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SECTIONS AND EXTENSIONS OF CONCAVE FUNCTIONS

by

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Introduction

In discussions of distributional ethics using the standard conceptualization of people as utility-maximizing agents, many seemingly paradoxical situations can arise when different agents are given suitably incompatible utility functions. Kolm [1972] attempted to go beyond the standard conceptualization through the notion of fundamental preferences: agents with apparently different preferences over a certain specified set of goods should be thought of as having the same preferences over a wider set of goods, some of which have already been allocated. A classical example of this phenomenon might be the folk observation that one's preferences for ideologies are strongly correlated with the size of one's endowment of money. To account for the many everyday differences among people with similar socio-economic situations the "goods" over which agents have equal fundamental preferences would presumably have to include aspects of our internal chemistry.

This paper is not concerned with the questions of what fundamental preferences should or might be. However, the main result (Theorem 2.2) we prove implies that, within the usual formal framework which represents preferences as concave, non-negative functions on the positive orthant in $\mathbb{R}^n$, the notion of fundamental preferences is mathematically feasible. A weaker result (Proposition 2.3) is used in Roemer [1985], and the results of this paper were motivated by conversations with John Roemer related to that paper.

1. Generalities on Concave Functions

For our main results we will construct sought-for functions as limits of decreasing families of functions satisfying appropriate conditions. A very pleasant technical aspect of these constructions is that the limit functions are automatically continuous. The first section demonstrates this automatic continuity. The minimal results necessary for our purposes could be gleaned from Section 10 of Rockafellar [1970] with slight adaptations, but the phenomenon involved seems interesting so we have developed it in more detail than in Rockafellar.

Let V be a real vector space of dimension $n < \infty$. Let $X \subseteq V$ be a closed convex body. Let ϕ be a concave, non-negative function on X. Define $G^-(\phi)$, the sub-graph of ϕ, as the subset of $V \times \mathbb{R}$ specified by

(1.1) $$G^-(\phi) = \{(x,t) : x \in X, 0 \leq t \leq \phi(x)\} .$$

If ϕ is not identically zero then $G^-(\phi)$ will be a convex body in $V \times \mathbb{R}$. We will say ϕ is semicontinuous if $G^-(\phi)$ is a closed subset of $V \times \mathbb{R}$. Observe that this is equivalent to the superlevel sets

(1.2) $$L^+(\phi,s) = \{x \in X : \phi(x) \geq s\} , \quad s \geq 0$$

being closed. Observe also that $L^+(\phi,s)$ is convex.

Conversely suppose for some $t > 0$ we have a closed convex set $Y \subseteq V \times [0,t]$ such that

(1.3a) $$Y \cap (V \times \{0\}) = X \times \{0\}$$

(1.3b) If $(y,r) \in Y$, then $(y, r') \in Y$ for $0 \leq r' \leq r$.

Then the recipe

$$\phi_Y(x) = \max\{r : (x,r) \in Y\} , \quad x \in X$$

defines a concave non-negative function on X , and

$$(1.4) \qquad Y = G^-(\phi_Y) .$$

Denote by SCNC(X) the set of semicontinuous non-negative concave functions on X . It is straightforward to check that the sum of two functions in SCNC(X) is again in SCNC(X). Also a positive scalar multiple of an element in SCNC(X) is again an element. Thus SCNC is a cone in the space of all real-valued functions on X . Also given a family $\{\phi_i\}_{i\in I}$ of functions in SCNC(X) (here the index set I may be infinite), we may form their infimum:

$$(1.5) \qquad \inf\{\phi_i\}(x) = \inf\{\phi_i(x) : i \in I\} .$$

It is well-known and easy to see that $\inf\{\phi_i\}$ is concave and non-negative. Moreover we clearly have

$$(1.6) \qquad G^-(\inf\{\phi_i\}) = \bigcap_i G^-(\phi_i)$$

so that $\inf\{\phi_i\}$ again belongs to SCNC(X).

Let $Z \subseteq X$ be an arbitrary (not necessarily convex) subset of X , and let f be an arbitrary real-valued function on Z . Consider the subset of SCNC(X) consisting of functions which dominate (take as least as large a value at each point) f on Z . Evidently the infimum of such functions will again dominate f . Thus if there are any elements of SCNC(X) dominating f on Z , there is a minimum one. In particular, given a point $x_0 \in X$, there is a minimum element of SCNC(X) taking the value 1 at x_0 .

