COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale Station

New Haven, Connecticut 06520

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ASYMPTOTIC RESULTS FOR GENERALIZED WALD TESTS[1]

Donald W. K. Andrews

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ABSTRACT

This note presents conditions under which a quadratic form based on a g-inverted weighting matrix converges to a chi-square distribution as the sample size goes to infinity. Subject to fairly weak underlying conditions, a necessary and sufficient condition is given for this result. The result is of interest, because it is needed to establish asymptotic significance levels and local power properties of generalized Wald tests (i.e., Wald tests with singular limiting covariance matrices). Included in this class of tests are Hausman specification tests and various goodness of fit tests, among others. The necessary and sufficient condition is relevant to procedures currently in the econometrics literature, because it illustrates that some results stated in the literature only hold under more restrictive assumptions than those given.

1. Introduction

This note is concerned with the convergence in distribution of quadratic forms based on g-inverted weighting matrices. The latter are being used increasingly in econometrics and statistics. Examples include testing procedures for the rank deficient linear model (e.g., see Mitra (1980)), Hausman specification tests (see Hausman and Taylor (1981), Holly (1982), and Duncan (1983)), goodness of fit tests (see Andrews (1985a, b), Heckman (1984), and Moore (1977)), generalized method of moments specification tests (see Newey (1984)), and more generally, all Wald tests based on statistics that have singular asymptotic normal distributions, i.e., generalized Wald tests (see Moore (1977)).

The purpose of this note is to provide a general asymptotic result for g-inverse quadratic forms that can be applied to numerous existing and prospective inferential procedures. This result is relevant to procedures currently in the literature, since it illustrates that some results stated in the literature only hold under more restrictive assumptions than are presented.

The g-inverse of an estimator A_n of a singular covariance matrix A often is used to replace in a quadratic form some g-inverse A^- of the covariance matrix A. Commonly, the claim then is made that if A_n is consistent for A, the use of A_n^- rather than A^- will not affect the asymptotic distribution of the quadratic form in question. This claim is important because it is used to determine the critical regions of various test procedures (such as those listed above). In some cases, it is justified by the second claim that if A_n is consistent for A, then A_n^+ is consistent for A^+ (where $(\cdot)^+$ denotes the Moore-Penrose inverse). In this note we show that neither claim is true in general, but that both are true under additional conditions.

2. Asymptotic Distributions of g-Inverse Quadratic Forms

We consider the asymptotic distribution of a g-inverse quadratic form given by $X_n'A_n^-X_n$, where X_n is an asymptotically normal real random L-vector with singular covariance matrix A , and A_n is a conformable real random matrix that converges in probability to A .

If the limit matrix A were non-singular, then A_n would be non-singular with probability that goes to one as $n \to \infty$, and by straightforward application of the continuous mapping theorem, the quadratic form $X_n'A_n^-X_n$ would have an asymptotic chi-square distribution. Since A is singular, however, the continuous mapping theorem does not apply, because g-inverses are not continuous.

As the following example illustrates, a problem arises when A is singular, beyond that of just the method of proof. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $X_n = X = (X_1, 0)'$ where $X_1 \sim N(0,1)$, and $A_n = \begin{bmatrix} 1 & 1/n \\ 1/n & \tau/n^2 \end{bmatrix}$, for some constant $\tau \in R$. For the case $\tau > 1$, we have $A_n \xrightarrow{n\to\infty} A$, A_n is symmetric and positive definite, for all n , and X_n is in the column space of both A_n and A , for all n . The example is quite "regular," except that A is singular. Notice that $A_n^{-1} = \frac{1}{\tau-1}\begin{bmatrix} \tau & -n \\ -n & n^2 \end{bmatrix}$, and so, $X_n'A_n^{-1}X_n = \left(\frac{\tau}{\tau-1}\right)X_1^2$. Thus, $X_n'A_n^{-1}X_n$ does not have an asymptotic chi-square distribution for any $\tau \in R$. In fact, its asymptotic distribution depends crucially on the sequence of matrices $\{A_n\}$. Depending on the value of τ , its mean could be any number in $(-\infty,\infty)$.

