

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut 06520

COWLES FOUNDATION DISCUSSION PAPER NO. 778

Note: Cowles Foundation Discussion Papers are preliminary materials circulated to stimulate discussion and critical comment. Requests for single copies of a Paper will be filled by the Cowles Foundation within the limits of the supply. References in publications to Discussion Papers (other than acknowledgment that a writer had access to such unpublished material) should be cleared with the author to protect the tentative character of these papers.

WALRASIAN INDETERMINACY AND KEYNESIAN MACROECONOMICS

J. D. Geanakoplos and H. M. Polemarchakis

October 1985

WALRASIAN INDETERMINACY AND KEYNESIAN MACROECONOMICS

J. D. Geanakoplos** and H. M. Polemarchakis***

July 1982

Revised October 1985

Overlapping generations models with or without production or a portfolio demand for money display a fundamental indeterminacy.

Expectations matter; and they are not, in the short run, constrained by the hypotheses of agent optimization, rational expectations, and market clearing. No short run policy analysis is possible without some explicit understanding of how agents expect the economy to respond to the policy.

In this framework of perfect foresight and market clearing prices, it is possible to make Keynesian assumptions about the rigidity of money wages and the exogeneity of "animal spirits" of investors, to use the standard IS-LM apparatus, and to derive Keynesian conclusions about the short run effectiveness of policy. Alternatively, starting from different but no less rational expectations, one can derive the "new classical" neutrality propositions.

*This work was supported by National Science Foundation Grants SES 82-10034 and SES-8411149. An earlier version appeared as Discussion Paper No. 8247, C.O.R.E., Universite Catholique de Louvain, July 1982; a still earlier version was presented at Cambridge, England, in November 1981. Geanakoplos would like to acknowledge the important contribution of his remarkable research assistant, Michael Mandler.

**Cowles Foundation, Yale University

***Graduate School of Business, Columbia University, and C.O.R.E., Universite Catholique de Louvain.

Keynesian macroeconomics is based in part on the fundamental idea that changes in expectations, or animal spirits, can affect equilibrium economic activity, including the level of output and employment. It asserts, moreover, that publicly announced government policy also has predictable and significant consequences for economic activity, and that therefore the government should intervene actively in the marketplace if investor optimism is not sufficient to maintain full employment.

The Keynesian view of the indeterminacy of equilibrium and the efficacy of public policy has met a long and steady resistance, culminating in the sharpest attack of all, from the so-called new classicals, who have argued that the time-honored microeconomic methodological premises of agent optimization and market clearing, considered together with rational expectations, are logically inconsistent with animal spirits and the non-neutrality of public monetary and bond financed fiscal policy.

The foundation of the new classical paradigm is the Walrasian equilibrium model of Arrow-Debreu, in which it is typically possible to prove that all equilibria are Pareto optimal and that the equilibrium set is finite (see Debreu [9]); at least locally, the hypothesis of market clearing fixes the expectations of rational investors. In that model, however, economic activity has a definite beginning and end. Our point of view is that for some purposes economic activity is better described as a process without end. In a world without a definite end, there is the possibility that what happens today is underdetermined, because it depends on what people expect to happen tomorrow, which in turn depends on what people tomorrow expect to happen the day after tomorrow, etc.

We construct a perfect foresight overlapping generations macroeconomic model, in which there is a two dimensional continuum of equilibria. This

immediately presents the logical possibility that changes in animal spirits can effect output and employment. This proposition holds if we allow the labor market not to clear ("Keynesian equilibria"), or if we require full Walrasian equilibrium but allow for elastic labor supply. But this suggests an important puzzle: if accepting Keynes means accepting a multiplicity of equilibria, what sense can be made of comparative statics, the bread and butter of Keynesian economics?

We approach comparative statics by considering an equilibrium path beginning at date $t = 0$, and then evaluating the consequences of a small policy perturbation. There is then a two-dimensional parameterizable set of possible equilibrium continuations. The plethora of equilibrium continuation does not make comparative statics logically impossible; instead it forces us to fix two parameter values. From this point of view, the most important contribution of this paper is to show that Keynesian macroeconomics has an essential unity: fixing the two variables (animal spirits and current nominal wages) that Keynes argued were psychologically and institutionally sticky under monetary and fiscal perturbations, gives rise to Keynesian policy predictions and comparative statics. Fixing other variables can give rise to other policy theories, including the new classical neutrality theory. Notice, however, that new classical theory has staked out the stronger claim, that the hypotheses of rationality and equilibrium make Keynesian conclusions impossible.

The overlapping generations model with two period lived agents holds a great advantage for Keynesian economics, in that optimizing behavior in any period (given the history to that point) depends only on current prices and next period's prices. The IS-LM apparatus, with its use of a single interest rate, has a natural home there. We conduct all

our comparative statics analysis in terms of the IS-LM diagram. Our model is therefore in the long tradition (represented recently by the French disequilibrium school--see [4], [5], [11], and [24]) of giving microeconomic foundations to Keynesian macroeconomics. In our dynamic model, expectations are explicitly modeled and rationalized, investment and capital accumulation are accounted for, and all individual and government budget constraints are met.

The paper is divided into three parts. In the first section we recall that an overlapping generations economy with one commodity per period and an initial stock of money can display a one-dimensional continuum of equilibria. By reinterpreting this exchange economy as a constant returns to scale production economy where labor is the only factor input, we exhibit the purest case of rational but volatile animal spirits. But this model is far too simple to express the central aspects of any macroeconomic theory, Keynesian or monetarist. In Section 2 we introduce a neoclassical production function, endogenous investment, capital accumulation, bonds, and a portfolio demand for money. In Section 2.1 we show that this richer model, built on explicit microeconomic foundations, also possesses a one-dimensional continuum of Walrasian equilibria, although, with labor inelastically supplied, initial period output and employment is always the same.

In Section 2.2 we examine "Keynesian equilibria" of the same model, in which we permit the period 0 labor market not to clear. We find that there is a two-dimensional set of Keynesian equilibria, which can be indexed by expected price growth and current nominal wages. When these two variables are taken as institutionally fixed, the effects of policy can be faithfully described by IS-LM analysis and Keynesian predictions are borne out. Moreover, since the model is explicitly intertemporal, it is possible, at least in principle, to trace out the medium and long run

effects of policy as well.

In Section 2.3 we once again insist on full market clearing, as well as rational expectations. We specialize our analysis to a specific utility function and production function, but we rely on the general principle that the more endogenous variables per period, the higher is the potential dimension of indeterminacy. Introducing the labor-leisure choice, and so an elastic supply of labor, increases the dimension of the equilibrium set to two. It is therefore possible to validate sticky wages and indeterminate expectations even in a market clearing model. Moreover, under Keynesian institutional assumptions, the effects of policy can once again be analyzed by an IS-LM apparatus, and, at least for fiscal policy, Keynesian predictions obtain. In Section 3 we compare the source of the second dimension of indeterminacy in the Keynesian equilibrium and elastic labor supply models, attributing the latter to a lack of market clearing at infinity.

Since Samuelson [27] introduced the consumption loan model in 1958 it has been clear (see for example, Diamond [10], Tirole [31], etc.) that overlapping generations economies are fertile ground for macroeconomic theorizing. Since Gale [12] it has also been clear that a one good overlapping generations exchange economy with money could display a one-dimensional continuum of Walrasian equilibria. Azariadis [1] showed that such a model might also possess a continuum of stationary sunspot equilibria, and Grandmont [17] showed that it was possible for the equilibrium set to include paths which are cyclical of any period. Neither of these latter models incorporates the possibility of investment and capital accumulation, nor is the comparative statics of unanticipated policy considered there.

The structure of the equilibrium set of OLG models with multiple

commodities has been analyzed in Kehoe-Levine [21], and Geanakoplos-Brown [15, 16]. There it is shown that the indeterminacy of equilibrium continuations arising after a small perturbation is typically parameterizable. The principle is established that the greater is the number of endogenous commodities, the higher is the potential dimension of indeterminacy.

1. Keynesian Indeterminacy in an Exchange Economy

Indeterminacy of expectations is a feature of the simplest overlapping generations economies. Even in an exchange economy, this allows us to justify some of the basic claims of Keynesian policy analysis, while maintaining individual optimization, market clearing and rational expectations. Consider the following variant of the Samuelson consumption-loan model. Discrete time begins at time 0 and extends indefinitely into the future. There is one perishable consumption good available at each point in time with its quantity denoted by x . There is also a second commodity, money, denoted by m , which can be carried from period to period. One generation, consisting of a single agent, is born each period and lives for two periods. The generation born in period t can be described by the strictly quasi-concave, twice differentiable utility function $u(x_t^t, x_{t+1}^t)$ and the endowment vector of consumption goods $e = (e_1, e_2) = (e_t^t, e_{t+1}^t)$. (Superscripts denote period of birth; subscripts denote period of consumption.) The utility function and the endowment vector are identical for each generation $t \geq 0$. However at $t = 0$, there is also one agent (called $t = -1$) who will live only one period and whose endowment consists of e_2 plus the entire stock of money, M . This agent simply wants to maximize consumption in the single period of his or her life. The price of money is one in each period. (We will only describe equilibria in which money has positive value.) The remainder of the price system is a positive vector $p = (p_0, p_1, \dots)$ of consumption good prices. For each generation $t \geq 0$, the following problem is solved:

$$\begin{aligned}
\max \quad & u(x_t^t, x_{t+1}^t) \\
\text{s.t.} \quad & m^t + p_t x_t^t \leq p_t e_1 \\
& p_{t+1} x_{t+1}^t \leq m^t + p_{t+1} e_2 \\
& (x_t^t, x_{t+1}^t) \geq 0 .
\end{aligned}$$

In addition the one generation born in the past solves:

$$\begin{aligned}
\max \quad & x_0^{-1} \\
\text{s.t.} \quad & p_0 x_0^{-1} \leq M + p_0 e_2 \\
& x_0^{-1} \geq 0 .
\end{aligned}$$

The demand of this agent for the consumption good is clearly $e_2 + M/p_0$. Assuming that for each generation $t \geq 0$ desired money holdings satisfy $m^t > 0$, we can combine the two constraints to yield the standard constraint $p_t x_t^t + p_{t+1} x_{t+1}^t = p_t e_1 + p_{t+1} e_2$. We define the relative price $q_t = p_{t+1}/p_t$. Each generation $t \geq 0$ solves the first maximization problem above, yielding a market demand function denoted by $f(q_t) = [f_1(q_t), f_2(q_t)]$ which can be shown to be differentiable. A perfect foresight equilibrium is $p = (p_0, p_1, \dots) = (p_0, q)$ such that:

$$f_1(q_0) + \frac{M}{p_0} = e_1$$

$$f_2(q_{t-1}) + f_1(q_t) = e_1 + e_2, \quad t \geq 1 .$$

PROPOSITION 1:¹ Let $\bar{q}_0 = \frac{\partial u(e_1, e_2)}{\partial x_2} / \frac{\partial u(e_1, e_2)}{\partial x_1}$, and suppose that $\bar{q}_0 > 1$. Then there is a $\bar{p}_0 > 0$ such that for all $p_0 \in (\bar{p}_0, \infty)$, there is an equilibrium (p_0, p_1, \dots) for the OLG exchange economy $E = (u, (e_1, e_2), M)$ described above with first period price p_0 . Moreover, all these equilibria are distinct in the sense that real consumption for any agent is different across all of them.

PROOF: By hypothesis

$$\begin{aligned} \max \quad & u(x_1, x_2) \\ \text{s.t.} \quad & x_1 + \bar{q}_0 x_2 \leq e_1 + \bar{q}_0 e_2 \end{aligned}$$

is solved at $(x_1, x_2) = (e_1, e_2)$, i.e. $f_1(\bar{q}_0) = e_1$. Consider now the equation $f_1(q_0) = e_1 - \frac{M}{p_0}$. If p_0 is sufficiently large, say bigger than \bar{p}_0 , then by the implicit function theorem there must be q_0 that solves this equation, since $\frac{\partial f_1}{\partial q} \Big|_{\bar{q}_0} \neq 0$ (this coincides with the pure substitution effect, since at \bar{q}_0 there is zero excess demand, and hence the derivative cannot be zero). Let $p_1 = q_0 p_0$. Thus p_0 will clear the market at time 0, provided that p_1 is expected to prevail at time 1. Furthermore, if p_0 is sufficiently big, q_0 is very near \bar{q}_0 . But by hypothesis $\bar{q}_0 > 1$, hence $q_0 > 1$, and $p_1 = q_0 p_0 > p_0$. Thus the whole argument can be repeated for the equation $f_1(q_1) = e_1 - \frac{M}{p_1}$, getting $q_1 > 1$, $p_2 = q_1 p_1 > p_1 > p_0$, and by induction, for all $t \geq 1$. Q.E.D.