Proposition 1.1. Denote by $E_X(x_0, x)$ the value at x of the function in SCNC(X) which is minimum among all elements of SCNC(X) taking the value 1 at x_0. Then we have the formula

$$(1.7)\qquad E_X(x_0, x) = \sup\{(t-1)/t : x_0 + t(x - x_0) \in X\} .$$

Proof. In $V \times \mathbb{R}$, let $C(X, x_0)$ denote the closed convex hull of the points $(x,0)$, for $x \in X$, and the point $(x_0, 1)$. Since X is convex, the convex hull of $X \times \{0\}$ and $(x_0, 1)$ is the set

$$\{(sx_0 + (1-s)y, s) : y \in X, 0 \leq s \leq 1\}$$

and $C(X, x_0)$ will be the closure of this set. Suppose

$$(x,r) = (sx_0 + (1-s)y, s)$$

Then $r = s$, and

$$y = x_0 + (1-s)^{-1}(x - x_0)$$

belongs to X. Setting $t = (1-s)^{-1}$ we have

$$r = s = 1 - t^{-1} = (t-1)/t .$$

See Figure 1.

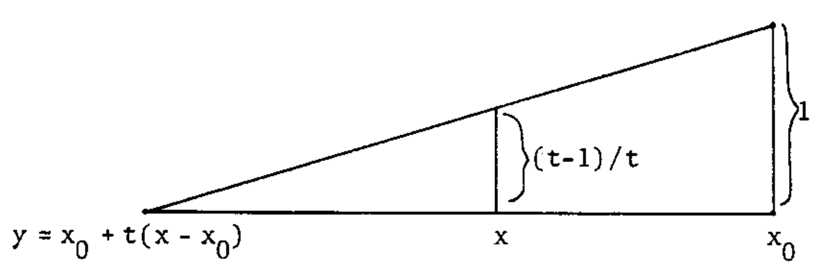

FIGURE 1

From Figure 1 and the convexity of X it is clear that if (x,r) is in $C(X, x_0)$, then so is (x, r') for $0 \leq r' \leq r$. Hence $C(X, x_0)$ satisfies conditions (1.3), and comparing (1.4) with definition (1.7) shows

$$C(X, x_0) = G^{-}(E_X(x_0, \cdot)) .$$

Furthermore, if ϕ is any function in SCNC(X) such that $\phi(x_0) \geq 1$ then obviously $G^{-}(\phi) \supseteq C(X, x_0)$, whence $\phi(x) \geq E_X(x_0, x)$. Thus Proposition 1.1 is proved.

Given a point x_0 in X, we say X is <u>conical at</u> x_0 if there exist

i) a neighborhood U of x_0 in V and

ii) a closed convex cone $C \subseteq V$

such that

(1.8) $$X \cap U = (C + x_0) \cap U .$$

That is, near x_0, the set X looks like a translated cone. Note we do not require C be a proper cone. In particular, C could be all of V; thus X is conical at all of its interior points.

<u>Proposition 1.2</u>. a) If $E_X(x_0, \cdot)$ (cf. formula (1.7)) is continuous at x_0, then all functions in SCNC(X) are continuous at x_0.

b) The function $E_X(x_0, \cdot)$ is continuous at x_0 if and only if X is conical at x_0.

<u>Proof.</u> a) Suppose $E_X(x_0, \cdot)$ is continuous at x_0. This means that given $\varepsilon > 0$ there is a neighborhood U of x_0 such that $E_X(x_0, x) > 1-\varepsilon$ for x in $U \cap X$. Consider $\phi \in$ SCNC(X). To show ϕ is continuous at x_0

we must be able to find a neighborhood U' such that $\phi(x_0) - \varepsilon < \phi(x) < \phi(x_0) + \varepsilon$ for x in U'. By semicontinuity the superlevel set $L^+(\phi, \phi(x_0) + \varepsilon)$ is closed, and it clearly does not contain x_0, so $U'' = V - L^+(\phi, \phi(x_0) + \varepsilon)$ is a neighborhood of x_0. If $\phi(x_0) = 0$, then U'' will serve for U' since ϕ is non-negative. If $\phi(x_0) \neq 0$, then by scaling ϕ we may assume without loss of generality that $\phi(x_0) = 1$. Then set $U' = U \cap U''$. Then for x in $U' \cap X$ we have $\phi(x) \geq E_X(x_0, x) > 1-\varepsilon = \phi(x_0) - \varepsilon$. Thus we have found the desired neighborhood U'. This proves part a).

