COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

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COWLES FOUNDATION DISCUSSION PAPER NO. 747

SOME THREE PERSON GAMES IN COALITIONAL FORM FOR TEACHING AND EXPERIMENTATION

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May 1985

SOME THREE PERSON GAMES IN COALITIONAL FORM FOR TEACHING AND EXPERIMENTATION*

by

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1. INTRODUCTION

Since 1973 I have used several three person games in coalitional form for both teaching and experimental purposes. They have been run in primarily a normative mode. The individuals have been asked to act as judges called upon to recommend a division of assets among three players.

The basic use of these games has been to help to raise questions about context and solution concepts in cooperative game theory. Since 1980 the three basic games noted below have been used with five more or less similar groups of students at Yale. The games, their didactic purpose and the results from the normative suggestions as to how the players should be rewarded are noted here and then these results are compared with previous games. Furthermore some extra sensitivity analysis problems are noted.

*This work relates to Department of the Navy Contract N00014-77-C-0518 issued by the Office of Naval Research under Contract Authority NR 047-006. However, the content does not necessarily reflect the position or the policy of the Department of the Navy or the Government, and no official endorsement should be inferred.

2. THE GAMES AND THEIR PURPOSE

Three three person games in characteristic function form have been used. Games 1a and 1b were used before students had been taught the concept of a core. The third game (Game 6 in a series of games in strategic, extensive and coalitional form used for teaching) was used about three weeks later when the students knew about the core but not about other solutions. (In some instances, as emerged from written comments a few students had been exposed to more game theory and introduced an extra knowledge bias, but this was less than 10% of the respondents.)

The way the games were presented is shown below:

GAME 1

a. Three individuals can, by cooperating on a job earn $400 altogether; if A and B cooperate they earn $100, if A and C cooperate they earn $200, if B and C cooperate they earn $300. The man left out earns nothing. This information can be summarized as follows:

$$V(A) = V(B) = V(C) = 0$$

$$V(AB) = 100 \quad V(AC) = 200 \quad V(BC) = 300$$

$$V(ABC) = 400$$

You are the judge, the three tell you that they have decided to work together to earn $400 but they cannot decide how to split the income. Write down three numbers a_1, a_2, a_3 such that:

$$a_1 + a_2 + a_3 = 400$$

This is your split of the income, given your reasons for choosing it.

b. Suppose that instead of the above productivity C was vital to the job and only one of A or B were really needed. The earnings are summarized as follows:

$$V(A) = V(B) = V(C) = 0$$

$$V(AB) = 0 \quad V(AC) = 400 \quad V(BC) = 400$$

$$V(ABC) = 400$$

State how you would split the $400 here and give your reasons.

GAME 6

You originally acted as judge in the division of proceeds between 3 individuals with different productivity who had agreed to cooperate. You are asked to act as judge once more in a somewhat different case. The values of the coalitions are given below:

$$V(A) = V(B) = V(C) = 0$$
$$V(AB) = 250 \quad V(AC) = 300 \quad V(BC) = 350$$
$$V(ABC) = 400$$

1. Write down how you divide the $400. Indicate why.
2. What is fundamentally different between this situation and the two previous ones?

