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REAL INDETERMINACY WITH FINANCIAL ASSETS

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I. Introduction.

Assets can be thought of as being of two kinds: those that specify contingent deliveries of real goods (real assets) and those that specify contingent deliveries of units of account (financial assets). The type of assets introduced by Arrow (1953) in his seminal work (delivery takes place in only one state) can be looked at either way but much of the general equilibrium research on assets (see Radner, 1972, Hart, 1975) has followed the real asset approach. Only recently attention has turned to financial assets (J. Werner, 1985, D. Cass, 1984, D. Duffie, 1985).

It should be expected that the equilibria of a world with financial assets will exhibit some degree of indeterminacy. The reason is clear: by arbitrarily fixing the price of a commodity in different states of the world we can make the financial asset correspond to a continuum of distinct real asset configurations. This indeterminacy can be real (i.e. affect the allocation of real goods) and not just nominal. Indeed, D. Cass (1984a) has exhibited a two states - one bond model where there is a one dimensional continuum of equilibria.

But the indeterminacy need not be real. For example, it will be purely nominal in any of the following cases:

- (a) complete markets (i.e., generically, as many bonds as states).
- (b) there are only Arrow securities.
- (c) there is only one trader.
- (d) there are no bonds.
- (e) the endowment allocation is a Pareto optimum.

The purpose of this paper, which takes up after D. Cass (1984a, 1984b), is to find the degree of real indeterminacy inherent in models with purely financial assets. We solve the problem for the case where there are enough traders (precisely, the number of traders is larger than the number of bonds) and the asset returns structure is in general position. We find that if the number of bonds is non-zero and fewer than the number of states then, generically, the number of dimensions of real indeterminacy is $S-1$, one less than the number of states.

There is something of a surprise in the above result, namely the dimension of real indeterminacy does not depend on the number of bonds (except in the two limit cases). Indeed, one initial conjecture was $S-B$. This points to an intriguing qualitative discontinuity at the complete market configuration. If markets are financially complete then the model is determinate. Let just one bond be missing and the model becomes highly indeterminate. Thus, in this sense, the complete markets hypothesis lacks robustness.

We also consider the possibility that some assets are real and others nominal. It is natural to suppose that the presence of real assets would reduce the amount of indeterminacy, but in fact we shall show that if there is enough incompleteness (more than twice as many states as bonds) and at least two financial assets, then, generically, there are still $S-1$ dimensions of real indeterminacy. We also point out similar results for mixed securities paying units of account in some states and commodities in others (a possible model for instruments such as collateralized loans). In summary: when markets are incomplete, the presence of financial assets creates an indeterminacy in competitive equilibrium allocations of a degree that does not depend on the absence of real assets.

Finally, we wish to mention a recent paper of D. Cass (1985) which has been developed independently and which investigates similar issues.

II. The Model And Result.

There are $L+1$ physical commodities ($\ell = 0, \dots, L$) and two dates. Spot trade tomorrow will take place under any of S states ($s = 1, \dots, S$). Today there is trade on current goods and on B financial bonds ($b = 1, \dots, B$). Bonds pay in units of account. We express their payoff by an $S \times B$ matrix with generic entry r_{sb} . We assume that R is in general position, i.e. every set of B rows from R are linearly independent.

There are $H+1$ consumers ($h = 0, \dots, H$). Every consumer h has a utility function u^h defined on $R_{++}^{(L+1)(S+1)}$ and satisfying the standard differentiability, monotonicity, curvature and boundary conditions needed to get a well defined C^r differentiable excess demand function (see, for example, Mas-Colell, 1985, Chp.2). Note: the degree of differentiability r is large enough for the subsequent transversality arguments to be justified. Every consumer also has an initial endowment vector $\omega^h \in JCR^{(L+1)(S+1)}$ where J is a closed rectangle with non-empty interior. In this section, when we say that a property of economies $E = (R, u^h, \omega^h; h \in H)$ is generic, we mean that for any \bar{R} and \bar{u}^h there is an open, dense subset \mathcal{D} of $\prod_{h \in H} J$ whose complement has Lebesgue measure zero

and such that the property applies to all economies $(\bar{R}, \bar{u}^h, \omega^h; h \in H)$ with endowment chosen in \mathcal{D} .