b) Let U be a convex (open) neighborhood of the origin in V, with compact closure $\overline{U}$. (One may think of a Euclidean ball.) Then any neighborhood of x_0 contains a set of the form

$$x_0 + \delta U = \{x_0 + \delta u : u \in U\}$$

for suitably small $\delta > 0$.

Let $\partial U = \overline{U} - U$ be the boundary of U. If C is any closed convex cone in V, then we have

$$C = \bigcup_{s \geq 0} s(C \cap \partial U) .$$

Suppose $E_X(x_0, \cdot)$ is continuous at x_0. Then we can find $\delta > 0$ such that $E_X(x_0, x) > 1/2$ for x in $(x_0 + \delta U) \cap X$. Set

$$B = (x_0 + \delta(\partial U)) \cap X$$

and

$$C = \bigcup_{s \geq 0} s(B - x_0) .$$

Then C is a cone (a union of rays), and clearly

$$(C + x_0) \cap (x_0 + \delta U) \subseteq X \cap (x_0 + \delta U)$$

I claim that in fact this inclusion is an equality. To verify this, consider a point y in $(x_0 + \delta U) \cap X$. Assume $y \neq x_0$. For suitable $t \geq 1$, the point

$$z = x_0 + t(y - x_0)$$

will be in $x_0 + \delta(\partial U)$. We will show that $z \in X$, and the claim will be established. Since X is convex there is some a, with $0 < a \leq 1$ such that

$$x_0 + r(z - x_0)$$

belongs to X for $0 \leq r \leq a$, and does not belong to X for $a < r \leq 1$. If $a = 1$, then $z \in X$. If $a < 1$, then $E_X(x_0, x_0 + a(z - x_0)) = 0$. But since $x_0 + a(z - x_0) \in X \cap (x_0 + \delta U)$, this is a contradiction. Thus we have shown that if $E_X(x_0, \cdot)$ is continuous at x_0, then X is conical at x_0.

Conversely, suppose X is conical at x_0. Suppose that U is a convex neighborhood of the origin, and C is a closed convex cone such that

$$X \cap (x_0 + U) = x_0 + C \cap U. \qquad (1.9)$$

Then for $\varepsilon > 0$, the set

$$X \cap (x_0 + \varepsilon U) = x_0 + \varepsilon(C \cap U)$$

is a neighborhood of x_0 in X, and from formula (1.7) and relation (1.9) we see that $E_X(x_0, y) \geq 1-\varepsilon$ for y in $X \cap (x_0 + \varepsilon U)$. Hence $E_X(x_0, \cdot)$ is continuous at x_0. This concludes Proposition 1.2.

A convex set $X \subseteq V$ is called polyhedral if it is defined by a finite number of linear inequalities

$$\lambda(v) \geq b$$

where λ is a linear functional on V and b is a number.

Proposition 1.3. If $X \subseteq V$ is a polyhedral closed convex body, then X is conical at each of its points. Consequently every element of SCNC(X) is continuous on X.

Proof. The second statement follows from the first and Proposition 1.2. Let the polyhedral set X be defined by inequalities

$$\lambda_i(x) \geq b_i \ , \quad 1 \leq i \leq m \ .$$

for suitable linear functionals λ_i on V and numbers b_i. Given $x_0 \in X$, we may suppose, perhaps after permuting the indices i, that for some $\ell \leq m$ we have

$$\lambda_i(x_0) = b_i \ , \quad 1 \leq i \leq \ell$$

$$\lambda_i(x_0) > b_i \ , \quad \ell < i \leq m \ .$$

Define the cone C by

$$C = \{v : \lambda_i(v) \geq 0, \ 1 \leq i \leq \ell\}$$

and define the neighborhood U of x_0 by

$$U = \{u : |\lambda_i(u - x_0)| < (\lambda_i(x_0) - b_i)/2\} \ , \ \ell < i \leq m \ .$$

Then it is clear that

$$U \cap X = (x_0 + C) \cap U \ .$$

This proves the proposition.

