c < 0$ and T is finite, $0 < a < 1$ and the model is evidently stable over a finite stretch of data. When c is close to zero, a is close to unity and the model may be thought of as being nearly nonstationary. In this case (1) and (2) comprise a nearly nonstationary AR(1) of the type considered by Ahtola and Tiao (1984). However, Ahtola and Tiao require the sequence of innovations $\{u_t\}_1^\infty$ in (1) to be iid and Gaussian, so that their model is a conventional Gaussian AR(1). Much weaker conditions on $\{u_t\}_1^\infty$ will be imposed in this paper. As a result, the asymptotic theory that we develop here will apply to rather a wide class of nearly nonstationary processes.

The parameter c in (2) may be conveniently regarded as a noncentrality parameter. When c is fixed and $T \dagger \infty$ we have $a \dagger 1$, so that in the limit (1) has a unit root. However, the rate of approach to unity is controlled at $O(T^{-1})$ so that the asymptotic theory which we develop is noncentral. This reflects the presence of the near unit root in (2) and features the noncentrality parameter c . Thus, the constant c may be used to measure the effects of departures from the hypothesis of a unit root in (1) on the limiting distribution theory.

Our theory will allow for explosive ($c > 0$) as well as stationary ($c < 0$) alternatives to a unit root in (1). We shall further examine the behavior of the limiting distributions as $|c| \uparrow \infty$. The case $c \uparrow \infty$ and $c \downarrow -\infty$ then provide a means by which our own results may be related to the theory that is presently known to apply in the stable and the unstable AR(1). The latter was first developed in the research of White (1958, 1959) and of Anderson (1959).

As indicated above, the asymptotic theory that we develop in this paper will permit much more general specifications than the AR(1) with iid errors. Our approach is to require that the sequence of innovations $\{u_t\}_1^\infty$ in (1) satisfy some rather general moment and weak dependence conditions. Under these conditions $\{u_t\}$ may be generated by a wide variety of models, including all Gaussian and many other finite order ARMA models.

Finally, to complete the specification of (1) we shall allow either of the commonly used initial conditions: (i) $y_0 = c$, a constant (with probability one); or (ii) $y_0 = \text{random}$ with a certain specified distribution. Proofs are given in the Mathematical Appendix.

2. SOME PRELIMINARY THEORY

We define the partial sum process $S_t = \sum_{j=1}^t u_j$ and set $S_0 = 0$. We shall make the following assumption about the innovation sequence $\{u_t\}_1^\infty$.

ASSUMPTION 2.1

- (a) $E(u_t) = 0$ all t ;
- (b) $\sup_t E|u_t|^{\beta+\epsilon} < \infty$ for some $\beta > 2$ and $\epsilon > 0$;
- (c) $\sigma^2 = \lim_{T \rightarrow \infty} E(T^{-1}S_T^2)$ exists and $\sigma^2 > 0$;

(d) $\{u_t\}_1^\infty$ is strong mixing with mixing coefficients α_m that satisfy

$$(3) \quad \sum_1^\infty \alpha_m^{1-2/\beta} < \infty .$$

Condition (d) imposes a form of asymptotic weak dependence on the sequence of innovations $\{u_t\}_1^\infty$. The reader is referred, for example, to Hall and Heyde (1980) for the definition of strong mixing and the mixing coefficients. The summability requirement (3) on the mixing coefficients is satisfied when $\alpha_m = O(m^{-\lambda})$ for some $\lambda > \beta/(\beta-2)$. Condition (b) controls the allowable heterogeneity in the sequence $\{u_t\}_1^\infty$ in relation to the mixing decay rate prescribed by (3). Thus, as β declines towards 2 the moment condition (b) weakens and the probability of outliers rises. On the other hand, the mixing decay rate (measured by $\beta/(\beta-2)$) increases as β approaches 2 and the effect of outliers is required by condition (3) to wear off more quickly.

Condition (c) is a convergence condition on the average variance of the partial sum S_T . It is a common requirement in much central limit theory although it is not strictly a necessary condition. However, if $\{u_t\}$ is weakly stationary then

$$(4) \quad \sigma^2 = E(u_1^2) + 2\sum_{k=1}^\infty E(u_1 u_k)$$

and the convergence of the series is implied by the mixing condition (3) (theorem 18.5.3 of Ibragimov and Linnik (1971)). Even in this case, however, it is still conventional to require $\sigma^2 > 0$ to exclude degenerate results.

Assumption 2.1 is quite weak. It allows the innovation sequence $\{u_t\}_1^\infty$ to be heterogeneously distributed and weakly dependent over time.

This includes a wide variety of possible data generating mechanisms such as all Gaussian and many other finite order ARMA models under very general conditions on the underlying errors (see Withers (1981)). The requirement that $\{u_t\}_1^\infty$ be strong mixing does, however, exclude some linear processes (see Andrews (1984)). It could be weakened further by working with functions of mixing processes as in Billingsley (1968, Sec. 21). However, we shall not make this further extension since it would divert us from the main purpose of the present paper.