Definition 1. An allocation (\bar{x}, \bar{y}) of goods and bonds is a financial asset equilibrium for $E = (R, u^h, \omega^h; h \in H)$ if:

- (i) $\sum_h \bar{x}^h = \sum_h \omega^h$, $\sum_y \bar{y}^h = 0$, and
- (ii) there is a price system $p \in R_{++}^{(L+1)(S+1)}$, $q \in R^B$ such that for every h , (\bar{x}^h, \bar{y}^h) maximizes $u^h(x^h)$ on

$$B^h(p, q) = \{x^h, y^h: p_0 \cdot x_0^h + q \cdot y^h \leq p_0 \cdot \omega_0^h, \text{ and } p_s \cdot x_s^h \leq p_s \cdot \omega_s^h + \sum_b y_b^h r_{sb}, \text{ all } s\}$$

With financial assets there is, in general, some indeterminacy in the equilibrium allocations.

If the indeterminacy affects only the holdings of bonds, y^h , then we call it nominal indeterminacy. Otherwise, we call it real indeterminacy. We are interested in the degree of real indeterminacy.

Theorem 1. *If $H \geq B$ and $0 < B < S$ then, generically, there are $S-1$ dimensions of real indeterminacy, i.e. the set of equilibrium allocations of commodities x contains the image of a C^1 , one-to-one function with domain \mathbb{R}^{S-1} .*

III. Proof Of Theorem 1.

The proof proceeds according to three main steps. The first introduces the notion of a real numeraire asset equilibrium and shows that the set of financial asset equilibria can be parameterized by an $S-1$ dimensional family of real numeraire asset equilibria. The second step proves a key technical lemma about B dimensional subspaces of $S>B$ dimensional spaces. The third step then shows that there is some $S-1$ dimensional open subset of the real numeraire asset equilibria for which all the commodity allocations x are distinct.

Step 1. Given the prices p_{s0} of the zero commodity (or, equivalently, the price of money $\lambda_s = 1/p_{s0}$ in terms of good 0) our system of financial assets is equivalent to a system of real numeraire assets where each asset pays in (equivalent worth) of the zero commodity. More precisely, given the matrix $\bar{R} = (\bar{r}_{sb})$, representing the payoffs of real assets in the numeraire commodity for each state s , let us define an allocation (\bar{x}, \bar{y}) of goods and real assets to be a real numeraire asset equilibrium if (i) and (ii) of Definition 1 are satisfied, but with respect to the budget set

$$\bar{B}^h(p, q) = \{(x, y) : p_0 \cdot x_0 + q \cdot y \leq p_0 \cdot \omega_0^h \text{ and } p_s \cdot x_s \leq p_s \cdot \omega_s^h + p_{s0} \cdot \sum_b y_b \bar{r}_{sb} \text{ for all } s\}.$$

We have just argued that (\bar{x}, \bar{y}) is a financial asset equilibrium, with asset return R , if and only if (\bar{x}, \bar{y}) is a real numeraire asset equilibrium with $p_{s0} = 1$ for all $s \in S$ and asset return matrix $\bar{R} = \Lambda R$, where Λ is some diagonal $S \times S$ matrix having $\lambda_s > 0$ for all s .

It has been shown in Geanakoplos-Polemarchakis [1985] that for each economy $\bar{E} = (\bar{R}, u^h, \omega^h; h \in H)$ satisfying our assumptions, a real numeraire asset equilibrium exists with $p_{s0} = 1$ for all s . Thus it suffices to show that, fixing $\lambda_1 = 1$, as $(\lambda_2, \dots, \lambda_S)$ varies over some open subset R_{++}^{S-1} we can take a differentiable selection of equilibrium allocations which are all distinct.

