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FRACTIONAL MATRIX CALCULUS AND THE DISTRIBUTION OF MULTIVARIATE TESTS

by

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ABSTRACT

Fractional matrix operator methods are introduced as a new tool of distribution theory for use in multivariate analysis and econometrics. Earlier work by the author on this operational calculus is reviewed and to illustrate the use of these methods we give an exact distribution theory for a general class of tests in the multivariate linear model. This distribution theory unifies and generalizes previously known results, including those for the standard F statistic in linear regression, for Hotelling's T^2 test and for Hotelling's generalized T_0^2 test. We also provide a simple and novel derivation of conventional asymptotic theory as a specialization of exact theory. This approach is extended to generate general formulae for higher order asymptotic expansions. Thus, the results of the paper provide a meaningful unification of conventional asymptotics, higher order asymptotic expansions and exact finite sample distribution theory in this context.

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1. INTRODUCTION

The purpose of this paper is to provide a short review of some new methods I have been working with recently in the field of econometric distribution theory. These methods have turned out to be surprisingly useful in furnishing solutions of a rather general nature to a wide range of problems that occur in finite sample econometrics. Since these problems are very similar to those that arise naturally in other areas of statistical theory, notably multivariate analysis, I hope that the methods I have been developing will be of some interest to mathematical statisticians who are working in these related fields.

The methods rely on the concept of matrix fractional differentiation and therefore belong to an operational calculus. At an abstract level the techniques may be interpreted within the framework of pseudo-differential operators on which there is a large mathematical literature (see, for example, Treves (1980)). At the algebraic and purely manipulative level it is hard to find any references in the literature beyond those which apply to scalar methods of fractional calculus. Even here most attention is concentrated on the Riemann-Liouville definition of a fractional integral (or derivative). Whereas in applications to statistical distribution theory, I have found that a form of Weyl calculus yields the simplest and most direct results. It is also the most amenable to matrix generalizations. For an introduction to scalar fractional operators of this type the reader is referred to the books by Ross (1974a), Spanier and Oldham (1974) and the review article by Lavoie, Osler and Tremblay (1976).

The use of an operational calculus in problems of distribution theory has many natural advantages. In the first place, seemingly difficult problems may often be solved quite simply with rather elegant general solution formulae. The latter usually avoid the complications of series representations, including those that are expressed in terms of zonal or invariant polynomials which many researchers find daunting and difficult for numerical work. Second, the routine manipulation of operators frequently leads to simplifications which are not otherwise obvious. Both these advantages arise, of course, in other applications of operator methods. However, I have discovered that there are some advantages to operational methods which are peculiar to their use in statistical distribution theory.

Perhaps the most important of these is that the methods provide a simple means of unifying limiting distribution theory, asymptotic expansions and exact distribution theory. This is because the operator representation of the exact finite sample distribution often lends itself to the immediate derivation of the asymptotic distribution and associated expansions about the asymptotic distribution. Thus, all three forms of distribution theory may often be derived from the same general formula. An example will be studied later in the paper.

A further special advantage of operational methods is that they help to resolve mathematical problems for which existing techniques of distribution theory are quite unsuited. One of the more prevalent of these in multivariate models, at least in the present stage of the development of the subject, arises from the presence of random matrices (usually sample covariance matrices) that are embedded in tensor formations. These tensor formations inhibit the use of conventional techniques such as change of variable methods. Prominent examples of such problems occur in econometrics with seemingly unrelated regression equations, and systems estimation methods like three stage least squares. In multivariate analysis many multivariate tests, such as the Wald test for testing coefficient restrictions in the multivariate linear model come into this category. Since this particular test includes so many commonly occurring statistics such as the F test, Hotelling's T^2 and the T_0^2 statistic we shall use it as the focus of our attention in this paper as a prototypical application of the operational method. For other examples and related work the reader may refer to some other papers by the author (1984a, 1984b, 1985, 1986).

2. FRACTIONAL OPERATORS

Historically, the concept of a fractional operator arose from the attempt by classical mathematicians, principally Leibnitz, Euler, Liouville and Riemann, to extend the meaning of the operation of differentiation (to an integral order) to encompass differentiation of an arbitrary order. These classical mathematicians addressed the following question: given the operator $D = d/dx$ and rules for working with D^n to the integer order n what, if any, meaning may be ascribed to D^α where α is fractional or possibly even complex? An interesting historical study of the evolution of ideas in this field is provided by Ross (1974b), who traces the origin of this search for an extended meaning of the differential operator to correspondence between Leibnitz and L'Hospital in 1695.

