COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

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COWLES FOUNDATION DISCUSSION PAPER NO. 750

THE NUMERAIRE, MONEY AND THE MISSING DEGREE OF FREEDOM

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May 23, 1985

THE NUMERAIRE, MONEY AND THE MISSING DEGREE OF FREEDOM*

by

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It is part of the common casual knowledge of general equilibrium theorists that any commodity can be used as a numeraire. Suppose that there is an exchange economy with n types of traders trading in m commodities. A representative trader of type i has an endowment of $(a_1^i, a_2^i, \ldots, a_m^i)$ and a utility function of $\varphi_i(x_1^i, x_2^i, \ldots, x_m^i)$. At equilibrium there are m prices $(p_1, p_2, \ldots, p_m)$. But as price is a ratio between two commodities only $m-1$ independent prices are needed to describe the equilibrium price system. We may set $\sum_{j=1}^{m} p_j = 1$ or alternatively we could set some $p_k = 1$.

When an exchange economy is modeled as a strategic market game an explicit price formation mechanism is given. Prices are formed as a result of the strategies employed. Prices could even be specified as part of strategies, thus it may become necessary to distinguish between *ex post* and *ex ante* prices or between proposed prices and realized prices at equilibrium. In particular this holds true when prices (or proposed

*This work relates to Department of the Navy Contract N00014-77-C-0518 issued by the Office of Naval Research under Contract Authority NR 047-006. However, the content does not necessarily reflect the position or the policy of the Department of the Navy or the Government, and no official endorsement should be inferred.

prices) are part of the strategies employed to determine final prices.

There are two basic explicit price forming mechanisms for strategic market games. The first is associated with Cournot's oligopoly model and the second with the work of Bertrand and Edgeworth on duopoly. The models of Cournot, Bertrand and Edgeworth were all limited to partial equilibrium or open models but they can be generalized for closed exchange economies. Two natural types of closed models may be constructed. The first is intrinsically symmetric and corresponds to the Arrow-Debreu model of complete markets. For m commodities the strategic market game model will have $m(m-1)/2$ markets. The second is intrinsically non-symmetric. One commodity is selected in advance as a means of exchange and in total there are $m-1$ markets.

As the prime purpose of this article is to illustrate some basic problems in the modeling of strategic market games with a single means of exchange or with complete markets a simple example will serve to make the points clear. In particular we consider three types of traders trading in three commodities.

1. COMPLETE MARKETS OR ONE MONEY

Let every trader have the same utility function $\varphi(x^i, y^i, z^i)$. A trader of type 1 (2, 3) has endowments (A,B,B) , (B,A,B) , (B,B,A) . With only three commodities an economy with one money has two markets as is shown in Figure 1a; an economy with complete markets has three markets as is indicated in Figure 1b.

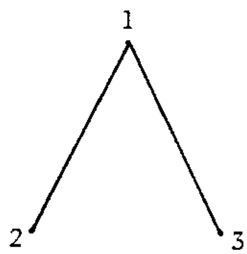

FIGURE 1a

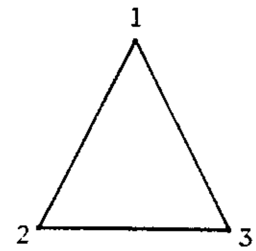

FIGURE 1b

1.1. The Bid-Offer Models: Complete Markets

Let there be n traders of each type.

An offer by trader i of type 1 (2, 3) for good k in terms of good g is w^i_{kg} (y^i_{kg}, z^i_{kg}) .