This example illustrates that problems can arise when forming test statistics using g-inverse quadratic forms. Since the use of g-inverse quadratic forms is becoming frequent in econometrics, it seems useful to establish precisely those conditions under which such problems do and do not arise. This is done below. Before discussing such results, however, we consider the question of whether such problems actually may arise in econometric applications. The following example illustrates that they may.

We consider a Wald test of a nonlinear restriction in a linear regression model with an intercept and one regressor. Suppose the regressor and the error are independent for each observation, and are independent identically distributed over observations, with the error having mean zero and variance σ^2, and the regressor having variance σ_X^2. The parameter vector θ equals $(\alpha, \beta, \sigma^2)'$, where α and β are the coefficients on the intercept and regressor, respectively. We wish to test the restriction $\alpha\beta = 0$. A Wald test is formed using the least squares estimator $\hat{\theta}$. The test statistic is $W_n = nh(\hat{\theta})'(\hat{H}\hat{V}\hat{H}')^{-1}h(\hat{\theta})$, where $\hat{H} = H(\hat{\theta}) = \frac{\partial}{\partial\theta}h(\hat{\theta})$ and $\hat{V}$ is a consistent estimator of the asymptotic covariance matrix of $\hat{\theta}$.

For the above restriction, $H(\theta_0) = 0$ for any null parameter vector of the form $\theta_0 = (0, 0, \sigma_0^2)'$, for some σ_0^2. Thus, $\hat{H}\hat{V}\hat{H}'$ has a singular probability limit (viz., zero) under θ_0, and the problem exemplified above may occur. In fact, it is not hard to show that $nh(\hat{\theta})^2$ converges in distribution to a random variable that equals zero with probability zero, and $\hat{H}\hat{V}\hat{H}'$ converges in probability to zero, under θ_0. Thus, the Wald statistic does not have a chi-square asymptotic distribution under θ_0, and further, the test rejects with probability that converges to one as $n \to \infty$. In addition, for null parameter vectors for which (α,β) does not equal $(0,0)$, but is close to $(0,0)$, we would expect the test statistic to converge very slowly to a chi-square distribution, and hence, to exhibit undesirable finite sample properties in all but very large samples. To conclude, this example serves to illustrate that the potential problems associated with the limit behavior of g-inverse quadratic forms, which are investigated below, are not strictly pathological, but may create difficulties in actual econometric applications.

We now give conditions under which $X_n'A_n^-X_n$ does, and does not, have an asymptotic chi-square distribution. For this purpose, we introduce the following conditions:

D1 $X_n \xrightarrow{d} X \sim N(\mu,A)$,

D2 $A_n \xrightarrow{p} A$,

D3 $P(A_n$ is symmetric and psd$) \to 1$ as $n \to \infty$,

D4 $P(X_n \in M(A_n)) \to 1$ as $n \to \infty$,

D5 $P(A_n$ is symmetric$) \to 1$ as $n \to \infty$,

where " $\xrightarrow{d}$ " denotes convergence in distribution as $n \to \infty$, " $N(\mu,A)$ " denotes the normal distribution with mean vector μ and covariance matrix A , " $\xrightarrow{p}$ " denotes convergence in probability as $n \to \infty$, " psd " abbreviates positive semi-definite, and $M(A_n)$ denotes the column space of the matrix A_n .

Let $r = rk[A]$ and $r_n = rk[A_n]$, where $rk[\cdot]$ denotes the rank of the matrix $\cdot$. By the singular value decomposition, the $L \times L$ real matrix A_n can be written as $B_nD_nC_n'$, where $D_n = \begin{bmatrix} \Delta_n & 0 \\ 0 & 0 \end{bmatrix}$ is an $L \times L$ diagonal matrix, Δ_n is an $r_n \times r_n$ full-rank diagonal matrix with diagonal elements non-increasing in absolute value from the upper left, and B_n and C_n are each $L \times L$ orthogonal matrices. Let $(\delta_{n1}, \ldots, \delta_{nr_n})$,