¹A similar proposition has been established long ago by Gale [1973].

Note that the essence of the proof is that the price p_0 that clears the market at time 0 depends on the expectations of the agents about the price p_1 . But the market clearing price p_1 for period 1 commodities, in turn depends on the expectations consumers at time 1 have about prices in period 2, etc. Since there is no end to time we can always fix p_{t+1} to clear the goods market at time t . A finite Walrasian economy arbitrarily declares there to be an end to time, consequently preventing this argument from working. In OLG economies, however, the hypothesis of optimization, rational expectations, and market clearing are consistent with indeterminacy.

Let us observe that the above argument does not depend in an essential way on there being only one agent born in every period. Nor does it depend on the fact that we assume every generation is identical.¹ This point is important since the problem of indeterminacy in OLG economies is often dismissed with the claim that the independence from time of agent characteristics naturally singles out of the possible continuum a steady state equilibrium. (Thus in our model if there is some (\bar{c}_1, \bar{c}_2) , $\bar{c}_1 + \bar{c}_2 = e_1 + e_2$, $\bar{c}_1 < e_1$, with $\frac{\partial u(\bar{c}_1, \bar{c}_2)}{\partial x_1} = \frac{\partial u(\bar{c}_1, \bar{c}_2)}{\partial x_2}$ then there is a "Samuelson" steady state with $p_t = M/(\bar{c}_2 - e_2)$ for all $t \geq 0$.)

¹ Let the representative consumer from generation t have utility $u^t(x_t^t, x_{t+1}^t)$ and endowment $e^t = (e_t^t, e_{t+1}^t)$, and suppose that there is some $\alpha > 1$ with $\frac{\partial u^t(e_t^t, e_{t+1}^t)}{\partial x_1} / \frac{\partial u^t(e_t^t, e_{t+1}^t)}{\partial x_2} \geq \alpha$. Then, if all the second derivatives $\frac{\partial^2 u^t(e_t^t, e_{t+1}^t)}{\partial x_i \partial x_j}$ are uniformly bounded, exactly the same proof demonstrates the same kind of indeterminacy.

The claim that the economy naturally "chooses" the Samuelson steady state equilibrium is of course suspect in a decentralized framework. Moreover, when there is intergenerational heterogeneity there typically are no equilibrium steady states among the continuum of equilibria. Finally, let us note that the indeterminacy we derived does not depend on there being an arbitrary starting point for the economy: one can demonstrate the existence of the same kind of indeterminacy when time goes from $-\infty$ to ∞ . We can interpret the model by supposing that all agents expect one of the equilibrium price paths to occur -- we might call this the state of long-term rational expectations. In other words, there is a vector of expected prices $\bar{p} = (\bar{p}_0, \bar{p}_1, \dots)$; each agent at time t , having observed $(\bar{p}_0, \bar{p}_1, \dots, \bar{p}_{t-1})$, expects $(\bar{p}_t, \bar{p}_{t+1}, \dots)$. Given a \bar{p} , during each time period t the market clearing price level $p_t = \bar{p}_t$ will be unique. This economy is distinctive, of course, in that there is a one dimensional set of vectors \bar{p} . Agents can choose any \bar{p} in the equilibrium set; then if there is unanimity among all agents \bar{p} will in fact be the equilibrium price vector that is revealed through time.

It is helpful to reinterpret the model as a simple production economy. Imagine that the endowment e_1 in the first period of life is actually labor, ℓ , which can be transformed into output, y , according to the production function, $y = \ell$. We would then think of any purchases of goods by the old generation as demand for real output to be produced by the young. The young in turn now derive utility from leisure in their youth and consumption in their old age.¹ Notice that the quantity equation, $p_t y_t = M$, holds for this economy. (Velocity equals one.)

¹This interpretation is due to Azariadis [1].

The indeterminacy of expectations has the direct implication that optimistic expectations by themselves can cause the economy's output to expand or contract. In short the economy has an inherent volatility. The Keynesian story of animal spirits causing economic growth or decline can be told without invoking irrationality or non-market-clearing. Moreover, the Keynesian claim is that exogenous changes in expectations cannot be relied upon to increase output, and that therefore the government should actively intervene.

In fact, the indeterminacy of equilibrium expectations is especially striking when seen as a response to public (but unanticipated) policy changes. Suppose the economy is in a long-term rational expectations equilibrium \bar{p} , when at time 0 the government undertakes some expenditures, financed either by lump sum taxation on the young or by printing money. How should rational agents respond? The environment has been changed, and there is no reason for them to anticipate that $(\bar{p}_1, \bar{p}_2, \dots)$ will still occur in the future. Indeed in models with more than one commodity (such as we will shortly consider) there may be no equilibrium $(\hat{p}_0, \hat{p}_1, \hat{p}_2, \dots)$ in the new environment with $\hat{p}_1 = \bar{p}_1$, $\hat{p}_2 = \bar{p}_2$, etc. There is an ambiguity in what can be rationally anticipated.

We shall argue that it is possible to explain the differences between Keynesian and monetarist policy predictions by the assumptions each makes about expectational responses to policy, and not by the one's supposed adherence to optimization, market clearing, and rational expectations, and the other's supposed denial of all three.

Consider now the government policy of printing a small amount of money, ΔM , to be spent on its own consumption of real output--or equivalently to be given to generation $t = -1$ to spend on its consumption. Imagine that agents are convinced that this policy is not inflationary, i.e. that \bar{p}_0 will remain the equilibrium price level during the initial period of the

new equilibrium. This will give generation $t = -1$ consumption level $(\bar{M} + \Delta M) / \bar{p}_0$. As long as ΔM is sufficiently small and the initial equilibrium was one of those described in proposition 1, there is indeed a new equilibrium price path \hat{p} beginning with $\hat{p}_0 = \bar{p}_0$. Output has risen by $\Delta \bar{M} / \bar{p}_0$, and in fact this policy is Pareto improving. On the other hand imagine that agents are convinced that the path of real interest rates

$(\bar{q}_0 - 1, \bar{q}_1 - 1, \dots) \equiv \left(\frac{\bar{p}_1}{\bar{p}_0} - 1, \frac{\bar{p}_2}{\bar{p}_1} - 1, \dots \right)$ will remain unchanged. In this

economy, price expectations are a function of p_0 . Agents expect $\hat{p}(p_0) = (\bar{q}_0 p_0, \bar{q}_1 \bar{q}_0 p_0, \dots)$. Recalling the initial period market-clearing equation, $(\bar{M} + \Delta M) / p_0 + f_1(p_1 / p_0) = e_1$, it is clear that if $\hat{p}_1 / \hat{p}_0 = \bar{p}_1 / \bar{p}_0$ then $\hat{p}_0 = \bar{p}_0 (1 + \Delta M / \bar{M})$, i.e. that prices rise proportionally to the growth in the money stock. The result is "forced savings"; output is unchanged and generation $t = -1$ must pay for the government's consumption. If the government's consumption gives no agent utility, the policy is Pareto worsening.

We can also ask about the effects of a "balanced budget" increase in government spending and taxes. If the government taxes the initial young generation and spends the output on its own consumption, what will be the change in output under the Keynesian expectations assumption, $\hat{p}_0 = \bar{p}_0$? The new total output simply equals the sum of government consumption and the old generation's consumption, $G + M / \hat{p}_0$, where G is the real level of government spending. If $\hat{p}_0 = \bar{p}_0$, M / p_0 is unchanged and the balanced budget multiplier equals one, precisely as in Keynesian policy analysis.

This model is only a crude approximation of the differences between Keynesian and monetarist assumptions about expectations and policy. It is quite possible to argue, for example, that holding p_1 / p_0 (the inflation rate) fixed is the natural Keynesian assumption to make. This ambiguity

is unavoidable when there is only one asset into which the young can place their savings. We are thereby prevented from distinguishing between the inflation rate and the interest rate. Our model must be enriched before we can perform satisfactory policy analysis. Nevertheless the model conveys the general principle that expected price paths are not locally unique. There is consequently no natural assumption to make about how expectations are affected by policy. A sensible analysis is therefore impossible without externally given hypotheses about expectations. These can be Keynesian, monetarist, or perhaps some combination of the two.

It is important to understand that despite indeterminacy, this economy, in its design, is an immediate extension of Walrasian economies with finite numbers of agents. In particular the notion of equilibrium is conventional: the price vector (p_0, p_1, \dots) is parametric to agents and equilibrium occurs where aggregate excess demand is zero.

The presence of money, however, calls for some comment. First as we noted earlier, for generations $t \geq 0$, the money stock falls out of the budget constraint. Second, if the additional commodity were another perishable consumption good rather than money, or if markets stretched infinitely into the past and there were no second commodity at all, the indeterminacy result would recur. Finally, the inclusion of money in this model in no way departs from the typical description of a Walrasian economy. Obviously, money does not enter the utility functions. Indeed it may be misleading to consider the extra commodity to be money at all, since we could equally well consider it to be a durable capital good. (We could also let the commodity depreciate at a known rate over time.) It is the fact that generations overlap and do so indefinitely in time which gives this commodity value, and not the description of technology or tastes.

The limited role that money plays in the model circumscribes the number of applications to standard macroeconomic questions. For Keynes, holding money had a precautionary motivation, and also a transactions purpose. In our model there is no uncertainty, and exchanges take place instantaneously, so neither of these aspects of money is present. Since introducing uncertainty and the institutional detail necessary to capture the transactions role of money would greatly complicate our analysis, we shall sometimes put real money balances directly into the utility function. This is not a satisfactory solution, but it is necessary if we are to retain the tractability of a simple macroeconomic model. Since the qualitative features of our analysis remain the same whether we write $u(x_t, x_{t+1}, m/p_t, m/p_{t+1})$, or more simply $u(x_t, x_{t+1}, m/p_t)$, we shall choose the latter form. One can verify at once that as long as the marginal utility of real balances is not too high at the point $(e_1, e_2, 0)$ then exactly the same proof of indeterminacy applies with this more general utility function.

2. A Macroeconomic Model with Keynesian Indeterminacy

We have shown in a simple pure exchange overlapping generations model that there is a continuum of rational expectations equilibria. This immediately leads to the conclusion that changes in expectations, or animal spirits, can affect the course of the economy. Furthermore this creates an ambiguity in the effects of policy which on the one hand might appear Keynesian, and at the other extreme, monetarist. So far, however, our model is not rich enough (even with its production interpretation) to express the central aspects of either macroeconomic theory. Without capital, for example, it is impossible to discuss the volatility of investment. We have already pointed out that the lack of a second asset in which to hold savings creates some difficulty in deciding what constitutes a Keynesian expectations assumption; we certainly cannot model at present the idea that money demand involves a tradeoff between the returns to holding capital and the benefits of cash in hand. In particular we cannot yet describe open market operations. Nor can we describe in the exchange model the Keynesian claim that lowering real wages (and raising employment) can be the equilibrium consequence of government policy. Finally, let us note that the model considered in the last section showed one dimension of indeterminacy; by definition it is impossible to consider simultaneous and independent changes in both the "animal spirits" of the agents and the nominal wage while maintaining equilibrium.

We will present two models that repair these deficiencies. Both models include the possibility of capital accumulation and a neoclassical production function depending on variable inputs of capital and labor. In the first model, which corresponds most strongly to standard Keynesian macroeconomic models, we will place real balances in the utility function and

agents will be endowed with a fixed labor supply, the use of which does not affect their utility. We will study "Keynesian" equilibria in which the labor market is permitted not too clear. This allows for involuntary unemployment and for government policies and expectational changes that can affect the level of employment. The IS-LM apparatus is ideally suited to describe the comparative statics of this model.

However such a model is subject to the now standard criticism that it is methodologically unsound in that it does not give an optimizing explanation for its assumed lack of market clearing. Therefore we also consider a second model in which all markets clear but where labor services are elastically supplied (as in Keynes' General Theory). To make this second model more tractable, we will use specific functional forms to describe utility and production. To make the model both more simple and more orthodox, we remove real balances from the utility function. We find that in the elastic labor supply model there is a continuum of Walrasian equilibrium levels of employment, and that the IS-LM apparatus is still appropriate for analyzing the effects of government policy on output and employment.

There is a central advantage to the elastic labor supply model. If labor supply at time $t+1$ is known by a firm with certainty at time t , the firm need only know the real interest rate to decide on an equilibrium level of capital investment. The interest rate tells the firm what the appropriate capital-labor ratio is; the added information of equilibrium labor supply at $t+1$ tells the firm the absolute level of capital required. However with elastic labor supply, we must include in the model some quantity signal indicating expected output next period. This variable plays a large role in the General Theory, since it signifies entrepreneurial expectations.