From the sequence of partial sums $\{S_t\}_1^T$ we construct the random element

$$(5a) \quad X_T(r) = T^{-1/2} \sigma^{-1} S_{[Tr]} = T^{-1/2} \sigma^{-1} S_{j-1} ; \quad (j-1)/T \leq r < j/T \quad (j = 1, \dots, T)$$

$$(5b) \quad X_T(1) = T^{-1/2} \sigma^{-1} S_T$$

where $[b]$ denotes the integral part of b . $X_T(r)$ lies in $D = D[0,1]$, the space of real valued functions on the interval $[0,1]$ that are right continuous and have finite left limits. It will be sufficient for our purpose if we endow D with the uniform metric defined by

$$\|f-g\| = \sup_r |f(r) - g(r)| \quad \text{for any } f, g \in D .$$

Under very general conditions the random element $X_T(r)$ obeys a central limit theory on the function space D . We shall make use of the following result, which is due to Herrndorf (1984):

LEMMA 2.2. *If $\{u_t\}_1^\infty$ satisfies Assumption 2.1 then as $T \rightarrow \infty$ $X_T(r) \Rightarrow W(r)$, a Wiener process on $C[0,1]$.*

The notation " \Rightarrow " that is used in the statement of Lemma 2.2 and elsewhere in the paper signifies weak convergence of the associated probability measures. In this case, the probability measure of $X_T(r)$ converges weakly to the probability measure (here Wiener measure) of the random function $W(r)$. The result is a functional central limit theorem or invariance principle (see Billingsley (1968) or Pollard (1984) for further discussion). The limit process $W(r)$ is popularly known as standard Brownian motion. The sample paths of $W(r)$ lie almost surely (Wiener measure) in $C = C[0,1]$, the space of real valued continuous functions on $[0,1]$.

DEFINITION 2.3. A time series $\{y_t\}_1^\infty$ that is generated by (1) and (2) with $c \neq 0$ and where $\{u_t\}_1^\infty$ satisfies Assumption 2.1 is called a near-integrated process. When $c = 0$ in (2) $\{y_t\}_1^\infty$ will be called an integrated process.

The terminology we employ here for an integrated process corresponds to usage popularized by Box and Jenkins (1970) when $\{u_t\}_1^\infty$ is generated by a stationary ARMA model. The above definition actually extends the terminology to include time series whose first differences are not necessarily stationary processes and may be generated, for example, by finite order ARMA models whose innovations are non identically distributed. When $c \neq 0$, the specification (2) allows us to introduce the closely related concept of a near-integrated process. The latter includes alternatives which are strongly autoregressive ($c < 0$) or mildly explosive ($c > 0$) in finite samples of data.

The following result is very useful in the development of our asymptotic theory.

LEMMA 2.4. If $W(r)$ is a standard Wiener process and

$$J_c(r) = \int_0^r e^{(r-s)c} dW(s) \quad \text{then:}$$

$$(6) \quad J_c(1)^2 = 1 + 2c \int_0^1 J_c(r)^2 dr + 2 \int_0^1 J_c(r) dW(r) ;$$

$$(7) \quad J_c(r) = W(r) + c \int_0^r e^{(r-s)c} W(s) ds .$$

In this Lemma $J_c(r)$ is a linear functional of the Gaussian process $W(s)$, $0 \leq s \leq r$, and is therefore Gaussian also. In fact, by a simple calculation we obtain

$$(8) \quad J_c(r) = \int_0^r e^{(r-s)c} dW(s) \equiv N(0, (e^{2rc} - 1)/2c)$$

where we use the notation " \equiv " to signify equality in distribution. We remark that the integral $\int_0^1 J_c(r) dW(r)$ that appears in (6) is a stochastic integral. Moreover, when $c = 0$ we have $J_c(r) = W(r)$ and (6) reduces to the commonly occurring formula:

$$\int_0^1 W(r) dW(r) = (1/2)(W(1)^2 - 1) .$$

3. ASYMPTOTICS FOR NEAR-INTEGRATED PROCESSES

Our first step is to find the relevant asymptotic theory for the sample moments of data generated by (1) and (2). As in the case of integrated processes, the limiting distribution theory is most conveniently expressed in terms of functionals of the Wiener process $W(r)$ on C . The next Theorem has all the results we shall need for the development of our regression theory.

THEOREM 3.1. If $\{y_t\}_1^\infty$ is a near-integrated process generated by (1) and (2) then as $T \uparrow \infty$:

$$(a) \quad T^{-1/2} y_T \Rightarrow \sigma J_c(1) ;$$

$$(b) \quad T^{-3/2} \Sigma_1^T y_t \Rightarrow \sigma \int_0^1 J_c(r) dr ;$$

$$(c) \quad T^{-2} \Sigma_1^T y_t^2 \Rightarrow \sigma^2 \int_0^1 J_c(r)^2 dr ;$$

$$(d) \quad T^{-1} \Sigma_1^T y_{t-1} u_t \Rightarrow \sigma^2 \int_0^1 J_c(r) dW(r) + (1/2)(\sigma^2 - \sigma_u^2) ;$$

where $J_c(r) = \int_0^r e^{(r-s)c} dW(s)$, $W(r)$ is a standard Wiener process and $\sigma_u^2 = \lim_{T \rightarrow \infty} T^{-1} \Sigma_1^T E(u_t^2)$.