Using the integral representation of the gamma function a very simple intuitive approach to fractional (complex) operators may be developed. Thus, if $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(z) > 0$ we have:

$$(1) \qquad z^{-\alpha} = \Gamma(\alpha)^{-1}\int_0^\infty e^{-zt}t^{\alpha-1}dt \ .$$

This formula, which is extensively used in applied mathematics, provides a simple mechanism for replacing an awkward power of a complex variable that occurs in a denominator by an integral involving an exponent which is much simpler to deal with. In a certain sense, this simple idea is the key to much of the subject and to its multivariate extensions that we shall examine below.

If we now consider replacing z in (1) by the operator $D = d/dx$ we note that whereas $D^{-\alpha}$ is difficult to interpret e^{-Dt} is not. The operator e^{-Dt} yields Taylor series representations for analytical functions and may be regarded as a simple shift operator. Thus

$$(2) \qquad e^{-Dt}f(x) = f(x-t)$$

for f analytic. This suggests that we may formally write:

$$(3) \qquad D^{-\alpha}f(x) = \Gamma(\alpha)^{-1}\int_0^{\infty} f(x-t)\,t^{\alpha-1}dt \ .$$

Then if the right side of (3) is absolutely convergent it may be used as a definition for the fractional integral $D^{-\alpha}f(x)$. Quite general operators with complex powers such as D^{μ} may now be defined by writing

$$D^{\mu}f(x) = D^{-\alpha}\{D^{m}f(x)\}$$

where $\mu = m-\alpha$, $m = +\text{ve}$ integer and $\mathrm{Re}(\alpha) > 0$. Operators of this type obey the law of indices and are commutative, although this is not true of general matrix extensions, of course. At an abstract level, these operators may be used to form algebraic systems such as operator algebras, which may in turn be used to justify routine manipulations of the operators as algebraic symbols.

After a change of variable on the right side (3) may be written as:

$$(4) \qquad D^{-\alpha}f(x) = \Gamma(\alpha)^{-1}\int_{-\infty}^{x} f(s)(x-s)^{\alpha-1}ds$$

which corresponds to one form of the Weyl fractional integral (see, for example, Miller (1974)).

It is easy to show with this definition that:

$$(5) \qquad D^{-\alpha}e^{ax} = e^{ax}a^{-\alpha} .$$

This may be proved using (3) for $Re(\alpha) > 0$, $Re(a) > 0$. The result (5) then holds by analytic continuation for all complex $a \neq 0$ and for all complex α . Similar results extending the rules for differentiating elementary functions may be obtained in the same way. Another rule which is quite useful is:

$$(6) \qquad D^{\mu}(1-x)^{-\beta} = \frac{\Gamma(\beta+\mu)}{\Gamma(\beta)}(1-x)^{-\beta-\mu}$$

$$Re(\beta) > 0 , \quad Re(\beta+\mu) > 0 .$$

(5) and (6) illustrate the great advantage that the Weyl operator (3) has over the Riemann-Liouville operator defined by

$$(7) \qquad D_{x_0}^{-\alpha}f(x) = \Gamma(\alpha)^{-1}\int_{x_0}^{x} f(s)(x-s)^{\alpha-1}ds$$

for $Re(\alpha) > 0$. The finite limit of integration x_0 in (7) allows us to admit a much wider class of functions into the definition (avoiding the conditions of convergence required by the improper integral involved in the Weyl definition (4)). However, when (7) is applied to elementary functions the results are usually much more complex than (5) and (6). For example, in the case of (5) we have

$$D_{x_0}^{-\alpha} e^{ax} = a^{-\alpha} e^{ax} \Gamma(\alpha)^{-1} \Gamma(\alpha,\ a(x - x_0))$$

where $\Gamma(\alpha, z)$ is the incomplete gamma function. This complication turns out to be a significant drawback to the Riemann Liouville operator in multivariate extensions and in applications to distribution theory. I have, therefore, found it most useful in my own work to employ (3) and its various generalizations rather than (7).

Multivariate extensions follow from the matrix gamma integral:

$$(8) \qquad (\det Z)^{-\alpha} = \Gamma_n(\alpha)^{-1} \int_{S>0} \operatorname{etr}(-SZ)(\det S)^{\alpha-(n+1)/2} dS$$

where Z is an $n \times n$ matrix with $\operatorname{Re}(Z) > 0$ and $\operatorname{Re}(\alpha) > (n-1)/2$. $\Gamma_n(\alpha)$ is the multivariate gamma functions which may be evaluated as $\Gamma_n(\alpha) = \pi^{n(n-1)/4} \Pi_{i=1}^{n} (\alpha - (i-1)/2)$. The integral (8) is extensively used in multivariate analysis. Its significance was first brought into prominence in the remarkable paper by Herz (1955).