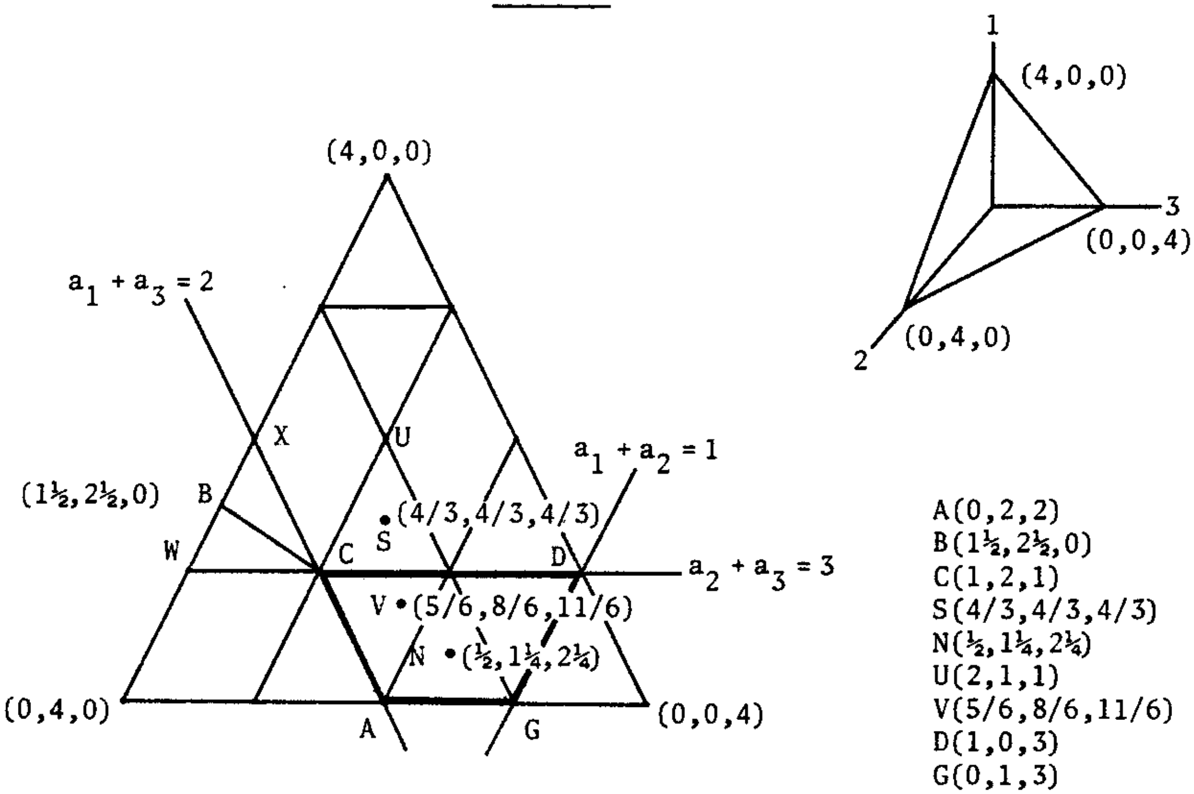
There are many solution concepts which have been suggested as ways to decide who gets how much in an n-person cooperative game. Among the better known are (1) the core, (2) the value, (3) the nucleolus, (4) the kernel, (5) the bargaining set and (6) the stable set solution of a cooperative game. If a game is a market game then there is a relationship between the imputation selected by the efficient price system in the related exchange economy and the core of the game. The price selected imputation will always be in the core. Games 1a and 1b are market games. Game 6 is not.

The use of the three games was primarily to raise questions concerning the plausibility and acceptability of different solution concepts, in particular, the core, the value, the nucleolus and the efficient price system. The core of a game consists of those imputations which are socially rational for any sized group of individuals. This can be expressed mathematically (for Game 1a) as

$$a_1 \geq 0 \ ; \quad a_2 \geq 0 \ ; \quad a_3 \geq 0$$
$$a_1 + a_2 \geq 100 \ ; \quad a_1 + a_3 \geq 200 \ , \quad a_2 + a_3 \geq 300$$
$$a_1 + a_2 + a_3 = 400$$

The core may be regarded as reflecting countervailing power. In Figure 1 it consists of all imputations in the set bounded by the figure ACDB.

GAME 1a



$v(1) = v(2) = v(3) = 0$

$v(12) = 1,\ v(13) = 2,\ v(23) = 3$ (all × 100)

$v(123) = 4$

FIGURE 1

As there are many different exchange economies all of which would have the same characteristic function shown in Game 1a, without specifying more economic detail we cannot select any particular imputation in the core as a price system. Shapley and Shubik (1976) have shown that the union of all price imputations of exchange economies representable by the same characteristic function will cover the core; i.e., any point in the core can be associated with some price system.

The value (Shapley, 1951) can be viewed as a fair division procedure. It was originally derived from fair division axioms and rewards each player with with his expected marginal contribution to every coalition he could join. The amount assigned to Player i is:

$$\varphi_i = \sum_{i \in S} \sum_{S \subseteq N} \frac{(n-s)!(s-1)!}{n!}[v(s) - v(s/i)] .$$

Applying this to Game 1a we obtain (5/6, 8/6, 4/6).

