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EXISTENCE, REGULARITY, AND CONSTRAINED SUBOPTIMALITY OF COMPETITIVE ALLOCATIONS WHEN THE ASSET MARKET IS INCOMPLETE

by

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Abstract

Let assets be denominated in an a priori specified numeraire.

Whether or not the asset market is complete, a competitive equilibrium exists as long as arbitrage is possible when assets are free. Generically, the set of competitive equilibria is finite, and the equilibrium prices and allocations in the commodity spot markets are uniquely determined by the asset allocation in a neighborhood of a competitive equilibrium.

If the asset market is incomplete, a competitive equilibrium allocation is generically constrained suboptimal: there exists an arbitrarily small reallocation of the existing assets, which leads to a Pareto improvement in welfare when prices and allocations in the commodity spot markets adjust to maintain equilibrium.

EXISTENCE, REGULARITY, AND CONSTRAINED SUBOPTIMALITY OF COMPETITIVE ALLOCATIONS WHEN THE ASSET MARKET IS INCOMPLETE

In 1951, Kenneth Arrow and Gerard Debreu independently established what is still today probably the central argument of economic theory: they proved that any competitive equilibrium of an "Arrow-Debreu" economy is Pareto optimal. In 1954, they jointly discovered a sub-class of economies, now referred to as convex Arrow-Debreu economies, for which competitive equilibria always exist. In 1970 Debreu introduced the techniques of differential topology into economics and proved that the competitive equilibria of smooth, convex Arrow-Debreu economies are generically locally unique. In this paper we extend the study of the existence, optimality and local uniqueness of competitive equilibria to a wider class of economies: the class of economies, suggested by Arrow in 1953, in which the asset market is possibly incomplete. We show that when assets pay off in some numeraire commodity, equilibria exist and are typically locally unique, even when the asset market is incomplete. However, when the asset market is incomplete, competitive equilibrium allocations are typically Pareto suboptimal in a strong sense: the market does not make efficient use of the existing assets.

The old proof of Pareto optimality (see, for example, Lange (1942)) essentially assumed differentiable utilities and strictly positive competitive equilibrium allocations. One considered the problem of how to divide the aggregate endowment to maximize the utility of one individual subject to the constraint that no other individual suffer a loss in utility. It was pointed out that the first order necessary conditions for a strictly positive allocation, $\vec{x}$, to be a local maximum to this problem are satisfied if the allocation is a competitive equilibrium. Moreover, it is now commonly known that in a concave programming problem the first order necessary conditions are also sufficient for global optimality. Thus, if we add the hypothesis that all the utilities are concave and apply this argument for $\vec{x}$ in turn to each individual, this old approach yields that competitive equilibria are Pareto optimal.

The new argument for Pareto optimality introduced by Arrow (1951) and Debreu (1951) does not require the differentiability of utilities, or even utility representations at all, nor does it require convexity of preferences or strict positivity of the equilibrium allocation. It depends on the fact that in a competitive equilibrium all potential allocations can be unambiguously valued at the equilibrium prices. If an alternative allocation makes an individual better off without harming anyone else, then the bundle allocated to this individual must cost strictly more than his endowment, while the bundle allocated to any other individual must cost at least as much as his endowment (given local nonsatiation). But this contradicts the feasibility of the alternative allocation. This new argument for Pareto optimality is analytically simple and requires a minimum of assumptions; it suffices, roughly speaking, that the preferences of individuals not display local satiation. What is demanding is the appropriate interpretation of the notion of a commodity or a market: as argued in Debreu (1959), it is required that commodities be differentiated, not only by qualitative, but by temporal and contingent characteristics as well, and that there be at the initial period, with all individual present, a single market for the exchange of all commodities so specified. If the market in contingent commodities is thus complete, individuals optimize under one budget constraint and one price system. Economic activity past the initial contracting period is limited to the execution of contracts; further trading in subsequent periods is not permitted, nor is it necessary.

It was Arrow's (1953) contribution to suggest an alternative to a complete market in contingent commodities: a complete asset market. If, for each time period and realization of uncertainty, a pure security exists which yields one unit of "revenue" or of a numeraire commodity at that date-event pair and zero otherwise, any allocation obtained as a competitive equilibrium with a complete market in contingent commodities at the initial period can be alternatively obtained as a competitive equilibrium with a complete market in pure securities in the initial period, and subsequent spot markets.

In Arrow's formulation, and in the more general model of Radner (1968) in which there may not be a complete system of assets, markets in commodities and assets are active at distinct periods and under alternative realizations of uncertainty. In such a framework of sequential exchange, if the asset market is indeed incomplete, individuals face a non-trivial multiplicity of budget constraints and price systems and the Arrow-Debreu argument for the Pareto optimality of competitive equilibrium fails. Indeed, so does the Arrow-Debreu proof of the existence of equilibrium, as Radner (1972) showed in a general model, where assets are thought of as claims on a vector of commodities indexed by states. Hart (1975) gave an explicit example in such a general framework of an economy with incomplete markets that has no competitive equilibrium.

In this paper we consider economies in which the assets are real; but they yield payoffs denominated in a single numeraire commodity -- such as gold. (Others, Cass (1984), Duffie (1985), Geanakoplos-Mas-Collel (1985) and Werner (1985), have considered the case of financial securities, where asset yields are in terms of units of account.) Individuals can buy or sell short any amount of each of the numeraire assets, in some limited collection. After the state of nature is revealed, they trade in the spot market with income derived from the sale of their initial endowments, plus the deliveries of the numeraire good they receive or make as a result of their portfolio holdings. Individuals are assumed, in equilibrium, to have perfect conditional foresight: they may not agree on the probability of the states, but they all understand what spot prices will prevail conditional on the state. In addition we postulate that in equilibrium all the asset and spot markets clear; in particular, there is no bankruptcy.

Let us consider a concrete example with three states of nature $s = 0, 1, 2$, $L + 1$ commodities and only two assets $a = 0, 1$ with numeraire payoffs in the three states given by the matrix $R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$. We can think of state 0 as period 0 and $s = 1, 2$ as states 1 and 2 in period 1. The above asset structure allows individuals to save from period 0 to period 1, by holding a negative amount of asset 0 and a positive amount of asset 1, but not to insure between consumption in states 1 and 2. One can see at once that a competitive equilibrium for such an economy is not likely to be Pareto optimal. The question we address in section 7 of this paper is to what extent the market uses efficiently the limited assets that do exist.

Before our analysis of the constrained optimality question, we prove in section 2 that competitive equilibria always exist as long as preferences are monotonic for the numeraire good in each state, and arbitrage _is_ possible when assets are free. The first difficulty to overcome in the proof is that since individuals are allowed to sell short assets, there is no a priori lower bound that can be assumed for asset demands. As Hart (1975) showed, this is fatal for existence when asset payoffs are in multiple commodities, or when asset payoffs depend more generally on prices (Hart (1974); see however Cass (1984), Duffie (1985), and Werner (1985) for the financial asset case). Second, since pure insurance involves assets that pay positive and negative amounts, some equilibrium asset prices may be negative; since asset demand is not continuous at prices equal to zero, we must carefully choose the space of asset prices on which to apply a fixed point argument.

In section 3 we show that under smoothness conditions, demand is differentiable, and in section 6 we use the same transversality techniques introduced first by Debreu (1970), which we explain in section 5, to prove that equilibria are generically locally unique. This is in contrast to the study of Geanakoplos and Mas-Colell (1985), which shows that there is an essential real indeterminacy when assets pay off in units of account.

The question of what is the appropriate definition of constrained optimality when the asset market is incomplete has been vexing at least since Hart (1975) gave an example of an incomplete markets economy which has two competitive equilibria that are Pareto comparable or, alternatively, in which a further reduction in the set of assets traded leads to a Pareto improvement at equilibrium. Newbery and Stiglitz (1979) provided a second example and intuition. Grossman (1977) gave a definition of constrained optimality under which the first two theorems of welfare analysis still apply, but, as Hart has remarked, it seems absurd to say that the econonomy is using its markets efficiently at one equilibrium when there is another equilibrium in which everyone is better off. A satisfactory definition of constrained optimality which recognizes that the underlying reasons for the incompleteness of markets may also limit a central planner has only recently been given by Stiglitz (1982) and Newbery-Stiglitz (1982). These latter papers, and also Greenwald-Stiglitz (1984), suggest that constrained suboptimality is a general phenomenon. They do not, however, develop the formal arguments and analytical apparatus needed to prove that claim.

We say that the asset allocation at a competitive equilibrium is constrained suboptimal if a reallocation of assets alone can lead to a Pareto improvement when prices and allocations in the commodity spot markets adjust to maintain equilibrium.[1]

We show in section 7 that, when the asset market is incomplete, the portfolio allocation at a competitive equilibrium is generically constrained suboptimal; this is true even if the portfolio reallocations are required to satisfy every agent's budget constraint at the original equilibrium prices. In the context of our earlier example, this means that if only the individuals could have been induced to save different amounts, they could all have been made better off. The intuition for this result is as follows: an asset reallocation in any economy has two effects on an individual's utility; a direct effect from the income transfer, and an indirect effect due to the relative price change in the commodity spot markets. When markets are complete, the income reallocation caused by the price change can be decomposed into a combination of

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[1]We show in section 6 that, generically, in a neighborhood of a competitive equilibrium, the equilibrium prices and allocations in the commodity spot markets are uniquely determined by the asset allocation.

assets that have already been priced by the market. When the asset market is incomplete, it is generically the case that the price changes will cause an income redistribution which the market itself could not directly implement. In essence, the central planner has access to a wider class of assets than those directly traded. In section 7.2 we discuss some special, nongeneric cases where competitive equilibria nevertheless can be constrained optimal.

Our method of proof is reminiscent of the old necessary conditions for local optimality, dressed up in modern matrix notation. We consider the matrix of utility effects caused by the various portfolio reallocations, and we prove that if the asset market is incomplete, for a generic economy, this matrix has full row rank. It follows immediately that there is a portfolio reallocation that makes everyone better off.

1. The Economy

Transactions occur in real securities called assets before the state of nature is known, and then subsequently in commodities, after the state of nature is known. States of nature are $s \in S = \{0, 1, \ldots, S\}$.[2]

Commodities are $\ell \in L = \{0, 1, \ldots, L\}$. Commodity ℓ in state s is denoted $\ell(s)$. A consumption plan is a vector $x = (\ldots, x(s), \ldots) \in E_+^{(S+1)(L+1)}$. At state s, commodity 0(s) is the designated numeraire commodity.