There are three markets 12, 23, 31, and any trader could be on both sides of the market, or less. Thus a strategy for a trader i of type 1 is a vector of dimension 6, $(w^i_{12}, w^i_{21}, w^i_{13}, w^i_{31}, w^i_{23}, w^i_{32})$. The payoff to an individual i of type 1 is:

$$(1) \quad \varphi\left(\left[A - w^i_{21} - w^i_{31} + w^i_{12}\left\{\frac{w_{12}+y_{21}+z_{21}}{w_{12}+y_{12}+z_{12}}\right\} + w^i_{13}\left\{\frac{w_{31}+y_{31}+z_{31}}{w_{13}+y_{13}+z_{13}}\right\}\right],\right.$$

$$\left[B - w^i_{12} - w^i_{32} + w^i_{21}\left\{\frac{w_{12}+y_{12}+z_{12}}{w_{21}+y_{21}+z_{21}}\right\} + w^i_{23}\left\{\frac{w_{32}+y_{32}+z_{32}}{w_{23}+y_{23}+z_{23}}\right\}\right],$$

$$\left.\left[B - w^i_{13} - w^i_{23} + w^i_{31}\left\{\frac{w_{13}+y_{13}+z_{13}}{w_{31}+y_{31}+z_{31}}\right\} + w^i_{32}\left\{\frac{w_{23}+y_{23}+z_{23}}{w_{32}+y_{32}+z_{32}}\right\}\right]\right)$$

where w_{kg} $(y_{kg}, z_{kg}) = \sum_{j=1}^{n} w^j_{kg}$ (etc.).

We must specify what happens if there is no bid or offer in any market. A reasonable convention is that if there is either no bid or offer or both there is no trade in that market, no price is formed and any bid or offer is returned.

In order for this game to have an equilibrium with all markets active we require that there be at least two active traders on each side of every market. Thus we might require as many as $2m(m-1)$ individuals if all markets are to be active.

If there were a continuum of all types of traders in all markets then expression (1) would simplify to:

$$(2)\quad \varphi\left(\left(A - w^i_{21} - w^i_{31} + w^i_{12}p_{12} + w^i_{13}p_{13}\right),\ \left(B - w^i_{12} - w^i_{32} + w^i_{21}\left(\frac{1}{p_{12}}\right) + w^i_{23}p_{23}\right),\right.$$
$$\left.\left(A - w^i_{13} - w^i_{23} + w^i_{31}\left(\frac{1}{p_{13}}\right) + w^i_{32}\left(\frac{1}{p_{23}}\right)\right)\right) .$$

At equilibrium $(p_{12})(p_{23}) = p_{13}$ must hold, otherwise an arbitrage situation would be available which is inconsistent with equilibrium.

1.2. The Bid-Offer Model: A Money Selected

Suppose that the third commodity is selected as a money and there are only markets 13 and 23. A strategy for a trader i of type 1 is a vector of dimension 4 $(b^i_1, q^i_1; b^i_2, q^i_2)$ where

$$b^i_1 + b^i_2 \leq a^i_3 \ ; \quad b^i_j \geq 0 \ ; \quad 0 \leq q^i_j \leq a^i_j \ , \quad j = 1, 2 .$$

Thus (1) becomes:

$$(3)\quad \varphi\left((A - q^i_1 + b^i_1\{q_1/b_1\}),\ (B - q^i_2 + b^i_2\{q_2/b_2\}),\ (B - b^i_1 - b^i_2 + q^i_1p_1 + q^i_2p_2)\right)$$

where $b_1 = \sum_j b^j_1$, $p_i = b_i/q_i$.

1.3. An Example

For illustration we specialize the utility functions to $\log(x,y,z)$. Thus with complete markets given endowments (A,B,B), (B,A,B) and (B,B,A) where $A > B$ we may guess and then establish that there is an equilibrium point where the strategies are of the form: $(w^i_{21}, 0, w^i_{31}, 0, 0, 0)$, $(0, w^j_{12}, 0, 0, w^j_{32}, 0)$, $(0, 0, 0, w^k_{13}, 0, w^k_{23})$. If this is so, then:

$$(4) \qquad \Pi_1 = \log(A - w^i_{21} - w^i_{31}) + \log\left(B + \frac{w_{21}}{w^i_{21} + w^{-i}_{21}}(w_{12})\right) + \log\left(B + \frac{w^i_{31}}{w^i_{31} + w^{-i}_{31}}(w_{13})\right)$$

where w^{-i}_{21} = sum of offers of all traders of type 1 except i of good 1 for 2,

w_{21} = sum of offers of all traders of type 1 of good 1 for 2.

Similar expressions can be derived for Π_2 and Π_3.