Step 2. For any $S \times B$ matrix A , let us denote by $sp[A]$ the linear subspace of \mathbb{R}^S spanned by the B columns of A .

Lemma 1. Let R be an $S \times B$ matrix (with $B < S$) such that every set of B rows are linearly independent. Let Λ_1 and Λ_2 be diagonal invertible matrices, and suppose that $\text{sp}[\Lambda_1 R] = \text{sp}[\Lambda_2 R]$. Then $\Lambda_1 = \alpha \Lambda_2$ for some scalar $\alpha \neq 0$.

Proof. Note first that if $\text{sp}[\Lambda_1 R] = \text{sp}[\Lambda_2 R]$, then $\text{sp}[\Lambda_2^{-1} \Lambda_1 R] = \text{sp}[R]$, hence it suffices to consider $\Lambda_1 = \Lambda$ and $\Lambda_2 = I$, where Λ has λ_s as generic diagonal elements and I is the identity.

Suppose that $\text{sp}[AR] \subset \text{sp}[R]$. This means precisely that there is some $B \times B$ matrix Y with $AR = RY$. The s row R_s of R is thus seen to be an eigenvector of Y with eigenvalue λ_s . Take any s with $B+1 \leq s \leq S$. From the hypothesis that R is in general position, we can find non-zero scalars μ_1, \dots, μ_B with $R_s = \mu_1 R_1 + \dots + \mu_B R_B$. If say $\mu_1 = 0$ then rows $s, 2, \dots, B$ would be linearly dependent. It follows that $R_s Y = \lambda_s (\mu_1 R_1 + \dots + \mu_B R_B)$ or, $(\mu_1 R_1 + \dots + \mu_B R_B) Y = \lambda_1 \mu_1 R_1 + \dots + \lambda_B \mu_B R_B$. Hence $\mu_1 (\lambda_s - \lambda_1) R_1 + \dots + \mu_B (\lambda_s - \lambda_B) R_B = 0$; since $\mu_i \neq 0$ for all $i = 1, \dots, B$, that is possible only if $\lambda_s = \lambda_1 = \dots = \lambda_B$. ■

We can see at once how important this lemma is in showing that the real asset equilibrium allocations are distinct for distinct Λ .

Lemma 2. Let (x, y) and (\hat{x}, \hat{y}) be real numeraire asset equilibria for, respectively, $E = (AR, u^h, \omega^h; h \in H)$ and $\hat{E} = (\hat{\Lambda}R, u^h, \omega^h; h \in H)$. Suppose that $\dim \text{sp}[y^1, \dots, y^H] = \dim \text{sp}[\hat{y}^1, \dots, \hat{y}^H] = B$. Then $x \neq \hat{x}$ unless $\Lambda = \alpha \hat{\Lambda}$ for some $\alpha > 0$.

Proof. Consider the vectors $(ARy^h; h \in H)$ and $(\hat{\Lambda}R\hat{y}^h; h \in H)$. By the hypothesis and Lemma 1 there is some h such that $\hat{\Lambda}R\hat{y}^h \neq ARy^h$. Suppose that $x^h = \hat{x}^h$. From the smoothness and boundary conditions on u^h , we must have that the goods equilibrium prices p and \hat{p} are equal. But by Walras Law, which holds state by state, that implies $\hat{\Lambda}R\hat{y}^h =$

$\Lambda R y^h$. ■

Step 3. Let M be the set of diagonal matrices with $\lambda_1 = 1$. We will now conclude the proof by showing that generically there is an open non-empty subset VCM and a C^1 parameterization of allocations $x(\Lambda), y(\Lambda), \Lambda \in V$, such that $(x(\Lambda), y(\Lambda))$ is a numeraire asset equilibrium with return matrix ΛR , and $y(\Lambda)$ satisfies the full dimension condition of Lemma 2 (and, therefore, if $\Lambda, \Lambda' \in V$ and $\Lambda \neq \Lambda'$ then $x(\Lambda) \neq x(\Lambda')$).