Individuals, $h \in H = \{0, 1, \ldots, H\}$ have preferences over

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[2]We use the same symbol to denote a set and also its last element -- no confusion should arise.

consumption plans represented by utility functions $W^h : E_+^{(S+1)(L+1)} \to E$.[3,4] For some problems we shall assume that W^h is representable as a von Neumann-Morgenstern expected utility, $W^h = \sum_{s \in S} \pi^h(s) U^h(s)$. Individuals have endowments $e^h \in E_+^{(S+1)(L+1)}$.

Assets $a \in A = \{0, 1, \ldots, A\}$ yield returns, denominated in the numeraire. The return of asset a in state s is $r_a(s)$, which may be negative or positive; the vector of asset returns at s is $r(s) = (\ldots, r_a(s), \ldots)$; the matrix of asset returns or asset return structure is $R = \{r_a(s)\}_{s \in S}^{a \in A}$, an $(S + 1) \times (A + 1)$ matrix. A portfolio is $y \in E^{A+1}$; again, the sign of y_a may be positive or negative -- there are no restrictions on short sales. The asset market is called complete if and only if R also has full row rank; equivalently, if and only if any distribution of revenue across states can be attained by a suitable choice of a portfolio. If $S > A$, the asset market is necessarily incomplete.

We allow free disposal of commodities and hence we only need to consider non-negative commodity prices: $p = (\ldots, p(s), \ldots) \in E_+^{(S+1)(L+1)}$. Asset prices are $q \in E^{A+1}$; they may be positive or negative.

Given prices $(q,p) \in E^{A+1} \times E_+^{(S+1)(L+1)}$, the budget set of individual h is defined as

$B^h(q,p) = \{(y,x) \in E^{A+1} \times E_+^{(S+1)(L+1)} | q \cdot y = 0$ and $p(s) \cdot (x(s) - e^h(s)) \leqq p_0(s)(r(s) \cdot y)$, for all $s \in S\}$.

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[3] E denotes the real numbers, E_+ the nonnegative reals, and E_{++} the positive reals. Similarly, we use E^k, E_+^k, E_{++}^k.

[4] Given two vectors y and z, we write $y \geqq z$; $y > z$; $y >> z$ to mean $y_i \geqq z_i$ for all i; $y_i \geqq z_i$ for all i and $y \neq z$; and finally $y_i > z_i$ for all i, respectively.

A commodity allocation is $\vec{x} = (x^h : h \in H)$ such that

$$\sum_{h \in H} (x^h - e^h) \leqq 0 .$$

A portfolio or asset allocation is $\vec{y} = (y^h : h \in H)$ such that

$$\sum_{h \in H} y^h = 0 .$$

An allocation is a pair $(\vec{y},\vec{x})$.

A competitive equilibrium for the economy with real, numeraire assets is a 4-tuple of asset prices, commodity prices, an asset allocation and a commodity allocation, $(q,p,\vec{y},\vec{x})$ which satisfies

- $\sum_{h \in H} (x^h - e^h) = 0$, if $p_\ell(s) > 0$ for all $s \in S$, $\ell \in L$;

- (y^h,x^h) is maximal in $B^h(q,p)$ for all $h \in H$.

For a given asset allocation $\vec{y}$ a spot market competitive equilibrium is a pair $(p,\vec{x})$ which satisfies

- $\sum_{h=0}^{H} (x^h - e^h) = 0$ if $p_\ell(s) > 0$ for all $s \in S$, $\ell \in L$;

- x^h is maximal in $B^h(p; y^h)$ for all $h \in H$.

The budget set $B^h(p; y^h)$ is defined by $B^h(p; y^h)$
$= \{x \in E_+^{(S+1)(L+1)} | p(s) \cdot (x(s) - e^h(s)) \leqq p_0(s)(r(s) \cdot y^h)$ for all $s \in S\}$.
That is, a spot market equilibrium allows individuals to optimize in their choice of consumption bundles but not in their portfolio choice.

A commodity allocation $\vec{x}$ is suboptimal if there exists an alternative commodity allocation $\vec{x}'$ such that

- $W^h(x'^h) \geqq W^h(x^h)$ for all $h \in H$, with some strict inequality.

An allocation $(\vec{y},\vec{x})$ is constrained suboptimal if there exists an alternative commodity allocation $\vec{x}''$ such that

- $W^h(x''^h) \geqq W^h(x^h)$ for all $h \in H$, with some strict inequality.

- there exists a commodity price vector p and a portfolio allocation $\vec{y}''$ such that $(p,\vec{x}'')$ is a spot market competitive equilibrium for the portfolio allocation $\vec{y}''$.

Constrained suboptimality is a stronger notion; it recognizes the constraints imposed by the asset structure, which suboptimality does not. Note that by the second welfare theorem the two notions coincide if the asset market is complete.

This appears to be the most natural, fully tractable model with real assets.[5] Radner (1968, 1972) gave a general formulation in which it is impossible to prove in general the existence of a full equilibrium. Hart (1975), for example, showed that, if assets pay off in multiple commodities, then it is possible to construct an economy with no competitive equilibria. In the next section we show that under mild conditions competitive equilibria with real numeraire assets always exist. One might think of our formulation as an abstraction of the theory of general equilibrium with possibly incomplete markets under the "gold standard." Geanakoplos and Mas-Colell (1985) show how the "gold standard equilibria" can be used to understand the set of "financial equilibria" when "money"

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[5]It suffices, more generally, that for each state there be an a priori specified bundle of commodities in which the returns of all assets are denominated.

is not tied to any standard. There, these equilibria are more precisely called financial securities competitive equilibria.

As early as 1953 Arrow defined a financial securities competitive equilibrium for the special financial security structure $\tilde{R}$ equal to the identity. In a general financial securities equilibrium, there is a given securities structure $\tilde{R}$, where $\tilde{r}_b(s)$ gives the payoff of financial security (bond) b in state s in units of account. The budget set $\tilde{B}^h(q,p)$ is now defined as $\tilde{B}^h(q,p) = \{(y,x) \in E^{A+1} \times E_+^{(L+1)(S+1)} | q \cdot y = 0$ and $p(s) \cdot (x(s) - e^h(s)) \leqq r(s) \cdot y$ for all $s \in S\}$ and (q,p) is allowed to vary over $E^{A+1} \times E_+^{(L+1)(S+1)}$. Arrow showed that, when $\tilde{R} = I$ (the case of a complete set of "Arrow securities"), an allocation is part of a financial securities competitive equilibrium if and only if it is an Arrow-Debreu equilibrium allocation. The object of study of Geanakoplos-Mas-Colell (1985) is the set of financial securities equilibrium allocations for general $\tilde{R}$, when the rank $\tilde{R} < (S + 1)$.

We return to equilibria in which the asset payoffs $r_a(s)$ are in units of the numeraire good 0(s), $a \in A$, $s \in S$. When R has full row rank and the asset market is thus complete, the set of equilibrium allocations is identical to the set of Arrow-Debreu equilibrium allocations. At first this might seem surprising, since, for example, an asset a with payoff $r_a(0) = -r_a(1) = 1$ appears to fix the rate of exchange between commodity 0 in state 0 and commodity 0 in state 1. But this is misleading, for the rate of exchange also depends on the asset prices q and the asset return structure. To see this, let $(\bar{p}, \vec{x})$ be an Arrow-Debreu equilibrium with $\bar{p} >> 0$. One can define the vector $v >> 0$ by $v(s) = \bar{p}_0(s)$ $s \in S$, and q to be $q = R'v$. Then $(q,\bar{p},\vec{y},\vec{x})$ is a competitive equilibrium with real numeraire assets for some $\vec{y}$. The argument in the reverse direction requires Farkas' lemma and is given in the next section.

Although we have indexed the spot commodity markets by referring to different states of nature, these could be reinterpreted, given the proper asset structure R, as time and state indexed; the same analysis thus covers a broad range of problems. One case of central importance occurs when state 0 is interpreted to be period 0 and state s, for $s = 1, \ldots, S$, is interpreted to be period 1 in state s. If asset 0 is given by $r_0(0) = 1$, $r_0(s) = 0$ for $s = 1, \ldots, S$, and all other assets pay $r_a(0) = 0$ for $a = 1, \ldots, A$, then we can interpret our model as one in which individuals decide at the same time on consumption in period 0 and their holdings of risky assets before knowing the state that will be revealed in period 1.

Another interesting special case occurs when the set of date-event consumption nodes can be represented as a tree (see Debreu [1959]). Consumers are supposed to know where in the tree they are when they act, but not which of the successor nodes will occur. Utilities are given by $W^h(x) = \sum_{s \in S} \pi^h(s) U^h(x(s); s)$, and agents update by Bayes' Law. Suppose also that at every date-event node s consumers can trade all future commodities contingent on which successor node s′ will occur. This is a situation of complete markets in the sense of Debreu. It can be represented in our model by an asset structure R that associates to every node s in the tree an asset a(s) such that $r_{a(s)}(s) = 1 = -r_{a(s)}(P(s))$, where P(s) is the predecessor of s, and $r_{a(s)}(s') = 0$ for $s' \notin \{s, P(s)\}$.

With these introductory remarks out of the way, we now state our assumptions:

A1) W^h is continuous and quasi-concave on $E_+^{(S+1)(L+1)}$; the range of W^h can be extended to $E \cup \{-\infty\}$.

A2) $e^h >> 0$.

A3) W^h is monotonic in commodity 0(s) for each $s \in S$: Let $\tilde{x}$ be any consumption plan which is nonnegative in every coordinate, and strictly positive at 0(s) for some $s \in S$. Then for all $x \in E_+^{(L+1)(S+1)}$, $W^h(x + \tilde{x}) > W^h(x)$.

A4) When securities are free there is arbitrage: there is $\bar{y} \in E^{A+1}$ with $R\bar{y} > 0$.

D1) R has full column rank.

D2) W^h is twice continuously differentiable, $DW^h >> 0$, and D^2W^h is negative definite on $E_{++}^{(S+1)(L+1)}$.

D3) The closure of the indifference curves of W^h do not intersect the boundary of $E_+^{(L+1)(S+1)}$.

S) The asset market is incomplete: $A < S$.

CS) Every set of A + 1 rows of R are linearly independent, and there is a portfolio y with $r(s) \cdot y \neq 0$ for all $s \in S$.