This Theorem gives an asymptotic distribution theory for the sample moments of a near-integrated process. The results may be used to approximate the distributions of the sample moments of nearly nonstationary time series. Thus, since $J_c(r)$ is Gaussian it is easy to show by elementary calculations that

$$\int_0^1 J_c(r) dr \equiv N(0, v)$$

where

$$v = (1/c^2) + (1/2c^3)(e^{2c} - 4e^c + 3) .$$

Parts (a) and (b) of Theorem 3.1 therefore become:

$$(9) \quad T^{-1/2} y_T \Rightarrow \sigma J_c(1) \equiv N(0, (\sigma^2/2c)(e^{2c} - 1))$$

and

$$(10) \quad T^{-3/2} \Sigma_1^T y_t \Rightarrow \sigma \int_0^1 J_c(r) dr \equiv N(0, \sigma^2 v) .$$

When $c = 0$ (9) is $\sigma J_0(1) \equiv N(0, \sigma^2)$, which is the limiting distribution of the standardized sum $T^{-1/2} \sum_1^T u_t$ of the innovations in (1). The variance of this limiting distribution is $\sigma^2 = \lim_{T \rightarrow \infty} T^{-1} E(S_T^2)$. If $\{u_t\}_1^\infty$ is stationary we have $\sigma^2 = E(u_1^2) + 2 \sum_{k=1}^\infty E(u_1 u_{1+k})$ from (4) and we may write $\sigma^2 = 2\pi f_u(0)$ where $f_u(\lambda)$ is the spectral density of $\{u_t\}_1^\infty$. In this special case of (9) $\sigma J_0(1) \equiv N(0, 2\pi f_u(0))$, which is a general central limit theorem for stationary time series (e.g. Hannan (1970) theorem 11, p. 221).

When $c = 0$ in (10) a simple calculation gives

$$\sigma \int_0^1 J_0(r) dr \equiv N(0, \sigma^2/3)$$

which is the limiting distribution of the standardized sample mean of an integrated process.

Perhaps the most useful application of Theorem 3.1 is to the theory of regression for near-integrated time series. Suppose (1) is estimated by least squares giving the regression coefficient

$$\hat{a} = \sum_1^T y_t y_{t-1} / \sum_1^T y_{t-1}^2$$

and associated t-statistic

$$t_a = (\sum_1^T y_{t-1}^2)^{1/2} (\hat{a} - a) / s$$

where $s^2 = T^{-1} \sum_1^T (y_t - \hat{a} y_{t-1})^2$. The asymptotic theory of these regression statistics is given in:

THEOREM 3.2. If $\{y_t\}_1^\infty$ is a near-integrated process generated by (1) and (2) then as $T \uparrow \infty$:

$$(a) \quad T(\hat{a}-a) \Rightarrow \left\{ \int_0^1 J_c(r) dW(r) + (1/2)(1 - \sigma_u^2/\sigma^2) \right\} / \left\{ \int_0^1 J_c(r)^2 dr \right\}^{1/2}$$

$$(b) \quad \hat{a} \xrightarrow{p} 1, \quad s^2 \xrightarrow{p} \sigma_u^2$$

$$(c) \quad t_a \Rightarrow \left\{ \int_0^1 J_c(r) dW(r) + (1/2)(1 - \sigma_u^2/\sigma^2) \right\} / \left\{ \int_0^1 J_c(r)^2 dr \right\}^{1/2}$$

When $c = 0$ and $\{u_t\}_1^\infty$ is iid(0, σ^2) parts (a) and (c) of this Theorem reduce to the known asymptotic theory for a first order autoregression with a unit root (White (1958), Fuller (1976), Dickey and Fuller (1979)), viz

$$T(\hat{a}-1) \Rightarrow \left\{ \int_0^1 W(r)^2 dr \right\}^{-1} \left\{ \int_0^1 W(r) dW(r) \right\};$$

$$t_a \Rightarrow \left\{ \int_0^1 W(r)^2 dr \right\}^{-1/2} \left\{ \int_0^1 W(r) dW(r) \right\}.$$

In this case, we have $\sigma^2 = \sigma_u^2$ and $J_c(r) = W(r)$ in the formulae of the Theorem.

When $c \neq 0$ Theorem 3.2 delivers the noncentral asymptotic theory for the regression statistics \hat{a} and t_a . We note in particular that (a) and (c) imply that

$$T(\hat{a}-1) \Rightarrow c + \left\{ \int_0^1 J_c(r)^2 dr \right\}^{-1} \left\{ \int_0^1 J_c(r) dW(r) + (1/2)(1 - \sigma_u^2/\sigma^2) \right\}$$

and

$$t_1 = (\Sigma_1^T y_{t-1})^{1/2} T(\hat{a}-1)/s \Rightarrow c \left\{ \int_0^1 J_c(r)^2 dr \right\}^{1/2} + \left\{ \int_0^1 J_c(r)^2 dr \right\}^{-1/2} \left\{ \int_0^1 J_c(r) dW(r) + (1/2)(1 - \sigma_u^2/\sigma^2) \right\}.$$

These distributions give the asymptotic power functions for tests of the hypothesis of a unit root based on \hat{a} and t_1 under the sequence of local alternatives $a = e^{c/T} \sim 1 + c/T$.

4. THE STATIONARY AR(1) AS A LIMIT CASE AS $c \downarrow -\infty$

It is interesting to study the limiting behavior of the above results as the noncentrality parameter c approaches the limits of its domain of definition. Here, we shall consider the limit as $c \downarrow -\infty$. Note that if we allow $c \downarrow -\infty$ in (2) then a time series generated by the model (1) is no longer a near-integrated process. In fact, if $\{u_t\}_1^\infty$ is weakly stationary so is $\{y_t\}_1^\infty$ in this case. Thus, stationary alternatives may be regarded as a natural limit of the model when $c \downarrow -\infty$. We now show that the conventional asymptotic theory for stationary time series follows from the results of the previous Section under the limit $c \downarrow -\infty$. We first give the theory for sample moments.