We may now proceed as in the scalar case by introducing the matrix operator $\partial Z = \partial/\partial Z$. Whereas $(\det \partial Z)^{-\alpha}$ is difficult to interpret $\operatorname{etr}(-\partial ZS)$ is not. In fact, if $f(Z)$ is an analytic function of the matrix variate Z the operator $\operatorname{etr}(-\partial ZS)$ yields the matrix Taylor series representation

$$(9) \qquad \operatorname{etr}(-\partial ZS) f(Z) = f(Z-S)$$

generalizing (2). We may therefore define

$$(10) \quad D_Z^{-\alpha} f(Z) = \Gamma_n(\alpha)^{-1} \int_{S>0} f(Z-S)(\det S)^{\alpha-(n+1)/2} dS \ ; \quad D_Z = \det \partial Z$$

provided the integral is absolutely convergent and $\operatorname{Re}(\alpha) > (n-1)/2$. The general case of an arbitrary complex power of D_Z may be dealt with in the same way as the scalar case by setting

$$D_Z^{\mu} f(Z) = D_Z^{-\alpha}\{D_Z^m f(Z)\}$$

for $\mu = m-\alpha$ with m = +ve integer and $\mathrm{Re}(\alpha) > (n-1)/2$.

Elementary functions of matrix argument may be complex differentiated as before. Thus

$$(11) \qquad D_Z^{-\alpha} \mathrm{etr}(AZ) = \mathrm{etr}(AZ)(\det A)^{-\alpha}$$

generalizes (5) and may be proved for $\mathrm{Re}(A) > 0$, $\mathrm{Re}(\alpha) > (n-1)/2$ using (10). The formula (11) holds by analytic continuation for all non singular A and for all complex α . In a similar way, we find

$$(12) \qquad D_Z^{\mu} \det(I-Z)^{-\beta} = \frac{\Gamma_n(\beta+\mu)}{\Gamma_n(\beta)} \det(I-Z)^{-\beta-\mu}$$

$$\mathrm{Re}(\beta) > (n-1)/2 \ , \quad \mathrm{Re}(\beta+\mu) > (n-1)/2$$

generalizing (6).

It is also useful to work with more complicated operators than D_Z . For example, if R is a $q \times nm$ matrix of rank $q \leq nm$ and M is a positive definite $m \times m$ matrix, then we may define

$$(13) \quad [\det(R(\partial Z \otimes M)R')]^{-\alpha} f(Z)$$

$$= \Gamma_q(\alpha)^{-1} \int_{S>0} [\mathrm{etr}\{-R(\partial Z \otimes M)R'S\} f(Z)](\det S)^{\alpha-(q+1)/2} dS$$

if the integral converges absolutely. The exponent $R(\partial Z \otimes M)R'S$ in the integrand of (13) is linear in the operator ∂Z and we may write:

$$\mathrm{tr}[-R(\partial Z \otimes M)R'S] = \mathrm{tr}[-\partial Z Q(S)]$$

where the $n \times n$ matrix Q is linear in the elements of S. Thus, (13) has the form:

$$\Gamma_q(\alpha)^{-1} \int_{S>0} f(Z - Q(S))(\det S)^{\alpha-(q+1)/2} dS \ .$$

Extensions to more complex tensor formations of operators are possible in an analogous fashion. Some of these are given and applied in one of the author's papers (1985) on the subject. When $f(Z)$ is an elementary function like $\mathrm{etr}(ZA)$ one obtains extensions of rules such as (11):

$$(14) \quad \det(R(\partial Z \otimes M)R')^{-\alpha} \mathrm{etr}(AZ) = \mathrm{etr}(AZ)\det(R(A \otimes M)R')^{-\alpha} \ .$$

Once again (14) is proved for $\mathrm{Re}(A) > 0$ and $\mathrm{Re}(\alpha) > (q-1)/2$ and then analytically continued for all non singular A and all complex α.

3. MULTIVARIATE TESTS

To illustrate the use of these operator methods in distribution theory we shall consider some commonly occurring multivariate tests. What we present here will in large part be a review of work already done by the author in (1984, 1986) and the reader is referred to these papers for full details and generalizations. However, we shall present some new results on asymptotic expansions and exact distribution functions.

We shall be concerned with the multivariate linear model

$$(15) \qquad y_t = Ax_t + u_t \ ; \qquad (t = 1, \ldots, T)$$

y_t is a vector of n dependent variables, A is an $n \times p$ matrix of parameters, x_t is a vector of nonrandom independent variables and the u_t are i.i. $N(0,\Omega)$ errors with Ω positive definite. Let us suppose that we are interested in a general linear hypothesis involving the elements of A, which we write in null and alternative form as:

$$(16) \qquad H_0 : R \text{ vec } A = r \ , \quad H_1 : R \text{ vec } A - r = b \neq 0$$

where R is a $q \times np$ matrix of known constants of rank q, r is a known vector and $\text{vec}(A)$ stacks the rows of A.