Under assumptions A1-A4 we shall show in the next section that competitive equilibria exist. Note that under the von Neumann-Morgenstern hypothesis, $W^h(x) = \sum_{s \in S} \pi^h(s)U^h(x(s),s)$, (A3) implies $\pi^h(s) > 0$ for all $s \in S$. Under the additional hypotheses D1-D3, excess demand for assets and commodities is differentiable over the domain of "nonarbitrage" prices, which we shall specify. Assumptions A1-A4 and D1-D3 also suffice to guarantee regularity of the equilibrium set. Finally, we show in the last section that asset market equilibria are generically Pareto suboptimal when $A < S$ and even generically constrained suboptimal when CS holds.

2. Existence of Competitive Equilibria

The proof of the existence of an asset market competitive equilibrium turns out not to be much more complicated than the standard existence proof for Arrow-Debreu equilibria pioneered by Arrow-Debreu (1954), and refined by Debreu (1959). When the consumption sets are bounded, the demand correspondences of the individuals are upper hemi-continuous. The only difficulty is finding the appropriate convex, compact price space on whose boundary aggregate excess demand is properly behaved. Here it turns out to be the set of non-arbitrage prices.

The recent work of Werner (1985) and Cass (1985) shows that the difficulties with existence in the Radner-Hart models are absent in models of financial securities competitive equilibria. Here we show that the same is true for real numeraire assets, a convenient middle ground between the purely nominal securities in the latter models and complicated vector-commodity assets in the former models. Of course, an equilibrium with real numeraire return structure R is also a financial market equilibrium with return structure $\tilde{R} = R$. Our equilibrium existence proof thus implies the existence of financial equilibria. The importance of the distinction between financial securities and real numeraire assets is discussed in Geanakoplos-Mas-Colell (1985). There are some other, minor differences between Werner's existence result and ours. He specializes to the case where financial security 0 pays off only in state 0 and other financial securities pay nothing in state 0. We have a more general asset structure. In the Werner model, financial securities pay out nonnegative amounts in every state. Strictly speaking, insurance is a transfer of wealth from one state to another, and that is why we have not restricted asset payoffs to be non-negative. In practice what is typically called insurance is a transfer of wealth from the current period to some subset of future periods. It may thus be achieved by a combination of selling (going short) in the 0^{th} Werner financial security, and buying another financial security with the appropriate non-negative payoffs. Our more general asset structure introduces complications which we return to when we discuss A4 before the proof of lemma 2.

Lemma 1: Consider the truncated individual demand correspondence $d^h(q,p;\, K) = \{(\hat{y},\hat{x}) \in \operatorname{Arg\,Max}_{(y,x)\in K} \{W(x) | q \cdot y = 0$ and $p(s) \cdot (x\,(s) - e^h(s)) = p_0(s)(r(s) \cdot y)$ for all $s \in S\}$, where $K \subseteqq E^{A+1} \times E^{(S+1)(L+1)}$ is a closed rectangle with center at the origin. Under A1-A3, $d^h(p,q;\, K)$ is non-empty, compact, convex-valued and upper hemi-continuous at each $(q,p) \in E^{A+1} \times E_+^{(S+1)(L+1)}$ with $q \neq 0$ and $p(s) \neq 0$ for all $s \in S$.

Proof: By continuity and compactness $d^h(q,p;\, K)$ is non-empty; by concavity and convexity, it is convex-valued.

Let $(q_n,p_n) \underset{n}{\rightarrow} (q,p)$ and $(y_n,x_n) \underset{n}{\rightarrow} (y,x)$ where $(y_n,x_n) \in d^h(q_n,p_n;\, K)$. Suppose that $(\hat{y},\hat{x}) \in \{(y',x') \in K | q \cdot y' = 0$ and $p(s) \cdot (x'(s) - e^h(s)) = p_0(s)(r(s) \cdot y')\}$ and $W^h(\hat{x}) > W^h(x)$. We shall derive a contradiction. Take $\lambda < 1$ but sufficiently large so that $W^h(\lambda\hat{x}) > W^h(x)$; by continuity, it also follows that for n large $W^h(\lambda\hat{x}) > W^h(x_n)$. Define $\hat{y}_n = \operatorname{Arg\,Min}_{y'} \{\|\hat{y} - y'\| \mid q_n \cdot y' = 0\}$. Note that for $q \neq 0$ $\hat{y}_n \underset{n}{\rightarrow} \hat{y}$. Now consider that $p(s) \cdot (\lambda\hat{x}(s) - e^h(s)) - p_0(s)(r(s) \cdot \lambda\hat{y}_n) < 0$ for all s, for n large, since $p(s) \cdot e^h(s) > 0$ for all s and $\lambda < 1$. But this is a contradiction, for $(\lambda y_n, \lambda x)$ is preferred to (y_n,x_n) for n large.

Q.E.D.

Let $d(q,p;\, K) = \sum_{h=0}^{H} d^h(q,p;\, K)$ be the aggregate truncated demand correspondence. Recall that the sum of upper hemi-continuous, non-empty, compact, convex-valued correspondences is upper hemi-continuous, and non-empty, compact, convex-valued.

If $(y^h,x^h) \in d^h(q,p,K)$, then $q \cdot y^h = 0$ and $p(s) \cdot (x^h(s) - e^h(s)) = p_0(s)(r(s) \cdot y^h)$, for all $s \in S$. Hence if $(y,x) \in d(q,p;\, K)$, then $q \cdot y = 0$ and $p(s) \cdot (x(s) - \sum_{h=0}^{H} e^h(s)) = p_0(s)(r(s) \cdot y)$. This is the modified Walras Law that holds for economies with possibility incomplete asset markets.

We must look for the appropriate price space -- one that is convex and has an approximate boundary -- to apply a fixed point argument.

Commodity prices pose no problem. Since a separate budget constraint must be satisfied at every state s, we may restrict our attention to $p = (\ldots,p(s),\ldots) \in \Delta^{(S+1)L} = \times_{s \in S} \Delta^L$, where $\Delta^L = \{p(s) \in E_+^{L+1} \mid \sum_{\ell \in L} p_\ell(s) = 1\}$.

Since we have allowed securities to pay off negative amounts in some states, the equilibrium asset prices q may well have to have some negative components. On the other hand, since demand is not upper hemi-continuous at $q = 0$, it is not immediately obvious what price domain to limit attention.

Assumption A4 allows us to describe a convex, compact set of potential equilibrium asset prices, which we call nonarbitrage prices. The assumption is trivially satisfied when assets pay off in non-negative amounts, or when the asset market is complete. If there are no assets, or only one asset, then in equilibrium there can be no trade in assets. Thus both the no asset and one asset cases are isomorphic to models with one asset that pays a strictly positive amount in every state, and there

A4 applies. It is perhaps curious that we need to assume the possibility of arbitrage in order to find an equilibrium without arbitrage, but the idea is clear: to apply a standard fixed point argument, we must elimminate 0 as a potential equilibrium.

We say that the asset prices $q \in E^{A+1}$ do not permit arbitrage if and only if there is no $y \in E^{A+1}$ with $q \cdot y \leqq 0$ and $Ry > 0$.

Lemma 2. Suppose A4 and D1 hold. Let $\bar{C} = \{q \in E^{A+1} | q = R'v,\ v \in E_+^{(S+1)}\}$. Then $\bar{C}$ is a closed, convex cone. Furthermore, $q \in \text{Int } \bar{C}$ if and only if q does not permit arbitrage. In particular, if $\hat{q} \in \bar{C}/\text{Int } \bar{C}$, then there is $\hat{y}$ with $\hat{q} \cdot \hat{y} = 0$ and $R\hat{y} > 0$. Finally, there is $\hat{q} \in E^{A+1}/\bar{C}$.

Proof: $\bar{C}$ is by construction a finite cone. All finite cones are convex and closed. Notice that $\text{Int } \bar{C} \neq \phi$, for if $v >> 0$, then $q = R'v \in \text{Int } \bar{C}$, since R has linearly independent columns. Suppose $q \in \text{Int } \bar{C}$. Let $Ry > 0$. Since $q \in \text{Int } \bar{C}$, $q - \varepsilon y \in \text{Int } \bar{C}$ for small enough positive ε; that is, $R'v = q - \varepsilon y$ for some $v \geqq 0$. Now suppose that $q \cdot y \leqq 0$. Then $v'Ry = (q \cdot y) - \varepsilon(y \cdot y) < 0$. But then, if $\bar{v}_s = v_s + \delta$ for very small δ, $s \in S$, $\bar{v} >> 0$ and still $\bar{v}'Ry < 0$, contradicting $Ry > 0$. Thus if $q \in \text{Int } \bar{C}$, and $Ry > 0$, then $q \cdot y > 0$. So $\text{Int } \bar{C}$ consists only of no arbitrage prices.

Recall now that by hypothesis there is some $\bar{y}$ with $R\bar{y} > 0$. Hence it follows from the above that $0 \notin \text{Int } \bar{C}$, and hence that $\bar{C} \neq E^{A+1}$.

Take $\hat{q} \in E^{A+1}$, $\hat{q} \notin \text{Int } \bar{C}$. Since int $\bar{C}$ is convex, we can find a nontrivial hyperplane $(\hat{y} \neq 0)$ $H_{\hat{y}} = \{q \in E^{A+1} | q \cdot \hat{y} = \hat{q} \cdot \hat{y}\}$ through $\hat{q}$ with $q \cdot \hat{y} \geqq \hat{q} \cdot \hat{y}$ for all $q \in \bar{C}$. Since $0 \in \bar{C}$, $\hat{q} \cdot \hat{y} \leqq 0$ and $q \cdot \hat{y} \geqq 0$ for all $q \in \bar{C}$. Since $\hat{y} \neq 0$ and R has independent columns, this means

$R\hat{y} > 0$. So we conclude that if $\hat{q} \notin \text{Int } \bar{C}$, then there is a $\hat{y}$ which gives rise to arbitrage at prices $\hat{q}$. Notice that if $\hat{q} \in \bar{C}/\text{Int } \bar{C}$, then in the above demonstration we must have an arbitrage portfolio with $\hat{q} \cdot \hat{y} = 0$.

Q.E.D

For $q = R'v \in \bar{C}$, various interpretations of the vector v are possible. Since its components are all non-negative, one can think of v as a measure on S. With this measure, the price of each asset is the expectation of its future payments. In the language of finance, the asset ptices q display the martingale property.

Given any finite cone $\bar{C} \subsetneq E^{A+1}$ as above, there is a hyperplane H (of dimension A) in E^{A+1} such that $0 \neq q \in \bar{C}$ if and only if there is some $\lambda > 0$ with $\lambda q \in H \cap \bar{C}$.