Let $f(p, q, \Lambda, \omega)$ be the excess demand function from $P = \mathbb{R}_{++}^{L(S+1)} \times \mathbb{R}^B \times \mathbb{R}_{++}^{S-1} \times \mathbb{R}_{++}^{(L+1)(S+1)(H+1)}$ to $\mathbb{R}^{L(S+1)} \times \mathbb{R}^B$. Of course this function is not defined for all $q \in \mathbb{R}^B$ but only for those asset prices which satisfy a "non-arbitrage" condition.

Lemma 3. *f is a C^1 function on the (non-empty) interior of its domain of definition. Moreover, $f(p, q, \Lambda, \omega) = 0$ if and only if p, q are real numeraire asset equilibrium prices for $E = (\Lambda R, u^h, \omega^h; h \in H)$. Also, $f(p, q, \Lambda, \omega) = 0$ implies that $\text{rank } \partial_{\omega_0} f(p, q, \Lambda, \omega) = L(S+1) + B$.*

Proof. See Geanakoplos and Polemarchakis [1985]. ■

Define now $g: P \times S^{B-1} \rightarrow \mathbb{R}^{L(S+1)} \times \mathbb{R}^B \times \mathbb{R}^B$, where S^{B-1} is the $B-1$ sphere, by

$$g(p, q, \Lambda, \omega, z) = (f(p, q, \Lambda, \omega), \sum_{h=1}^B z_h y_1^h, \dots, \sum_{h=1}^B z_h y_B^h)$$

where y_b^h is the h -th consumer demand for bond b at (p, q, Λ, ω) .

Lemma 4. *If $g(p, q, \Lambda, \omega, z) = 0$ then $\text{rank } \partial_{\omega} g(p, q, \Lambda, \omega, z) = L(S+1) + 2B$.*

Proof. Let $(p, q, \Lambda, \omega, \bar{z}) \in g^{-1}(0)$. Because $\bar{z} \in S^{B-1}$ we know that $\bar{z}_h \neq 0$ for some h . Given Lemma 3 it suffices to show that for any $b = 1, \dots, B$ there is some perturbation Δ^h and Δ^0 of the endowments of consumers h and 0 that leaves f and $y_{h', b}$

unaffected for all $(h',b') \neq (h,b)$ but does change $y_{h,b}$. Let Δ^h be given by a decrease in ω_{00}^h of q_b and an increase in ω_{s0}^h of $\lambda_s R_{sb}$ for all $s = 1, \dots, S$. Let Δ^0 be given by an increase in ω_{00}^0 of q_b and a decrease in ω_{s0}^0 of $\lambda_s R_{sb}$. Then consumer h decreases his demand $y_{h,b}$ by one unit and aggregate f is unaffected. ■

By the Transversality Theorem (see e.g., Mas-Colell, 1985, Subsection 1.I) for a.e ω $f_\omega^{-1}(0)$ and $g_\omega^{-1}(0)$ are C^r -manifolds of respective dimensions $S-1$ and $S-2$. By Sard's theorem the projection of $f_\omega^{-1}(0)$ on M has a regular value $\hat{\Lambda}$. From Geanakoplos and Polemarchakis [1985] we know that $(\hat{p}, \hat{q}, \hat{\Lambda}, \omega) \in f^{-1}(0)$ for some \hat{p}, \hat{q} . Therefore, from the Implicit Function Theorem, there are open sets $P'_\omega \subset P_\omega$, $V' \subset M$ and a C^1 function $\xi: V' \rightarrow P'_\omega$ such that $(p, q, \Lambda, \omega) \in f^{-1}(0) \cap (P_\omega \times \{\omega\})$ if and only if $\xi(\Lambda) = (p, q, \Lambda, \omega)$. Let $\bar{P}'_\omega \subset P_\omega$ be the closure of P'_ω . Then the projection of $g_\omega^{-1}(0) \cap (\bar{P}'_\omega \times S^{B-1})$ on M is compact and so we can find a non-empty open set $V \subset V'$ which is disjoint from this projection. But this means that if $\Lambda \in V$ then the $(y^h)_{h=1}^B$ corresponding to $\xi(\Lambda)$ satisfy the spanning condition of Lemma 2. We have thus obtained the desired parameterization of equilibria. ■