Taking derivates and setting the results equal to zero to check first order conditions, if we assume that a type symmetric equilibrium will exist and that there are n (≥ 2) traders of each type we obtain:

$$(5) \qquad \frac{1}{A - 2x} = \frac{1}{B + x}\,\frac{(n-1)}{n} \quad \text{or}$$

$$(6) \qquad x = \frac{(n-1)A - nB}{3n - 2} \quad \text{where} \quad x = w^i_{jk} \quad \text{for all.}$$

For an active strategy we require $A > \frac{n}{n-1}B$.

If $n = 2$, we obtain $x = (A - 2B)/4$ and hence the final (not Pareto optimal) distribution is $\left(\frac{A}{2} + B, \frac{B}{2} + \frac{A}{4}, \frac{B}{2} + \frac{A}{4}\right)$ for traders of Type 1.

For $n \to \infty$, $x \to (A-B)/3$ giving the C.E. distribution $\left(\frac{A+2B}{3}, \frac{A+2B}{3}, \frac{A+2B}{3}\right)$.

If we cut down the number of markets to two the game becomes nonsymmetric. Suppose that the first commodity is selected as the money (see Figure 1a) then traders of Type 1 (A,B,B) are more liquid than all others.

Suppose that a trader i of Type 1 bids y_2^i and y_3^i in the market for goods 2 and 3. A trader j of Type 2 bids x_3^j and offers q_2^j. A trader k of Type 3 bids x_2^k and offers q_3^k. Thus strategies are of four dimensions and can be characterized as $(y_2^i, 0, y_3^i, 0)$, $(0, q_2^j, x_3^j, 0)$, $(x_2^k, 0, 0, q_3^k)$.

The payoffs can be written as

$$\Pi_1 = \log(A - y_2^i - y_3^i) + \log(B + y_2^i/p_2) + \log(B + y_3^i/p_3) \tag{7}$$

$$\Pi_2 = \log(B + p_2 q_2^j - x_3^j) + \log(A - q_2^j) + \log(B + x_3^j/p_3) \tag{8}$$

$$\Pi_3 = \log(B + p_3 q_3^k - x_2^k) + \log(B + x_2^k/p_2) + \log(B - q_3^k) \tag{9}$$

We see that Π_2 and Π_3 have the same symmetry, but Π_1 does not. First order conditions give

$$\frac{1}{A - 2y} = \frac{p}{pB + y}\frac{\partial}{\partial y}\{y/p\} = \frac{p}{pB + y}\left\{\frac{1}{p} - \frac{y}{p^2}\frac{\partial p}{\partial y}\right\}$$

but $\frac{\partial p}{\partial y} = \frac{1}{nq}$ hence

$$\boxed{\frac{1}{A - 2y} = \frac{1}{pB + y}\left\{1 - \frac{y}{npq}\right\}} \; . \tag{10}$$

This is derived from (7) thinking $\partial\Pi_1/\partial y_2^i = 0$, setting $y_2^i = y_3^i = y$ and $p_2 = p$ and taking into account n of each type.

From (8) we obtain

$$\frac{1}{B + pq - x}\left(p + q\frac{\partial p}{\partial q}\right) = \frac{1}{A - q} ,$$

but $\frac{\partial p}{\partial q} = - \frac{n(x+y)}{n^2q^2} = - \frac{x + y}{nq}$ hence

$$\text{(11)} \qquad \boxed{\frac{\left(p - \frac{x+y}{n}\right)}{B + pq - x} = \frac{1}{A - q}} \;.$$

This is derived from (8) taking $\partial \Pi_2/\partial q_2^i = 0$.

Also from (8) taking the variation of x_3^j we obtain

$$\frac{1}{B + pq - x} = \frac{1}{B + x/p}\frac{\partial}{\partial x}\{x/p\} \quad \text{giving}$$

$$\boxed{\frac{1}{B + pq - x} = \frac{1}{Bp + x}\left\{1 - \frac{x}{npq}\right\}} \;.$$

For $n \to \infty$ equations (10), (11) and (12) should give the competitive equilibrium. They simplify to:

(13) $\qquad pB + y = A - 2y$

(14) $\qquad p(A-q) = B + y$ and

(15) $\qquad Bp + x = B + y$.