Let $Q = H \cap \bar{C}$ be the set of normalized security prices. Q^0, the interior of Q relative to H, is the set of <u>normalized non-arbitrage security prices</u>. Q is closed, bounded, and convex.

We are now ready to state and prove our main proposition in this section.

<u>Proposition 1</u> (Existence): If A1-A4 are satisfied, a competitive equilibrium for the economy with real numeraire assets exists.

<u>Proof</u>: Without loss of generality, we may assume that D1 is satisfied, since, once we have priced a maximal set of linearly independent assets, the prices of the reamining assets are determined by arbitrage.

Consider a rectangle K with center at the origin in $E^{A+1} \times E^{(S+1)(L+1)}$ so large that it contains $(0, 2 \sum_{h=0}^{H} e^h)$. Let Q be as above. Consider the correspondence $\Phi^K : Q \times \Delta^{L(S+1)} \times K \to Q \times \Delta^{L(S+1)} \times K$ defined componentwise as follows: let $\Phi_3^K(q,p,(y,z)) = d(q,p;K) - (0, \sum_{h\in H} e^h)$ be the aggregate truncated excess demand correspondence. By lemma 2 this correspondence is upper hemi-continuous and non-empty, compact, convex-valued.

Define $\Phi_1(q,p,(y,z)) = \{\hat{q} \in Q | \hat{q} \in \underset{\bar{q}\in Q}{\text{Arg Max}}\ \bar{q} \cdot y\}$ and similarly let $\Phi_2 = (\ldots,\Phi_2(s),\ldots)$, $s = 0, 1, \ldots, S$, where $\Phi_2(q,p,(y,z))(s) = \{\hat{p}(s) \in \Delta^L | \hat{p}(s) \varepsilon \underset{\bar{p}\in\Delta^L}{\text{Arg Max}}\ \bar{p} \cdot z(s)\}$. Since Q and the Δ^L are compact and convex, it is trivial to verify that Φ_1 and Φ_2 are upper hemi-continuous and non-empty, compact, convex-valued.

By Kakutani's Fixed Point Theorem, there must be some point $(q^*,p^*,y^*,z^*) \in \Phi^K(q^*,p^*,y^*,z^*)$.

Note first that $Ry^* \leqq 0$, for if $r(s) \cdot y^* > 0$ for some s, then $q^* + (0, \ldots, 1, \ldots, 0)'R \equiv q$ satisfies $q \cdot y^* > 0$ and $\lambda q \in Q$ for some $\lambda > 0$, contradicting the maximality of $q^* \cdot y^* = 0$. By Walras' Law it now follows that $p^*(s) \cdot z^*(s) \leqq 0$ for all $s \in S$, and hence that $z_\ell^*(s) \leqq 0$ for all $s \in S$ and $\ell \in L$ (otherwise there would be $p \in \Delta^L$ with $p \cdot z^*(s) > 0 \geqq p^*(s) \cdot z^*(s)$). But since $x^h \geqq 0$, it follows from $z^* \leqq 0$ that $\|x^h\| \leqq \|\sum_{h'\in H} e^{h'}\|$ for all h.

Now let K_n be a sequence of successively larger rectangles. For each n there must be a fixed point $(q_n^*,p_n^*,y_n^* = \sum_{h\in H} y_n^{*h}, z_n^* = \sum_{h\in H} (x_n^{*h} - e^h))$ for Φ^{K_n}. From the above compactness of the $\|x_n^{*h}\|$ it follows that by passing to convergent subsequences we must have $q_n^* \underset{n}{\to} q^*$, $p_n^* \underset{n}{\to} p^*$, and for all h, $x_n^{*h} \underset{n}{\to} x^{*h}$. Suppose that $p_0^*(s) > 0$ for all $s \in S$. Then

$$r(s) \cdot y = \frac{1}{p_0^*(s)}[p^*(s)(x^{*h}(s) - e^h(s))] \quad \text{for all } s \in S$$

has a unique solution $y^{*h} = \lim_{n\to\infty} y_n^{*h}$. It is immediate that $(q^*,p^*,\vec{y}^* = (\ldots,y^{*h},\ldots),\ x^* = (\ldots,x^{*h},\ldots))$ is an asset equilibrium. For large n, (y^{*h},x^{*h}) is interior to K_n, and hence by continuity and convexity of W^h, (y^{*h},x^{*h}) is maximal in $B^h(*q,*p)$. By continuity, if $y^* \equiv \sum_{h\in H} y^{*h}$ then $q^* \cdot y^* = 0$ and from the first paragraph, $(Ry^*) \leqq 0$. But then $y^* = 0$, for by hypothesis R has independent columns, and if $Ry^* < 0$, then $R(-y^*) > 0$ and consumer h could better choose $y^h = y^{*h} + (-y^*)$, contradicting the maximality of (y^{*h},x^{*h}) in $B^h(q^*,p^*)$. Finally, if $y^* = 0$, then by Walras' Law $p^*(s)z^*(s) = 0$, and since $z^*(s) \leqq 0$ it must be that $z_\ell^*(s) = 0$ if $p_\ell^*(s) > 0$.

It only remains to check that it is impossible that $\lim_{n\to\infty} p_0^*(s)_n = p_0^*(s) = 0$ for some $s \in S$. Suppose $p_0^*(s) = 0$. Let $u_0(s)$ be the unit vector in $E^{(S+1)(L+1)}$ in the 0(s) direction. For any consumer h, let $k = \min_{\ell\in L} e_\ell^h(s)$; by monotonicity, $W^h(x^{*h} + ku_0(s)) > W^h(x^{*h})$, and by continuity of W^h, since $(p_0^*(s))_n \underset{n}{\to} 0$ and $x_n^{*h} \underset{n}{\to} x^{*h}$, $W^h((1 - (p_0^*(s))_n)x^{*h} + ku_0(s)) > W^h(x^{*h})$ for large enough n. But if $p_0^*(s)_n < 1$, then $((1 - (p_0^*(s))_n)y^{*h},\ ((1 - (p_0^*(s))_n)x_n^{*h} + ku_0(s)) \in B^h(q_n^*,p_n^*) \cap K$ contradicting the fact that $(y^{*h},x^{*h}) \in d^h(q^*,p^*;\ K\ )$. Q.E.D.

3. Continuous Differentiability of Demand

We have shown in the last section that the interior of $\bar{C}$ is the set of nonarbitrage asset prices, an open set in E^{A+1}. If $q \varepsilon \text{ Int } \bar{C}$, then there is no $y \neq o$ with $q \cdot y = 0$ and $Ry \geqq 0$, if R has independent columns. Let $(q,p) \in \text{Int } \bar{C} \times E_{++}^{(S+1)(L+1)}$. It follows that the demand

$d^h(q,p)$ is nonempty: with positive prices p there is only a compact set of y which satisfy $q \cdot y = 0$ and leave consumer h with nonnegative wealth in every state.

We shall now show that if A1-A4 and D1-D3 are satisfied, the individual demand $d^h(q,p)$ is a continuously differentiable function on Int $\bar{C} \times E_{++}^{(S+1)(L+1)}$. Clearly $d^h(q,p)$ is a singleton and lies in $E^{A+1} \times E_{++}^{(S+1)(L+1)}$.

Without loss of generality, $p_0(s) = 1$, $s \in S$. The necessary and sufficient first order conditions for an interior optimum are

$$\left.\begin{aligned} &D_{x(s)}W^h - \lambda(s)p(s) = 0 \\ &p(s) \cdot (x(s) - e^h(s)) - r(s) \cdot y = 0 \end{aligned}\right\} \quad \text{for all } s \in S$$

$$\lambda' R - \mu q = 0$$

$$q \cdot y = 0$$

for $\lambda = (\ldots,\lambda(s),\ldots) \in E_{++}^{S+1}$ and $\mu \in E$. Totally differentiating the first order conditions, we obtain the matrix of coefficients:

$$A = \begin{bmatrix} D^2W & P & 0 & 0 \\ P' & 0 & R & 0 \\ 0 & R' & 0 & q \\ 0 & 0 & q' & 0 \end{bmatrix}$$

From the implicit function theorem, continuous differentiability obtains if $|A| \neq 0$. Suppose that for some $\hat{z} = (\hat{x},\hat{\lambda},\hat{y},\hat{\mu})$, $A\hat{z} = 0$ and so $(\hat{z})'A(\hat{z}) = 0$. Multiplying out one finds that from the negative definiteness of D^2W^h, $\hat{x} = 0$. Since $p(s) \neq 0$ for all $s \in S$ and $q \neq 0$, it follows that $\hat{\lambda}(s) = 0$ for all $s \in S$ and $\hat{\mu} = 0$. Finally, since R has full colinear rank, $\hat{y} = 0$ as well. Hence $\hat{z} = 0$, which completes the argument.

The aggregate demand function $d(q,p) = (y(q,p), x(q,p))$ and the aggregate excess demand function $f(q,p) = (y(q,p), x(q,p)) - (0, \sum_{h \in H} e^h)$ are similarly well defined and continuously differentiable on $E^{A+1} \times E_{++}^{(S+1)L}$.

4. The Space of Economies

Rather than concentrating on an arbitrary fixed economy, we now consider a broader class of economies, within which we shall establish properties of "typical" economies. Consider an open set $I \subset E_{++}^{(S+1)(L+1)(H+1)}$ of possible endowments for each of the H + 1 agents. Assume that I is bounded and that the closure of I, $\bar{I}$, does not intersect any boundary: that is, assume all possible endowments are bounded away from zero for every commodity.

We have already described our hypothesis on W^h. What is important is that on some large, but bounded rectangle containing the largest possible aggregate endowment from I, the W^h are twice continuously differentiable and strictly concave in the interior, and satisfy the previously discussed boundary condition. Note that adding to such a W^h a sufficiently small multiple εf of any smooth function f (f must be smooth on the boundary as well) will produce another utility $V \equiv W^h + \varepsilon f$ also satisfying the same assumptions. We shall take our space of utilities W to be a finite dimensional manifold of utility functions as above that is sufficiently rich in perturbations. Explicitly, if $W \in W$ and εf is quadratic and separable between states, then for all sufficiently small

ε, $V = W + \varepsilon f \in \mathbb{W}$. Note that if W^h is of the form $W^h = \sum_{S=0}^{S} \pi^h(s)U^h(s)$ then so is $W + \varepsilon f$.

In the remainder of the paper we shall argue that certain properties hold for "nearly all" choices of economies $(\vec{e},\vec{W}) \in \mathbf{E} = I \times \mathbb{W}^{H+1}$. In section 5 section we describe the mathematical technique of transversality which we shall then use in sections 6 and 7.