From least squares estimation of (15) we have:

$$(17) \qquad A^* = Y'X(X'X)^{-1} \ , \quad \Omega^* = Y'(I - P_X)Y/N$$

where $Y' = [y_1, \ldots, y_T]$, $X' = [x_1, \ldots, x_T]$, $P_X = X(X'X)^{-1}X'$ and $N = T-p$. We take X to be a matrix of full rank $p \leq T$ and define $M = (X'X)^{-1}$.

The Wald statistic for testing the hypothesis (16) is

$$(18) \qquad W = (R \text{ vec } A^* - r)'\{R(\Omega^* \otimes M)R'\}^{-1}(R \text{ vec } A^* - r)$$
$$= N\ell'B\ell$$

where $\ell = R \text{ vec } A^* - r$ is $N(b,V)$ under H_1 with $V = R(\Omega \otimes M)R'$, and $B = \{R(C \otimes M)R'\}^{-1}$. $C = Y'(I - P_X)Y$ is central Wishart with covariance matrix Ω and N degrees of freedom.

We define $y = \ell'B\ell$ and write y in canonical form as

$$(19) \qquad y = g'Gg$$

where $g = V^{-1/2}\ell$ is $N(m, I_q)$, $m = V^{-1/2}b$ and

$G^{-1} = V^{-1/2}\{R(C \otimes M)R'\}V^{-1/2}$. With this notation we see that y and W are simply positive definite quadratic forms in normal variates, conditional on C . The distribution problem becomes one of integrating up this conditional distribution over the distribution of C .

Important special cases of the statistic W are as follows:

(i) The regression F statistic

Set $n = 1$, $A = a$, $H_0 : Ra = r$, $\Omega^* = s^2$ and then

$$W = (Ra^* - r)'[R(X'X)^{-1}R']^{-1}(Ra^* - r)/s^2$$

$$\equiv \text{const. } F_{q,N} \tag{20}$$

where $F_{q,N}$ denotes a variate with an F distribution with q and N degrees of freedom. We shall use the notation of (20) to signify here and elsewhere in the paper that W (or its distribution) is equivalent to (that of) a constant times an F variate with the designated degrees of freedom.

(ii) Hotelling's T^2 statistic

Set $R = R_1 \otimes r_2'$, $H_0 : R_1Ar_2 = r$ and then

$$W = (R_1A^*r_2 - r)'[R_1\Omega^*R_1']^{-1}(R_1A^*r_2 - r)/r_2'Mr_2$$

$$\equiv \text{const. } x'S^{-1}x \equiv \text{const. } F_{q,N-q+1} \tag{21}$$

with $x \equiv N(0, R_1\Omega R_1')$ and $S \equiv W_q(N, R_1\Omega R_1')$ under the null; x and S are of course independent.

(iii) <u>The</u> T_0^2 <u>statistic</u>

Set $R = R_1 \otimes R_2'$, $H_0 : R_1 A R_2 = r$ with R_1 $q_1 \times n$ and R_2 $m \times q_2$. Then

$$
\begin{aligned}
W &= \mathrm{vec}(R_1 A^* R_2 - r)'[R_1 \Omega^* R_1' \otimes R_2' M R_2]^{-1} \mathrm{vec}(R_1 A^* R_2 - r) \\
&= \mathrm{tr}[(R_1 A^* R_2 - r)'(R_1 \Omega^* R_1')^{-1}(R_1 A^* R_2 - r)(R_2' M R_2)^{-1}] \\
&= \text{const. } \mathrm{tr}(XX' S_2^{-1}) \\
&= \text{const. } \mathrm{tr}(S_1 S_2^{-1}) \qquad (22)
\end{aligned}
$$

with $X \equiv$ matrix $N(0, R_1 \Omega R_2' \otimes I_{q_2})$, $S_2 \equiv W_{q_1}(N, R_1 \Omega R_1')$ and $S_1 = W_{q_1}(q_2, R_1 \Omega R_1')$ under the null. Because of invariance to the covariance matrix in (22) we may treat S_1 as $W_{q_1}(q_2, I_{q_1})$ and S_2 as $W_{q_1}(N, I_{q_2})$; S_1 and S_2 are independent.

Interestingly, the exact distribution of the statistic $\mathrm{tr}(S_1 S_2^{-1})$ has not been found in the statistical literature, in spite of apparently substantial efforts by many researchers (see Pillai (1976, 1977) and Muirhead (1982) for reviews). Many conjectures have been made about the form of the exact density of this statistic. The classic article by Constantine (1966) which gives a series representation that is valid over the unit interval [0,1] is still perhaps the most general treatment. We shall show below how the distribution may be found in the general case quite simply by operator algebra. A full treatment is available in the author's paper (1984).