The nucleolus (Schmeidler, 1969) is obtained by adding a number ε to all coalitions (except N) until the core of the new game changes dimension. If it is more than a single point, a new ε' is added to the coalitions bounding the core and the process is repeated until a single point remains. The nucleolus may be viewed as the point which minimizes the maximum subsidy or bonus that any group obtains over their own claims. For games without a core the ε will be negative and eventually a core will appear. The nucleolus for Game 1a is at (1/2, 5/4, 9/4).

Although we do not discuss the other solutions we note that points in the core and points in the triangle WXC could belong to a stable set solution.

Turning to Game 1b we have a single point core at (0,0,4) but this must be the nucleolus and the imputation arising from any market game whatsoever which can be represented by this characteristic function. The value gives (2/3, 2/3, 8/3). These points are shown in Figure 2.

GAME 1b

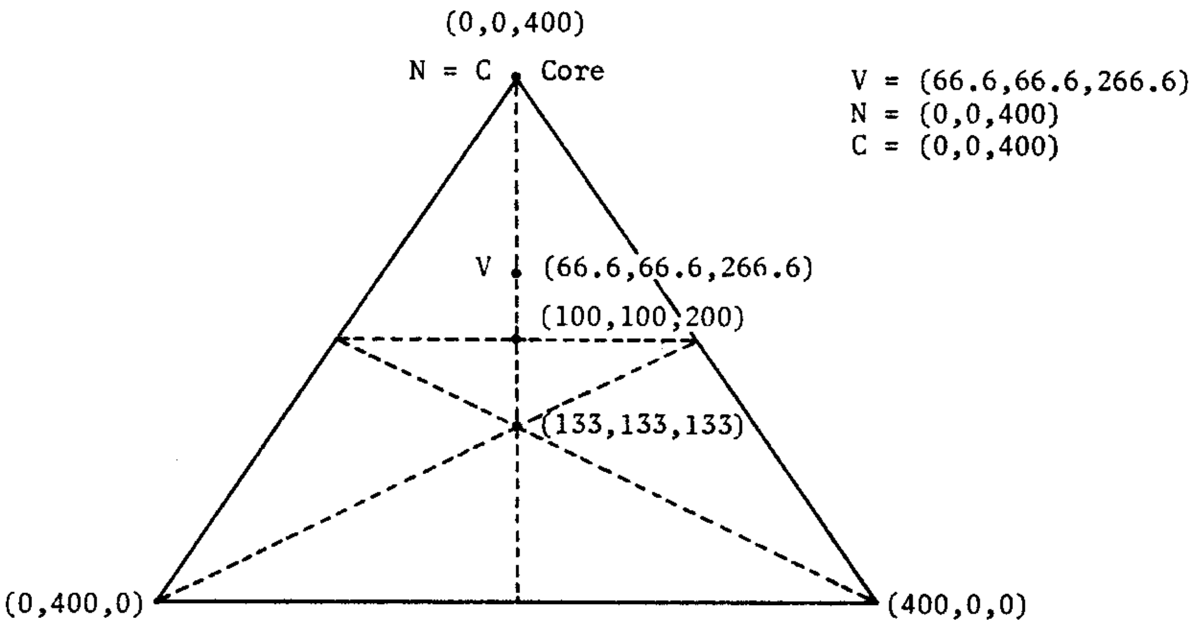


FIGURE 2

Game 6 has no core and hence no possible price system solution. The characteristic function could not have been derived from an exchange economy. A little later labor yields a value of (13/12, 16/12, 19/12) and a nucleolus of (5/6, 8/6, 11/6).

In all instances one could ignore the details about the power of coalitions and argue for an egalitarian split of the whole. This yields (4/3, 4/3, 4/3).

Table 1 shows the summary of the solutions to the three games.