Remark: Conversely one can show that if X is conical at every point, then X is "locally polyhedral," that is, any compact subset of X is contained in a polyhedral subset of X.

Let ϕ be a function in SCNC(X). Given a point $x_0 \in X$, the point $(x_0, \phi(x_0)) \in V \times \mathbb{R}$ is on the boundary of $G^-(\phi)$. Since $G^-(\phi)$ is convex, standard separation theorems (Rockafellar [1970]) allow us to find a supporting hyperplane to $G^-(\phi)$ at $(x_0, \phi(x_0))$. Thus we can find a linear functional $\lambda \in V^*$, and a number b, not both zero, such that

$$\lambda(x) + bt\phi(x) \leq \lambda(x_0) + b\phi(x_0) \ , \quad x \in X \ , \quad 0 \leq t \leq 1 \ .$$

Suppose x_0 is in the interior of X. Then necessarily $b \neq 0$, and taking $x = x_0$ and $t = 1/2$, we find $b > 0$. Thus we can rewrite this inequality to say

$$\phi(x) \leq \phi(x_0) + \lambda'(x - x_0) \tag{1.10}$$

where $\lambda' = -b^{-1}\lambda$. Thus we see that given ϕ in SCNC(X) and any $x_0 \in \text{int } X$, there is an affine (linear plus constant) function α such that $\alpha(x) \geq \phi(x)$ for all x with equality for $x = x_0$. In other words, any element of SCNC(X) is the infimum of all affine functions dominating it.

2. Sections and Extensions of Functions on Orthants

Let $\mathbb{R}^{n+}$ be the positive orthant in $\mathbb{R}^n$. We want to study $SCNC(\mathbb{R}^{n+})$. Recall an affine map $A : \mathbb{R}^m \to \mathbb{R}^n$ is a map of the form

$$A(x) = y_0 + L(x) \ , \quad x \in \mathbb{R}^m \tag{2.1}$$

where $y_0 \in \mathbb{R}^n$ and L is linear. If A is an affine map from $\mathbb{R}^m$ to $\mathbb{R}^n$ and $A(\mathbb{R}^{m+}) \subseteq \mathbb{R}^{n+}$ we call A positive. It is clear that if $A : \mathbb{R}^{m+} \to \mathbb{R}^{n+}$ is positive affine, then for any $\varphi \in SCNC(\mathbb{R}^{n+})$ the pull-back $\varphi \circ A$ is in $SCNC(\mathbb{R}^{m+})$.

Given $x, y \in \mathbb{R}^n$, we say $x \geq y$ provided $x-y \in \mathbb{R}^{n+}$. Given a real-valued function f on $\mathbb{R}^{n+}$ we say f is increasing provided $f(x) \geq f(y)$ whenever $x \geq y$.

Lemma 2.1. Any $\varphi \in SCNC(\mathbb{R}^{n+})$ is increasing.

Proof. Consider $x \geq y \in \mathbb{R}^{n+}$. Define $A : \mathbb{R} \to \mathbb{R}^n$ by

$$A(t) = y + t(x-y) \ , \quad t \in \mathbb{R}$$

Then A is positive and affine, so $\varphi \circ A \in SCNC(\mathbb{R}^+)$. Clearly $\varphi(x) \geq \varphi(y)$ if and only if $\varphi \circ A(1) \geq \varphi \circ A(0)$. Thus it is enough to prove the lemma when $n = 1$. But this is a straightforward matter which will be left to the reader.

We note also that $\mathbb{R}^{n+}$ is polyhedral so that elements of $SCNC(\mathbb{R}^{n+})$ are continuous. Thus in addition to being semicontinuous and non-negative, elements of $SCNC(\mathbb{R}^{n+})$ are increasing and continuous.

We are interested in the following question. Suppose $\varphi_1, \varphi_2, \ldots, \varphi_\ell$ belong to $SCNC(\mathbb{R}^{n+})$. Under what conditions on the φ's can we find an $m \geq n$, a function $\psi \in SCNC(\mathbb{R}^{m+})$, a positive linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ and points $y_i \in \mathbb{R}^{m+}$ such that $\varphi_i(x) = \psi(L(x) + y_i)$ for $x \in \mathbb{R}^{n+}$?