4. THE NULL DISTRIBUTION

It is shown in Phillips (1986, equation (32)) that the null density of W in the general case (18) is given by:

$$(23)\quad \mathrm{pdf}(w) = \frac{w^{q/2-1}}{N^{q/2}\Gamma\left(\frac{q}{2}\right)}\left[\det\left(L(\partial X\otimes I)L'\right)^{1/2}{}_0F_0\left(-L(\partial X\otimes I)L', w/N\right)\det(I-X)^{-N/2}\right]_{X=0} .$$

where

$$(24)\qquad L = [R(\Omega\otimes M)R']^{-1/2}R(\Omega^{1/2}\otimes M^{1/2}) .$$

The function ${}_0F_0(-L(\partial X\otimes I)L', w/N)$ in (23) is a linear operator which may be explicitly represented as:

$$\int_{V_{1,q}} \mathrm{etr}(-(w/N)L(\partial X\otimes I)L'hh')(\underline{dh})$$

where $(\underline{dh})$ denotes the normalized invariant measure on the sphere $V_{1,q} = \{h : h'h = 1\}$. An alternative representation in terms of an absolutely convergent operator power series is also available:

$$\sum_{j=0}^{\infty} \frac{(-1)^j (w/N)^j C_j(L(\partial X\otimes I)L')}{j!C_j(I)}$$

where $C_j(\ )$ denotes the top order zonal polynomial of degree j, for which explicit formulae were given by James (1964).

The simplicity of (23) is unusually striking. Yet, as we shall see, all existing exact distribution theory for the null case is embodied in this formula. Moreover, (23) also delivers the appropriate asymptotic theory and asymptotic expansions with little effort. In the following specializations we shall use the notational reductions detailed for these special cases in Section 3.

(i) The regression F statistic

$$\begin{aligned}
\mathrm{pdf}(w) &= \text{const. } w^{q/2-1}\Big[(\partial x)^{q/2} e^{-\partial x w/N}(1-x)^{-N/2}\Big]_{x=0} \\
&= \text{const. } w^{q/2-1}\Big[e^{-\partial x w/N}(1-x)^{-N/2-q/2}\Big]_{x=0} \\
&= \text{const. } w^{q/2-1}(1+w/N)^{-(N+q)/2} \\
&\equiv \text{const. } F_{q,N}\,.
\end{aligned}$$

The reductions in the second and third lines above follow directly from the rules (5) and (6) given earlier for fractional differentiation.

(ii) Hotelling's T^2 statistic

Noting that $L = L_1 \otimes \ell_2'$ with $L_1 L_1' = I_{q_1}$, $\ell_2'\ell_2 = 1$, $q_2 = 1$ and $q = q_1$, we find that the density of W is:

$$\begin{aligned}
\mathrm{pdf}(w) &= \text{const. } w^{q/2-1}\Big[\det(L_1\partial X L_1')^{1/2}\,{}_0F_0(-L_1\partial X L_1',\ w/N)\det(I-X)^{-N/2}\Big]_{X=0} \\
&= \text{const. } w^{q/2-1}\Big[(\det \partial X_{11})^{1/2}\,{}_0F_0(-\partial X_{11},\ w/N)\det(I-X_{11})^{-N/2}\Big]_{X=0} \\
&= \text{const. } w^{q/2-1}\Big[{}_0F_0(-\partial X_{11},\ w/N)\det(I-X_{11})^{-(N+1)/2}\Big]_{X_{11}=0} \\
&= \text{const. } w^{q/2-1}\Big[\int_{V_{1,q}} \operatorname{etr} -(w/N)\partial X_{11} hh'\ \underline{(dh)}\det(I-X_{11})^{-(N+1)/2}\Big]_{X_{11}=0} \\
&= \text{const. } w^{q/2-1}\int_{V_{1,q}} \det(I+(w/N)hh')^{-(N+1)/2}\underline{(dh)} \\
&= \text{const. } w^{q/2-1}(1+w/N)^{-(N+1)/2} \\
&\equiv \text{const. } F_{q,N-q+1}\,.
\end{aligned}$$

In the second line of this argument X_{11} is a $q_1 \times q_1$ matrix of auxiliary variates obtained from the $q \times q$ matrix X by transforming $X \to PXP'$ where $P' = [L_1', K']$ is orthogonal. Note that under this transformation $\partial X \to P'\partial XP$ and $L_1 \partial X L_1' \to \partial X_{11}$, giving the stated result.