$(b_{n1}, \ldots, b_{nL})$, and $(c_{n1}, \ldots, c_{nL})$ denote the diagonal elements of Δ_n , the columns of B_n , and the columns of C_n , respectively. Then, $(\delta_{n1}^2, \ldots, \delta_{nr_n}^2, 0, \ldots, 0)$ are the eigenvalues of both A_nA_n' and $A_n'A_n$; $(b_{n1}, \ldots, b_{nL})$ are eigenvectors of A_nA_n' ordered to correspond to its eigenvalues; and $(c_{n1}, \ldots, c_{nL})$ are eigenvectors of $A_n'A_n$ ordered analogously. If A_n is symmetric, we can take $B_n = C_n$. In this case, $(\delta_{n1}, \ldots, \delta_{nr_n}, 0, \ldots, 0)$ are the eigenvalues of A_n ordered in non-increasing absolute value, and $(b_{n1}, \ldots, b_{nL})$ are eigenvectors of A_n ordered to correspond to its eigenvalues.

Let

$$Q_n = \begin{cases} \sum_{j=r+1}^{r_n} (b_{nj}'X_n)(c_{nj}'X_n)/\delta_{nj} & \text{when} \quad r_n > r \\ 0 & \text{otherwise.}^2 \end{cases} \tag{1}$$

As the following Theorem indicates, the behavior of the random variable Q_n is crucial in determining whether $X_n'A_n^-X_n$ has an asymptotic chi-square distribution. Q_n is zero if the ranks of A_n and A are equal, or if X_n is orthogonal to those eigenvectors of A_nA_n' and $A_n'A_n$ that correspond to eigenvalues of $A_n'A_n$ that are non-zero, but converge in probability to zero (given their ordering in D_n). If A_n is symmetric, then $b_{nj} = c_{nj}$ and Q_n is a weighted sum of squared residuals from the projections of X_n onto the spaces spanned by those eigenvectors of A_n that correspond to eigenvalues of A_n that are non-zero, but that converge in probability to zero. If A_n is symmetric and psd, then $Q_n \geq 0$. This is important below for establishing a necessary condition for $X_n'A_n^-X_n$ to have an asymptotic chi-square distribution.

Let $\chi^2(A,\mu)$ denote a non-central chi-square random variable (or distribution) with $rk[A]$ degrees of freedom and non-centrality parameter $\mu' A^+ \mu$, and let iff abbreviate "if and only if."

The main result is the following:

Theorem 1. (a) *Suppose* D1-D2 *hold. Then,* $Q_n \xrightarrow{p} 0$ *implies* $X_n' A_n^+ X_n \xrightarrow{d} \chi^2(A,\mu)$.
(b) *Suppose* D1, D2, D4, *and* D5 *hold. Then,* $Q_n \xrightarrow{p} 0$ *implies* $X_n' A_n^- X_n \xrightarrow{d} \chi^2(A,\mu)$ *for all sequences of* g-*inverses* $\{A_n^-\}$ *of* $\{A_n\}$.
(c) *Suppose* D1-D3 *hold. Then,* $Q_n \xrightarrow{p} 0$ *iff* $X_n' A_n^+ X_n \xrightarrow{d} \chi^2(A,\mu)$.
(d) *Suppose* D1-D4 *hold. Then,* $Q_n \xrightarrow{p} 0$ *iff* $X_n' A_n^- X_n \xrightarrow{d} \chi^2(A,\mu)$ *for all sequences of* g-*inverses* $\{A_n^-\}$ *of* $\{A_n\}$.

Comments: 1. If $P(rk[A_n] = rk[A]) \to 1$ as $n \to \infty$, then $Q_n \xrightarrow{p} 0$. Thus, D1, D2, and $P(rk[A_n] = rk[A]) \to 1$ are sufficient for $X_n' A_n^+ X_n$ to have a chi-square asymptotic distribution. These are particularly convenient sufficient conditions. Note that given D1-D3, the condition $P(rk[A_n] = rk[A]) \to 1$ is the weakest sufficient condition that does not involve an interaction between X_n and A_n.

2. The Theorem shows that some results in the literature are not completely accurate. It is not sufficient to establish D1-D3 or D1-D4 in order to assert that a g-inverse quadratic form has an asymptotic chi-square distribution.

3. Conditions D1 and D2 alone are sufficient to yield an asymptotic chi-square distribution of $X_n' A_n^+ X_n$ when A is non-singular. Since $P(rk[A_n] = rk[A]) \to 1$ and $Q_n \xrightarrow{p} 0$ are necessarily satisfied when A is non-singular, Theorem 1 part (a) contains the general result for non-singular A as a special case. This is not true of part (c), because part

(c) imposes D3.