Suppose we can find such a ψ , such an L , and such y_i . Then in fact we can take $m = n+\ell$, we can take L to be the standard injection

$$j_{n,n+\ell}(x_1, \ldots, x_n)^T \to (x_1, x_2, \ldots, x_n, 0, 0, \ldots, 0)^T$$

of $\mathbb{R}^n$ into $\mathbb{R}^{n+\ell}$, and we can take $y_i = e_{n+i}$, the (n+i)-th standard basis vector of $\mathbb{R}^{n+\ell}$. For suppose ψ , L and y_i exist. Define

$$\tilde{L} : \mathbb{R}^{n+\ell} \to \mathbb{R}^m$$

by the formula

$$\text{(2.2)} \qquad \tilde{L}(x_1, \ldots, x_{n+\ell}) = L(x_1, \ldots, x_n) + \sum_{i=1}^{\ell} x_{n+i} y_i .$$

Then it is clear that

$$\tilde{L}(j_{n,n+\ell}(x) + e_{n+i}) = L(x) + y_i , \quad x \in \mathbb{R}^n .$$

So setting $\tilde{\psi} = \psi \circ \tilde{L}$, we can reduce to the standard situation described above. So in trying to determine whether ψ , L and the y_i exist we may as well reduce to the standard situation.

Consider $\varphi \in SCNC(\mathbb{R}^{n+})$. For $t \in \mathbb{R}^+$ define

$$\text{(2.3)} \qquad D_t\varphi(x) = t^{-1}\varphi(tx) .$$

Since for any $t \geq 1$

$$\varphi(x) \geq (1 - t^{-1})\varphi(0) + t^{-1}\varphi(tx)$$

by concavity, we see that $D_t\varphi(x)$ is decreasing (more precisely, non-increasing) as t increases. If $D_t\varphi = \varphi$ for all $t \in \mathbb{R}^+$, we say

φ is <u>homogeneous</u>. Because φ is concave, one can see that if $D_t\varphi = \varphi$ for a single $t > 1$, then φ is homogeneous.

In general, if φ is not homogeneous, the functions $D_t\varphi$ form a monotone decreasing family in SCNC($\mathbb{R}^{n+}$), so according to Section 1, we can take the limit

$$(2.4)\qquad \varphi_\infty'(x) = \inf_{t>0} D_t\varphi(x) = \lim_{t\to\infty}(\varphi(tx)/t)$$

and this will again belong to SCNC($\mathbb{R}^{n+}$). In particular it will be continuous. Replacing x by sx, for any $s > 0$, in equation (2.4) shows that φ_∞' is homogeneous. Evidently it is the largest homogeneous function on $\mathbb{R}^{n+}$ dominated by φ.

<u>Theorem 2.2</u>. Let $\varphi_1, \varphi_2, \ldots, \varphi_\ell$ belong to SCNC($\mathbb{R}^{n+}$). In order that there exist $\psi \in$ SNCN($\mathbb{R}^{(n+\ell)+}$) such that

$$(2.5)\qquad \varphi_i(x) = \psi(j_{n,n+\ell}(x) + e_{n+i}), \quad x \in \mathbb{R}^{n+}$$

(see discussion surrounding equation (2.2) for notation) it is necessary and sufficient that

$$(2.6)\qquad (\varphi_i)_\infty' = (\varphi_j)_\infty'$$

for all $i, j \leq \ell$.

<u>Proof</u>. The necessity of condition (2.6) is easy to establish. Set

$$\varphi_0 = \psi \circ j_{n,n+\ell}.$$

Then since ψ is increasing (Lemma 2.1) we see from formula (2.5) that $\varphi_0 \leq \varphi_i$ for $1 \leq i \leq \ell$. Hence $(\varphi_0)_\infty' \leq (\varphi_i)_\infty'$ for $1 \leq i \leq \infty$. On the other hand

$$\varphi_0(x) = \lim_{t\to 0} \psi(j_{n,n+\ell}(x) + te_{m+i})$$

$$\geq \lim_{t\to 0} t\psi(j_{n,n+\ell}(t^{-1}x) + e_{m+i})$$

$$= \lim_{t\to 0} t\varphi_i(t^{-1}x) = (\varphi_i)'_\infty(x)$$

Thus $(\varphi_0)'_\infty \geq (\varphi_i)'_\infty$, and so $(\varphi_0)'_\infty = (\varphi_i)'_\infty$ for $1 \leq i \leq \ell$. Equation (2.6) follows.