TABLE 1

	GAME 1a	GAME 1b	GAME 6
Core	"fat" ACDG	(0,0,4)	empty
Nucleolus	(1/2,3/2,2)	(0,0,4)	(5/6,8/6,11/6)
Price System	core	(0,0,4)	none
Value	(5/6,8/6,11/6)	(2/3,2/3,8/3)	(13/12,16/12,19/12)
Stable Sets	core + XWC	every imputation	every imputation

3. YALE GAME THEORY CLASSES

Five runs, one from 1980, 1981, 1983, 1984 and 1985 are noted. The data are presented in Tables 2, 3 and 4. Table 5 presents some summary data.

The evidence from Game 1a points clearly to the core as an attractive candidate as can be seen in Table 5. Outside of the core, even split appears to be the driving consideration. For Game 1b, however, the adherents to the core drop considerably whereas the even split hardly changes. In both games, even selecting a five percent range around the value yields no adherents. The same is true for the nucleolus in Game 1a, in Game 1b it coincides with the core.

In Game 6 no core exists; by including imputations as far as 6-7 percent away from the value and nucleolus a small percentage of imputations are included. However the even split selection, though it becomes larger also becomes more variable. Table 6 shows the averages over all imputations in Game 6. The sixth data set is from the Indian Institute of Management at Bangalore. The imputations in Table 6 have been normalized to add to 4. The value for this game is (1.08, 1.33, 1.58). The conclusions to be drawn from this coincidence should be approached with some caution.

Table 7 shows the most favored imputation in each game and the overall percentage of its selection.

The argument given for (100, 100, 200) is that a syndicate of 1 and 2 in Game 1b reduces it to a completely symmetric split of 400, the syndicate 1 and 2 should split evenly.

GAME 1a

POINTS IN THE CORE

	1980	1981	1983	1984	1985	Totals
(100,150,150)	29	20	15	7	19	90
(100,133,167)	5	1	2	-	7	15
V (83.3,133.3,183.3)						0
N (50,125,225)						0
(100,125,175)	2	-	1	2	1	6
(67,133,200)	5	3	7	-	2	17
Others	9	9	2	8	13	41
	50	33	27	17	42	169
Percent in Core	86	86.8	87	89.5	93	88.5

POINTS OUTSIDE THE CORE

	1980 #	1981 #	1983 #	1984 #	1985 #	Totals
(133, 133, 133)	6	4	3	2	3	18
Others	2	1	1	-	-	4
	8	5	4	2	3	22
Inefficient	1	1	-	2	-	4
Illogical (< 400 / > 400)	2	-	1	1	-	4
	3	2	1	3	-	9
Total	61	40	32	22	45	200

TABLE 2

GAME 1b

POINTS IN THE CORE

	1980 #	1981 #	1983 #	1984 #	1985 #	Totals
(0, 0, 400)	7	-	3	4	3	17
≈ (0, 0, 400)	8	2	4	-	5	19
	15	2	7	4	8	36
Percentage in core	28.3	5	21.8	19	20	19.3

POINTS OUTSIDE THE CORE

	1980 #	1981 #	1983 #	1984 #	1985 #	Totals
(100,100,200)	28	28	16	10	25	107
(133,133,133)	5	4	3	2	2	16
(75,75,250)	2	-	-	1	2	5
V (66.6,66.6,266.6)	-	-	-	-	-	-
(50,50,300)	-	2	3	1	2	8
Others	3	4	3	3	1	14
	38	38	25	17	32	150
Inefficient	2	-	-	-	-	2
Stumped	6	-	-	1	3	10
	8	0	0	1	3	12
Total	66	40	32	22	43	198

TABLE 3

GAME 6

	1980	1981	1983	1984	1985	Totals
(133,133,133)	14	6	2	1	1	24
(125,125,150)	5	2	3	2	1	13
(120,133,147)	0	5	0	0	3+	8
(117,132,152)	0	0	2	0	0	2
$\simeq$ V (110,130,160)	1	0	2	1	0	4
$\simeq$ V (100,133,167)	1	2	0	2	0	5
(100,125,175)	1	1	1	1	1	5
$\simeq$ N (89,133,178)	1	1	1	1	1	5
(75,175,150)	0	0	0	2	0	2
(67,133,200)	3	0	1	0	1	5
(50,175,175)	2	4	1	0	4	11
Others*	3	4	9	9	5	30
Total	31	25	22	19	17	144
Stumped or	4	0	0	0	4	8
Infeasible	4	1	2	0	1	11
	8	1	2	0	5	19
Total	39	26	24	19	22	133