We will see that it is not uniquely determined. An equilibrium model of Keynesian economics can come to fruition only in a model with elastic labor supply: in that case both current output and expected output (animal spirits) are indeterminate.

The model with fixed labor supply nevertheless serves a number of purposes. It provides a complete intertemporal version of a Keynesian unemployment equilibrium that can be analyzed accurately by the IS-LM device. Moreover many economists will view the presence of involuntary unemployment as a positive descriptive virtue of the model. Finally the model will serve as a general introduction to overlapping generations economies with production.

2.1. Fixed Labor Supply and Walrasian Equilibria

Suppose once again at every period t an agent is born and lives for two periods. Each agent has the quasi-concave utility $u(c_t, c_{t+1}, M/p_t)$ for consumption, and real balances, and an endowment of labor equal to one. In some periods, we will give agents the labor supply constraint, $\bar{\ell}_t$, which may be less than one. In contrast to our previous model of pure exchange, we shall now suppose that consumers can save in two different ways. As before, they can hold paper money M which always has a price of 1 and which in period $t+1$ will allow for the purchase of M/p_{t+1} units of the consumption good. By holding money, a consumer foregoes the chance of consuming M/p_t units of the consumption good in period t . The gross rate of return to holding money is therefore $(M/p_{t+1})/(M/p_t) = p_t/p_{t+1}$. Alternatively, consumers can save from period t to period $t+1$ by holding new capital, which in period t is a perfect substitute for the consumption good. If the capital k_{t+1} is invested (e.g. planted in the ground) in

period t , it becomes productive capital in period $t+1$. It can then be combined with labor, instantaneously producing the consumption good and disappearing. Productive capital is not a perfect substitute for output; its price r_{t+1} is endogenously determined in equilibrium. The agent who foregoes one unit of consumption in period t to hold capital, gains the power to purchase r_{t+1}/p_{t+1} consumption goods in period $t+1$. Thus r_{t+1}/p_{t+1} is the gross rate of return to capital. We can summarize these relations by writing the budget constraints:

$$(B) \quad p_t c_t + m + p_t k_{t+1} \leq w_t \bar{l}_t$$

$$p_{t+1} c_{t+1} \leq m + r_{t+1} k_{t+1}$$

$$c_t, c_{t+1}, m, k_{t+1} \geq 0 .$$

Notice that if $p_t/p_{t+1} > r_{t+1}/p_{t+1}$, the agents can earn a better return by holding money than by holding capital, and in addition get the satisfaction of holding real balances. Hence the only interesting case occurs when $p_t/p_{t+1} \leq r_{t+1}/p_{t+1}$. Indeed, let us suppose that at the prevailing $(p_t, p_{t+1}, w_t, r_{t+1})$ the consumer chooses to hold $k_{t+1} > 0$. Then we can combine the two budget constraints into one:

$$(B') \quad c_t + c_{t+1} \left(\frac{p_{t+1}}{r_{t+1}} \right) + \frac{m}{p_t} \left(1 - \frac{p_t}{p_{t+1}} \frac{p_{t+1}}{r_{t+1}} \right) \leq \bar{l}_t \left(\frac{w_t}{p_t} \right) .$$

Notice that the "price of money" is $1 - \frac{p_t}{p_{t+1}} \frac{p_{t+1}}{r_{t+1}}$, which, as we would expect, varies inversely with the return to money and directly with the return to capital. We define the income of the (young) agent at time t as $I_t^y = (w_t/p_t) \bar{l}_t$. The demand functions (or correspondences) for c_t ,

c_{t+1} , and M/p_t are determined by p_t/p_{t+1} , r_{t+1}/p_{t+1} , and I_t^y .

Using the prices defined by the single budget constraint above, we assume that the utility has the following gross substitutes property:

$$\begin{aligned} \text{(GSP) The maximization problem: } & \text{Max } u\left(c_0, c_1, \frac{m}{p}\right) \\ & \text{s.t. } \pi_0 c_0 + \pi_1 c_1 + \pi_m \frac{m}{p} \leq I \end{aligned}$$

gives rise to continuous, single-valued demands for c_0 , c_1 , and m/p . Each demand is decreasing in its own price, and increasing in the other prices and in income I . □

Needless to say, there is a large class of utilities which satisfy this gross substitute property (GSP). From this one assumption on utilities, we will later derive all of the classical properties of IS and LM curves. The GSP assumption has implications not only for the demand for consumption when young and for $s_t^m = m/p_t$, but also for capital holdings s_t^k , and total savings $s_t^k + s_t^m$. Clearly it implies that s_t^m varies directly with p_t/p_{t+1} , and inversely with r_{t+1}/p_{t+1} .

Observe that capital savings can be written $s_t^k = \left(c_{t+1} - \frac{m}{p_t} \frac{p_t}{p_{t+1}} \right) \frac{p_{t+1}}{r_{t+1}}$. Total savings are given by $s_t = s_t^k + s_t^m = I_t^y - c_t$. Using these definitions we can also conclude from GSP that s_t^k is decreasing in p_t/p_{t+1} , for if p_t/p_{t+1} rises, m/p_t rises and c_{t+1} falls. Furthermore since c_t is decreasing in r_{t+1}/p_{t+1} and p_t/p_{t+1} , s_t is increasing in those two variables. Given that demands for c_t , c_{t+1} , and m/p_t are increasing in I_t^y , we conclude that $s_t = I_t^y - c_t$ is also increasing in I_t^y (the marginal propensity to consume is less than one).

We can summarize the implications of GSP that we will use below by the following assumption (which also includes the continuity and limiting properties of the functions that are necessary for the subsequent proof).

(A1) The functions $s^k(I_t^y, r_{t+1}/p_{t+1}, p_t/p_{t+1})$ and $s^m(I_t^y, r_{t+1}/p_{t+1}, p_t/p_{t+1})$ are continuous; s^m is monotonically increasing in its own rate of return; s^m and s^k are both increasing in the competing rate of return. The function $s(I_t^y, r_{t+1}/p_{t+1}, p_t/p_{t+1}) = s^k(\cdot) + s^m(\cdot)$ is increasing in I_t^y , r_{t+1}/p_{t+1} , and p_t/p_{t+1} . For any fixed r_{t+1}/p_{t+1} and I_t^y , $\lim_{p_t/p_{t+1} \rightarrow 0} s^m(I_t^y, r_{t+1}/p_{t+1}, p_t/p_{t+1}) = 0$. \square

The last part of (A1) says that as the rate of inflation approaches infinity, desired real balances decline to 0. Obviously the reader can take the assumption on demand functions to be primitive rather than the assumption about the utility function.

Let us turn to the productive side of the economy. We assume that for each $t \geq 0$ there is a firm that can transform k_{t+1} units of capital (i.e. of the consumption good) invested at time t , together with ℓ_{t+1} units of labor applied at time $t+1$ into $F(k_{t+1}, \ell_{t+1})$ units of the consumption good at time $t+1$. Since there is no uncertainty we can imagine that the firm raises the resources to buy each unit of capital at time t by issuing a bond that pays r_{t+1} dollars at time $t+1$. The firm maximizes profits $p_{t+1}F(k_{t+1}, \ell_{t+1}) - r_{t+1}k_{t+1} - w_{t+1}\ell_{t+1}$. We shall assume

(A2) The production function $F(k_{t+1}, \ell_{t+1})$ has constant returns to scale and is strictly quasi-concave. Also $\lim_{k \rightarrow 0} \frac{\partial F(k, \ell)}{\partial k} = \lim_{\ell \rightarrow 0} \frac{\partial F(1, \ell)}{\partial \ell} = \infty$. It

follows that there is a uniquely defined, continuous, decreasing function (the factor price frontier) $w/p = \varphi(r/p)$, $\varphi : (0, \infty) \rightarrow (0, \infty)$, such that $(\frac{\partial F(k, \ell)}{\partial k}, \frac{\partial F(k, \ell)}{\partial \ell}) \in \text{graph } \varphi$ for all (k, ℓ) . Finally, we assume

$\ell \frac{\partial F(k, \ell)}{\partial \ell}$ is increasing in ℓ , for any fixed k . \square

With the exception of the last part, (A2) makes the standard assumptions used in neoclassical growth theory. The last part of (A2) accords with the Keynesian hypothesis that diminishing returns are "not too severe." If the firm knows ℓ_{t+1} , hypothesis (A2) implies that the prices $(r/p, \varphi(r/p))$ uniquely determine the capital demand at time t . We can describe this choice by the function $k(r/p, \ell)$. Since there are constant returns to scale, capital demand would be indeterminate without the quantity signal ℓ . We will sometimes use the expression $k(r/p) \equiv k(r/p, 1)$. Looking ahead to the next model with elastic labor supply, it is clear that we could also use a function $k(r/p, y^e)$ where y^e is the expected level of demand next period.

We need to make one more assumption, which relates the preferences of the consumer to the technology of the firm.

(A3) There is $(\bar{r}/\bar{p}) < 1$ such that $s^k(\bar{w}/\bar{p}, \bar{r}/\bar{p}, 0) - \bar{K} = 0$ where $\bar{w}/\bar{p} = \varphi(\bar{r}/\bar{p})$, $\bar{K} = k(\bar{r}/\bar{p}, 1)$. There is also $(\tilde{r}/\tilde{p}) > 1$ such that $s^k(\tilde{w}/\tilde{p}, \tilde{r}/\tilde{p}, 1) = \tilde{K}$, and $s^m(\tilde{w}/\tilde{p}, \tilde{r}/\tilde{p}, 1) > 0$, where $\tilde{w}/\tilde{p} = \varphi(\tilde{r}/\tilde{p})$, $\tilde{K} = k(\tilde{r}/\tilde{p}, 1)$. □

Assumption (A3) simply states that there is both a monetary and a non-monetary steady state. It is the analogue to the pure exchange hypothesis that there was some price ratio that supported a monetaryless equilibrium, and also that the Samuelson steady state equilibrium $p_t = \bar{p}$ for all $t \geq 0$ involved holding positive monetary balances.

The description of the economy is completed by a specification of the stock of productive capital k_0 on hand at time 0, and the stock M of paper money held by the old at time 0. We assume that the old at time 0 own all the capital k_0 , and desire only the consumption good c_0 .

A Walrasian equilibrium $\left((p_t, w_t, r_{t+1}^e, p_{t+1}^e, \bar{\lambda}_t)_{t \geq 0}, c_0^{-1}, r_0, (k_t, c_t, c_{t+1}^t)_{t \geq 0} \right)$ for the fixed labor supply economy is a sequence of prices, labor constraints and consumption choices such that

- 1) $(c_t^t, c_{t+1}^t, M/p_t, k_{t+1})$ maximizes $u(c_t^t, c_{t+1}^t, m/p_t)$ subject to the budget constraint, B , and the labor constraint $\bar{\lambda}_t$, for all $t \geq 0$,
- 2) $(k_t, \bar{\lambda}_t)$ maximizes $p_t F(k, \ell) - r_t k - w_t \ell$ for all $t \geq 0$,
- 3) $c_0^{-1} = M/p_0 + k_0 r_0/p_0$,
- 4) $c_t^{t-1} + c_t^t = F(k_t, \bar{\lambda}_t) - k_{t+1}$ for all $t \geq 0$, and
- 5) $p_{t+1}^e = p_{t+1}$, $r_{t+1}^e = r_{t+1}$, for all $t \geq 0$,
- 6) $\bar{\lambda}_t = 1$, for all $t \geq 0$.

In Walrasian equilibrium expectations are rationally held (there is perfect foresight), and all markets clear. We can now state the indeterminacy proposition for a market-clearing economy with production.

PROPOSITION 2: Let E be an economy meeting assumptions (A1)-(A3), with an initial level of the capital stock, k_0 , lying in the interval $[k(\tilde{r}/\tilde{p}), k(\bar{r}/\bar{p})]$. Then E possesses a one-dimensional family of equilibria indexed by the price of output at the initial point in time. This remains true under small perturbations of the economy at time 0, due for example to government policy interventions.

PROOF: See Appendix.

Q.E.D.