4. The difference between the sufficient conditions of parts (a) and (c) of the Theorem, is the additional condition D3 in the latter. D3 usually is not restrictive. For example, any asymmetric covariance matrix estimator can, and usually should, be symmetrized before being employed in a quadratic form. Similarly, most covariance matrix estimators are psd with probability one, for all n. In the presence of serial dependence, however, some covariance matrix estimators are not psd with probability one (see the discussion of Newey and West (1985)), and hence, could fail to satisfy D3 even though they satisfy D2. For such estimators, it is certainly easier, and may be necessary, to rely on part (a) rather than part (c).

5. The proof of parts (b) and (d) actually shows that $X_n'A_n^-X_n$ is numerically identical for all choices of g-inverse of A_n with probability that converges to one as $n \to \infty$, when D4 and either D3 or D5 hold.

6. The noncentrality parameter $\mu'A^+\mu$ of the limit distribution is invariant with respect to the choice of g-inverse of A, if $\mu \in \mathcal{M}(A)$. See Vuong (1986) for some results where $\mu \notin \mathcal{M}(A)$.

The proof of Theorem 1 uses a result that establishes the consistency of Moore-Penrose inverted matrices. Since this result helps one to understand the need for the condition $Q_n \xrightarrow{p} 0$, and since it may be of independent interest, we briefly discuss the problem and present the result. The proofs of Theorem 1 and Lemma 1 below are given in Section 3.

It is easy to see that for non-random matrices $H_n \to H$ does not necessarily imply $H_n^+ \to H^+$. Consider the scalar case with $H_n = 1/n$, for $n = 1, 2, \ldots,$ and $H = 0$. Then, $H_n^{-1} = n \to \infty \neq H^+ = 0$. The result $H_n^+ \to H^+$ fails in this case, and in others, because the Moore-Penrose inverse is not a continuous function. The following Lemma establishes the conditions under which this discontinuity does, and does not, affect the convergence of H_n^+ to H^+. For the case of real matrices, this Lemma already appears in the applied mathematics literature, see Stewart (1969). We extend Stewart's result slightly by considering complex matrices, which allows for applications in multivariate spectral analysis. More importantly, the proof given in Section 3 below is much shorter and more transparent than that given by Stewart.[3]

Let H_n, $n = 1, 2, \ldots,$ and H be nonrandom complex $V \times W$ matrices. Let $||\cdot||$ denote the Euclidean norm, and let ev., and i.o., abbreviate "eventually" (i.e., for all but a finite number of n), and "infinitely often" (i.e., for infinitely many n), respectively.

Lemma 1 (Stewart). *Suppose* $A_n \xrightarrow{n\to\infty} A$. *Then,* $A_n^+ \xrightarrow{n\to\infty} A^+$ *iff* $\mathrm{rk}[A_n] = \mathrm{rk}[A]$ *ev. iff* $\limsup_{n\to\infty} ||A_n^+|| < \infty$.

Lemma 1 can be used to establish necessary and sufficient conditions for weak and strong consistency of estimators of Moore-Penrose inverted matrices. Suppose $H_n : n = 1, 2, \ldots$ are random matrices, but otherwise are as above. Let a.s. abbreviate "almost surely" (i.e., with probability one). By standard arguments, Lemma 1 yields the following:

Theorem 2. (a) *Suppose* $H_n \xrightarrow{n\to\infty} H$ *a.s. Then,* $H_n^+ \xrightarrow{n\to\infty} H^+$ *a.s. iff* $\mathrm{rk}[H_n] = \mathrm{rk}[H]$ *ev. a.s. iff* $\limsup_{n\to\infty} ||H_n^+|| < \infty$ *a.s.* (b) *Suppose* $H_n \xrightarrow{p} H$. *Then,* $H_n^+ \xrightarrow{p} H^+$ *iff* $P(\mathrm{rk}[H_n] = \mathrm{rk}[H]) \xrightarrow{n\to\infty} 1$ *iff* $\{H_n^+ : n = 1, 2, \ldots\}$ *are stochastically bounded (i.e.,* $\exists M < \infty$ *such that* $P(||H_n^+|| \leq M) \xrightarrow{n\to\infty} 1$ *).*

Comments: 1. Theorem 2 shows that if the rank condition $rk[H_n] = rk[H]$ ev. a.s. (or $P(rk[H_n] = rk[H]) \xrightarrow{n\to\infty} 1$) does not hold, then not only is H_n^+ inconsistent for H^+, but the error of estimation is unboundedly large as $n \to \infty$.