(iii) <u>The</u> T_0^2 <u>statistic</u>

$L = L_1 \otimes L_2$, $L_1 L_1' = I_{q_1}$, $L_2 L_2' = I_{q_2}$, $q = q_1 q_2$ and the density of W is:

$$\text{pdf}(w) = \text{const. } w^{q/2-1}\left[\det(L_1 \partial X L_1')^{q_2/2} {}_0F_0(-L_1\partial XL_1' \otimes I_{q_2}, w/N)\det(I-X)^{-N/2}\right]_{X=0}$$

$$= \text{const. } w^{q/2-1}\left[{}_0F_0(-\partial X_{11} \otimes I_{q_2}, w/N)\det(I - X_{11})^{-(N+q_2)/2}\right]_{X_{11}=0}$$

(25) $$= \text{const. } w^{q/2-1}\left[\int_{V_{1,q}} \text{etr}\{-(w/N)(\partial X_{11} \otimes I)hh'\}(\underline{dh})\det(I - X_{11})^{-(N+q_2)/2}\right]_{X_{11}=0}$$

(26) $$= \text{const. } w^{q/2-1} \int_{V_{1,q}} \det\left(I + (w/N)Q\right)^{-(N+q_2)/2} (\underline{dh})$$

where $Q = \sum_{i=1}^{q_2} h_i h_i'$ and $h' = (h_1', \ldots, h_{q_2}')$. For $0 \le w/N < 1$ we may expand the determinantal expression in the integrand of (26) giving

$$\text{pdf}(w) = \text{const. } w^{q/2-1}\sum_{k=0}^{\infty} \frac{(-w/N)^k}{k!} \sum_{\kappa}\left(\frac{N+q_2}{2}\right)_{\kappa} \int_{V_{1,q}} C_{\kappa}(Q)(\underline{dh})$$

(27) $$= \text{const. } w^{q/2-1}\sum_{k=0}^{\infty} \frac{(-w/N)^k}{k!\left(\frac{q_1 q_2}{2}\right)_k} \sum_{\kappa}\left(\frac{N+q_2}{2}\right)_{\kappa}\left(\frac{q_2}{2}\right)_{\kappa} C_{\kappa}(I_{q_1})$$

which is the series obtained by Constantine (1966) for the null distribution. The integration over $V_{1,q}$ leading to (27) may be obtained quite simply using operator methods. The reader is referred to Phillips (1984) where full details are given.

An alternative everywhere convergent series is obtained by working from (25). Once again details are provided in Phillips (1984). We state only the final result here:

$$\text{(28)}\quad \mathrm{pdf}(w) = \frac{\text{const. } w^{q/2-1}}{(N+q_2w)^{q_1(q_2+N)/2}} \sum_{k=0}^{\infty} \frac{\left(\frac{w}{N+q_2w}\right)^k}{k!}$$

$$\cdot \sum_\theta \sum_\kappa \left\{ \frac{b_\theta^\kappa \left(\frac{q_1q_2-1}{2}\right)_\kappa}{\left(\frac{q_1q_2}{2}\right)_\kappa} \right\} \left(\frac{q_2+N}{2}\right)_\theta C_\theta(I_{q_1})$$

where the summations are over all partitions θ , κ of k into $\leq q_1$ parts, and the b_θ^κ are certain constants.

(iv) Asymptotic theory

We employ the simple asymptotic representation

$$\det(I-X)^{-N/2} \sim \mathrm{etr}(NX/2)$$

for $X \sim 0$ in (23) and deduce immediately that:

$$\text{(29)}\qquad \mathrm{pdf}(w) \sim \frac{w^{q/2-1}e^{-q/2}}{2^{q/2}\Gamma\left(\frac{q}{2}\right)} \equiv \chi_q^2 \ .$$

Thus, the asymptotic distribution appears as a special case of (23) in a single step.

(v) Higher order asymptotics

We transform $X \to Z = NX$ in (23) giving $\partial X = N\partial Z$ and:

$$(30)\quad \mathrm{pdf}(w) = \Gamma(q/2)^{-1} w^{q/2-1}\left[\det\left(L(\partial Z \otimes I)L'\right)^{1/2} {}_0F_0\left(-L(\partial Z \otimes I)L', w\right)\det(I-Z/N)^{-N/2}\right]_{Z=0}$$

We now expand the determinantal factor as $N \to \infty$:

$$\det(I - Z/N)^{-N/2} = \exp\left\{-\frac{N}{2}\ln \det\left(I - \frac{Z}{N}\right)\right\}$$

$$\sim \exp\left\{\frac{N}{2}\sum_{k=1}^{\infty} \frac{1}{k}\mathrm{tr}\left(\frac{Z}{N}\right)^{k}\right\}$$

$$= \mathrm{etr}\left(\frac{1}{2}Z\right)\exp\left\{\frac{1}{2}\sum_{j=1}^{\infty} \frac{\mathrm{tr}(Z^{j+1})}{(j+1)N^{j}}\right\}$$

$$(31)\quad = \mathrm{etr}\left(\frac{1}{2}Z\right)\left[1 + \sum_{k=1}^{\infty} \frac{1}{N^{k}} \sum_{\ell=1}^{k} \frac{(1/2)^{\ell}}{\ell!} \sum_{j_1,\ldots,j_\ell}^{*} \frac{\mathrm{tr}\, Z^{j_1+1}\ \mathrm{tr}\, Z^{j_2+1} \ldots \mathrm{tr}\, Z^{j_\ell+1}}{(j_1+1)(j_2+1)\ldots(j_\ell+1)}\right] .$$