First of all, we should note that this result is unrelated to the presence of money in the utility function. If we take money out of the utility function, the model then reduces to the Diamond [10] model with money, for which a parallel proposition could be proved. Although the proof of

Proposition 2 is also in the Appendix, it is worthwhile to discuss the simple idea that lies behind it. In the exchange economy proof we noticed that at $t = 0$ there were two relevant prices for current decision-makers (the young); p_0 and p_1 . There were two markets that existed at $t = 0$, the goods market and money market. But by Walras' law, only one of the market clearing conditions is independent. Thus it was unsurprising that we could choose p_0 freely and let p_1 adjust so as to clear the one remaining market. (As with all economies in this paper, we have already chosen money as numeraire.) We then proceeded by induction. In the present economy there are four markets (goods, bonds, labor, and money) and thus three independent market clearing conditions. There are, however, four relevant independent prices at $t = 0$: p_0 , w_0 , r_1^e , p_1^e . Hence we might expect there to be a one-dimensional indeterminacy.

2.2. IS-LM-IB

We can describe market clearing in any period by means of IS and LM curves. The equilibrium condition for the goods market in period 0 is, $c^Y(I_0^Y, r_1/p_1, p_0/p_1) + (r_0/p_0)k_0 + M/p_0 + k(r_1/p_1) = y_0$ where $c^Y(\)$ is the consumption of the young. The second two terms are the demand for consumption goods by the old and the last term is the demand for capital by firms. Firms rationally anticipate $\bar{\ell}_1 = 1$. Rearranging this equation, and using the fact that $y_0 - (r_0/p_0)k_0 = (w_0/p_0)\bar{\ell}_0$, we have $(w_0/p_0)\bar{\ell}_0 - c^Y(\) - M/p_0 = k(r_1/p_1)$. The first two terms are simply the aggregate savings by the young, and so we have $s(I_0^Y, r_1/p_1, p_0/p_1) - M/p_0 = k(r_1/p_1)$. This version of the goods market clearing condition simply states that the savings of the young less the dissaving of the old equals new capital demand; in the language of Keynesian economics, it is an IS

curve. Recall assumption (A2) that states in part that given a fixed k_t worker income (I_t^y) is increasing in \bar{x}_t , or equivalently increasing in y_t . In that case we can use the same symbol s and write

$$\text{IS: } s(y_0, r_1/p_1, p_0/p_1) - M/p_0 = k(r_1/p_1) ,$$

and we know that since $s^k(\cdot) + s^m(\cdot)$ is an increasing function of I_t^y , $s(\cdot)$ is an increasing function of y_0 . We can now draw the IS curve in $(y, r/p)$ space (Figure 1). Assumption (A1) states that $s(\cdot)$ is increasing in r_1/p_1 . Since $k(\cdot)$ is decreasing in the same variable, it is clear that y_0 must fall if r_1/p_1 rises; hence the downward slope of IS in Figure 1. Using the same substitution, money market equilibrium can be described by

$$\text{LM: } s^m(y_0, r_1/p_1, p_0/p_1) = M/p_0 .$$

Since $s^m(\cdot)$ is decreasing in r_1/p_1 , the LM curve must be upward sloping in Figure 1. Thus we can describe the equilibrium pair $(r_1/p_1, y_0)$ given the equilibrium values of p_0 and p_0/p_1 . We do not need to know the equilibrium w_0 to draw either the IS or LM curves. We know from Walras' law that if we have accounted for the goods and money markets, the bond market clearing condition must be redundant. We can confirm this by summing the IS and LM equations. Recalling that $s(\cdot) = s^k(\cdot) + s^m(\cdot)$, this yields,

$$\text{IB: } s^k(y_0, r_1/p_1, p_0/p_1) = k(r_1/p_1)$$

which is indeed the bond market equilibrium condition. The IB curve must therefore pass through the intersection of the IS and LM curves.

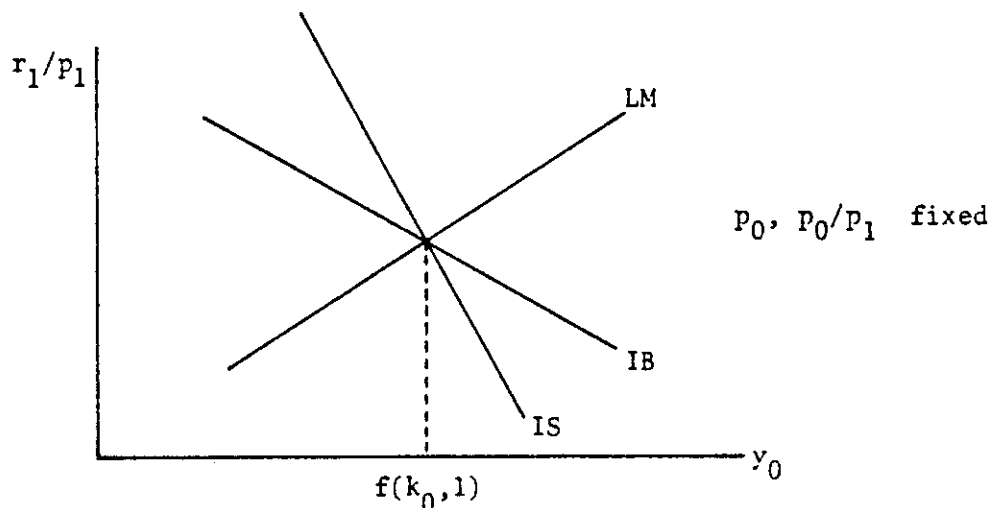


FIGURE 1: IS-LM-IB

The GSP assumption on utility is not sufficient to sign the slope of the IB curve. Thus although we could choose any two of the three curves IS, LM, IB to do comparative statics, the IS-LM pair gives the most information. Note, incidentally, that if the IB curve is downward sloping, as we have drawn it, it must be less steep than the IS curve.

Labor market clearing is achieved at $y_0 = F(k_0, 1)$. By shifting P_0/P_1 , we can move all three IS-LM-IB curves over to intersect simultaneously with the $y_0 = F(k_0, 1)$ curve, while keeping p_0 fixed.

2.3. Keynesian Equilibria and Comparative Statics

We have developed a model in which it is possible to pose the classical questions of macroeconomics. The difficulty is that with a neoclassical production function with given capital stock k_0 and inelastic labor supply, no government policy or aggregate demand shock can possibly change initial period, Walrasian equilibrium output.

The Keynesian hypothesis is that the labor market need not clear, at least in the short run, in period 0. We define a Keynesian equilibrium as a sequence of prices, expectations, etc. satisfying 1-6 of the last

section, except allowing $\bar{\ell}_0 \leq 1$ rather than $\ell_0 = 1$ (so that producers are still on their demand curve for labor, but not all workers who want jobs may be able to find them). A trivial extension of the proof of Proposition 2 now shows that there is a two-dimensional continuum of Keynesian equilibria, which is robust to small policy changes at period 0. If we also permitted the period 1 labor market not to clear, there would be three dimensions of indeterminacy.

As in the exchange economy, there is a large choice about how to analyze the consequences of policy interventions. Keynesian expectations and institutional assumptions are best described by fixing p_0/p_1 and w_0 and then asking how r_1/p_1 and y_0 are affected by policy or exogenous changes in p_0/p_1 or w_0 . So far, however, we have indexed our IS-LM-IB diagram by p_0 and p_0/p_1 . This could easily be changed, for if w_0 is fixed, we could solve for p_0 from the equation: $\frac{w_0}{p_0} = \frac{\partial F(k_0, \bar{\ell}_0)}{\partial \bar{\ell}_0}$. We could therefore find the monotonically increasing function $p_0(y_0)$ that relates these two variables, and use this function in the IS and LM equations. The only result would be to make both curves more steep; it would not change the sign of their slope. For now, we shall maintain p_0 and p_0/p_1 fixed.

Imagine that at a Keynesian equilibrium the government unexpectedly makes a public purchase of $\Delta M/p_0$ bonds with freshly minted money, promising to retire the money it will receive in interest payments in period 1. We know from Proposition 2 that there is a two-dimensional set of equilibrium continuations. If we assume that p_0 and p_0/p_1 do not change, there is a unique equilibrium continuation. The new LM equation is $s^m = (M + \Delta M)/p_0$ which constitutes a shift outwards of the LM curve, since at each r_1/p_1 , y_0 must be larger in order for money demand to equal

the larger supply. The IS equation $s^k + [s^n - M/p_0] = k(r_1/p_1)$ will be unaffected. Recall from the derivation of the IS equation that M/p_0 appears only insofar as it indicates the demand of the old (which does not change under this policy experiment). Shifting out the LM curve along a stationary IS curve obviously raises y_0 and lowers r_1/p_1 . (See Figure 2.) The medium run effects are also expansionary. Since r_1/p_1 falls, k_1 increases, and since labor input is fixed at 1, y_1 rises.¹

We can also examine open market operations in the face of a liquidity trap. In a liquidity trap, agents are so saturated with money holdings that additional money holdings provide no additional utility. This is precisely what is described when money holdings do not enter the utility function. In that case the rate of return on holding money must equal the rate of return on capital, i.e. $r_1/p_1 = p_0/p_1$. If this condition is met, distinct capital and money demands are not well-defined, since savers will then not care about the composition of their savings portfolio. Since p_0/p_1 is fixed, the LM curve must be horizontal at $r_1/p_1 = p_0/p_1$. The only equilibrium occurs where savers agree to hold the money stock that happens to be on the market. Increasing the money stock therefore merely alters the composition of private saving and effects no variable at $t = 0$. Just as in the Keynesian parable, the LM curve cannot shift out (Figure 3).

One can similarly analyze other standard macroeconomic policies. Consider a bond financed government expenditure, where the government raises the revenue to make its interest payment in period 1 by taxing the young born in period 1. This shifts out the IS curve, leaving the LM curve in

¹Note that no matter how the government promises to dispose of its interest payments in period 1, as long as it does not affect the budget constraint of the generation born at $t = 0$, the period 0 analysis will remain the same, although economic activity after period 1 may be affected.

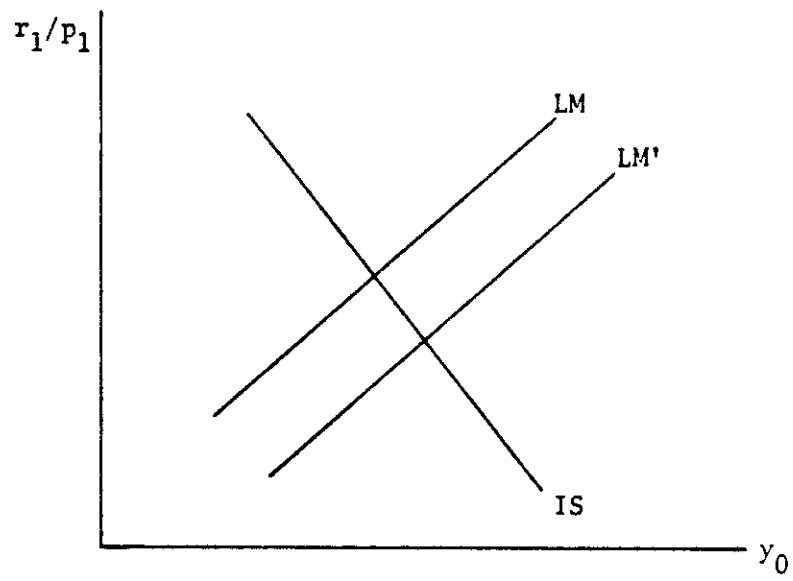


FIGURE 2: An Increase in M

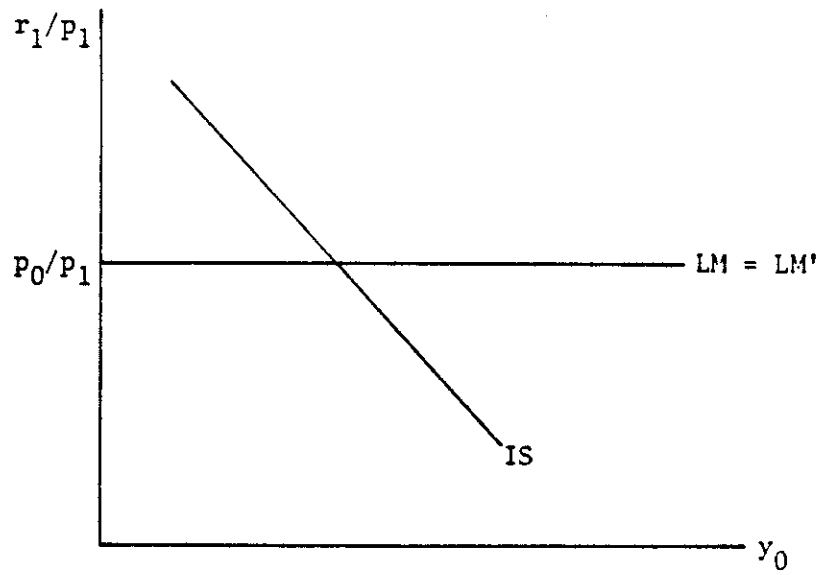


FIGURE 3: Liquidity Trap

place, in period 0. The new IS equation is $s^k + s^m - M/p_0 = k + \Delta G$, where ΔG is the increment to government expenditure. If we subtract the LM equation, we arrive at the correct IB equation $s^k = k + \Delta G$. As a result, y_0 goes up, r_1/p_1 goes up, crowding out some private investment, so that k_1 is lower than in the original equilibrium, as is next period output y_1 .