*Other

1980	1981	1983	1984	1985
V(108,133,158)	V(110,135,155)	(120,140,140)	(175,175, 50)	(130,130,140)
(126,133,141)	(100,150,150)	(150,100,150)	V(111,133,150)	N($83\frac{1}{3}$,$133\frac{1}{3}$,$183\frac{1}{3}$)
(110,133,155)	(75,150,175)	(115,135,150)	(120,140,140)	($116\frac{2}{3}$,$133\frac{2}{3}$,$142\frac{2}{3}$)
	(120,140,140)	V(111,130,156)	(160,140,100)	V(115,135,150)
		N (83,133,183)	(150,150,100)	(100,150,150)
		(125,135,140)	V(115,135,150)	
		(150,150,100)	(137,150,163)	
		(122,133,144)	(75,100,225)	
		(120,130,150)	V(108,133,158)	

Actually
+ 123 133 144
  122 133 144
  122 133 140

++the others involve failure of grand coalition to form

TABLE 4

TABLE 5

PERCENTAGE SELECTION OF VARIOUS SOLUTIONS

		1980	1981	1983	1984	1985	Totals
Core	1a	86	86.8	87	89.5	93	88.5
	1b	28.3	5	21.8	19	20	19.3
	6	--	--	--	--	--	--
Value	1a	0	0	0	0	0	0
	1b	0	0	0	0	0	0
	6	12.9	12	13.6	31.6	5.8	14.9
Nucleolus	1a	0	0	0	0	0	0
	1b	28.3	5	21.8	19	20	19.3
	6	3.2	40	91	5.3	11.8	6.1
Even Split	1a	10.3	10.5	9.7	10.5	6.7	9.4
	1b	9.4	10	9.4	9.5	5.0	8.6
	6	45.2	24	9.1	5.3	5.9	21.1

TABLE 6

GAME 6, AVERAGE IMPUTATION

	Sample Size	Average Imputation
1980	31	(1.14, 1.34, 1.52)
1981	25	(1.06, 1.39, 1.56)
1983	22	(1.14, 1.33, 1.54)
1984	19	(1.15, 1.39, 1.46)
1985	17	(.96, 1.43, 1.61)
IIM. 1979	9	(1.05, 1.29, 1.66)
	123	(1.09, 1.37, 1.55)

TABLE 7

		Most Favored Imputation Selection Percentage
Game 1a	(100, 150, 150)	49.7
Game 1b	(100, 100, 200)	57.5
Game 6	(133.3, 133.3, 133.3)	21.1

4. OTHER REPLICATIONS WITH DIFFERENT PLAYERS

The same games have been used at various times since 1973 in lectures of an expository nature on the theory of games. Six runs in Australia, one in the United States and seven in India have been discussed in a previous paper (Shubik, 1979). Six other runs are reported here. Four were with small groups of senior executives of an international corporation, one was at the University of Western Ontario and an important contrasting group consisted of professional game theorists at a conference.

Table 8 shows data on Game 1a which should be compared with Table 2. As the game theorists were a considerably different population from the others their data was not aggregated, but is presented in the last column of Tables 8 and 9.

The group C-1980 had an identical briefing to that of the first set of briefings noted, but for C-1981a and b and C-1982 100 was added to the payoff of each individual. The resultant game should be strategically equivalent to the previous game. The results have been renormalized back from $V(A) = 100$, etc.... $V(ABC) = 700$ to $V(A) = 0$, etc.... and $V(ABC) = 400$.

The reasons for the change in normalization was that in Game 1b Players 1 and 2 would be expected to get at least 100, rather than 0, hence would not appear to be left destitute.

Game 6 was not used with these groups.