5. Transversality

The crucial mathematical idea we use to establish our generic results is the so-called transversality theory. Debreu (1970) was the first to apply these ideas to the study of economics.

Suppose that M and N are smooth m and n dimensional manifolds, respectively, lying in some finite dimensional Euclidean space. Let $f : M \to N$ be a smooth map.[6] Let 0 be a point in N. We say that f is transverse to 0, and we write $f \pitchfork 0$, if for all $x \in M$ with $f(x) = 0$, $Df|_x$ has full rank n. Note that if $m < n$, then it is impossible that $Df|_x$ has rank n; in that case $f \pitchfork 0$ if and only if $f^{-1}(0)$ is empty.

If $f \pitchfork 0$, then $f^{-1}(0)$ is either an m-n dimensional manifold, or else empty. In particular, if $m = n$, and $f^{-1}(0)$ is not empty, then $f^{-1}(0)$ is a 0-dimensional manifold, and thus is a discrete set of points. If $K \subset M$ is any compact set, then $f^{-1}(0) \cap K$ consists of at most a finite number of points.

Consider now a third manifold L, of dimension ℓ, and a smooth function $f : L \times M \to N$. Suppose that $f \pitchfork 0$. We know from the preceding

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[6]For the transversality theorem which follows, smooth can be taken to mean max [1, m - n] times continuously differentiable.

discussion that $f^{-1}(0) \subset L \times M$ is an $(\ell + m - n)$-dimensional manifold, or else empty. Let x be any fixed element of L, and consider the smooth function $f_x : M \to N$ given by $f_x(y) \equiv f(x,y)$ for all $y \in M$. A natural question arises: is it likely that f_x is transverse to 0? Likely can be taken in two senses. With any manifold there is a natural definition of measure zero or null set. So we might say that likely means for all $x \in L$ except those x in some null set. Recall that any set whose complement is null must be dense. Alternatively, we might take likely to mean for all x in some open and dense set.

Transversality Theorem: Let L, M, N be manifolds of dimension ℓ, m, n, respectively. Let $f \pitchfork 0$, for a point $0 \in N$. Then except for a null set of $x \in L$, $f_x \pitchfork 0$. Moreover, for any compact set $K \subset M$, the set of x in L for which $f_x \pitchfork 0$, when f_x is restricted to K, is open in L. In particular, if $f^{-1}(0) \subset L \times K$, for some compact $K \subset M$, then the set of x in L such that $f_x \pitchfork 0$ is open, dense, and its complement is null.

We call any open, dense set with null complement a generic set. The two cases of most importance to us will be when $m = n$, or when $m < n$. Suppose $m = n$, and $f \pitchfork 0$, and $f^{-1}(0) \subset L \times K$ for some compact K. We can conclude that for a generic set $D \subset L$, if $x \in D$, then $f_x \pitchfork 0$ and $f_x^{-1}(0)$ is a finite set. If $m < n$ and $f \pitchfork 0$, then for a generic set $D \subset L$, if $x \in D$, then $f_x^{-1}(0) = \phi$.

Let us see how the transversality theorem can be used to show that an $n \times n$ matrix is generically invertible. This shall be the basis of our proof in section 7. The set $R^{n \times n}$ of matrices is certainly a manifold. Let e_{ij} be the matrix which is 0 everywhere except at the $(i,j)^{th}$ element, which is 1. The matrices e_{ij} form a basis for the manifold of $n \times n$ matrices. Consider also the compact n-1-dimensional manifold $S^{n-1} = \{z_1,\ldots,z_n \mid \sum_{i=1}^{n} z_i^2 = 1\}$. Let $f : R^{n\times n} \times S^{n-1} \to R^n$ be defined by $f(A,z) = zA$. Notice that f is smooth. Moreover, A is invertible if and only if $f_A^{-1}(0)$ is empty. Suppose that we could show $f \pitchfork 0$. Then by the transversality theorem we could conclude that for a generic matrix A, $f_A \pitchfork 0$. But f_A maps an (n-1)-manifold into a n-manifold, hence, as we have said, it can be transverse to 0 only if $f_A^{-1}(0)$ is empty, i.e., only if A is invertible.

Thus, to conclude the proof that a generic matrix is invertible, it suffices to show that $f \pitchfork 0$. Let e_j be the j^{th} unit vector in R^n. We must show that if $\bar{z}'\bar{A} = 0$, then for any direction e_j we can find some directional derivative of f, evaluated at $(\bar{A},\bar{z})$, that gives a multiple λe_j of e_j, $\lambda \neq 0$. Note that $\left.\frac{\partial f}{\partial e_{ij}}\right|_{\bar{A},\bar{z}} = \bar{z}_i e_j$. Since there must be at least one element $\bar{z}_i$ of $\bar{z} \in S^{n-1}$ that is nonzero, we conclude that indeed $\bar{D}f|_{\bar{A},\bar{z}}$ has full rank, and hence that $f \pitchfork 0$.

In fact our proof can be slightly modified to produce a stronger result. Suppose that we replace the set of all matrices with a smaller manifold, L, that has the property that for any $A \in L$, small symmetric perturbations of A are also in L. In other words, assume that we can have derivatives with respect to $e_{ij} + e_{ji}$. Then once again consider $f : L \times S^{n-1} \to R^n$, $f(A,z) = zA$. Let $\bar{z}'\bar{A} = 0$. Let e_j be given. If $\bar{z}_j \neq 0$, then $\left.\frac{\partial f}{\partial(e_{jj} + e_{jj})}\right|_{\bar{A},\bar{z}} = 2\bar{z}_j e_j$ gives a nonzero multiple of e_j. If $\bar{z}_j = 0$, take some i with $\bar{z}_i \neq 0$. Then $\left.\frac{\partial f}{\partial(e_{ij} + e_{ji})}\right|_{\bar{A},\bar{z}} = \bar{z}_i e_j + \bar{z}_j e_i$

$= \bar{z}_i e_j$, which is again a nonzero multiple of e_j.

One final fact that we use is that if $\mathbf{D}'$ is generic in $\mathbf{D}$, which is generic in E, then D′ is generic in E.

6. Regularity

In this section we prove the following two propositions:

Proposition 2 (Regularity): If A1-A4 and D1-D3 are satisfied, then for any choice of utilities $\vec{W} \in W^{H+1}$, there is a generic set $I(\vec{W})$ of endowments in I such that for every economy $(\vec{e},\vec{W})$, with $\vec{e} \in I(\vec{W})$, the set of competitive equilibria is finite. Furthermore, on $I(\vec{W})$ the set of competitive equilibria[7] is a continuously differentiable function of the endowment allocation $\vec{e}$.

Proposition 3 (Strong Regularity): If A1-A4 and D1-D3 are satisfied, there is a generic set of economies $\mathbf{D} \subset \mathbf{E}$ on which

- the set of competitive equilibria is finite, and is a continuously differentiable function of the endowment and utility assignment $(\vec{e},\vec{W})$;
- the spot market competitive equilibrium corresponding to any competitive equilibrium portfolio allocation is, locally, a continuously differentiable function of the portfolio allocation $\vec{y}$.

[7]A set is a differentiable function if all its elements are differentiable functions.

Proof of Generic Regularity: Without loss of generality we may assume that asset 0 satisifies $r(s) \geqq 0$ for all $s = 0, \ldots, S$. In other words, if $\bar{y}' = (1, 0, \ldots, 0)$, $R\bar{y} > 0$. It follows that for $q \in \text{Int } \bar{C}$, $q_0 \neq 0$ and hence we can normalize asset prices by setting $q_0 = 1$. Similarly, we take commodity $0(s)$ to be the numeraire in each state s. The price domain is thus $M = E^A \times E_{++}^{(S+1)L}$. This is an open set, and hence a manifold, of dimension $A + (S+1)L$. The excess demand function for assets and commodities other than the numeraire $a = 0$ and $0(s)$, $s \in S$, we denote by $f = (y,z) : \mathbf{E} \times M \to E^A \times E^{(S+1)L}$; no confusion should arise. Let us now fix the utility allocation $\vec{W}$. On account of Walras' Law if all but the numeraire markets clear then it does too, so $f_{\vec{W}}^{-1}(0)$ is the graph of the equilibrium correspondence. Let us check that $f_{\vec{W}} \pitchfork 0$. Given an element $(\bar{e},\bar{q},\bar{p}) \in f_{\vec{W}}^{-1}(0)$ and any spot commodity $j(s)$ we can look at the change in excess demand when agent 0's endowment of good $j(s)$ is increased by 1 and his endowment of good $0(s)$ is decreased by $p_j(s)$. Evidently demand stays the same, but now supply in the $j(s)$ market has increased by 1, so there is a net effect of $(0, \ldots, -1, \ldots, 0)$ on excess demand. Similarly, for any asset $a = 1, \ldots, A$, we can increase agent 0's endowment of good 0 in state s by $r_a(s) - q_a r_0(s)$, for each $s = 0, \ldots, S$. Clearly, the effect on agent 0's demand will be a decrease in asset a by 1, an increase in asset 0 by q_a, and nothing else. Thus, by varying over assets $a = 1, \ldots, A$, and commodities s, $s(j)$, $s = 0, \ldots, S$, $j = 1, \ldots, L$, we can show $f_{\vec{W}} \pitchfork 0$. Notice that we have shown that $D_{e^0} f_{\vec{W}}$ has full rank, not using prices, or endowments for agents $1, \ldots, H$ at all.

It is also easy to see, given that I is contained in some compact set, and hence has compact closure, that $f_{\vec{W}}^{-1}(0) \subset I \times K$ for some compact set K. Hence we can apply the full transversality theorem, deducing that for any utility allocation $\vec{W}$ and a generic endowment, $\vec{e} \in I(\vec{W})$, $f_{(\vec{e},\vec{W})} \pitchfork 0$. Q.E.D.

Proof of Generic Strong Regularity: The proof of generic strong regularity is almost as simple. Let $N = E \times M$ and let $f : N \to E^{A} \times E^{(S+1)L}$ be the non-numeraire excess demand function, this time with the utilities free to vary as well.