It is sensible in this model to consider the effect of a pure change in expectations, say of p_0/p_1 . If investors expect the rate of growth of prices to drop (i.e. p_0/p_1 to increase), then according to the GSP assumption, both monetary savings and aggregate savings will increase, and the IS and LM curves will shift back. The IB curve will also shift since $s^k(\)$ is a function of p_0/p_1 . Period 0 output will decline, and the economy will be moved onto a path in which investors' pessimism is justified. Hence this economy can grow or shrink merely due to the changing state of expectations. This captures the Keynesian idea that the economy is intrinsically volatile (see Figure 4). In order to give a completely Keynesian account of an economy receiving an expectations shock and the government policies that can remedy the change, we will treat w_0 as exogenous rather than p_0 . As we mentioned earlier, this will merely make the IS and LM curves more steep, but it will not change their orientation.

The analysis of the last paragraph still applies. One of Keynes' central arguments was that although a fall in w_0 could return y_0 to its original level (by shifting out the IS and LM curves), it is extremely unlikely to occur. It is more feasible to engage in government monetary and fiscal policy. We have seen that the two policies of bond financed government spending and open market operations can move the IS and LM curves independently. Therefore, with w_0 fixed, stabilization policy can move

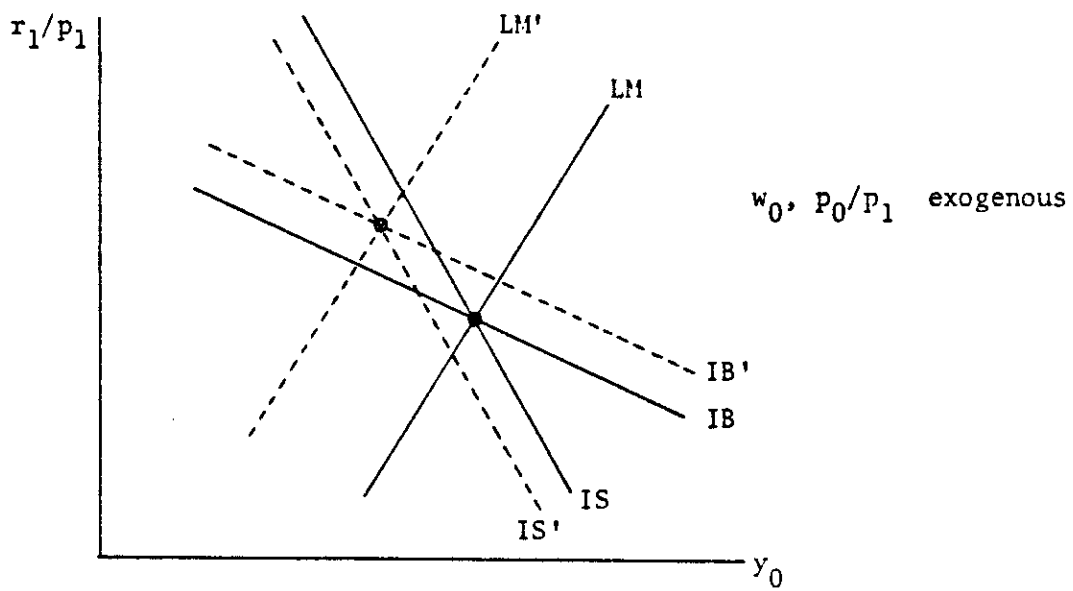


FIGURE 4: A Rise in P_0/P_1

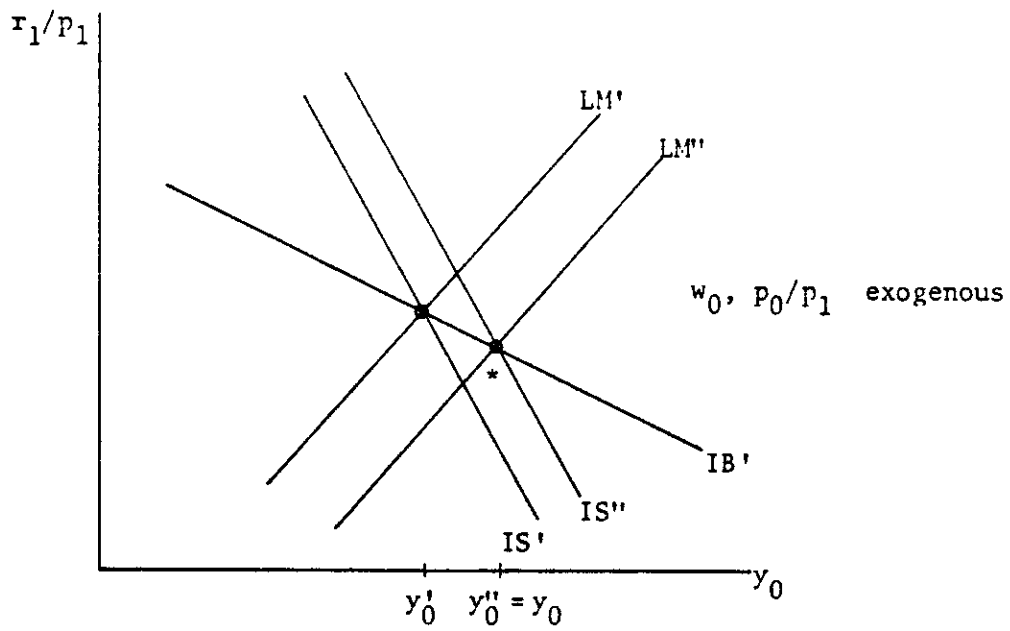


FIGURE 5: A Fall in w_0

the economy to the pre-shock levels of r_1/p_1 , y_0 and y_1 . Notice that wage-cutting cannot achieve this dual objective, since the IB equation, $s_k(\cdot) = k(\cdot)$, is not affected by wage cutting, and therefore wage cutting can only choose among the pairs $(y_0, r_1/p_1)$ on the new IB curve (see Figure 5). If the IB curve is downward sloping, then the original level of output y_0 can be achieved by cuts in w_0 alone only at the expense of a higher level of real interest rates r_1/p_1 , and hence ultimately with lower output y_1 .

One last interesting case to consider is a balanced budget increase G in government spending and taxes, with p_0 and p_0/p_1 (or w_0 and p_0/p_1) held fixed. Let us suppose that the taxes G are raised in a lump sum way from the old and the young in proportion to their share of total income y_0 . Suppose for simplicity that the income shares do not depend on output, i.e. that $F(k, \ell) = k^\alpha \ell^{1-\alpha}$ is Cobb-Douglas. The textbook Keynesian analysis suggests that if the LM curve is horizontal, then output rises by G , but if the LM curve is upward sloping, then the interest rate rises with the shift in the IS curve, crowding out some investment, and reducing the balanced budget multiplier. However, in our model, where money demand has a portfolio explanation, as it does in theories of liquidity preference as behavior towards risk, government spending and taxation affect wealth and so shift the LM curve as well as the IS curve. In fact, one can easily check that under the simplifying hypotheses we have adopted, the LM curve will shift out just far enough, with p_0 and p_0/p_1 (or w_0 and p_0/p_1) held fixed, so that output y_0 goes up by G ! The balanced budget multiplier is still one.

It is important to understand that the description of Keynesian dynamics above hinges on the adoption of Keynesian expectations and insti-

tutional assumptions. Imagine that we fix the ratio w_0/p_0 instead of either w_0 or p_0 . This embodies the monetarist notion that workers are not willing to accept a cut in real wages in order to gain greater employment. Further suppose agents still expect p_1/p_0 to be fixed. Notice that the IB equation, $s^k(r_1/p_1, y_0, p_0/p_1) = k(r_1/p_1)$, is unaltered by the new institutional assumption. Now consider an expansion of the money supply. Since w_0/p_0 is fixed, the level of output y_0 is fixed. Since the IB equation has not changed, we can then conclude that r_1/p_1 also cannot change. With r_1/p_1 and y_0 unchanged, and p_0/p_1 fixed, the demand for real balances, $s^k(r_1/p_1, p_0/p_1, y_0)$, cannot change and therefore in equilibrium the real money supply cannot change either. The monetarist model, properly specified with the correct institutional assumptions, yields the monetarist result that monetary expansion is inflationary, even though the labor market is not required to clear. It is tempting to believe that if we require the labor market to clear that there is in fact only one equilibrium w_0/p_0 . If labor supply enters the utility function, however, we shall see that this is not the case.

2.4. Elastic Labor Supply, Walrasian Indeterminacy, and Keynesian Macroeconomics

Keynesian policy analysis is most naturally represented in a model with two dimensions of indeterminacy. We saw that we could gain the second dimension of indeterminacy in our production model simply by allowing the labor market not to clear in period 0. This made output and employment variable in period 0, and gave unemployment an involuntary character.

It is a general property of overlapping generations economies (as explained in [21], [15], and [16]) that the greater are the number of endo-

genous variables per period, the higher is the dimension of possible indeterminacy of Walrasian equilibria. Thus in this section we make labor supply endogenous, letting the labor-leisure choice affect marginal utility, and we thereby obtain a second dimension of indeterminacy while maintaining labor market clearing in every period. We find that there is a continuum of levels of equilibrium employment and unemployment. Of course now the unemployment is all voluntary, in the sense that at the going wages, any worker who wants a job can find one. Recall, however, one definition of unemployment that Keynes gave. He wrote that it occurred at a given real wage w_0/p_0 if a rise in the nominal price level p_0 (presumably with w_0 held fixed) would lead to higher actual employment. This is precisely the phenomenon we shall now investigate.

The analysis of government policy interventions and comparative statics in models where the labor market always clears can also be undertaken through the IS-LM apparatus. In Section 2.1 there were four independent prices, p_0 , w_0 , p_1 , r_1 that affected behavior at $t = 0$. Since firms could predict \bar{x}_1 with certainty, they did not need a quantity signal at $t = 0$. Now, with \bar{x}_1 endogenous, firms need some quantity signal, which we take to be y_1^e . In this section we will hold both w_0 and y_1^e fixed, which is the most natural interpretation of the General Theory's institutional assumptions. Assuming that the price of future consumption affects the young's labor-leisure choice, we can always adjust p_0/p_1 to maintain labor market clearing, and so we can conduct the IS-LM analysis in the same $(y_0, r_1/p_1)$ space as before, adjusting the curves to take into account the fluctuations in p_0/p_1 . Moreover, we find that Keynes' policy predictions will then be borne out.

Let us consider a concrete example for which we can derive closed form

formulas.¹ Accordingly, let

$$u\left(\ell_t, c_t, c_{t+1}, \frac{m}{p_t}\right) = -\frac{1}{\beta}\ell_t^\beta + \frac{1}{\gamma}c_{t+1}^\gamma, \quad \beta \geq 1, \gamma < 1, \gamma \neq 0.$$

We have left real money balanced out of the utility, in order to make our model as classical as possible; we shall say more about this shortly. The absence of c_t from the utility function is of no importance--it makes more dramatic the difference between the marginal propensities to consume of the young and the old. Of crucial importance is the inclusion of labor as a choice variable; its presence in the utility function means that market clearing can be consistent with a range of different levels of employment.

Let the production function F be Cobb-Douglas:

$$F(k_t, \ell_t) = k_t^\alpha \ell_t^{1-\alpha}, \quad 0 < \alpha < 1 - \alpha < 1$$

Clearly the production function satisfies assumption (A2); in particular, the share of output y that goes to labor, $\ell \frac{\partial F(k, \ell)}{\partial \ell} = (1-\alpha)y$, is increasing in y , for any k .

A Walrasian equilibrium (given u , F , k_0 and M) is a price system $(p_t, w_t, r_t)_{t \geq 0}$ such that there exists a consumption allocation $c = (c_t^t, c_{t+1}^t)_{t \geq 0}$ and a production plan $(k_t, \ell_t)_{t \geq 0}$ such that for all $t \geq 0$

¹To demonstrate the existence of two dimensions of indeterminacy for a more general class of models requires a technical apparatus, such as can be found in [21] or [16], more advanced than we have introduced here.