To establish the sufficiency of condition (2.6), we consider the set

$$Z = \bigcup_{1\leq i\leq \ell} (j_{n,n+\ell}(\mathbb{R}^{n+}) + e_{m+i}) \subseteq \mathbb{R}^{(n+\ell)+}$$

Define a function φ on Z by

$$\varphi(j_{n,n+\ell}(x) + e_{m+i}) = \varphi_i(x) \ , \quad x \in \mathbb{R}^{n+} \ , \quad 1 \leq i \leq \ell \ .$$

We want to let ψ be the smallest element of $SCNC(\mathbb{R}^{(n+\ell)+})$ which dominates φ on Z . To prove the theorem, we need to show that ψ exists and that it satisfies equations (2.5). According to the discussion at the end of Section 1, to do this, it suffices to show that, for any $z_0 \in Z$ and any $\varepsilon > 0$ we can find an affine function α on $\mathbb{R}^{(n+\ell)+}$ such that $\alpha|Z \geq \varphi$, and $\alpha(z_0) < \varphi(z_0) + \varepsilon$.

Let $z_0 = j_{n,n+\ell}(x_0) + e_{m+i}$. We know we can find an affine function α on $\mathbb{R}^n$ such that $\alpha \geq \varphi_i$ on $\mathbb{R}^{n+}$ and $\alpha(x_0) < \varphi_i(x_0) + \varepsilon$. Write

$$\alpha(x) = b + \lambda(x)$$

where λ is linear. Then $\lambda = \alpha'_\infty$. Let μ be a linear function on $\mathbb{R}^n$ which is strictly increasing along all rays in $\mathbb{R}^{n+}$. (For example

$\mu(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i$ will do.) Since $\alpha \geq \varphi_i$ we have $\lambda = \alpha'_\infty \geq (\varphi_i)'_\infty$. By adding a very small multiple of μ to α , we may guarantee that in fact

(2.7) $\alpha'_\infty > (\varphi_i)'_\infty$

while maintaining $\alpha(x_0) < \varphi_i(x_0) + \varepsilon$.

From (2.6) and (2.7) we have, for any j , $1 \leq j \leq \ell$, that

$$\alpha'_\infty > (\varphi_j)'_\infty \ .$$

By Dini's Theorem (Kelley, 1955) the convergence of $D_t\varphi_j$ to $(\varphi_j)'_\infty$ as $t \to \infty$ is uniform on compact sets. Hence we have $\alpha(x) > \varphi_j(x)$ for x outside some compact set. Hence $\varphi_j(x) - \alpha(x)$ is bounded above. Choose a constant

$$c_j \geq \max\{\varphi_j(x) - \alpha(x), 0\} \ , \quad x \in \mathbb{R}^{n+}$$

for $1 \leq j \leq \ell$. We can and do take $c_i = 0$.

Define an affine function β on $\mathbb{R}^{(n+\ell)}$ by

$$\beta(j_{n,n+\ell}(x) + \sum_{j=1}^{\ell} s_j e_{n+j}) = \alpha(x) + \sum_{j=1}^{\ell} c_j s_j \ .$$

Then by our choice of α and the constants c_j we see that $\beta \geq \varphi$ on Z , but $\beta(z_0) < \varphi(z_0) + \varepsilon$. This finishes the proof of Theorem 2.2.

One can ask for refinements of Theorem 2.2. For example, given $\varphi_1, \ldots, \varphi_\ell$ as there, what additional conditions do the φ_i need to satisfy in order that for given $k < \ell$, there is $\psi \in \mathrm{SCNC}(\mathbb{R}^{(n+k)+})$ such that

$$\varphi_i(x) = \psi(j_{n,n+k}(x) + y_i)$$

for suitable $y_i \in \mathbb{R}^{(n+k)+}$. This is a more involved matter. Here we will only consider a simple case.