THEOREM 4.1. As $c \downarrow -\infty$:

- (a) $(-2c)^{1/2} J_c(1) \Rightarrow N(0,1)$;
- (b) $(-c) \int_0^1 J_c(r) dr \Rightarrow N(0,1)$;
- (c) $(-2c) \int_0^1 J_c^2(r) dr \xrightarrow{p} 1$;
- (d) $(-2c)^{1/2} \int_0^1 J_c(r) dW(r) \Rightarrow N(0,1)$;
- (e) $(-2c)^{1/2} \{(-2c) \int_0^1 J_c(r)^2 dr - 1\} \Rightarrow N(0,4)$.

These results may be shown to yield the usual asymptotic theory for the sample moments of a stationary time series. Thus, part (b) of Theorem 4.1 and part (b) of Theorem 3.1 imply the asymptotic approximation

$$(11) \quad (-c)T^{-3/2}\Sigma_1^T y_t \sim N(0, \sigma^2) .$$

But $a = e^{c/T}$ so that

$$(12) \quad -2c = (-2 \ln a)T \sim (1 - a^2)T \sim 2T(1-a)$$

for a close to unity. Combining (11) and (12) we deduce the asymptotic approximation

$$(13) \quad T^{-1/2}\Sigma_1^T y_t \sim N(0, \sigma^2/(1-a)^2) \equiv N(0, 2\pi f_y(0))$$

where $f_y(\lambda) = |1 - ae^{i\lambda}|^{-2} f_u(\lambda)$ is the spectral density of $\{y_t\}_1^\infty$ and $f_u(\lambda)$ is the spectral density of $\{u_t\}_1^\infty$. (13) is the usual asymptotic normal approximation delivered by conventional central limit theory for stationary processes. (See, for example, Hall and Heyde (1980), corollary 5.2, p. 135.)

In a similar way, from part (c) of Theorems 3.1 and 4.1 we deduce that:

$$(14) \quad (-2c)T^{-2}\Sigma_1^T y_t^2 \sim (1 - a^2)T^{-1}\Sigma_1^T y_t^2 \xrightarrow{p} \sigma^2 .$$

When (1) is a stationary AR(1) with iid(0, σ^2) innovations u_t , (14) yields the correct asymptotic formula

$$T^{-1}\Sigma_1^T y_t^2 \xrightarrow{p} \sigma^2/(1 - a^2) = E(y_t^2)$$

of conventional theory. Finally, in this case of iid innovations, we observe that $\sigma^2 = \sigma_u^2$ and from part (d) of Theorems 3.1 and 4.1 we obtain

$$(-2c)^{1/2}T^{-1}\Sigma_1^T y_{t-1} u_t \sim N(0, \sigma^4) .$$

In view of (12), this leads directly to the asymptotic approximation:

$$T^{-1/2} \sum_{t=1}^T y_{t-1} u_t \sim N(0, \sigma^4 / (1 - a^2)) .$$

Once again this approximation is the same as that delivered by conventional central limit theory when y_t is a stationary AR(1).

COROLLARY 4.2. As $c \rightarrow -\infty$:

$$(a) \quad (-2c)^{-1/2} \left\{ \int_0^1 J_c(r) dW(r) \right\} \left\{ \int_0^1 J_c(r)^2 dr \right\}^{-1} \Rightarrow N(0,1) ;$$

$$(b) \quad \left\{ \int_0^1 J_c(r) dW(r) \right\} \left\{ \int_0^1 J_c(r)^2 dr \right\}^{-1/2} \Rightarrow N(0,1) .$$

We deduce from this Corollary the usual asymptotic theory for a first order stationary autoregression. Thus, when y_t is generated by (1) with iid(0, σ^2) innovations u_t and $|a| < 1$, Theorem 3.2 and Corollary 4.2 imply the following asymptotic approximations:

$$(-2c)^{-1/2} T(\hat{a} - a) \sim N(0,1) ; \quad t_a \sim N(0,1) ;$$

that is:

$$(15) \quad T^{1/2}(\hat{a} - a) / (1 - a^2)^{1/2} \sim N(0,1) ; \quad t_a \sim N(0,1) .$$

Both results are well known from the traditional asymptotic theory for stationary autoregressions.

We observe that the asymptotic theory for the stationary AR(1) is obtained precisely when the innovation sequence $\{u_t\}_1^\infty$ of (1) is iid(0, σ^2). In this case, of course, $\sigma_u^2 = \sigma^2$. When $\{u_t\}_1^\infty$ is weakly dependent, as under Assumption 2.1, but is not iid then $\sigma_u^2 \neq \sigma^2$. Moreover, \hat{a} is no longer a consistent estimator of a ($|a| < 1$) and the conventional limiting

distribution theory, as given by (15), no longer applies. By contrast, when $\{y_t\}_1^\infty$ is an integrated or near-integrated process \hat{a} is consistent (for unity) and the general limiting distribution theory of Theorem 3.2 continues to apply. Thus, when we come to specialize the latter theory to the stationary case, as we have done above, we need to employ the additional requirement that u_t is $iid(0, \sigma^2)$. This requirement, which is usual in the traditional asymptotic theory for the stationary AR(1), ensures that $\sigma^2 = \sigma_u^2$ so that Theorem 3.2 actually gives us:

$$T(\hat{a}-a) \Rightarrow \left\{ \int_0^1 J_c(r)^2 dr \right\}^{-1} \left\{ \int_0^1 J_c(r) dW(r) \right\}$$

and

$$t_a \Rightarrow \left\{ \int_0^1 J_c(r)^2 dr \right\}^{-1/2} \left\{ \int_0^1 J_c(r) dW(r) \right\}$$

in this case.