In the final expression (31) the summation Σ^* is over all ℓ-tuples of positive integers $(j_1, \ldots, j_\ell)$ satisfying

$$\sum_{i=1}^{\ell} j_i = k\ , \quad j_i = 1, 2, \ldots, k\ ; \quad (1 \le i \le \ell)\ .$$

We deduce from (30) and (31) the following general form for the asymptotic expansion of the density of W to an arbitrary order as $N \to \infty$:

$$(32)\quad \mathrm{pdf}(w) \sim \frac{w^{q/2-1}e^{-w/2}}{2^{q/2}\Gamma(q/2)} + \sum_{k=1}^{\infty} \frac{1}{N^k} \sum_{\ell=1}^{k} \frac{(1/2)^{\ell}}{\ell!} \sum_{j_1,\ldots,j_\ell}^{*} \frac{w^{q/2-1}}{\Gamma(q/2)(j_1+1)\ \ldots\ (j\ +1)}$$

$$\left[\det\left(L(\partial Z \otimes I)L'\right)^{1/2} {}_0F_0\left(-L(\partial Z \otimes I)L', w\right)\mathrm{etr}\left(\tfrac{1}{2}Z\right)\mathrm{tr}(Z^{j_1+1})\,\mathrm{tr}(Z^{j_2+1})\ldots\mathrm{tr}(Z^{j_\ell+1})\right]_{Z=0}$$

To $O(N^{-1})$ we have:

$$(33)\quad \mathrm{pdf}(w) = \chi_q^2 + \frac{w^{q/2-1}}{2N\ (q/2)}\left[\det\left(L(\partial Z \otimes I)L'\right)^{1/2} {}_0F_0\left(-L(\partial Z \otimes I)L', w\right)\mathrm{etr}\left(\tfrac{1}{2}Z\right)\mathrm{tr}\ Z^2\right]_{Z=0}$$

$$+\ o(N^{-1})\ .$$

The correction term of $O(N^{-1})$ in (33) may be evaluated using the rules of operator calculus given earlier. The final result may be shown to correspond to the expression obtained by more conventional methods in Phillips (1984c).

5. THE DISTRIBUTION FUNCTION

We may also derive the cdf of the null distribution of W . We shall use the incomplete gamma integral:

$$\int_0^Y e^{-y\zeta}y^{\alpha-1}dy = \alpha^{-1}Y^{\alpha}\,{}_1F_1(\alpha,\ \alpha+1;\ -Y\zeta)$$

where $\mathrm{Re}(\alpha)$, $\mathrm{Re}(\zeta) > 0$ (Erdeyli (1953), p. 266). We have:

$$\mathrm{cdf}(w) = P(W \le w)$$

$$= [N^{q/2}\Gamma(q/2)]^{-1}\int_0^w y^{q/2-1}\left[\det\left(L(\partial X\otimes I)L'\right)^{1/2}{}_0F_0\left(-L(\partial X\otimes I)L', y/Z\right)\det(I-X)^{-N/2}\right]_{X=0} dy$$

$$= [N^{q/2}\Gamma(q/2)]^{-1}\int_0^w \int_{V_{1,q}} y^{q/2-1}\left[\det\left(L(\partial X\otimes I)L'\right)^{1/2}\right.$$

$$\left.\cdot\exp\{-(y/N)h'L(\partial X\otimes I)L'h\}(\underline{dh})\det(I-X)^{-N/2}\right]_{X=0} dy\ .$$

Interchanging the orders of operation in the above expression, which is permissible in view of the continuity of the integrand and the compactness of the domains of integration, we obtain:

$$\mathrm{cdf}(w) = \frac{w^{q/2}}{N^{q/2}\Gamma(q/2+1)}\left[\det\left(L(\partial X\otimes I)L'\right)^{1/2}\right.$$

$$\left.\cdot\int_{V_{1,q}} {}_1F_1\left(q/2,\ q/2+1;\ -(w/N)h'L(\partial X\otimes I)L'h\right)(\underline{dh})\det(I-X)^{-N/2}\right]_{X=0}$$

$$(34)\quad = \frac{(w/N)^{q/2}}{\Gamma(q/2+1)}\left[\det\left(L(\partial X\otimes I)L'\right)^{1/2}{}_1F_1^{(q)}\left(q/2,q/2+1;-(w/N),L(\partial X\otimes I)L'\right)\det(I-X)^{-N/2}\right]_{X=0}$$

In (34), ${}_1F_1^{(q)}$ is a confluent hypergeometric function with two matrix arguments (see James (1964)). In the present case one of the arguments is scalar and the function admits a series representation in terms of top order zonal polynomials.