Consider for each $h \in H$ the function $\hat{z}^{h} : E^{A+1} \times E_{++}^{(S+1)L} \to E^{(S+1)L}$ given by $\hat{z}^{h}(y^{h},p)$, the excess demand by agent h for commodities $\ell(s)$, $s \in S$, $\ell = 1, \ldots, L$, given that his portfolio is fixed at y^{h}. Let $\hat{z} : E^{(H+1)(A+1)} \times E_{++}^{(S+1)L} \to E^{(S+1)L}$ be given by $\hat{z}(\vec{y},p) = \hat{z}(y^{0}, \ldots, y^{H}, p) = \sum_{h \in H} \hat{z}^{h}(y^{h},p)$. We must show that for a generic economy (W,e) in $E = W^{H+1} \times I$, if $\vec{y}$ is an equilibrium portfolio allocation and if p is the corresponding vector of commodity prices, then $D_{p}\hat{z}(\vec{y},p)$ is invertible.

The essential idea of the proof is borrowed from demand theory. We know that for any consumer, say h = 0, if we perturb the second derivatives $D^{2}W^{h}$, without disturbing the first derivatives DW^{h} at the point of demand x^{h}, given prices p and income specified by y^{h}, then the demand $\hat{z}^{h}(y^{h},p)$ will be unaffected at the old (y^{h},p), but the derivative $D_{p}\hat{z}^{h}$ can be altered by any symmetric matrix K. (Recall the Slutsky equations with a single budget constraint: $D_{p}z = K - vz'$, where K is symmetric and negative definite, but otherwise K and v can be chosen arbitrarily by varying $D^{2}W$, leaving p and z' fixed -- see Geanakoplos-Polemarchakis (1980)).

Consider now the function $G : N \times S^{(S+1)L-1} \to E^A \times E^{(S+1)L} \times E^{(S+1)L}$ given by $G(\eta,r) = (f(\eta),\ r'D_p\hat{z})$ where $\eta = (\vec{e},\vec{W},q,p)$ and where $\hat{z} = \hat{z}(\vec{e},\vec{W},\vec{y}(\eta),p)$ and $D_p\hat{z}$ is the jacobian with respect to the last argument, holding $\vec{y}$ fixed. The vector r is any element of the sphere $S^{(S+1)L-1}$ of dimension (S+1)L-1.

From the foregoing proof that $f \pitchfork 0$ and from the remarks on the generic invertibility of matrices admitting symmetric perturbations given in section 5, together with our observation that by varying the utility W^0 of agent 0 in the right way, it is possible to perturb $D_p\hat{z}$ without changing f, it follows that $G \pitchfork 0$. Hence, by the transversality theorem, for a generic choice $(\vec{e},\vec{W}) \in E$, $G_{(\vec{e},\vec{W})} \pitchfork 0$. But then since the domain of $G_{(\vec{e},\vec{W})}$ has dimension one less than the range, $G^{-1}_{(\vec{e},\vec{W})}(0) = \phi$. Thus for a generic $(\vec{e},\vec{W}) \in E$, $(\vec{e},\vec{W})$ is a regular economy and at each equilibrium $r'D_p\hat{z} = 0$ has no solution, i.e. $D_p\hat{z}$ is invertible and $(\vec{e},\vec{W})$ is strongly regular.

& Q.E.D.

7. Suboptimality

Let us suppose that there are at least two assets. We can imagine 2H independent allocations of the assets. Let $a^{h'}$ be the amount of asset a that individual 0 gives individual h', a = 0, ..., A, h' = 1, ..., H. If $a^{h'}$ is small enough, then by our strong regularity theorem, after the transfer is made, it is possible to calculate the unique small change in the spot market allocation and prices that will clear those markets. Thus it is possible to calculate the effect on the final utility levels of all the individuals resulting from such a reallocation of assets between two of them. One can imagine the (H + 1) × (H × (A+1)) matrix expressing the derivatives of W^h with respect to $a^{h'}$:

$$\begin{array}{c} \\ W^0 \\ W^1 \\ \vdots \\ W^H \end{array} \begin{array}{c} \cdots\ 0^1 \quad 1^1 \quad 1^2 \cdots 1^H \ \cdots \\ \left[\begin{array}{c} \\ \\ A = \{a_h^{h'}\}_{h\in H}^{h'\in H/\{0\}} \\ \\ \\ \end{array} \right] \end{array}$$

Notice in the diagram we have listed explicitly only H + 1 of the possible columns. Clearly there are many more.

The remarkable property is this: to prove generic constrained suboptimality, it suffices to show that the matrix A has full row rank! For then in particular there is an infinitesimal change $da^{h'}$ in the various $a^{h'}$ that increases each W^h. This matrix formulation of suboptimality was first articulated by Smale (1979).

Let us consider more carefully whether we should expect this matrix to have full rank or not. Note first that A is the sum of two matrices, A = T + P, where T is the direct transfer effect on utilities and P is the effect on utilities caused by the price redistribution. Let us concentrate first on T:

$$\begin{array}{c} \cdots\ 0^1 \cdots 1^1 \cdots 1^2 \cdots 1^3 \cdots 1^H \cdots \end{array}$$

$$T = \begin{array}{c} W^0 \\ W^1 \\ W^2 \\ W^3 \\ \vdots \\ W^H \end{array} \left[\begin{array}{ccccc} -q_0 & -q_1 & -q_1 & -q_1 & -q_1 \\ q_0 & q_1 & 0 & 0 & 0 \\ 0 & 0 & q_1 & 0 & 0 \\ 0 & 0 & 0 & q_1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & q_1 \end{array} \right]$$

We have normalized the utilities by the appropriate marginal utility of income. One sees at once that the matrix T does not have full row rank: indeed letting u be the vector in E^{H+1} consisting of all 1's, $u'T = 0$.

The redistribution of assets alone, without any further change in the spot markets, will not cause a Pareto improvement. On the other hand, looking at the columns $1^1, \ldots, 1^H$ (assuming $q_1 \neq 0$), it is clear that the rank of T is already H.

7.1 Suboptimality

Proposition 4 (Suboptimality): If the asset market is incomplete (S), and if A1-A4 and D1-D3 are satisfied, then for any economy $(\vec{e},\vec{W}) \in D$, a generic set, all competitive equilibria are suboptimal.

Proof: Consider another asset α not in the span of R, and now replace column 0^1 in T with α^1, getting the matrix $\hat{T}$. We can easily show that for a generic economy, at each equilibrium the corresponding $\hat{T}$ has full row rank. In fact, by inspection of $\hat{T}$ it is obvious that if the sum of elements in the column α^1 is not equal to zero, then $\hat{T}$ has full row rank.

Let $G : N \to E^A \times E^{(S+1)L} \times E$ be given by $G(\eta) = (f(\eta), \sum_{h\in H} \alpha_h^1)$ where α_h^1 is the h^{th} element in the column α^1, evaluated at assignment of goods given by $f(\eta)$, where $\eta = (\vec{e},\vec{W},q,p)$. We shall show now that it is always possible to find a linear perturbation of individual 0's utility W^0 that will leave f and α_h^1, h = 1, ..., H, unaffected, but will change α_0^1. Since we have already shown that $f \pitchfork 0$, this will imply that $G \pitchfork 0$, and that $G^{-1}(0) = \phi$; hence $\sum_{h\in H} a_h' \neq 0$ whenever $f(\eta) = 0$.

Perturb W^0 by the linear function $x \to \Delta \cdot x$ given as follows. Since the asset α payoff is not in the span of R, we can find $\Delta_0 = (\Delta_0(s);\ s \in S)$ such that $\Delta_0'R = 0$ and yet $\Delta_0 \cdot \alpha \neq 0$. That is, if we perturb the $0(s)^{th}$ marginal utilities of W^0 by $\Delta_0(s)$, $s \in S$, then the marginal utilities to agent 0 of assets a = 0, 1, ..., A are all unchanged, while that of the hypothetical asset does change. Hence α_0^1 must be different. On the other hand, if we let $\Delta_\ell(s) = p_\ell(s)\Delta_0(s)$ $s \in S$, $\ell = 1, \ldots, L$, then individual 0's demand for commodities and existing assets (given that α is unavailable) is left unaltered. Q.E.D.

7.2 Constrained Suboptimality

To prove the generic constrained suboptimality of competitive equilibrium allocations when the asset market is incomplete is of course much more difficult than proving Pareto suboptimality, because it must be done with the existing assets. For ease of notation, and not because it makes the proof easier, we shall assume W^h is separable:

$$W^h(x) = \sum_{s\in S} \pi^h(s)\ U^h(x(s);\ s)\ .$$

A central role in our analysis is played by the vector of marginal utilities of the numeraire good over the different states of nature: $\nabla^h = (\ldots, \nabla^h(s), \ldots)$, where $\nabla^h(s) = \dfrac{\partial U^h(x(s);\ s)}{\partial x_0(s)} \Big/ \mu^h$, for $s = 0, 1, \ldots, S$. Recall that at an equilibrium $(q, p, \vec{y}, \vec{x})$, we must have $R'\nabla^h = q$ for all $h \in H$. We have already seen that when asset markets are incomplete, then generically the vectors ∇^h are not equal.

Consider a strongly regular economy $(\vec{W},\vec{e})$ and one of its competitive equilibria, $(q, p, \vec{y}, \vec{x})$. Recall that by definition the associated excess demand function, with the portfolios y^h held fixed, for the $(S+1)L$ non-numeraire commodities given by $\hat{z}(\vec{y},p)$ has $D_p\hat{z}$ invertible. We can thus calculate the change in equilibrium spot prices when $\vec{y}$ is perturbed from the implicit function theorem:

(i) $$dp = -(D_p\hat{z})^{-1}(D_{\vec{y}}\hat{z})d\vec{y}$$

This expression can be made more explicit after we introduce some notation. Let $V^h \in E^{(S+1)L}$ be the vector of income effects. We can think of V^h as a (S+1) list of L-vectors, where the L-vector $V^h(s)$ gives the additional consumption that would occur in the L goods, $\ell(s) = 1(s), \ldots, L(s)$ given an extra unit of state s income. Notice that the separability hypothesis allows us to assume that an extra dollar in state s has no effect on consumption in state $s' \neq s$. Define a list α as an ordered collection of (S+1) $i \times j$ matrices. Suppose β is a list of (S+1) $j \times k$ matrices. Then we shall write $\alpha \times \beta$ to mean the list of (S+1) $i \times k$ matrices obtained by the componentwise product of α and β.