5. THE EXPLOSIVE AR(1) AS A LIMIT CASE AS $c \uparrow \infty$

When $c > 0$, $a = e^{c/T}$ gives local alternatives to unity in the direction of explosive roots of (1). The limit $c \uparrow \infty$ may therefore be regarded as a natural boundary for the noncentrality parameter corresponding to an explosive root $a > 1$. We shall examine the behavior of the general asymptotic theory of Section 3 at this boundary. We start with the theory for sample moments.

THEOREM 5.1. As $c \uparrow \infty$

- (a) $(2c)^{1/2} e^{-c} J_c(1) \Rightarrow N(0,1)$;
- (b) $(2c^3)^{1/2} e^{-c} \int_0^1 J_c(r) dr \Rightarrow N(0,1)$;
- (c) $(2c)^2 e^{-2c} \int_0^1 J_c(r)^2 dr \Rightarrow \eta^2$;
- (d) $(2c) e^{-c} \int_0^1 J_c(r) dW(r) \Rightarrow \xi \eta$.

where ξ and η are independent $N(0,1)$ variates.

These results yield an asymptotic theory for the sample moments of an explosive time series (1) with fixed $a > 1$. With some qualifications, which we shall discuss, the theory corresponds with earlier analyses by Anderson (1959) and White (1958, 1959). Thus, in the case of part (a) for instance, we deduce from Theorems 5.1 and 3.1 that:

$$(16) \quad (2c)^{1/2} e^{-c} T^{-1/2} y_T \sim N(0, \sigma^2) .$$

But $a = e^{c/T}$ so that

$$(17) \quad 2c = (2 \ln a)T = (\ln a^2)T \sim (a^2 - 1)T$$

and $e^c = a^T$. Thus, (16) implies that

$$(18) \quad a^{-T} y_T \sim N(0, \sigma^2 / (a^2 - 1)) .$$

From (1) we note that:

$$(19) \quad y_T = u_T + a u_{T-1} + \dots + a^{T-1} u_1 + a^T y_0 .$$

Where $y_0 = 0$ and $\{u_t\}$ is iid(0, σ^2) we have $\text{var}(y_T) = \sigma^2 (a^{2T} - 1) / (a^2 - 1)$,

so that $\text{var}(a^{-T}y_T) \rightarrow \sigma^2/(a^2 - 1)$ as $T \uparrow \infty$ ($a > 1$). If, in addition, the u_t are normally distributed then $a^{-T}y_T \Rightarrow N(0, \sigma^2/(a^2 - 1))$ corresponding to (18) above. This result was given in Anderson (1959, Theorem 2.6). However, as emphasized by Anderson (1959, p. 680), normality of the u_t 's is necessary for $a^{-T}y_T$ to have a limiting normal distribution. This is because component variates in the sum such as u_1 and y_0 have a non negligible influence on the asymptotic distribution of $a^{-T}y_T$, as is clear from (19).

These requirements ($y_0 = 0$ and $\{u_t\}_1^\infty$ iid $N(0, \sigma^2)$) are not brought out in our own approach to (18). This is explained by the fact that the results of Theorem 5.1 are deduced as a limiting case of a theory in which the sample moments are already asymptotically distributed as functionals of a Gaussian process (see Theorem 3.1). As we have seen, these functionals are invariant to the distributional and temporal dependence properties of the innovation sequence $\{u_t\}_1^\infty$ within the general requirement Assumption 2.1; and they hold irrespective of the initial conditions on y_0 . These invariance properties persist for all finite c and hence hold for all near integrated processes. However, the invariance properties do not continue to hold at the boundary $c = \infty$. In fact, as the above example makes clear, the limiting case $c \uparrow \infty$ yields the correct asymptotic theory for an explosive time series with fixed $a > 1$ when $y_0 = 0$ and when $\{u_t\}_1^\infty$ is iid $N(0, \sigma^2)$.

A major application of the limiting case $c \uparrow \infty$ is to the asymptotic theory for the regression statistics \hat{a} and t_a in the explosive case.

We have:

THEOREM 5.2. As $c \uparrow \infty$:

$$(a) \quad (2c)^{-1} e^c \left\{ \int_0^1 J_c(r)^2 dr \right\}^{-1} \left\{ \int_0^1 J_c(r) dW(r) + (1/2)(1 - \sigma_u^2/\sigma^2) \right\} \Rightarrow \text{Cauchy} ;$$

$$(b) \quad \left\{ \int_0^1 J_c(r)^2 dr \right\}^{-1/2} \left\{ \int_0^1 J_c(r) dW(r) + (1/2)(1 - \sigma_u^2/\sigma^2) \right\} \Rightarrow N(0,1) .$$

Since $(2c)^{-1} e^c \sim a^T / \{T(a^2 - 1)\}$ we deduce immediately from part (a) of Theorems 3.2 and 5.2 that

$$(20) \quad \{a^T / (a^2 - 1)\} (\hat{a} - a) \sim \text{Cauchy}$$

for large T and $a > 1$. This corresponds to the asymptotic theory developed by White (1958) directly for the explosive case. Moreover, from part (b) of Theorems 3.2 and 5.2 we obtain the asymptotic approximation:

$$(21) \quad t_a \sim N(0,1)$$

which also corresponds to the asymptotic theory derived in White (1959) and Anderson (1959).

As discussed above the limiting case $c \uparrow \infty$ does not share the invariance properties of the general theory for near integrated processes with c finite. In fact, as shown by White (1958, 1959), result (20) applies in explosive models with fixed $a > 1$ when $y_0 = 0$ and $\{u_t\}_1^\infty$ is iid $N(0, \sigma^2)$; and (21) applies when $\{u_t\}_1^\infty$ is iid $N(0, \sigma^2)$ independent of the initial condition y_0 . These results were also demonstrated by Anderson (1959).