By accident the wrong characteristic function for Game 1b was given out at the University of Western Ontario. It was

$$V(A) = 0 \quad V(B) = 0 \quad V(C) = 0$$

$$V(AB) = 0 \quad V(AC) = 0 \quad (BC) = 400$$

$$V(ABC) = 400$$

GAME 1a

POINTS IN THE CORE

	C-1980	C-1981a	C-1981b	C-1982	UWO 1982	Totals	1983 Game Theory
(100,150,150)	4	4	5	6	4	23	3
(100,133,167)	3	4	6	5	1	19	
V (83.3,133.3,183.3)	1	-	-	1	-	2	14
N (50,125,225)							1
(67,133,200)		1			1	2	
Others	4	8	6	6	9	33	8
	12	17	17	18	15	79	26
Percent in Core	100	94.4	89.5	100	93.8	95.2	100

POINTS OUTSIDE THE CORE

	C-1980	C-1981a	C-1981b	C-1982	1982	Totals	1983 Game Theory
(133.3,133.3,133.3)	-	1	2	-	1	4	-
Others	-	-	-	-	-	-	-
	-	1	2	-	1	4	-
Inefficient	-	-	1	-	-	1	-
Illogical/other	2	2	-	-	20	24	15
	2	2	1	-	20	25	15
Total	14	20	20	18	36	108	41

TABLE 8

GAME 1b

POINTS IN THE CORE

	C-1980	C-1981a	C-1981b	C-1982	Totals	1983 Game Theory
(0,0,400)	1	1	-	5	7	1
(0,0,400)						3
	1	1	-	5	7	4
Percentage in Core	12.5	5.6	0	27.8	10.9	15.4

POINTS OUTSIDE THE CORE

	C-1980	C-1981a	C-1981b	C-1982	Totals	1983 Game Theory
(100,100,200)	3	9	7	6	25	5
(133,133,133)	-	2	1	-	3	2
(75,75,250)	-	-	2	-	2	-
V (66.6,66.6,266.8)	1	-	-	-	1	11
(50,50,300)	2	-	4	-	6	-
Others	1	6	6	7	20	4
	7	17	20	13	57	22
Inefficient	-	-	-	-	-	-
Illogical/other	7	2	-	-	9	15
Total	15	20	20	18	73	41

TABLE 9

Out of 16 responses 12 were (0, 200, 200) and 4 (0, 400-x, x). For this game the value and nucleolus coincide at (0, 200, 200) and the core is (0, 400-x, x) where $0 \leq x \leq 400$.

The most favored imputation in Game 1a was (100, 150, 150) with 27.7 percent selecting it; and in Game 1b (100, 100, 200) with 43.9 percent selecting it. As can be seen from Tables 8 and 9 many game theorists like the value.

Table 10 compares the results for Game 1a across all previous runs. These include three sets of data from Australia A1 - Australian social scientists, A2 - mathematicians and A3 - economists with an economic scenario to the games. India all aggregated together and all students with some faculty, then one game theory class at Yale, G where the games were run after reading on solution concepts. Then Y stands for the aggregate of the Yale students, GT for the game theory conference and C for the corporate executives plus the University of Western Ontario.

TABLE 10

PERCENTAGE OF REPLIES

	A1	A2	A3	India	G	GT	Y	C
Core	66.6	81.3	67.9	94	100	100	88.5	95.2
Value	7	12.5	17.9	0	22.2	53.8	0	2.4
Nucleolus	10.5	18.7	3.6	0	33.3	3.8	0	0
(400/3,400/3,400/3)	31.6	18.7	32.1	4.2	0	0	9.4	4.8
(100,150,150)	14.0	0	0	18.5	16.7	11.5	47.1	27.7
Respondents*	57	16	28	168	18	26	191	83

*Only those who replied without clear error.

5. CONCLUSIONS

There appeared to be a cultural difference between the Australian and all the other data. The straight even split appears to be substantiall: higher. Prior training in game theory biases the normative views of many towards the value.

The core is highly attractive when "fat" and not too nonsymmetric. The one point core is only acceptable to a few. A strong bias towards (100, 100, 200) appears.

REFERENCES

Schmeidler, D. (1969), "The nucleolus of a characteristic function game," SIAM Journal of Applied Mathematics, 17: 1163-1170.

Shapley, L. S. (1951), "The value of an n-person game," RAND publication RM-670.

Shapley, L. S. and M. Shubik (1976), "Competitive outcomes in the cores of market games," International Journal of Game Theory, 4,4: 229-337.

Shubik, M. (1979), "Cooperative game solutions: Australian, Indian and U.S. Opinions," CFDP #517.