Let us return to our expression for dp. Recall that $a^{h'}$ is the gift of one unit of asset a by agent 0 to agent h'. Let R_a be the a^{th} column of R. Think of R_a as a list of 1×1 matrices; then for column $a^{h'}$,

(ii) $$dp_{a^{h'}} = -(D_p\hat{z})^{-1}(V^{h'} - V^0) \times R_a da^{h'} .$$

Let $\hat{z}^h$ be the (S+1)L vector of excess demands of agent h. We can think of $\hat{z}^h$ as an (S+1) list of $1 \times L$ matrices. We can now write the expression for the change in utility dW^h given a change in the portfolio allocation $d\vec{y}$ (using the envelope theorem):

(iii) $$dW^h = \nabla^h \cdot Rdy^h - \nabla^h \cdot (\hat{z}^h \times D_{\vec{y}}p)$$

From this expression it is easy to derive the matrices A = T + P. To calculate $\partial W^h/\partial a^{h'}$, for $h' \neq 0$, set $dy_a^{h'} = 1 = -dy_a^0$ and $dy_{a'}^{h''} = 0$ for all other h'' and a'.

The first term in (iii), $\nabla^h \cdot Rdy^h$ is the direct effect of the transfer of assets dy^h, assuming all prices remain constant. It can equivalently be written $q \cdot dy^h$. The second term is the effect on utility produced by the change in relative prices p stemming from the portfolio change $d\vec{y}$.

We must now show that for a generic economy, there is no vector $r \in S^H$, the H dimensional sphere, with $r'A = 0$. By combining expressions (i), (ii), (iii) we see that there are a number of special cases where $r' = (1, 1, \ldots, 1)$ solves $r'A = 0$, and hence allows for constrained optimality. (Most of these cases have been pointed out by Stiglitz (1982).)

7.2.1. Some Non-Generic Examples where Constrained Optimality is Possible

First, suppose that markets are complete, so A = S. From Arrow's (1953) analysis, we know that we must be at a full Pareto optimum. We can see that locally, in our analysis, by checking that if R is square, then $R'\nabla^h = q$ has a unique solution ∇^h, so all the ∇^h are the same, and we must be at a full local Pareto optimum. Of course any Pareto optimum must be a constrained optimum. Indeed, taking $r = (1, \ldots, 1)$ allows $r'A = 0$. For the first term of (iii), corresponding to the T matrix, we already saw that $(1, \ldots, 1)\ T = 0$, no matter what the ∇^h, since $\sum_{h \in H} dy^h = 0$ and $R'\nabla^h = q$ all h. As for the second term, when all the ∇^h are identical, we get $\sum_{h \in H} \nabla^h \cdot (\hat{z}^h \times dp) = \nabla^h \cdot (\sum_{h \in H} \hat{z}^h \times dp)$ $= \nabla^h \cdot 0 \cdot dp = 0$, since in equilibrium $\sum_{h \in H} \hat{z}^h = 0$.

There are other cases when we do not have full optimality, and yet we may have constrained optimality. Suppose that A < S, and that the ∇^h are not colinear at some equilibrium. If at the equilibrium there is no trade, $\hat{z}^h = 0$ for all h, then the second expression disappears, and again we must have $(1, \ldots, 1)\, M = (1, \ldots, 1)\, T = 0$.

Alternatively, if $V^{h'} = V^0$ for all h′, then we see from (ii) that once again the second term in (iii) disappears. Relative spot prices will not readjust when income is redistributed among the agents. The standard examples of equality among income effects occur when they are all equal to zero. For example, if there is only one good, good 0, in each state, then of course nothing is ever spent on goods 1, ..., L. This is the case studied by Diamond (1967), for which he proved constrained optimality. Another example is where $W^h = \sum_{s \in S} \pi^h(s) U^h(x(s); s)$ and each $U^h(s)$ can be written $U^h(x_0(s), x_1(s), \ldots, x_L(s)) = x_0(s) + \hat{U}^h(x_1(s), \ldots, x_L(s); s)$, i.e., the case of constant marginal utility of "money" (the numeraire good).

A final case to consider is where all individuals h have identical and homothetic von Neumann-Morgenstern utilities, $W^h = \sum_{s \in S} \pi^h(s) U(s)$.

In the next section we show that all of the above cases are fortuitous accidents. If we are allowed to perturb the endowments, typically there will be trade in equilibrium. If we are allowed quadratic perturbations of the utilities, then typically the income effect terms V^h will be distinct.

Before moving to the formal proof, let us note that if the portfolio reallocation $d\vec{y}$ were required to satisfy the budget constraint $q \cdot dy^h = 0$ for all h, then the direct effect would entirely disappear. The second term would be slightly altered, giving:

$$\text{(iv)} \qquad \frac{\partial W^h}{\partial a^{h'}} = \nabla^h \cdot \{\hat{z}^h \times [-(D_p \hat{z})^{-1} (V^{h'} - V^0) \times (R_a - \frac{q_a}{q_0} R_0)]\} .$$

The same special cases of constrained optimality apply here as well.

7.2.2. Proof of Constrained Suboptimality

Proposition 5 (Constrained Suboptimality): Suppose $0 < 2L \leqq H < SL$. If the asset market is incomplete, S, and if A1-A4, D1-D3, and CS are satisfied, then for any economy $(\vec{e},\vec{W}) \in D$, a generic set, all competitive equilibria are constrained suboptimal as long as there are at least two assets, $(A + 1) \geqq 2$. If $(A + 1) \geqq 3$, this remains true even if the reallocation of assets must satisfy the asset budget constraint for each individual at the equilibrium asset prices.

Remark: An upper bound on H is necessary if Pareto improvement is to take place through reallocations which satisfy the original budget constraint. From (iv) we know that we can write

$$D_{\vec{y}}W^h = (\nabla^h \times \hat{z}^h) \cdot D_{\vec{y}}p ,$$

and there cannot be more than (S + 1)L linearly independent vectors $(\nabla^h \times \hat{z}^h)$.

Proof of Constrained Supoptimality: Suppose that $0 < 2L \leqq H < SL$ and that D is a generic set of economies such that all economies in D are regular and strongly regular, and each of their equilibria satisfy:

(1) For the H + 1 different portfolio reallocations $d\vec{y}$ given by $d0^1, d1^1, \ldots, d1^H$, the (S+1)L dimensional vectors $D_{\vec{y}}p\ d0^1, \ldots, D_{\vec{y}}p\ d1^H$ are linearly independent, even if

attention is restricted to any SL of their coordinates (by ignoring an arbitrary state), and

(2) Given any vector $r \in S^H$, it is possible in at least S of the S + 1 states s to arbitrarily perturb $\sum_{h\in H} r^h \nabla^h(s)\hat{z}^h(s)$ by perturbing the economy $(\vec{e},\vec{W})$ in D without changing aggregate excess demand, or the derivative of aggregate excess demand $D_p\hat{z}$ or the income effect terms V^h for any $h \in H$, all evaluated at the given equilibrium $(q, p, \vec{y}, \vec{x})$ for $(\vec{e},\vec{W})$.

Notice that condition 1 is at least possible if $H + 1 \leqq SL$, and condition 2 is possible if $H > L$. Notice that (2) implies that it is possible to alter the weighted sum $\sum_{r\in H} r^h \nabla^h(s) z^h(s)$ without altering the sum $\sum_{h\in H} z^h(s)$ of excess demands. We shall show that in fact there is a generic D where both 1 and 2 hold. Clearly 2 could not hold for $r = (1, \ldots, 1)$ if all the ∇^h were identical under all perturbations of (W,e).

Given (1) and (2), let $N = D \times E^A \times E^{(S+1)L}_{++}$ and consider the map $F : N \times S^H \to E^A \times E^{(S+1)L} \times E^{H+1}$ given by $F(\eta,r) = (f(\eta), r'A)$, where η is an economy and equilibrium, f is the non-numeraire excess demand function, $r \in S^H$, and A is the matrix T + P of utility effects of the H + 1 portfolio reallocations described in the last section. Since we have already shown that $f \pitchfork 0$, it suffices to show that by perturbations in D that leave f unaffected, we can make the function $\mu(\eta,r) = r'A$ transverse to 0. We know from the last section that for each column $a^{h'}$ of A, we can write the corresponding entry of $r'P = r'(A - T)$ as:

$$\mu(\eta,r) = \Big(\sum_{h\in H} (r^h \nabla^h \times \hat{z}^h)\Big) \cdot (D_{\vec{y}}p \; da^{h'}) \; .$$

From hypothesis (2) we know that for SL coordinates we can perturb the vector $\gamma = (\sum_{h \in H} r^h V^h \times \hat{z}^h))$ arbitrarily, and from hypothesis (1), the vectors $dp_{a^{h'}} = (D_{\vec{y}} p da^{h'})$ are linearly independent even when restricted to those SL coordinates. Hence for each column of the P matrix there is a perturbation of the economy $(\vec{e},\vec{W})$ which changes γ in a way which has nonzero dot product with $dp_{a^{h'}}$, and zero with all the other H vectors of price changes. At the same time this perturbation leaves the aggregate excess demands, hence f unchanged, and also the derivatives $D_p\hat{z}$. But recall that each column $dp_{a^h} = (D_{\vec{y}}p)da^{h'} = -(D_p\hat{z})^{-1}(V^{h'} - V^0) \times R_a da^{h'}$, and by hypothesis (2) none of this is changed by the perturbation, nor is $r'T$.

Thus we have shown that by perturbing $(\vec{e},\vec{W})$ we can change any element of $r'A$, without changing the others, and without changing f. This implies that $F \pitchfork 0$. But then for a generic set of economies $D' \subset D \subset E$, if $(W,e) \in D'$, then $F_{(\vec{e},\vec{W})} \pitchfork 0$. But the domain of $F_{(\vec{e},\vec{W})}$ has one less dimension than the range, hence $F^{-1}_{(W,e)}(0) = \phi$. Thus, for a generic $(\vec{e},\vec{W})$ in **D**, if $f(\vec{e},\vec{W},q,p) = 0$, then $r'A$ has no solution. In other words, for a generic $(\vec{e},\vec{W}) \in$ **E**, all its equilibria are constrained suboptimal. Thus to verify our proof, we need only check that there is a generic set **D** of strongly regular economies that satisfy conditions (1) and (2).

Before proceeding to this part of the proof, notice that we have made no use of the direct utility effects of the transfer. Exactly the same theorem would hold if in addition we required that $q \cdot dy^h = 0$ for all $h \in H$. (We would then need to require at least three assets.)