Theorems 3.2 and 5.2 do suggest one generalization of the asymptotic theory of White and Anderson for explosive models. We note that the effect of temporal weak dependence in the innovation sequence $\{u_t\}_1^\infty$ affects the limiting distributions given in Theorem 3.2 through the presence of the

constant $(1/2)(1 - \sigma_u^2/\sigma^2)$ in the numerator of the limiting random variates. However, in the limiting case $c \uparrow \infty$ this term vanishes and the effects of temporal dependence in $\{u_t\}$ disappear. This suggests that (20) and (21) remain valid for quite general weakly dependent Gaussian innovations, such as those generated by stationary ARMA models.

6. FINAL REMARKS

The theory for near-integrated time series presented in Section 3 seems likely to be most useful in the development of a noncentral asymptotic distribution theory for the analysis of the power properties of statistical tests in the vicinity of a unit root. The conventional approach here would suggest local alternatives of the form $a = 1 + c/T$. This was, in fact, considered by the author (1985b) but it was found that alternatives of the form (2) lead, in general, to much simpler derivations. The final results are, of course, the same with either approach.

MATHEMATICAL APPENDIX

Proof of Lemma 2.2. See Herrndorf (1984).

Proof of Lemma 2.4. We define the process $\xi(r)$ by the differential $d\xi(r) = e^{-rc}dW(r)$. Then, from the Ito formula for stochastic differentiation applied to the function $\xi(r)^2$ we obtain:

$$d(\xi(r)^2) = e^{-2rc}dr + 2e^{-rc}\xi(r)dW(r) .$$

Multiplying by e^{2rc} and integrating over $[0,1]$ we deduce (6) immediately. (7) follows from integration by parts, which applies here because $e^{(r-s)c}$ is continuous.

Proof of Theorem 3.1. From (1) and (2) we have

$$y_t = \sum_{j=1}^t e^{(t-j)c/T} u_j + e^{tc/T} y_0$$

and thus

$$\begin{aligned} T^{-1/2} y_T &= \sigma \sum_{j=1}^T e^{(1-j/T)c} \int_{(j-1)/T}^{j/T} dX_T(s) + O_p(T^{-1/2}) \\ &= \sigma \sum_{j=1}^T \int_{(j-1)/T}^{j/T} e^{(1-s)c} dX_T(s) + O_p(T^{-1/2}) \\ (A1) \quad &= \sigma \int_0^1 e^{(1-s)c} dX_T(s) + O_p(T^{-1/2}) . \end{aligned}$$

We use integration by parts on the first term of (A1), which is valid since $e^{(1-s)c}$ is continuous and $X_T(s)$ is increasing and of bounded variation.

(A1) becomes

$$\sigma \left\{ \chi_T(1) + c \int_0^1 e^{(1-s)c} \chi_T(s) ds \right\} + O_p(T^{-1/2})$$

$$\Rightarrow \sigma W(1) + c \int_0^1 e^{(1-s)c} W(s) ds = \sigma J_c(1) ; \text{ as } T \uparrow \infty$$

by Lemma 2.2 and the continuous mapping theorem (e.g., Billingsley (1968), p. 30). This proves part (a). To prove (b) we note that

$$\begin{aligned} T^{-3/2} \sum_{i=1}^T y_t &= T^{-3/2} \sum_{i=1}^T \sum_{j=1}^T e^{(i-j)c/T} u_j + T^{-3/2} \sum_{i=1}^T e^{ic/T} y_0 \\ &= \sigma T^{-1} \sum_{i=1}^T \sum_{j=1}^T e^{(i-j)c/T} \int_{(j-1)/T}^{j/T} d\chi_T(s) + O_p(T^{-1/2}) \\ &= \sigma \sum_{i=1}^T \sum_{j=1}^i \int_{(i-1)/T}^{i/T} dr \int_{(j-1)/T}^{j/T} e^{(i-j)c/T} d\chi_T(s) + O_p(T^{-1/2}) \\ &= \sigma \sum_{i=1}^T \sum_{j=1}^i \int_{(i-1)/T}^{i/T} dr \int_{(j-1)/T}^{j/T} e^{(r-s)c} d\chi_T(s) + O_p(T^{-1/2}) \\ &= \sigma \int_0^1 dr \int_0^r e^{(r-s)c} d\chi_T(s) + O_p(T^{-1/2}) \\ &= \sigma \int_0^1 \left\{ \chi_T(r) + c \int_0^r e^{(r-s)c} \chi_T(s) ds \right\} dr + O_p(T^{-1/2}) \\ &\Rightarrow \sigma \int_0^1 J_c(r) dr , \end{aligned}$$

again by Lemma 2.2 and direct application of the continuous mapping theorem. This proves part (b). The proof of part (c) is entirely analogous and is, therefore, omitted.

To prove part (d) we note by squaring (1) that

$$y_t^2 = e^{2c/T} y_{t-1}^2 + 2e^{c/T} y_{t-1} u_t + u_t^2 .$$

In view of part (a), $T^{-1/2}y_t = O_p(1)$ for all $t \leq T$. Thus,

$$y_t^2 - y_{t-1}^2 = (2c/T)y_{t-1}^2 + u_t^2 + 2y_{t-1}u_t + O_p(T^{-1/2})$$

and summing over $t = 1, \dots, T$ we obtain

$$T^{-1}y_T^2 = 2cT^{-2}\sum_1^T y_{t-1}^2 + T^{-1}\sum_1^T u_t^2 + 2T^{-1}\sum_1^T y_{t-1}u_t + O_p(T^{-1/2}).$$

We note that $T^{-1}\sum_1^T u_t^2 \xrightarrow{\text{a.s.}} \sigma_u^2$ by the strong law of large numbers for weakly dependent sequences (see, in particular, McLeish (1975), Theorem 2.10).