IV. Conclusion.

We conclude by formulating a number of observations in the form of remarks. Only remarks 4 and 9 are given formal demonstration.

Remark 1. The assumption that R is not only of full rank, but also in general position, is used in an important way in the proof. It rules out the case where there are $B < S$ pure Arrow securities, for the rows corresponding to states with no payoffs cannot be linearly independent. Indeed, in the case where all the assets are pure Arrow securities, there are typically a finite number of equilibria. The restriction to generic endowments is also necessary; for pareto optimal endowment allocations, for example, no trade will take place no matter what the asset structure.

Remark 2. Suppose that $B < H+1 \leq S-1$. In that case there are at least as many dimensions of indeterminacy as there are individual types. One would expect very often to find Pareto comparable financial equilibria. To put it differently, the monetary standard to which assets could be tied may be Pareto comparable.

Remark 3. The conclusion of the theorem implies that the set of equilibrium real allocations x contains a non-empty $S-1$ topological (i.e. C^0) manifold. The conclusion can be strengthened to C^1 manifold (one only needs to show that the derivative of the parameterization has full rank everywhere). Because nothing of economic substance is involved we skip the extra technical work.

Remark 4. The conclusion of the Theorem can be sharpened when $H \geq SB$. In this case the entire set of equilibrium real allocations can be expressed as the differentiable one-to-one image of an $S-1$ C^1 manifold (the observation parallel to Remark 3 also applies here). For the proof one considers the function $g: P \times S^{S(B-1)} \rightarrow R^{L(S+1)} \times R^B \times R^{SB}$, where $S^{S(B-1)}$ is the $S(B-1)$ sphere, defined by

$$g(p, q, A, \omega, z^1, \dots, z^S) \\ = (f(p, q, A, \omega), \sum_{h=1}^B z_h^1 y_1^h, \dots, \sum_{h=1}^B z_h^1 y_B^h, \dots, \sum_{h=1}^B z_h^S y_1^{(S-1)B+h}, \dots, \sum_{h=1}^B z_h^S y_B^{(S-1)B+h}).$$

Exactly as before one shows that 0 is a regular value of g , hence for a generic ω , 0 is a regular value of g_ω . But this is impossible unless $g_\omega^{-1}(0) = \emptyset$ because the range of g_ω

has greater dimension than its domain. If ω is generic for f and g we have then that $f_{\omega}^{-1}(0)$ is an $S-1$ manifold and because Lemma 2 applies the real allocations corresponding to any two points in $f_{\omega}^{-1}(0)$ are necessarily distinct.

Remark 5. We have considered an economy with only two time periods. This is more general than it may appear at first sight. We could imagine an economy with many time periods, as in Debreu [1959], where time and uncertainty resolve themselves as in a tree. But in an equilibrium with perfect conditional foresight (and von-Neumann-Morgenstern utilities) the tree model can be regarded as a special case of our two period model with as many states of nature as there are nodes in the tree (less one for date 0). The number of states of nature, hence the degree of indeterminacy, can grow geometrically with the length of the time horizon.