These together with $p = (x+y)/q$ yield

$$3y(A+2B) = (A+2B)(A-B) \quad \text{or} \quad y = (A-B)/3$$

and $x = (A-B)/3$, $p = 1$ and $q = 2(A-B)/3$.

We note that if $(A-B)/3 > B$ or $A > 4B$ then $x > B$ and this would require that the model of trade permit credit as the expenditures of x by traders of type 2 and 3 would exceed B, their amount of cash at hand. This constraint is not present with complete markets; A and B could be of any relative size.

If $A > 4B$ and there is no credit, the monied players even in the limit with many players are benefitted. A simple example illustrates this. Given $x = B$ instead of (15) we obtain $p = (B+y)/q$ which can be substituted into (14) to yield

(16) $$q = A/2 .$$

Then from (13)

(17) $$y = \frac{A^2 - 2B^2}{3A + 2B} .$$

A quick check shows that for $A = 4B$, $x = y = B$ and $q = 2B$. But, for example, suppose $A = 6B$, then:

$$x = B , \quad q = 3B , \quad y = 1.7B \quad \text{and} \quad p = .9 .$$

Thus the final distribution is $(2.6B, 2.\dot{8}B, 2.\dot{8}B)$, $(2.7B, 3B, 2.\dot{1}B)$ and $(2.7B, 2.\dot{1}B, 3B)$ which favors the monied players.

1.4. Discussion of the Example

If we regard the competitive exchange economy as the limit of a strategic market game then as long as there is enough money it does not matter whether we model the economy with one money and $m-1$ markets or complete ($m(m-1)/2$ markets) or for that matter one money and any extra number of markets we wish. All of these strategic market games yield the competitive equilibrium in the limit. The example above shows this for $(m-1)$ markets and complete markets. But the contrast between a strategic market game with a single money and with complete markets raises several problems as indicated in Table 1.

	Complete Markets	A Single Money
numbers of markets	m(m-1)/2	m-1
enough money	always	not always (may need credit)
existence of active N.E.	no existence proof - conjectured yes	existence proof
numeraire	ex ante, ex post problem	naturally defined by price of money

TABLE 1

The problem of enough money has been illustrated by example 1.3 and discussed elsewhere (Shubik and Wilson, 1977 and Shubik and Dubey, 1978). The definition of what is meant by "enough" can be intuitively seen as enough to guarantee an interior N.E. (noncooperative equilibrium), but some care is required to make this precise.

The existence of an active pure strategy N.E. in a strategic market game with a single money is well known (Shapley, 1981, Dubey and Shubik, 1979). But these proofs do not apply to the strategic market game with complete markets. The example in 1.3 shows an equilibrium with all markets active. It is conjectured that an equilibrium with all markets active will exist if there are at least two players on each side of every market who have enough endowments and want the other goods.

The last point concerns the selection of a numeraire which appears to be a more or less trivial afterthought in the study of a price system. This is discussed in Section 2.

2. THE NUMBER OF INDEPENDENT PRICES

The game described in Section 1.1 is intrinsically symmetric in the strategy sets and all $m(m-1)/2$ prices are determined by strategic behavior.

In equilibrium as no profitable arbitrage will be possible only $m-1$ independent prices will remain. Thus when an equilibrium has been reached a numeraire can be selected.

If one attempts to select a numeraire before trade takes place this is tantamount to imposing $(m-1)$ constraints on the game. The prices of all commodities are given in terms of the numeraire but until equilibrium has eliminated all arbitrage there are $m(m-1)/2$ not $m-1$ independent prices.

The game described in Section 1.2 is intrinsically nonsymmetric in the strategic behavior. Thus there is a natural normalization which has no influence *ex ante*. The single money can also serve as the numeraire, the setting of $p_m = 1$ has no influence on the strategy sets. The physical meaning of price in a strategic market game with a single money is:

$$p_j = b_j/q_j \quad \text{for} \quad j = 1, \ldots, m-1 .$$

Price of nonmoney j is the ratio of the amount of money given in exchange for the quantity q_j . Its dimensions are quantity of money divided by quantity of good. The price of money is a pure number. Its dimensions being quantity of money divided by quantity of money.