From parts (a) and (c) and the continuous mapping theorem we now deduce that as $T \uparrow \infty$:

$$\begin{aligned} 2T^{-1}\sum_1^T y_{t-1}u_t &\Rightarrow \sigma^2 J_c(1)^2 - 2c\sigma^2 \int_0^1 J_c(r)^2 dr - \sigma_u^2 \\ &= 2\int_0^1 J_c(r) dW(r) + \sigma^2 - \sigma_u^2 \end{aligned}$$

in view of (6). Part (d) of the Theorem follows immediately.

Proof of Theorem 3.2. To prove part (a) we note that

$$\begin{aligned} T(\hat{a}-a) &= (T^{-2}\sum_1^T y_{t-1}^2)^{-1} (T^{-1}\sum_1^T y_{t-1}u_t) \\ &\Rightarrow \left\{ \int_0^1 J_c(r)^2 dr \right\}^{-1} \left\{ \int_0^1 J_c(r) dW(r) + (1/2)(1 - \sigma_u^2/\sigma^2) \right\} \end{aligned}$$

by direct application of the continuous mapping theorem and Theorem 3.1. Moreover, this implies that $\hat{a} = a + O_p(T^{-1}) = 1 + O_p(T^{-1})$ so that part (b) also follows. Part (c) is an immediate consequence of Theorem 3.1, the continuous mapping theorem and part (b).

Proof of Theorem 4.1. Note that $J_c(r)$ is Gaussian with $E\{Z_c(r)\} = 0$ and $\text{var}\{J_c(r)\} = (1 - e^{2rc})/(-2c)$ for all c . It follows that for fixed $r > 0$ $(-2c)^{1/2}J_c(r) \Rightarrow N(0,1)$ as $c \rightarrow -\infty$. Part (a) now follows immediately. In a similar way, part (b) follows from (10) and the fact that $(-c)^2 v \rightarrow 1$ as $c \rightarrow -\infty$.

To prove parts (c) and (d) we shall employ a different approach. We shall first find the limiting distribution of $(T^{-1}\Sigma_1^T y_{t-1} u_t, T^{-2}\Sigma_1^T y_{t-1}^2)$ when the innovation sequence $\{u_t\}_1^\infty$ is iid $N(0,1)$. In this case, of course, $\sigma^2 = \sigma_u^2 = 1$ and by Theorem 3.1 the limiting distribution is that of the functional $(\int_0^1 J_c(r) dW(r), \int_0^1 J_c(r)^2 dr)$. However, by the invariance principle this distribution is not dependent on the normality assumption made about the innovation sequence. The assumption is simply a device which facilitates the extraction of the mathematical form of the distribution. Moreover, relaxation of the independence assumption about the u_t leads, under Assumption 2.1, only to the additional presence of the constants σ^2 and σ_u^2 in the limiting distributions (see parts (c) and (d) of Theorem 3.1 in particular). Thus, we may extrapolate easily from the limiting distribution obtained under iid $N(0,1)$ innovations to the general case of Theorem 3.1.

We note that, when u_t is iid $N(0,1)$, $(T^{-1}\Sigma_1^T y_{t-1} u_t, T^{-2}\Sigma_1^T y_{t-1}^2)$ is a pair of quadratic forms in normal variates. The joint moment generating function (mgf) of these forms may be obtained in precisely the same way as in White (1958), allowing for the representation $a = e^{c/T}$. The limit of this function as $T \rightarrow \infty$ is then the mgf of $(\int_0^1 J_c(r) dW(r), \int_0^1 J_c(r)^2 dr)$. Simple calculations along lines identical to those of White (1958) yield the following limiting mgf as $T \rightarrow \infty$:

$$(A2) \quad M_c(w, z) = \left\{ (1/2)(c^2 + 2cw - 2z)^{-1/2} e^{c+w} \left[\{(c^2 + 2cw - 2z)^{1/2} - (c+w)\} e^{(c^2 + 2cw - 2z)^{1/2}} + \{(c^2 + 2cw - 2z)^{1/2} - (c+w)\} e^{-(c^2 + 2cw - 2z)^{1/2}} \right] \right\}^{-1/2}.$$

This expression holds for all c and will be used later in our derivations for explosive ($c \uparrow \infty$) alternatives. For our present purpose (with $c < 0$) we note that the joint mgf of $((-2c)^{1/2} \int_0^1 J_c(r) dW(r), (-2c) \int_0^1 J_c(r)^2 dr)$ is

$$(A3) \quad L_c(p, q) = M_c((-2c)^{1/2} p, (-2c)q).$$

We observe that for large negative c we have the binomial expansion:

$$(A4) \quad \left\{ c^2 - 2^{3/2}(-c)^{3/2}p + 4cq \right\}^{1/2} = (-c) - 2^{1/2}(-c)^{1/2}p - p^2 - 2q + o(|c|^{-1/2}).$$

Using (A4) in (A3) we deduce that as $c \rightarrow -\infty$:

$$L_c(p, q) \rightarrow e^{p^2/2+q} = \text{mgf}(N(0,1), 1).$$

It follows that

$$(-2c)^{1/2} \int_0^1 J_c(r) dW(r) \Rightarrow N(0,1)$$

$$(-2c) \int_0^1 J_c(r)^2 dr \xrightarrow{p} 1$$

as $c \rightarrow -\infty$, proving parts (c) and (d).