6. THE NON NULL DISTRIBUTION

Analysis of the non null distribution of W proceeds along similar lines. The derivations are more complicated and the reader is referred to the author's paper (1986) for details. The final result for the density may be expressed as:

$$\text{(35)} \quad \mathrm{pdf}(w) = \frac{w^{q/2-1}e^{-m'm/2}}{N^{q/2}\Gamma(q/2)}\left[\det\left(L(\partial X\otimes I)L'\right)^{1/2}\int_{V_{1,q}} \exp\{-(w/N)h'L(\partial X\otimes I)L'h\}\right.$$

$$\left.\cdot {}_0F_1\left(\frac{1}{2}, \frac{w}{2N}h'L(\partial X\otimes I)L'mm'h\right)(\underline{dh})\det(I-X)^{-N/2}\right]_{X=0} .$$

An alternative series representation of (35) is possible in terms of top order invariant polynomials (Davis (1979)) with two matrix argument operators. Specializations to the non null distributions of the statistics in Section 4 and to the asymptotic theory of W under local alternatives are also given in Phillips (1986).

7. CONCLUSIONS

There seems to be considerable scope for applying the methods outlined here to other problems of distribution theory in multivariate analysis. The author (1984a) has used similar methods in studying the distribution of the Stein-rule estimator in linear regression. The latter results have recently been extended by Knight (1986) to nonnormal errors.

The technique of developing general formulae for asymptotic expansions from exact theory also seems to be very promising. This approach avoids much of the tiresome algebraic manipulation that is a feature of the traditional work on Edgeworth expansions. Moreover, the final formulae are simpler in form and may be used to obtain expansions to an arbitrary order, which is very difficult with the traditional approach.

Here and elsewhere in the application of these methods to problems of distribution theory it would be helpful to have a glossary of results on fractional and matrix fractional calculus. Until now I have been developing rules for working with these operators as the need for them arose. With a systematic set of formulae for the action of matrix fractional operators on elementary and commonly occurring special functions as well as rules for operation on products and compositions of functions of matrix argument it should be possible to make progress on many presently unsolved problems of multivariate distribution theory.

REFERENCES

Constantine, A. G. (1966). "The distribution of Hotelling's generalized T_0^2," Annals of Mathematical Statistics, 37, 215-225.

Erdêyli, A. (1953). Higher Transcendental Functions, Vol. 1. New York: McGraw Hill.

Herz, C. W. (1955). "Bessel functions of matrix argument," Annals of Mathematics, 61, 474-523.

James, A. T. (1964). "Distribution of matrix variates and latent roots derived from normal samples," Annals of Mathematical Statistics, 35, 475-501.

Knight, J. L. (1986). "The distribution of the Stein-rul estimator in a model with non-normal disturbances," Econometric Theory, 2 (to appear).

Lavoie, J. L., T. J. Osler and R. Tremblay (1976). "Fractional derivatives and special functions," SIAM Review, 18, 240-268.

Miller, K. S. (1974). "The Weyl fractional calculus," in B. Ross (ed.), Fractional Calculus and Its Applications. Berlin: Springer-Verlag.

Muirhead, R. (1982). Aspects of Multivariate Statistical Theory. New York: Wiley.

Phillips, P. C. B. (1984a). "The exact distribution of the Stein rule estimator," Journal of Econometrics, 25, 123-131.

_________ (1984b). "An everywhere convergent series representation of the distribution of Hotelling's generalized T_0^2," Cowles Foundation Discussion Paper No. 723, Yale University, September.

_________ (1984c). "Finite sample econometrics using ERA's," Journal of the Japan Statistical Society, 14, 107-124.

_________ (1985). "The exact distribution of the SUR estimator," Econometrica, 53, 745-756.

_________ (1986). "The exact distribution of the Wald statistic," Econometrica (to appear).

Pillai, K. C. S. (1976). "Distributions of characteristic roots in multivariable analysis: Null distributions," Canadian Journal of Statistics, 4, 157-186.

_________ (1977). "Distributions of characteristic roots in multivariate analysis: Non-null distributions," Canadian Journal of Statistics, 5, 1-62.

Ross, B. (1974a). Fractional Calculus and Its Applications. Berlin: Springer-Verlag.

________ (1974b). "A brief history and exposition of the fundamental theory of fractional calculus," in B. Ross (ed.), Fractional Calculus and Its Applications. Berlin: Springer-Verlag.

Spanier, J. and K. B. Oldham (1974). The Fractional Calculus. New York: Academic Press.

Treves, S., (1980) An Introduction to Pseudodifferential and Fourier Integral Operators. New York: Plenum Press.