- (i) (ℓ_t, k_t) maximizes $p_t F(k, \ell) - w_t \ell - r_t k$;
- (ii) $(\ell_t, c_t^t, c_{t+1}^t, M/p_t)$ maximizes $u(\ell, c_1, c_2, m/p_t)$ subject to the budget constraints $p_t c_1 + m + p_t k_{t+1} \leq w_t$ and $p_{t+1} c_2 \leq m + r_{t+1} k_{t+1}$;
- (iii) $p_t = r_{t+1}$;
- (iv) $(1-\alpha)F(k_t, \ell_t) - M/p_t = k_{t+1}$.

Conditions (i) and (ii) represent firm profit maximization and agent utility maximization. Condition (iii), an LM relationship, is required for clearing the money market, since money and capital are perfect substitutes. (We encountered the same condition when we considered the liquidity trap in Section 2.2.) Condition (iv) is the IS relationship which assures clearance of the goods market. Savings by the young, $s(\)$ in the previous sections, are now $(1-\alpha)F(k_t, \ell_t)$, given that the young do not consume in the first period of life, and that the production function is Cobb-Douglas. Conditions (i) and (ii) imply labor market clearing.

Let us consider the market clearing conditions for the initial period alone. As we know, there are three independent markets. The money and goods market clearing equations, (iii) and (iv) above, we now repeat for period 0.

$$\text{LM: } r_1/p_1 = p_0/p_1$$

$$\text{IS: } (1-\alpha)y_0 - M/p_0 = k(r_1/p_1, y_1^e) .$$

We have replaced $F(k_0, \ell_0)$ by y_0 and given k_1 the appropriate functional notation. Lastly we consider the labor market. The demand for labor, given k_0 , is $\ell_0^d = (1-\alpha)^{1/\alpha} k_0 (w_0/p_0)^{-1/\alpha}$. The supply of labor, derived from utility maximization is $\ell_0^s = (w_0/p_1)^{\gamma/\beta-\gamma} = [(w_0/p_0) \cdot (p_0/p_1)]^{\gamma/\beta-\gamma}$.

Equilibrium in the labor market is obtained when

$p_0/p_1 = A(w_0/p_0)^{-(\alpha\gamma+\beta-\gamma)/\alpha\gamma}$, where $A > 0$ is a constant depending on k_0 , α , β and γ . For fixed k_0 , w_0/p_0 is a decreasing function of y_0 . Using this and combining LM and the labor market clearing equation above, we can calculate:

$$\text{LML: } r_1/p_1 = B y_0^{(\beta+\alpha\gamma-\alpha)/\gamma(1-\alpha)}$$

where again $B > 0$. We now have a simple system of three equations, IS, LM, and LML, in five unknowns, p_0 , p_1 , w_0 , r_1 , y_1^e , that describe a static equilibrium for period 0. The next proposition assures us, however, that any static equilibrium is part of a dynamic Walrasian equilibrium involving full optimization and market clearing in the complete intertemporal model described above.

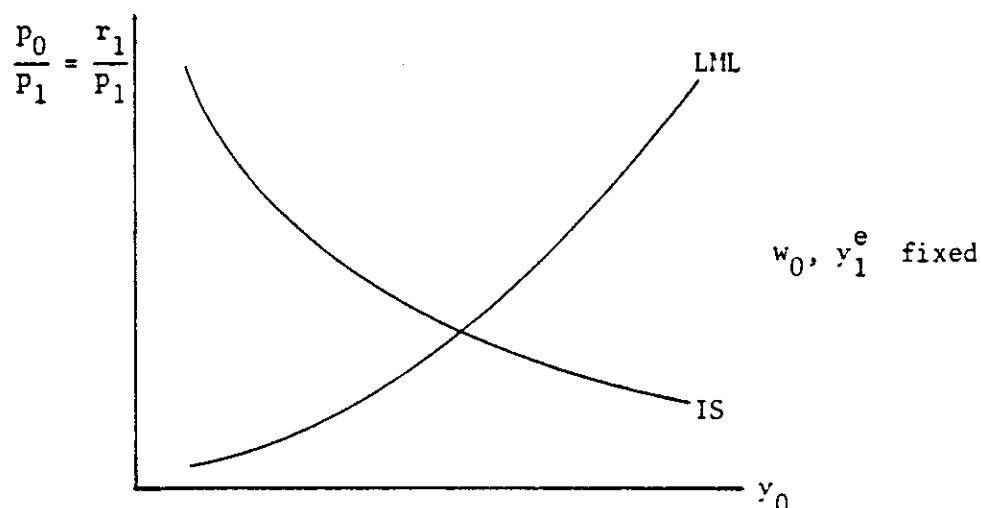
PROPOSITION 3: Let $k_0 > 0$ and let $M > 0$ be given. Then there is a nontrivial 5-dimensional rectangle R such that any choice of the variables $(w_0, p_0, r_1, p_1, y_1^e)$ in R that clears the IS, LM, and LML markets can be extended to a Walrasian equilibrium for the economy $E = (u, F, k_0, M)$. Moreover, the same is true after any policy perturbation at time 0.

Proof: See Appendix.

Proposition 3 demonstrates once again that the effects of policy are indeterminate unless one knows the institutional arrangements which shape the expectational responses of rational agents. We make the Keynesian hypotheses that w_0 and y_1^e are fixed. We shall show that these Keynesian institutional hypotheses lead to Keynesian policy predictions, while other assumptions can lead to monetarist or "new classical" policy predictions.

For fixed w_0 and y_1^e , we can think of p_0 as an increasing function of y_0 , calculated to maintain $w_0/p_0 = \partial F/\partial \ell$. Then the IS equa-

tion describes a downward sloping curve in $(y_0, r_1/p_1 = p_0/p_1)$ space. For as y_0 increases, w_0/p_0 decreases, and with w_0 held fixed, that means p_0 must increase. Thus both terms on the left of the IS equation increase with y_0 and so r_1/p_1 must decrease to maintain the balance. The LML equation describes an upward sloping curve in $(y_0, r_1/p_1 = p_0/p_1)$ space, if $0 < \gamma < 1$. We draw both:



At the intersection of the two curves all three markets, output, labor, and money, must clear.

Consider an increase in "animal spirits" y_1^e . That increases the firm's investment demand at any r_1/p_1 , and hence shifts out the IS curve. Current output y_0 rises, as does the real interest rate r_1/p_1 in order to maintain market clearing in period 0. According to Proposition 3 the increased optimism on the part of investors can be entirely validated in a new equilibrium, in which of course the higher expectations y_1^e will be confirmed. Animal spirits can be whimsical and rational.

Suppose that new money ΔM is printed and given to the old at time 0. This is precisely the policy experiment analyzed by Lucas [23] in an overlapping generations economy. As more money chases the same number

of goods, p_0 will presumably tend to rise. Lucas argued that if the increase in the stock of money is public knowledge, then a rational young consumer must understand that the rise in p_0 does not signal a change in relative prices, but only a rise in the general price level. He will not be tempted to alter the amount he works or his consumption. It is only, Lucas suggested, if the rise in price p_0 cannot be entirely attributed to a rise in M (if, for instance, the agent is not informed of changes in the money stock and if there are other influences on p_0) that the agent may suspect a change in relative prices p_1/p_0 and alter his behavior. Indeed in our model, if p_1/p_0 and y_1^e are held fixed, then p_0 will rise proportionately to $\Delta M/M$ and real output will remain the same. A Keynesian however, can maintain that even an agent perfectly informed about the money supply could rationally suspect that his wages would not keep pace with the increase in prices, and that he would therefore alter his spending and labor supply. Firms expecting the same thing would try to hire more workers, and output would rise. Indeed if we hold y_1^e and w_0 fixed in our model, the increase in ΔM shifts out the IS_0 curve, giving a higher period 0 market clearing output y_0 and real interest rate r_1/p_1 . Workers are willing to work more at the lower real wage since they know that their savings are earning a higher rate of return; in fact their utility rises. By Proposition 3 this effect can be fully validated in equilibrium.

One can similarly analyze other policy changes. Balanced budget spending increases (where the tax is raised from the young) are similar to the money financed spending above; they both move out the IS curve, with w_0 and y_1^e held fixed. Both policies have persistent consequences, since they affect the rate of capital accumulation. In fact it is easy to see that both policies increase employment l_1 in the medium run. For with

y_1^e held fixed and r_1/p_1 higher, it must be that ℓ_1 is higher too. Alternatively, if in addition to y_1^e we hold either w_0/p_0 or r_1/p_1 fixed (the two natural choices for monetarist assumptions) then both have neutral effects in period 0 (though not necessarily for period 1); their potency is sapped by an increase in current prices p_0 . A balanced budget increase in G and T shifts the IS curve out; but the LML curve is unaffected, and with w_0/p_0 (hence y_0) or r_1/p_1 fixed, output cannot rise. Hence prices rise, reducing the wealth and consumption of the old, to bring the IS curve back to its original position. The result is that government spending has crowded out exactly the same amount of consumption by the old. It also reduces the real money stock and the real value of savings of the young. In period 1 output is the same, but with lower real balances, $k_2 = (1-\alpha)y_1 - M/p_1$ will increase.

The fact that we have not permitted money to enter the utility function in our example limits the analysis somewhat by restricting the government to fiscal policy. Substituting money for bonds obviously can have no effect, since at the prevailing $r_1/p_1 = p_0/p_1$ savers are willing to change the composition of their portfolios. As a result, any intervention that raises output y_0 must also raise the real interest rate r_1/p_1 (if $0 < \gamma < 1$). If we had let money enter the utility function, then we could still have shown that there are two dimensions of indeterminacy. In that model, with w_0 and y_1^e held fixed, both monetary and fiscal policy would be effective.

There is, nonetheless, a potentially significant difference (beyond the voluntary vs. involuntary nature of unemployment) between the Keynesian policy analysis of Section 2.2, in which the period 0 labor market was permitted not to clear, and the elastic labor supply model of the present

section. When the LS and LM curves in $(r_1/p_1, y_0)$ space are combined with the labor market clearing equation, making p_0/p_1 an endogenous variable, adjusting to clear the labor market, with w_0 and y_1^e fixed, the resulting LML, IS curves may not have the natural Keynesian slopes. Thus in the simple example of this section, if we had chosen the utility parameter γ to be less than -1, instead of between 0 and 1, then the LML curve would have been downward sloping. We would still have found that market clearing, rational expectations, and full optimization are compatible with the volatile character of animal spirits. Moreover, if the nominal wages w_0 and animal spirits y_1^e do not change when government policy changes, then that policy will have nonneutral effects, and if LML is less steep than IS, the multipliers giving the effects of policy on output still have the Keynesian sign.¹

¹One can verify this easily from the usual diagram. There is one appealing property the model has when $\gamma \ll -1$, namely that near the monetaryless steady state, equilibrium is completely stable, in the sense that every policy change will put the economy on a path which converges back to the monetaryless steady state. (See Appendix.) We also show in the Appendix that when α is small and $\gamma \ll -1$, the IS curve is steeper than the LML curve.

3. Conclusion

Expectations matter. And they are not, in the short run, constrained by the hypotheses of rationality or market clearing (or lack of market clearing). No short run policy analysis can be sensible, even with complete knowledge of all the preferences and technologies of every agent and firm in the economy, if it is not based on some explicit understanding of how those agents expect the economy to respond to the policy.

Throughout, we have restricted our attention to the four fundamental macroeconomic quantities: money, output, capital, and labor. In the last section we made labor supply elastic and noted that when we counted equations and unknowns in the initial period, there were two degrees of freedom. Proposition 3 stated that there were then two degrees of freedom in the full intertemporal model. We then specified institutional hypotheses and considered the effect of those assumptions on government policy.

Clearly we can reverse this procedure. If we are concerned exclusively with the effects of policy on just two variables, say the short run interest rate r_1/p_1 and current output y_0 , and if labor supply is endogenous and if money is not a perfect substitute for bonds, then one can imagine in our model institutional and psychological rigidities to justify any policy prediction. This refutes the claim that, as a matter of pure logic, the methodological axioms of full optimization, market clearing, and rational expectations lead inexorably to a view of government policy interventions as neutral or ineffectual. Of course it is equally true, as a matter of logic, that it is possible to conceive of rational agents in the economy who form their expectations in a way that does render policy neutral. We have shown from this point of view, however, that Keynesian analysis has an important unity: if one follows Keynesian assumptions about the short run

stickiness of money wages, and the exogenous volatility of animal spirits, then one can be led to Keynesian policy predictions.