Let us begin by reviewing what can be changed by perturbations of the utilities. Note that (as shown for example in Geanakoplos-Polemarchakis (1980)) the jacobian (with portfolios held fixed) of non-numeraire excess demand $\hat{z}^h(p)$ at a point p in state s may be written $\lambda^h(s)K^h(s) - V^h(s)z^h(s)'$ where K is symmetric and negative definite, but otherwise $K^h(s)$ and $V^h(s)$ can be perturbed arbitrarily by altering the second derivatives $D^2U^h(s)$ at $x^h(s) = e^h(s) + z^h(s)$ without changing the demand at p. To verify that (1) generically holds, we must show that $(V^1 - V^0) \times R_0$ and $(V^{h'} - V^0) \times R_1$ for $h' \in H/\{0\}$ are linearly independent when restricted to any SL coordinates at every equilibrium of a generic economy. Note first that, on account of CS, perhaps by relabeling assets, we may assume that $r_1(s) \neq 0$ for all $s \in S$. We also know from CS that there is another asset, say 0, such that restricted to any S states R_0 and R_1 are linearly independent. To verify condition (1) then it suffices to use our standard trasversality argument. Take $F = (f,g)$, where f is excess demand and g = 0 only if (1) is not met. Then, since we can perturb the V^h, and hence g, without disturbing f, generically $F^{-1}(0) = \phi$. We have given this argument too many times to repeat it again formally here.

As for condition (2), note again that if $H \geqq L$, it can be shown in the usual manner that for any set of L + 1 distinct individuals, $h_0, \ldots, h_L$, the differences $V^{h_0}(s) - V^{h_1}(s), \ldots, V^{h_0}(s) - V^{h_L}(s)$ are all linearly independent for all s at any equilibrium of a generically chosen economy. In particular it follows that any vector $\Delta \in E^L$ may be written as a linear combination of these L differences. From now on D shall denote the open, dense, full measured (i.e., generic) set of economies which are regular, strongly regular, to which condition (1) applies, and for which at any equilibrium, and state s, and collection of $L + 1$ individuals, $\{V^{h'} - V^{h_0}\}$ span E^L.

We now want to show that we can perturb a weighted sum of individual excess demands without disturbing the aggregate excess demand (unweighted sum) or its derivatives. It is not difficult to increase individual 0's excess demand and decrease some other individual's by the same amount thus changing the weighted but not the unweighted sum. This, however, is not enough because from the Slutzky equation we know that such a change would affect the derivatives of the excess demand. Our method consists in choosing changes in the excess demands that produce symmetric changes in the jacobian of the excess demand which can then be undone by simultaneously perturbing the utilities.

Consider now an arbitrary perturbation Δ of the endowment of agent h_0's non-numeraire L commodities in state s at some equilibrium (q,p) of an economy in D. Perturb the endowment $e_0^h(s)$ by just the right amount $\Delta_0(s)$ to keep h's income constant at the equilibrium spot prices p(s), $\Delta_0(s) + p(s) \cdot \Delta = 0$. Choose another consumer $h' \neq h_0$ and perturb the h' endowment $(e_0^{h'}(s), e^{h'}(s))$ by $-(\Delta_0(s), \Delta)$. Then aggregate demand is unchanged though individual h_0's excess demand has changed by $-\Delta$. But we can do even better than that, since the perturbation to h_0 and h' alone may change $D_p z$. Let us write $\Delta = \sum_{i=1}^{L} \Delta^i$, where each Δ^i is a scalar multiple of $V^{h_i}(s) - V^{h_0}(s)$ for the $L + 1$ distinct individuals $h_0, h_1, \ldots, h_L$. If we altered the endowments of agents h_i, $i = 1, \ldots, L$, by $-(\Delta_0^i(s), \Delta^i)$, where $\Delta_0^i(s) = -p(s) \cdot \Delta^i$, then once again agent h_0's excess demand in state s has been changed arbitrarily by $-\Delta$, without changing aggregate excess demand. Furthermore, from the Slutsky decomposition $D_{p(s)}\hat{z}^h(s) = \lambda^h(s)K^h(s) - V^h(s)z^h(s)'$, we see that the derivative of aggregate excess demand has been altered by $\sum_{i=1}^{L} (V^{h_i}(s) - V^{h_0}(s)(\Delta^i)'$, which is symmetric. Hence by simultaneously perturbing some agent's K matrix, we can also maintain the derivative $D_{p(s)}\hat{z}\,(s)$ of aggregate excess demand in state s.

Suppose finally that the numbers m_s^h are given, and that we are interested not in $\Delta + \sum_{i=1}^{L} (-\Delta^i)$, but in $\beta = m^{h_0}(s)\Delta + \sum_{i=1}^{L} m^{h_i}(s)(-\Delta^i)$. When by choosing Δ arbitrarily and then constructing Δ^i as above, can we be sure to obtain any vector β? Clearly, if all the m^{h_i} are the same for $i = 1, \ldots, L$, it is impossible to achieve any β other than 0, since by construction $\Delta - \sum_{i=1}^{L} \Delta^i = 0$. On the other hand, we now show that if $m^{h_i}(s) \neq m^{h_0}(s) \neq 0$ then it is possible. Let B be the $L \times L$ matrix whose i^{th} column is $V^{h_i} - V^{h_0}$, and let m be the $L \times L$ diagonal matrix with i^{th} diagonal element equal to $m^{h_i}(s)$. Then we must solve $\beta = [m^{h_0}(s)I - BmB^{-1}]\Delta$. Clearly, this has a solution for all β if and only if the matrix BmB^{-1} does not have an eigenvalue equal to $m^{h_0}(s) \neq 0$, i.e. if and only if $m^{h_i}(s) \neq m^{h_0}(s) \neq 0$ for all $i = 1, \ldots, L$.

In summary, we have shown so far that given $L + 1$ numbers $m^{h_0}(s), \ldots, m^{h_L}(s)$, if $m^{h_0}(s) \neq 0$ and $m^{h_0}(s) \neq m^{h_i}(s)$, $i = 1, \ldots, L$, it follows that by perturbations of the economy $(\vec{e},\vec{W})$ in D it is possible to perturb $\sum_{i=0}^{L} m^{h_i}(s)\hat{z}^{h_i}(s)$ at a given equilibrium arbitrarily, without affecting aggregate excess demand, in any state, or the derivative of excess demand, or the income effect terms V^h of any individual h at the given equilibrium.

We are interested finally in the case where $m^h(s) = r^h \nabla^h(s)$, where $r \in S^H$ can be arbitrary. Suppose that $\nabla^h(s)/\nabla^h(s') \neq \nabla^{h'}(s)/\nabla^{h'}(s')$ for any pairs $h \neq h'$, $s \neq s'$ and that $\nabla^h(s) \neq 0$ for all h, s. We will show that in at least all but one of the states there are individuals $h_0, h_1, \ldots, h_L$ such that $0 \neq r^{h_0}\nabla^{h_0}(s) \neq r^{h_i}\nabla^{h_i}(s)$ for $i = 1, \ldots, L$. To see this, take h_0 so that $r^{h_0} \neq 0$ (there must be one since $r \in S^H$). Take any s; by hypothesis $m^{h_0} = r^{h_0}\nabla^{h_0}(s) \neq 0$, since ∇^{h_0} has no zero elements. Suppose it is impossible to find L other agents with $r^h\nabla^h(s) \neq m^{h_0}(s)$. Then if $H \geqq 2L$, it must be that for at least L of them, $m^h(s) = m^{h_0}(s)$. But then by the hypothesis on the $\nabla^h(s)$, $0 \neq m^{h_0}(s) \neq m^{h'}(s')$ for all other states s′, for all of these individuals.

Thus to conclude the proof we need only check that for a generic set of economies in D, at each equilibrium $\nabla^h(s)/\nabla^h(s') \neq \nabla^{h'}(s)/\nabla^{h'}(s')$, if $h \neq h'$ and $s \neq s'$. If asset markets are incomplete by 2, $(A \leqq S - 2)$, then there is some $\nabla \in E^{S+1}$ with $\nabla(s) \neq 0 = \nabla(s')$ such that $\nabla' R = 0$. By adding ∇ to ∇^h but not to $\nabla^{h'}$, we can change $\nabla^h(s)/\nabla^h(s')$ but not excess demand or $\nabla^{h'}(s)/\nabla^{h'}(s')$. A familiar transversality argument will then show that for a generic economy in **D**, all its equilibria satisfy the above gradient conditions.

If $A = S - 1$, then there is a unique ∇ with $\nabla R = 0$, and it may happen that $\nabla^h(s)/\nabla^h(s') = \nabla(s)/\nabla(s')$, so we must give a slightly longer argument. Perturb all the ∇^h by adding ε in the s^{th} position. Add $\varepsilon r(s)$, the s^{th} row of R, to q. Then still $R'\nabla^h = q$ for all h. Change each y_0^h by dy^h to maintain, at the new prices q, $q \cdot y^h = 0$. Change each consumer's zeroeth endowment in every state by $de_0^h(s) = r_0(s)dy^h$. This does not change aggregate excess demand ($\sum_{h\in H} dy^h = 0 = \sum_{h\in H} e_0^h(s)$), but it does change all the $\nabla^h(s)/\nabla^h(s')$. Thus generically we cannot have $\nabla^h(s)/\nabla^h(s') = \nabla(s)/\nabla(s')$ in equilibrium. But now perturb ∇^h by adding ∇, without changing $\nabla^{h'}$. Q.E.D.

Conclusion

The normative appeal of competitive equilibrium rests nowadays on its Pareto optimality. Yet, the analysis of this paper shows, as Stiglitz has suggested, that when the asset market is incomplete the existing opportunities for trade are typically not efficiently used in a competitive equilibrium. To be sure, there are exceptions: if all consumers have identical income effects or if there is no trade, incomplete markets competitive equilibria can be constrained Pareto optimal. But we believe that for general economies the notion of the efficiency of the market must be reexamined. Kenneth Arrow, among others, has for a long time asserted that markets are incomplete and therefore to move towards Pareto optimality might require active (government) intervention in the economic arena. The preceding analysis strengthens that assertion by showing that even if the government is limited to the same assets as the market, it can still typically effect Pareto improvements.

When markets are incomplete, knowledge of demand functions is not sufficient information to recover preferences. (Think, for example, of the case of two states of nature and no assets; in that case it is impossible to deduce the relative importance to a consumer of consumption in the two states from a knowledge of the demand functions.) Whether it is generically possible to effect a Pareto improving portfolio reallocation when knowledge of investors' preferences is limited to market demand functions is an open question.

To be sure, in order to effect a Pareto improving reallocation of assets the government must be able to forecast all the resulting adjustments in spot market prices and their effects on individual's utilities. This is an enormous information burden, which it may be argued the government cannot carry. But such an argument against market intervention, based on the presumed ignorance of the government, is radically different from the standard argument for Pareto optimality which does not rely on any lack of information.

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