We also note that from the Ito formula (6) we have

$$\begin{aligned} (-2c) \int_0^1 J_c(r)^2 dr &= 1 + 2 \int_0^1 J_c(r) dW(r) - J_c(1)^2 \\ &= 1 + o_p(|c|^{-1/2}) + o_p(|c|^{-1}) \end{aligned}$$

in view of parts (a) and (d). We may deduce from this expression that

$$\begin{aligned} (-2c)^{1/2} \left\{ (-2c) \int_0^1 J_c(r)^2 dr - 1 \right\} &= 2(-2c)^{1/2} \int_0^1 J_c(r) dW(r) + O_p(|c|^{-1/2}) \\ &\Rightarrow N(0,4) \end{aligned}$$

as $c \rightarrow \infty$, proving part (e).

Proof of Corollary 4.2. This follows immediately from the results of Theorem 4.1 by application of the continuous mapping theorem.

Proof of Theorem 5.1. $J_c(1) \equiv N(0, e^{2c} - 1)/2c$. It follows that $(2c)^{1/2} e^{-c} J_c(1) \Rightarrow N(0,1)$ as $c \rightarrow \infty$, proving part (a). Similarly, $\int_0^1 J_c(r) dr \equiv N(0,v)$ and, from (10), $(2c^3) e^{-2c} v \rightarrow 1$ as $c \rightarrow \infty$ so that $(2c^3)^{1/2} e^{-c} \int_0^1 J_c(r) dr \Rightarrow N(0,1)$ as required for part (b).

We shall prove parts (c) and (d) together, using the invariance principle method explained in the proof of Theorem 4.1 above. In particular, we know that the joint mgf of $(\int_0^1 J_c(r) dW(r), \int_0^1 J_c(r)^2 dr)$ is given by (A2). It follows that the joint mgf of $(2ce^{-c} \int_0^1 J_c(r) dW(r), (2c)^2 e^{-2c} \int_0^1 J_c(r)^2 dr)$ is:

$$(A5) \quad K_c(p,q) = M_c(2ce^{-c}p, (2c)^2 e^{-2c}q) .$$

Now

$$\begin{aligned} &\left[c^2 + (2c)^2 e^{-c}p - 2(2c)^2 e^{-2c}q \right]^{1/2} \\ &= c \left[1 + 4e^{-c}p - 8e^{-2c}q \right]^{1/2} \\ (A6) \quad &= c \left[1 + 2e^{-c}p - 4e^{-2c}q - 2(e^{-c}p - 2e^{-2c}q)^2 + O(e^{-3c}) \right] \end{aligned}$$

for large positive c . Substituting (A6) into (A5) we deduce after a little calculation that as $c \rightarrow \infty$:

$$(A7) \quad K_c(p,q) \rightarrow (1 - p^2 - 2q)^{-1/2} .$$

Setting $\rho = 0$ in (A7) we have the marginal mgf $K(0,q) = (1 - 2q)^{-1/2}$. This is the mgf of a χ_1^2 variate, proving part (c). Setting $q = 0$ in (A7) we have $K_\infty(p,0) = (1 - p^2)^{-1/2}$, which is the mgf of a product of independent $N(0,1)$ variates (see, for instance, Kendall and Stuart (1969), p. 269). This proves part (d).

A simple calculation shows that $K_\infty(p,q) = (1 - p^2 - 2q)^{-1/2}$ is the joint mgf of $(\xi\eta, \eta^2)$ where ξ and η are independent $N(0,1)$ variates.

Thus we also have the joint weak convergence of:

$$(A8) \quad \left((2c)^2 e^{-2c} \int_0^1 J_c(r)^2 dr, 2ce^{-c} \int_0^1 J_c(r) dW(r) \right) \Rightarrow (\eta^2, \xi\eta), \text{ as } c \uparrow \infty.$$

Proof of Theorem 5.2

$$\begin{aligned} & (2c)^{-1} e^c \left\{ \int_0^1 J_c(r)^2 dr \right\}^{-1} \left\{ \int_0^1 J_c(r) dW(r) + (1/2)(1 - \sigma_u^2/\sigma^2) \right\} \\ &= \left\{ (2c)^2 e^{-2c} \int_0^1 J_c(r)^2 dr \right\}^{-1} \left\{ (2c) e^{-c} \int_0^1 J_c(r) dW(r) + ce^{-c}(1 - \sigma_u^2/\sigma^2) \right\} \\ &\Rightarrow \xi\eta/\eta^2 = \xi/\eta \equiv \text{Cauchy} \end{aligned}$$

as $c \uparrow \infty$ by (A8) and the continuous mapping theorem. This proves part

(a). In a similar way we find

$$\begin{aligned} & \left\{ \int_0^1 J_c(r)^2 dr \right\}^{-1/2} \left\{ \int_0^1 J_c(r) dW(r) + (1/2)(1 - \sigma_u^2/\sigma^2) \right\} \\ &= \left\{ (2c)^2 e^{-2c} \int_0^1 J_c(r)^2 dr \right\}^{-1/2} \left\{ (2c) e^{-c} \int_0^1 J_c(r) dW(r) + ce^{-c}(1 - \sigma_u^2/\sigma^2) \right\} \\ &\Rightarrow \xi\eta/\eta = \xi \equiv N(0,1) \end{aligned}$$

as $c \uparrow \infty$, proving part (b).

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