In Keynesian analysis the assumption that the labor market need not clear has at least a three-fold significance, which it is perhaps important to sort out. Lack of labor market clearing makes it possible to conceive of (Keynesian) equilibria with different levels of output and employment. It makes the system of demand and supply underdetermined, so that endogenous variables like animal spirits (i.e. expectations) which are normally fixed by the equilibrium conditions can be volatile. It creates unemployment that is involuntary. We have tried to argue here that one need not rely on what has seemed to many an ad hoc assumption about the labor market in order to get at least the first two desiderata of Keynesian analysis.

Evidently if both the supply and demand for labor are elastic, then it is logically possible to conceive of differing levels of equilibrium employment and unemployment, unless the equilibrium condition itself determines the outcome uniquely. In our example, workers are concerned about their purchasing power in the future, as well as their real wage in terms of current commodities, and thus labor market clearing does not determine a unique w_0/p_0 . In general, when labor supply is dependent on a vector of prices, indeterminacy of equilibrium, and hence indeterminacy of equilibrium output and employment will arise if any market is allowed not to clear.

The indeterminacy and Pareto suboptimality of Walrasian equilibria in overlapping generations economies can indeed be attributed to a lack of market clearing--not of the initial period labor market--but of the markets "at infinity."¹ Let us reconsider our static one-period model.

¹This is given a rigorous interpretation in [15].

Indeterminacy arises there because market clearing in period 0 is affected by prices in period 1, corresponding to markets we do not worry about clearing in the static model. To extend the static model to periods zero and one, we would have to specify prices through period two. Again there would be prices in period 2 influencing the model, corresponding to markets that need not be cleared in the model. As we extend the static model farther, we push farther back the period during which we do not require market clearing; in the limit, it is the market at infinity that need not clear. Thus we have explained how the overlapping generations model maintains the strictly Walrasian hypotheses of agent optimization, market clearing, and rational expectations, and at the same time allows for the sub-optimality and indeterminacy of equilibrium that Keynesian analysis achieved by dropping the requirement that the labor market clear in the initial period. Note that the introduction of elastic labor supply in period zero alone does not by itself increase the dimension of indeterminacy. It is the elastic supply of labor in periods 1, 2, ... that allows the labor market not to clear "at infinity" which accounts for the extra dimension of indeterminacy.

Through the introduction of elastic labor into the overlapping generations model, we showed that the comparative statics of the Keynesian model of Section 2.2 and the strictly Walrasian model of Section 2.3 were formally nearly identical. Each model allows for changes in employment, has two dimensions of indeterminacy, and can be analyzed in an IS-LM diagram. There remains the difference, that in the former model unemployment is involuntary, and in the latter, voluntary.

REFERENCES

- [1] Azariadis, C. (1981). "Self-fulfilling Prophecies," Journal of Economic Theory, 25(3), 380-396.
- [2] Balasko, Y., D. Cass and K. Shell (1980). "Existence of Competitive Equilibrium in a General Overlapping-Generations Model," Journal of Economic Theory, 23(3), 307-322.
- [3] Balasko, Y. and K. Shell (1980-81). "The Overlapping Generations Model, I-III," Journal of Economic Theory, 23(3), 281-306; 24(1), 112-142; 24(1), 142-152.
- [4] Barro, R. J. and H. Grossman (1971). "A General Disequilibrium Model of Income and Employment," American Economic Review, 61(1), 82-93.
- [5] Benassy, J. P. (1975). "Neo-Keynesian Disequilibrium in a Monetary Economy," Review of Economic Studies, 42(3), 502-523.
- [6] Calvo, G. (1978). "On the Indeterminacy of Interest Rates and Wages with Perfect Foresight," Journal of Economic Theory, 19(2), 321-337.
- [7] Cass, D. (1972). "On Capital Overaccumulation in the Aggregative, Noeclassical Model of Economic Growth," Journal of Economic Theory, 4(2), 200-223.
- [8] Clower, R. W. (1965). "The Keynesian Counterrevolution: A Theoretical Appraisal," in F. Hahn and F. Brechling (eds.), The Theory of Interest Rates. London: McMillan.
- [9] Debreu, G. (1970). "Economies with a Finite Set of Equilibria," Econometrica, 38, 387-392.
- [10] Diamond, P. A. (1965). "National Debt in a Neo-classical Growth Model," American Economic Review, 55(5), 1126-1150.
- [11] Dreze, J. (1975). "Existence of an Exchange Equilibrium under Price Rigidities," International Economic Review, 16(2), 301-320.
- [12] Gale, D. (1973). "Pure Exchange Equilibrium of Dynamic Economic Models," Journal of Economic Theory, 6(1), 12-36.
- [13] Gale, D. (1982). "Some Remarks on Overlapping Generations Models and Dynamic Models in General," Discussion Paper No. A 237 0182, Laboratoire d'Econometrie, Ecole Polytechnique.
- [14] Geanakoplos, J. (1978, 1980). "Sraffa's Production of Commodities by Means of Commodities: Indeterminacy and Suboptimality in Neo-classical Economics," RIAS Working Paper (1978), also Ch. 3 of "Four Essays on the Model of Arrow-Debreu," Ph.D. dissertation, Harvard University (1980).
- [15] Geanakoplos, J. and D. Brown (1982). "Understanding Overlapping Generations Economies as Lack of Market Clearing at Infinity," Yale mimeo.

- [16] Geanakoplos, J. and D. Brown (1985). "Comparative Statics and Local Indeterminacy in OLG Economies: An Application of the Multiplicative Ergodic Theorem," Cowles Foundation Discussion Paper No.
- [17] Grandmont, J.-M. (1985). "Endogenous, Competitive Business Cycles," Econometrica, 53(5), 995-1046.
- [18] Hahn, F. (1966). "Equilibrium Dynamics with Heterogeneous Capital Goods," Quarterly Journal of Economics, 53(4), 633-646.
- [19] Hahn, F. (1982). Money and Inflation. Oxford: Basil Blackwell.
- [20] Hicks, J. (1973). "Mr. Keynes and the Classics," in J. Hicks, Critical Essays in Money Theory. Oxford: Clarendon Press.
- [21] Kehoe, T. J. and D. K. Levine (1985). "Comparative Statics and Perfect Foresight in Infinite Horizon Models," Econometrica, 53(2), 433-453.
- [22] Keynes, J. M. (1936). The General Theory of Employment, Interest and Money. London: McMillan.
- [23] Lucas, R. E. (1972). "Expectations and the Neutrality of Money," Journal of Economic Theory, 4(2), 102-121.
- [24] Malinvaud, E. (1977). The Theory of Unemployment Reconsidered. Oxford: Basil Blackwell.
- [25] Peeters, M. (1982). "Inflation, Unemployment and Rational Expectations," Economic Theory Discussion Paper No. 62, Cambridge University.
- [26] Phelps, E., et al. (1970). The Microeconomic Foundations of Inflation and Employment Theory. New York: Norton.
- [27] Samuelson, P. A. (1958). "An Exact Consumption Loan Model of Interest, with or without the Social Contrivance of Money," Journal of Political Economy, 66(5), 467-482.
- [28] Shell, K. and J. Stiglitz (1967). "The Allocation of Investment in a Dynamic Economy," Quarterly Journal of Economics, 81(4), 592-609.
- [29] Shell, K. (1971). "Notes on the Economics of Infinity," Journal of Political Economy, 79(5), 1002-1011.
- [30] Sraffa, P. (1960). The Production of Commodities by Means of Commodities. Cambridge: Cambridge University Press.
- [31] Tirole, J. (1983). "Asset Bubbles and Overlapping Generations," Econometrica, 53(5), 1071-1100.
- [32] Tobin, J. (1980). Asset Accumulation and Economic Activity. Chicago: University Chicago Press.
- [33] Wilson, C. (1981). "Equilibrium in Dynamic Models with an Infinity of Agents," Journal of Economic Theory, 24(1), 95-111.

APPENDIX

Proof of Proposition 2

Let $\tilde{k} < \bar{k}$ be chosen as in (A3) to satisfy:

$$s^k\left(\frac{\tilde{w}}{\tilde{p}}, \frac{\tilde{r}}{\tilde{p}}, 1\right) = \tilde{k}, \quad s^m\left(\frac{\tilde{w}}{\tilde{p}}, \frac{\tilde{r}}{\tilde{p}}, 1\right) = \frac{M}{\tilde{p}} > 0$$

for $\tilde{w}/\tilde{p} = \varphi(\tilde{r}/\tilde{p}) = \partial F(\tilde{k}, 1)/\partial \lambda$, $\tilde{r}/\tilde{p} = \partial F(\tilde{k}, 1)/\partial k$ and $s^k(\bar{w}/\bar{p}, \bar{r}/\bar{p}, 0) = \bar{k}$, $s^m(\bar{w}/\bar{p}, \bar{r}/\bar{p}, 0) = 0$ where $\bar{w}/\bar{p} = \varphi(\bar{r}/\bar{p}) = \partial F(\bar{k}, 1)/\partial \lambda$.

We begin by proving a preliminary result about the range of money demand: Take $(p, I, r/p)$ satisfying $p \geq \tilde{p}$, $\tilde{w}/\tilde{p} \leq I \leq \bar{w}/\bar{p}$, $\bar{r}/\bar{p} \leq r/p \leq \tilde{r}/\tilde{p}$. Then $s^m(I, r/p, \gamma) = M/p_0$ has a unique solution $\gamma < 1$. Moreover the resulting function $\vec{\gamma}(p, I, r/p)$ is continuous, increasing in r/p , and decreasing in I . To prove this, note that if γ exists it must be unique, since by (A1) s^m is increasing in γ . If $(I_n, (r/p)_n, (p_0)_n, \gamma_n) \rightarrow (I, r/p, p_0, \gamma)$ and $s^m(I_n, (r/p)_n, \gamma_n) = M/(p_0)_n$, then by continuity of s^m , $s^m(I, r/p, \gamma) = M/p_0$. Thus if γ always exist, the fact that it lies in a bounded set and is unique implies that the function $\vec{\gamma}$ is continuous. The fact that $\vec{\gamma}$ is increasing in r/p and decreasing in I follows immediately from the properties of s^m . Finally, to show that γ exists to solve $s^m(I, r/p, \gamma) = M/p_0$ for $p \geq \tilde{p}$, $\tilde{w}/\tilde{p} \leq I \leq \bar{w}/\bar{p}$ and $\bar{r}/\bar{p} \leq r/p \leq \tilde{r}/\tilde{p}$, observe that $s^m(I, r/p, 0) - M/p \leq 0 = s^m(\tilde{w}/\tilde{p}, \tilde{r}/\tilde{p}, 1) - M/\tilde{p} \leq s^m(I, r/p, 1) - M/p$, so for some $0 \leq \gamma \leq 1$, $s^m(I, r/p, \gamma) - M/p = 0$.

We are now ready to prove the indeterminacy of the equilibrium. Let k_0 satisfy $\tilde{k} < k_0 < \bar{k}$, and let $p_0 > \tilde{p}$. We will construct an equilibrium

with first period output price p_0 . Let $r_0/p_0 = \partial F(k_0, 1)/\partial k$, let $w_0/p_0 = \partial F(k_0, 1)/\partial \ell$, and consider the function $\vec{\gamma}(p_0, w_0/p_0, r_1/p_1)$ derived in the last paragraph. No matter what real interest rate r_1/p_1 we choose, if we put $p_0/p_1 = \vec{\gamma}(p_0, w_0/p_0, r_1/p_1)$ then we will clear the money market. All that remains is to find the correct r_1/p_1 to simultaneously clear the savings-investment market.

Recall that the demand for new capital, $k^d = \bar{k}^d(r_1/p_1)$ satisfies $r_1/p_1 = \partial F(k^d, 1)/\partial k$, and the function \bar{k}^d is continuous and decreasing in r_1/p_1 . Consider that $s^k(w_0/p_0, \bar{r}/\bar{p}, \gamma(p_0, w_0/p_0, \bar{r}/\bar{p})) - \bar{k}^d(\bar{r}/\bar{p}) < s^k(\bar{w}/\bar{p}, \bar{r}/\bar{p}, 0) - \bar{k}^d(\bar{r}/\bar{p}) = 0$ since s^k is increasing in income and decreasing in p_1/p_0 . But $s^k(w_0/p_0, \tilde{r}/\tilde{p}, \gamma(p_0, w_0/p_0, \tilde{r}/\tilde{p})) - \bar{k}^d(\tilde{r}/\tilde{p}) \geq s^k(\tilde{w}/\tilde{p}, \tilde{r}/\tilde{p}, 1) - \bar{k}^d(\tilde{r}/\tilde{p}) = 0$, since s^k is increasing in I and decreasing in p_1/p_0 and $\gamma(p_0, w_0/p_0, \tilde{r}/\tilde{p}) \leq 1$. Hence there is r_1/p_1 , p_0/p_1 with $\tilde{r}/\tilde{p} \leq r_1/p_1 \leq \bar{r}/\bar{p}$ such that $s^k(w_0/p_0, r_1/p_1, p_0/p_1) = \bar{k}^d(r_1/p_1)$ and $s^m(w_0/p_0, r_1/p_1, p_0/p_1) = M/p_0$. It follows that $\tilde{k} \leq k_1 \leq \bar{k}$, and that $p_1 \geq p_0 \geq \tilde{p}$. Thus the whole process can be repeated with these starting values. Q.E.D.