In England the numeraire for medical doctors' fees, solicitor's and tailors' fees was the guinea rather than the pound (a guinea equaled 21 shillings, a pound equaled 20 shillings). Its price, like that of the pound is still a pure number and equal to 1 in terms of itself or 1.05 in terms of the pound. This change in name of the numeraire has no strategic impact.

If instead of using the money as a numeraire we wished to use cabbages or another nonmonetary commodity as the unit of account the strategic market game will be influenced. We must distinguish between _ex ante_ and _ex post_ prices and the meaning of strategy in terms of a promise to pay. This is best illustrated using the strategies in the price-quantity game (Shubik, 1981, Dubey and Shubik, 1980, Dubey, 1982). A strategy in a price-quantity game with m commodities one of which (say the m^{th}) is money is a vector of $4(m-1)$ dimensions $(p_1^i q_1^i, \tilde{p}_1^i \tilde{q}_1^i, \ldots, p_{m-1}^i q_{m-1}^i, \tilde{p}_{m-1}^i \tilde{q}_{m-1}^i)$. In each market j an individual i states a selling price p_j^i or more at which he will sell q_j^i units or less; and a buying price $\tilde{p}_j^i$ or less at which he will buy $\tilde{q}_j^i$ units or less. There is some mechanism which takes all $2n$ prices and quantities in each market and produces a final price and set of trades. A simple and natural way is shown in Figure 2 where aggregate bid and offer curves have been drawn. We could determine an overall market price by taking the price at the intersection of the two curves.

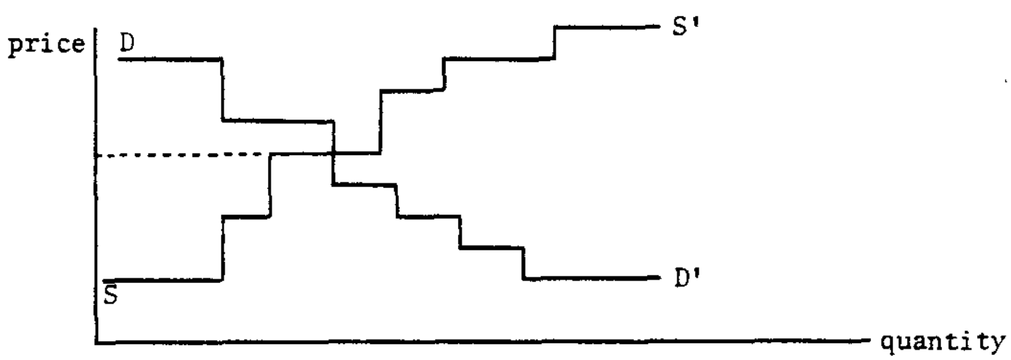

FIGURE 2

The meaning of a bid or offer is a promise to pay or an offer to accept an amount of money $\tilde{p}_j^i$ or p_j^i per unit for commodity j , where $p_m = 1$ has been fixed in advance. But if the means of payment is commodity m and the numeraire is say commodity 1 the meaning of a strategy is somewhat forced. $\tilde{p}^i_j$ stands for I promise to pay for one unit of j, some amount of m. But this amount cannot be specified until the price of m in terms of 1 has been determined.

3. CONCLUDING REMARKS

The usual casual discussion of the role of a numeraire is implicitly given in an ex post context, i.e., where prices already exist. If one attempts to construct a playable game for an exchange economy in which prices are formed strategically then the choice of a numeraire before prices are formed is not feasible without influencing strategy sets in the strategic market game with complete markets. The game with one commodity serving as a money is nonsymmetric and the selection of the money as a numeraire is natural and does not influence the strategic structure of the game.

In a strategic market game with a money used as a means of payment an attempt to utilize a different commodity as a numeraire without it being a means of payment has no operational significance. Its price must be formed by the game and is not available as a measure before the game is played.

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