2. It is the potential for inconsistency of Moore-Penrose inverted matrices, as demonstrated by Theorem 2, that necessitates the condition $Q_n \xrightarrow{p} 0$ in Theorem 1 above. Note, however, that $A_n \xrightarrow{p} A$ plus the condition $P(rk[A_n] = rk[A]) \to 1$ as $n \to \infty$ is both necessary and sufficient for consistency of A_n^+ for A^+, but is only sufficient for $Q_n \xrightarrow{p} 0$. Thus, there exist cases where the g-inverse quadratic form $X_n'A_n^+X_n$ has an asymptotic chi-square distribution even though $A_n^+ \not\xrightarrow{p} A^+$.

3. Proofs

Proof of Theorem 1. Write the real $L \times L$ matrix A_n as in Section 2, i.e., $A_n = B_nD_nC_n'$. Let $\tilde{\Delta}_n$ denote the diagonal matrix formed by the upper left $r \times r$ block of Δ_n. With probability that goes to one, $r_n \geq r$ and $\tilde{\Delta}_n$ is well-defined. Let $\tilde{D}_n$ be an $L \times L$ matrix defined by $\tilde{D}_n = \begin{bmatrix} \tilde{\Delta}_n & 0 \\ 0 & 0 \end{bmatrix}$ if $\tilde{\Delta}_n$ is well-defined, and $[0]$ otherwise. Let $\tilde{A}_n = B_n\tilde{D}_nC_n'$.

To prove parts (a) and (c) of Theorem 1 we write

$$X_n'A_n^+X_n = X_n'\tilde{A}_n^+X_n + X_n'(A_n^+ - \tilde{A}_n^+)X_n , \tag{2}$$

and show that the first summand has asymptotic chi-square distribution, and the second summand equals Q_n.

By construction, $A_n - \tilde{A}_n = B_n(D_n - \tilde{D}_n)C_n' \xrightarrow{p} 0$. Thus, $\tilde{A}_n \xrightarrow{p} A$.

Since $P(\mathrm{rk}[\tilde{A}_n] = \mathrm{rk}[A]) \to 1$ as $n \to \infty$, Theorem 2 implies $\tilde{A}_n^+ \xrightarrow{p} A^+$. The continuous mapping theorem now gives

$$X_n'\tilde{A}_n^+X_n \xrightarrow{d} X'A^+X \sim \chi^2(A,\mu) \ , \tag{3}$$

where $X'\tilde{A}^+X \sim \chi^2(A,\mu)$ follows by Theorem 9.2.3 of Rao and Mitra (1971).

Since $\tilde{A}_n^+ = C_n\tilde{D}_n^+B_n'$, simple algebra yields

$$X_n'(A_n^+ - \tilde{A}_n^+)X_n = X_n'C_n(D_n^+ - \tilde{D}_n^+)B_n'X_n = Q_n \ . \tag{4}$$

Equations (2), (3), and (4) combine to give Theorem 1 part (a). If A_n satisfies D3, then $Q_n \geq 0$ (with probability that goes to one). In this case, (2), (3), and (4) combine to show that $Q_n \xrightarrow{p} 0$ is both necessary and sufficient for the asymptotic chi-square result of Theorem 1 part (c).

Part (b) follows from part (a) and part (d) follows from part (c) in Theorem 1 by noting that $X_n \in M(A_n)$ implies the existence of a vector Z_n such that $X_n = A_nZ_n$, and so, $X_n'A_n^-X_n = Z_n'A_n'A_n^-A_nZ_n = Z_n'A_nZ_n$, using the symmetry of A_n. Thus, the quadratic form $X_n'A_n^-X_n$ is invariant with respect to g-inverse. □

<u>Proof of Lemma 1</u>. Without loss of generality, assume $\mathrm{rk}[H_n] = g$, $\forall n$, for some integer $g > 0$. Then, either $g = \mathrm{rk}[H]$ or $g > \mathrm{rk}[H]$.