Proposition 3: The economy E possesses a 2-dimensional family of equilibria indexed by the price, p_0 , and the real wage, w_0/p_0 (or, equivalently, the real rate of interest, r_0/p_0) at the initial period $t = 0$. (Proposition 3 in the text can be immediately deduced from this.)

Proof: The initial capital stock is given and is equal to k_0 . Choose p_0 and w_0/p_0 arbitrarily but such that $k_0^{(1-\alpha)^{1/\alpha}} (w_0/p_0)^{(\alpha-1)/\alpha} \geq (1+\delta)M/p_0$, where $\delta = \alpha/(1-2\alpha)$. To show that there exists a Walrasian equilibrium price system (p^*, w^*, r^*) with p_0 and w_0 exogenously specified, we proceed as follows. First we choose values for p_1^* and w_1^*/p_1 which

guarantee that the period 0 markets clear; then we show that the construction can be repeated ad infinitum.

From the factor price frontier equation $\phi\left(\frac{w}{p}\right) = \alpha(1-\alpha)^{(1-\alpha)/\alpha}\left(\frac{w}{p}\right)^{(\alpha-1)/\alpha}$, the real rate of interest is determined.

Given the real wage or, equivalently, the real return to capital, the capital/labor ratio desired by the firm is determined:

$(k_0/\ell_0) = \alpha^{1/(1-\alpha)}(r_0/p_0)^{1/(\alpha-1)}(1-\alpha)^{1/\alpha}(w_0/p_0)^{1/\alpha}$; since the supply of capital is given and equal to k_0 , the demand for labor necessary for equilibrium in the capital market is $\ell^d(w_0^*, k_0) = (1-\alpha)^{1/\alpha}k_0(w_0^*/p_0)^{-1/\alpha}$.

The supply of labor is derived from the maximization of

$u(\ell_0, c_1, c_2) = -\frac{1}{\beta}\ell_0^\beta + (1/\gamma)c_2^\gamma$ subject to the budget constraint

$(w_0, \ell_0) = c_2 \cdot p_1$; it is thus $\ell^s(w_0, p_0, p_1) = (w_0/p_0)^{\gamma/(\beta-\gamma)}(p_0/p_1)^{\gamma/(\beta-\gamma)}$.

Equilibrium in the labor market is thus obtained by setting

$$(1) \quad p_1 = \left[(1-\alpha)^{(\beta-\gamma)/\alpha\gamma} k_0^{(\beta-\gamma)/\gamma} w_0^{-(\beta+\alpha\gamma-\gamma)/\alpha\gamma} \right]^{-1} p_0$$

The supply of period 1 capital is now determined as the difference between the real savings of the young during period 0 and the amount carried over in real money balances. The former are $s(w_0, p_0, p_1) = (w_0/p_0)(w_0, p_0, p_1) = (w_0/p_0)^{\beta/(\beta-\gamma)}(p_0/p_1)^{\gamma/(\beta-\gamma)}$; equilibrium in the money market requires that the latter equal M/p_0 . Substituting for p_1 we obtain

$$(2) \quad k_1 = (1-\alpha)^{1/\alpha} k_0 (w_0/p_0)^{(\alpha-1)/\alpha} - M/p_0.$$

To guarantee equilibrium in all period 0 markets, it only remains to make sure that the returns to money and capital coincide. This is equivalent to $r_1/p_1 = (p_0/p_1)$. Substituting, we obtain

$$(3) \quad \left(\frac{w_1}{p_1} \right) = \alpha \frac{\alpha}{1-\alpha} (1-\alpha) \frac{\beta-2\gamma+\alpha\gamma}{\gamma(\alpha-1)} \frac{\alpha(\beta-\gamma)}{k_0^{\gamma(\alpha-1)}} \left(\frac{w_0}{p_0} \right)^{\frac{\beta-\gamma+\alpha\gamma}{\gamma(1-\alpha)}} .$$

To complete the proof, it remains to demonstrate that the construction can be repeated ad infinitum without violating the boundary positivity constraints; equivalently, that if $(1-\alpha)^{1/\alpha} k_0 (w_0/p_0)^{(\alpha-1)/\alpha} \geq (1+\delta)M/p_0$, it is also the case that $(1-\alpha)^{1/\alpha} k_1^* (w_1^*/p_1)^{(\alpha-1)/\alpha} \geq (1+\delta)M/p_1^*$. Substituting, we see that this is equivalent to $[(1-\alpha)^{1/\alpha} k_0 (w_0/p_0)^{(\alpha-1)/\alpha} - M/p_0] \geq [(\alpha/(1-\alpha))(1+\delta)M/p_0]$. But by the choice of p_0 and w_0 it is sufficient that $\delta \geq (\alpha/(1-\alpha))(1+\delta)$, which in turn is equivalent to $\delta \geq \alpha/(1-2\alpha)$, which is indeed the case. Q.E.D.

Remarks are in order:

◦The economy has a unique monetary stationary Walrasian equilibrium with $\bar{k} = (1-\alpha)^{\gamma/(\beta-\gamma)} \alpha^{(\alpha\gamma+\beta-\gamma)/(1-\alpha)(\beta-\gamma)}$, $\bar{r}/\bar{p} = 1$, and $M/\bar{p}^{**} = \bar{k} (1-2\alpha)/\alpha$.

◦The economy has also a unique non-monetary stationary Walrasian equilibrium with $M/\bar{p} = 0$, $\bar{w}/\bar{p} = (1-\alpha)^{1/(1-\alpha)}$, $\bar{k} = \alpha^{\gamma/(\beta-\gamma)} (1-\alpha)^{(\beta-\gamma+\alpha\gamma)/(1-\alpha)(\beta-\gamma)}$.

◦In the definition of equilibrium we have not required that the level of employment remain bounded. This is formally correct, since we have not assumed any such lower bound on the individual consumption sets. On the other hand, the imposition of a lower bound does not decrease the dimension of the set of equilibria, at least for some values of the preference and technology parameters α and γ , respectively. To see this, consider the non-monetary stationary equilibrium. It is an equilibrium of the non-linear dynamical system $(p, k, w/p)$ defined by equations (1), (2), and (3).

If we rewrite (1) in terms of $1/p_1$ instead of p_1 , and write the system in terms of $(1/p_0, k, w/p)$ near $(0, \bar{k}, \bar{w}/\bar{p})$, the eigenvalues of the linearized system at the non-monetary equilibrium are $\lambda_1 = \alpha/(1-\alpha)$ and $\lambda_2, \lambda_3 = \left(\beta \pm \sqrt{\beta^2 - 4\alpha\beta(1-\alpha)\gamma} \right) / (2\gamma(1-\alpha))$. Since $0 < \alpha < 1$, for $\gamma \ll -\beta < 0$ all eigenvalues are less than 1 in absolute value, and the equilibrium is locally asymptotically stable. It follows that if the initial values $(1/p_0, k_0, w_0/p_0)$ are sufficiently close to $(0, \bar{k}, \bar{w}/\bar{p})$, then the level of employment will stay bounded along all equilibrium paths, converging to \bar{l} , the monetaryless steady state employment.

We can also check the slopes of the IS and LML curves near the monetaryless steady state. A simple calculation shows that the two curves are always defined (in the absence of policy), for fixed y_1^e and w_0 , by.

$$\text{IS: } \frac{r_1}{p_1} = \frac{\alpha y_1^e}{(1-\alpha)y_0 - M/p_0}$$

$$\text{LML: } \frac{r_1}{p_1} = (1-\alpha)^{-1} k_0^{\frac{\alpha\beta}{\gamma(\alpha-1)}} y_0^{\frac{(\beta+\alpha\gamma-\gamma)}{\gamma(1-\alpha)}}$$

It is clear that the IS curve is always negatively sloped. We claim that at the monetaryless steady state, when α is near 0 and $\gamma \ll 0$, the LML curve is also negatively sloped, but less steep than the IS curve.

At the Monetaryless Steady State:

$$\frac{\bar{r}}{\bar{p}} = \frac{\alpha}{1-\alpha}, \quad k = \alpha^{\frac{\gamma}{\beta-\gamma}} (1-\alpha)^{\frac{\beta+\alpha\gamma-\gamma}{(1-\alpha)(\beta-\gamma)}}, \quad y = \alpha^{\frac{\gamma}{\beta-\gamma}} (1-\alpha)^{\frac{\alpha\beta}{(\beta-\gamma)(1-\alpha)}}$$

$$\text{IS: } \left. \frac{d(r_1/p_1)}{dy_0} \right|_{\bar{y}} = - \frac{\alpha}{1-\alpha}$$

$$\begin{aligned}
\text{LM: } \left. \frac{d(r_1/p_1)}{dy_0} \right|_{\bar{y}, \bar{k}} &= \frac{\beta + \gamma(\alpha - 1)}{\gamma(1 - \alpha)} (1 - \alpha)^{-1} k^{\frac{\alpha\beta}{\gamma(\alpha - 1)}} y^{\frac{\beta - 2\gamma(1 - \alpha)}{\gamma(1 - \alpha)}} \\
&= \frac{\beta + \gamma(\alpha - 1)}{\gamma(1 - \alpha)^2} \alpha^{\frac{\alpha\beta}{(\beta - \gamma)(\alpha - 1)}} (1 - \alpha)^{\frac{\alpha\beta(\beta + \alpha\gamma - \gamma)}{(1 - \alpha)(\alpha - 1)\gamma(\beta - \gamma)}} \\
&\quad \frac{\beta - 2\gamma(1 - \alpha)}{\alpha(\beta - \gamma)(1 - \alpha)} (1 - \alpha)^{\frac{-\alpha\beta(\beta - 2\gamma(1 - \alpha))}{(1 - \alpha)(\alpha - 1)\gamma(\beta - \gamma)}} \\
&= \frac{\beta + \gamma(\alpha - 1)}{\gamma} \alpha^{\frac{\alpha\beta - \beta + 2\gamma - 2\alpha\gamma}{(\beta - \gamma)(\alpha - 1)}} (1 - \alpha)^{\frac{-\alpha\beta + 2\beta - 2\gamma + 2\alpha\gamma}{(\alpha - 1)(\beta - \gamma)}} \\
&= \frac{\beta + \gamma(\alpha - 1)}{\gamma} \left(\frac{\alpha}{1 - \alpha} \right)^{\frac{\alpha\beta - \beta + 2\gamma - 2\alpha\gamma}{(\beta - \gamma)(\alpha - 1)}} (1 - \alpha)^{\frac{\beta}{(\alpha - 1)(\beta - \gamma)}}
\end{aligned}$$

For stability--at $\alpha = 0$ --it is necessary and sufficient that $\gamma < -1$.

It follows that both the IS and LM curves are negatively sloped.

We want to guarantee that at their intersection the IS is steeper than the LM, i.e.

$$-\left. \frac{dr_1}{dy_0} \right|_{\text{IS}} > -\left. \frac{dr_1}{dy_0} \right|_{\text{LM}} .$$

Equivalently,

$$\frac{\alpha}{1 - \alpha} > \frac{\beta - \gamma(1 - \alpha)}{-\gamma} \frac{\alpha}{1 - \alpha} \frac{\alpha^{\frac{\alpha\beta - \beta + 2\gamma - 2\alpha\gamma}{(\beta - \gamma)(\alpha - 1)}}}{(1 - \alpha)^{\frac{\beta}{(\alpha - 1)(\beta - \gamma)}}}$$

or

$$\left(\frac{\alpha}{1 - \alpha} \right)^{\frac{\gamma}{\beta - \gamma}} > \frac{\beta - \gamma(1 - \alpha)}{-\gamma} (1 - \alpha)^{\frac{\beta}{(\alpha - 1)(\beta - \gamma)}}$$

But this is indeed the case for $\alpha \sim 0$ and $\gamma \ll 0$; the l.h.s. tends to ∞ and the r.h.s. to $1 - \alpha$.