Proposition 2.3. Let φ_1 , φ_2 be two functions in $SCNC(\mathbb{R}^{n+})$. In order that there exist $\psi \in SCNC(\mathbb{R}^{(n+1)+})$ and numbers $a \geq b \geq 0$ such that

$$\varphi_1(x) = \psi(x,a) \ , \quad \varphi_2(x) = \psi(x,b)$$

it is necessary and sufficient that

$$(2.8) \qquad \varphi_1(x) \geq \varphi_2(x) \geq (\varphi_1)'_\infty(x) = (\varphi_2)'_\infty(x) \ , \quad x \in \mathbb{R}^{n+} \ .$$

If one wishes to specify a and b , then the corresponding condition is

$$(2.9) \qquad \varphi_1 \geq \varphi_2 \geq D_{a/b}(\varphi_1) \ .$$

Proof. We use the same strategy as in Theorem 2.2. The necessity of $(\varphi_1)'_\infty = (\varphi_2)'_\infty$ is seen exactly as in that proof. The necessity of $\varphi_1 \geq \varphi_2$ is implied by Lemma 2.1 and the second inequality in (2.9) is implied by the fact that $D_t\psi$ is decreasing in t :

$$\varphi_1(x) = \psi(x,a) = \left(\frac{a}{b}\right)\left(\frac{b}{a}\right)\psi\left(\frac{a}{b}\left(\frac{b}{a}x,\ b\right)\right) = \left(\frac{a}{b}\right)D_{a/b}\psi\left(\frac{b}{a}x,\ b\right)$$

$$\leq \left(\frac{a}{b}\right)\psi\left(\frac{b}{a}x,\ b\right) = \left(\frac{a}{b}\right)\varphi_2\left(\frac{b}{a}x\right) = D_{b/a}\varphi_2(x) \ .$$

Now assume φ_1 and φ_2 satisfy inequalities (2.8) or (2.9). Let ψ be the smallest element among elements $\tilde{\psi}$ of $SCNC(\mathbb{R}^{(n+1)+})$ satisfying

$$(2.10) \qquad \tilde{\psi}(x,a) \geq \varphi_1(x) \ , \quad \tilde{\psi}(x,b) \geq \varphi_2(x) \ .$$

The function $\tilde{\psi}_1(x,t) = \varphi_1(x)$ satisfies the inequalities (2.10) , with equality in the first one, so ψ will exist and $\psi(x,a) = \varphi_1(x,a)$. If

$b > 0$, define

$$\widetilde{\psi}_2(x,t) = D_{b/t}\varphi_2(x) .$$

It is straightforward to check that $\widetilde{\psi}_2$ is in $SCNC(\mathbb{R}^{(n+1)+})$, and that $\widetilde{\psi}_2(x,b) = \varphi_2(x)$. If inequality (2.9) holds, then so does (2.10). Hence in the case of conditions (2.9) the proposition is proved.

Suppose we know only the weaker facts (2.8). Then we may have to take $b = 0$. In this case fix $x_0 \in \mathbb{R}^{n+}$ and $\varepsilon > 0$. Let α be an affine function on $\mathbb{R}^n$ such that $\alpha \geq \varphi_2$ on $\mathbb{R}^{n+}$, and $\alpha(x_0) < \varphi_2(x_0) + \varepsilon$. As in Theorem 2.2 we may assume $\alpha'_\infty > (\varphi_2)'_\infty$. Then by (2.8) we have $\alpha'_\infty > (\varphi_1)'_\infty$, hence $\varphi_1 - \alpha$ is bounded above. If

$$c = \max\{\varphi_1(x) - \alpha(x), 0\} .$$

Then

$$\widetilde{\psi}(x,t) = \alpha(x) + ct/a$$

is an affine function on $\mathbb{R}^{(n+1)+}$ satisfying inequalities (2.10), and such that $\widetilde{\psi}(x_0, 0) < \varphi_2(x_0) + \varepsilon$. Hence the remarks at the end of Section 1 again tell us that in this case we will have $\psi(x,0) = \varphi_2(x)$ for $x \in \mathbb{R}^{n+}$, as desired. This establishes the proposition.

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