Remark 6. Financial assets in our model yield payoffs in what might be called "inside money". The aggregate endowment of each asset, and the aggregate payoff in each state, is zero. It is, of course, of central importance to the indeterminacy that we find in financial assets markets, that the payoffs not directly enter any consumer's utility; indeed, this is what distinguishes financial assets from real assets. In any equilibrium for a finite horizon model outside money cannot be positively priced. However in an infinite horizon model, like the overlapping generations model, it is possible to have non-trivial outside money. One could easily introduce uncertainty and financial assets that have non-zero aggregate supply into an overlapping generations economy. Indeed, what is called money in that model is the archtypical financial asset.

Remark 7. The possibility of combining remarks 5 and 6 is intriguing. One is irresistibly lead to conjecture that in an overlapping generations economy with repeated moves of nature and incomplete financial markets there will be an infinity of dimensions of indeterminacy!

Remark 8. Although our theorem only holds for a generic set of endowments, one can guess that there are economies where across states the endowments and von Neumann-Morgenstern utilities are identical, and yet if markets are incomplete, the presence of financial assets creates $S-1$ dimensions of real indeterminacy, i.e. $S-1$ dimension of "sunspot" equilibria.

Remark 9. One imagines that in actuality there are both nominal and real assets. It seems reasonable to conjecture that the larger is the proportion of real assets, the smaller is the indeterminacy associated with financial assets. However, we now show that the dimension of real indeterminacy is robust to the introduction of real securities, as long as markets are sufficiently incomplete.

In order to avoid the difficulties with existence that are known to plague models with real assets which yield vector-valued payoffs (see Hart, 1975), we shall confine our attention to real numeraire assets, i.e. real assets that, for each state $s \in S$, pay only in commodity 0.

Let R be the $S \times B$ matrix representing the B financial assets and let \bar{R} be the $S \times A$ matrix representing the real numeraire assets. Thus r_{sb} is the number of dollars paid by financial asset b in state s , and \bar{r}_{sa} is the amount of good 0 paid by real asset a in state s .

The definition of an equilibrium is now a triple (x, y, \bar{y}) satisfying (i) and (ii) of Definition 1 where the budget set $B^h(p, q, \bar{q})$ is now defined as

$$((x, y, \bar{y}): p_0 \cdot x_0 + q \cdot y + \bar{q} \cdot \bar{y} \leq p_0 \cdot \omega_0^h \quad \text{and}$$

$$p_s \cdot x_s \leq p_s \cdot \omega_s^h + \sum_{b=1}^B y_b^h r_{sb} + p_s 0 \sum_{a=1}^A y_a^h \bar{r}_{sa} \quad \text{for all } s).$$

Theorem 2. Suppose that $B \geq 2$, $S > 2(A+B)$ and $H > A+B$. Then for a generic choice of matrices R and \bar{R} , there is a generic set of endowments such that each of the corresponding economies has $S-1$ dimensions of real indeterminacy (in the sense of Theorem 1).

Proof. By following the logic of the proof of Theorem 1, it suffices to show that for a generic choice of matrices R and \bar{R} , there is no diagonal matrix $\Lambda \neq \sigma I$ with $\Lambda R \subset \text{sp}[R, \bar{R}]$.

Let \mathcal{D} be the set of $S \times (B+A)$ matrices $W = (R, \bar{R})$ which have rank $B+A$, satisfying $r_{sb} \neq 0$ for $b \in \{1, 2\}$ and $s \in S$, and have $r_{s1}/r_{s'1} \neq r_{s2}/r_{s'2}$ for all $s \neq s' \in S$. Clearly, \mathcal{D} is an open, dense subset of the set of all $S \times (B+A)$ matrices, whose complement has null

measure. We shall show that there is a generic subset $\mathcal{D}' \subset \mathcal{D}$ of matrices (R, \bar{R}) for which only diagonal matrices Λ that are multiples of the identity satisfy $\Lambda R \subset \text{sp}[R, \bar{R}]$.