By the singular value decomposition (e.g., see Rao (1973, pp. 42-3)), we can write $H_n = R_n\Omega_nS_n^*$, for $\Omega_n = \begin{bmatrix} \Lambda_n & 0 \\ 0 & 0 \end{bmatrix}$, where R_n and S_n are $V \times V$ and $W \times W$ unitary complex matrices, respectively (i.e., $R_n^*R_n = I_V$), Λ_n is a $g \times g$ nonsingular diagonal matrix whose diagonal elements squared equal the eigenvalues of $H_n^*H_n$, and $(\cdot)^*$ denotes the conjugate transpose.

We use the result: $H_n^+ \xrightarrow{n\to\infty} H^+$ iff every subsequence of H_n^+ has a sub-subsequence that converges to H^+. Let $\{n_k\}$ be an arbitrary subsequence of $\{n\}$. Since the elements of R_n, S_n, and Ω_n are bounded above uniformly in n, there is a subsequence $\{n_m\}$ of $\{n_k\}$ such that R_{n_m}, S_{n_m}, and Λ_{n_m} converge to some matrices R, S, and Λ as $m \to \infty$. Form the $V \times W$ matrix $\Omega = \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix}$. It is easy to show that $R\Omega S^* = H$, R and S are unitary, Λ is nonsingular when $g = rk[H]$, and Λ is singular when $g > rk[H]$.

If $g = rk[H]$, then $\Omega_{n_m}^+ = \begin{bmatrix} \Lambda_{n_m}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \xrightarrow{m\to\infty} \begin{bmatrix} \Lambda^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \Omega^+$, and $\lim_{m\to\infty} H_{n_m}^+ = \lim_{m\to\infty} S_{n_m} \Omega_{n_m}^+ R_{n_m}^* = S\Omega^+ R^* = H^+$, as desired. Alternatively, if $g > rk[H]$, then for some $j \le g$, $[\Lambda_{n_m}]_{jj} \xrightarrow{m\to\infty} 0$, $||\Lambda_{n_m}^{-1}|| \xrightarrow{m\to\infty} \infty$, and $\lim_{m\to\infty} ||H_{n_m}^+|| = \lim_{m\to\infty} ||S_{n_m} \Omega_{n_m}^+ R_{n_m}^*|| = \infty$. □

4. Conclusion

The use of g-inverse quadratic forms is becoming frequent in econometrics. This note provides results concerning the asymptotic distributions of such statistics. Sufficient conditions are given for a g-inverse quadratic forms to have a non-central chi-square asymptotic distribution. Also, subject to weak underlying conditions, a necessary and sufficient condition is given for this result.

FOOTNOTES

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[2]When A_n is symmetric, Q_n is uniquely defined provided $\delta_{nr} \neq \delta_{nr+1}$, even though b_{nj}, for $j = r+1, \ldots, r_n$, may not be uniquely defined. Thus, under D2 and D3 or D5, Q_n is uniquely defined with probability that goes to one as $n \to \infty$.

When A_n is asymmetric with probability bounded away from zero for infinitely many n, however, Q_n is not necessarily "uniquely defined with probability that goes to one as $n \to \infty$." Non-uniqueness may result if $\delta_{nj} = \delta_{nj+1}$ for some $j \in (r, r_n)$. Unlike the symmetric A_n case, the choice of which eigenvector to match with δ_{nj} and which to match with δ_{nj+1} may affect the value of Q_n when A_n is asymmetric. And, the probability of non-distinct elements δ_{nj}, for $j = r+1, \ldots, r_n$, does not necessarily go to zero as $n \to \infty$.

[3]A special case of Stewart's (1969) result is given by Tyler (1981, Lemma 2.2) for symmetric real matrices. Tyler's proof is quite brief, due to his application of well-developed spectral results for symmetric matrices. Unfortunately, his result precludes applications in asymmetric cases, as occurs, for example, in covariance matrix estimation when the covariance matrix is of the form $A^{-}B(A^{-})'$, for A asymmetric. It also precludes applications in multivariate spectral analysis.

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