Suppose in particular that $\Lambda R^1 \in \text{sp}[R, \bar{R}]$ and that $\Lambda R^2 \in \text{sp}[R, \bar{R}]$, where R^1, R^2 are, respectively, the first and second column of R . Since we can rewrite

$$\begin{bmatrix} \lambda_1 & & & 0 \\ & \ddots & & \\ 0 & & & \\ & & & \lambda_S \end{bmatrix} \begin{bmatrix} r_{11} \\ \vdots \\ r_{S1} \end{bmatrix} = \begin{bmatrix} 1/r_{11} & & & 0 \\ & \ddots & & \\ 0 & & & \\ & & & 1/r_{S1} \end{bmatrix}^{-1} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_S \end{bmatrix}$$

we must have two $(A+B)$ -dimensional vectors z and \hat{z} with

$$\begin{bmatrix} 1/r_{11} & & & 0 \\ & \ddots & & \\ 0 & & & \\ & & & 1/r_{S1} \end{bmatrix} \begin{bmatrix} R \\ \bar{R} \end{bmatrix} z = \begin{bmatrix} 1/r_{12} & & & 0 \\ & \ddots & & \\ 0 & & & \\ & & & 1/r_{S2} \end{bmatrix} \begin{bmatrix} R \\ \bar{R} \end{bmatrix} \hat{z}.$$

We know, of course, that $z = (\lambda, 0, \dots, 0)$, $\hat{z} = (0, \lambda, 0, \dots, 0)$ is always a solution for any λ . We show that for a generic choice of R and \bar{R} , there is no other choice of z and \hat{z} that constitute a solution.

Note first that if $S > 2(A+B)$, then there are more equations to satisfy than there are unknowns. It suffices to show, therefore, that eliminating from the domain the previous special configuration of z and \hat{z} , the above system of equations has zero as a regular value. That is, it suffices that given any R, \bar{R} and solution $z \neq (\lambda, 0, \dots, 0)$, $\hat{z} \neq (0, \lambda, 0, \dots, 0)$, we can perturb any equation s by changing the R, \bar{R}, z, \hat{z} in such a manner that the remaining equations are not disturbed. A routine application of the transversality theorem would then finish the proof.

Suppose that for a solution R, \bar{R}, z, \hat{z} , there is some k , $3 \leq k \leq A+B$ with either $z_k \neq 0$ or $\hat{z}_k \neq 0$ (or both). It follows that $z_k/r_{s1} \neq \hat{z}_k/r_{s2}$ for at least $S-1$ of the S states. For any such state s , a small perturbation of w_{sk} (if $k \leq B$, $w_{sk} = r_{sk}$; if $B < k \leq A+B$, $w_{sk} = \bar{r}_{s, A+B-k}$) will change the s -th equality without disturbing the rest. For the remaining state s_0 , one can change z_1 . That will affect every equality, including s_0 ;

but this is clearly a perturbation with an affect which is independent of the other $S-1$ perturbations.

Suppose alternatively that $z_k = \hat{z}_k = 0$ for all $k \geq 3$. Then we are back to exactly the framework of Lemma 1, with only two financial assets, and we know that there are no solutions to the system of equations except for $z = (\lambda, 0, \dots, 0)$, $\hat{z} = (0, \lambda, 0, \dots, 0)$, which we have already excluded from the domain. ■

Remark 10. More generally, we could also consider mixed assets which pay both in real commodities and in money. Once again there will be natural sufficient conditions guaranteeing the dimension of indeterminacy is $S-1$. For example, suppose that for each asset the states can be divided into those in which the asset pays in units of account and those in which the asset pays in numeraire commodities. Loans with collateral are of this type: there is a specified financial payment and a real collateral payoff in case of default (which here should be thought of as an exogenous event). One could also think of firm-issued debt in similar terms. Let A be the total number of mixed assets. Suppose that for every $s \in S$ there are two assets and a collection $F(s)CS$ of at least $2A+1$ states (including s) on which the two assets both pay in units of account. Then the proof of Theorem 2 easily yields that there are $S-1$ dimension of real indeterminacy.

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