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Abstract: We report a generalization of recent results on the existence of marginal cost pricing equilibria (MCPE) in economies with an increasing returns to scale industry. Our result makes no ad hoc assumptions which force the equilibrium to be on the efficiency frontier of the aggregate production possibility set. We also present an additional condition under which our MCPE are productively efficient in the aggregate.

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1. Introduction

There have been several recent papers on the existence of marginal cost pricing equilibria (MCPE) in economies with an increasing returns to scale industry; see Beato [1982], Brown-Heal [1982] and Cornet [1982]. All of these papers make assumptions which guarantee that the equilibria satisfy aggregate productive efficiency. However, it is clear from Beato-Mas-Colell [1982] that such results have to be based on **ad hoc** assumptions since there exist examples of "well-behaved" economies none of whose MCPE exhibit aggregate productive efficiency. The existence of MCPE in economies which are general enough to accommodate possibilities of aggregate productive inefficiency has remained an important and elusive open question. We answer this question here.

Beato-Mas-Colell [1983] and Dierker-Guesnerie-Neuefeind [1983] have independently obtained results similar to those reported here and we begin by making brief comments on each.

The Beato-Mas-Colell result assumes: (i) upper hemi-continuity and convex-valuedness of the marginal cost correspondence and (ii) that each production set can be written as the negative orthant and a compact set. The result that we report here dispenses with (i) through the use of the Clarke-Rockafeller tangent cones and with (ii) through the Hurwicz-Reiter theorem on the compactness of attainable sets of non-convex economies. It is not clear to us how the Hurwicz-Reiter theorem can be used in the Beato-Mas-Colell argument without a substantial strengthening of their income hypothesis.[1] It is worth pointing out, however, that Beato-Mas-Colell allow for many increasing returns to scale firms while we can only do so under additional assumptions.

The Dierker-Guesnerie-Neuefeind [1983] result when specialized to marginal cost pricing does not apply to the following important cases of increasing returns to scale: (i) Firms with a fixed cost. Such firms violate their assumption (P2). (ii) Firms with decreasing average costs such as Cobb-Douglas technologies. Such firms under marginal cost pricing violate their assumption (PR2). Both of these important cases are covered by our result.

In addition to our existence result. Theorem 1 below, we also present in Theorem 2 a sufficient condition under which our MCPE are productively efficient in the aggregate. This condition, though technically not surprising, is attractive from the intuitive point of view. It is essentially the requirement that the aggregate production plan be "non-tight" in the sense that there exist corresponding individual production plans such that any feasible ε-perturbation of the aggregate production plan can be attained by suitable feasible ε-perturbation of the individual plans. Our condition has the added advantage that it makes explicit the connection between decentralization and the constraint qualification in nonlinear programming.

The model and results are presented in Section 2, and the proofs are relegated to Section 3 and the Appendix.

2. The Model and Results

We consider an economy consisting of m consumers and $(\ell+1)$ producers. Consumers are indexed by i and each of them has a consumption set[2] $X^i \subseteq R^n$ and a utility function $u^i(\cdot)$ defined over X^i. Each producer has a production set Y^j where j runs over I and 1 to ℓ where I denotes the increasing returns industry. Aggregate endowment w is assumed be strictly positive. The income distribution is given in terms of a fixed structure of revenues (a^i), where $a^i > 0$ for all i and $\sum_i a^i = 1$, so that consumer i's income is $M^i(p,(y^j),y^I) = a^i p\,(\sum_{j=1}^{\ell} y^j + y^I + w)$. We shall make the following assumptions on the underlying parameters of our economy.

A1. For all i, X^i is a closed, convex subset of R^n_+ and contains 0. $u^i(\cdot)$ are quasi-concave and exhibit local non-satiation.

A2. For all j, Y^j is closed, contains 0 and $Y^j - R^n_+ \subseteq Y^j$. For all $j \neq I$, Y^j is convex.

A3. Let $Y = \sum_{j \neq I} Y^j$, $\hat{Y} = Y^I + Y$. Then

(i) $\hat{Y}$ is closed.

(ii) $\mathcal{A}(\tilde{Y}) \cap -\mathcal{A}(\hat{Y}) = \{0\}$, where $\mathcal{A}(\hat{Y})$ denotes asymptotic cone of $\hat{Y}$.

Before we introduce our equilibrium concepts, we need to develop formally the notion of marginal rates of transformation corresponding to a point on the efficiency frontier of an arbitrary closed production set. For this we shall need the cone

of normals as developed by Clarke [1975] and Rockafellar [1979]. This concept was first applied to our problem by Cornet [1982] but our presentation, in relying on **tangent cones** as the primitive concept, will be marginally different from his.[3] The use of **tangent cones** allows us to formalize the notion of marginal rates of transformation even at inward kinks of a production set. The **cone of interior displacement** as used by Beato [1982] does not share this property.

For any $x \in C \subset R^{\ell}$, let $x^k \overset{C}{\to} x$ denote a sequence chosen from C which tends to x and $t^k \downarrow 0$ denote a sequence of positive numbers converging to zero. We now have

D.1 For any non-empty closed subset C of R^n, and for $x \in C$, the **tangent cone** $T_C(x)$ consists of all $y \in R^n$ such that for all sequences $x_k \overset{C}{\to} x$, $t_k \downarrow 0$, there exists a sequence $y_k \to y$ such that for all k large enough, $x_k + t_k y_k \in C$. The **normal cone** is given by $N_C(x) = \{y \in R^{\ell} | (z,y) \leq 0 \quad \forall z \in T_C(x)\}$.

Proposition:

(a) Let C be a C^1 manifold. Then $N_C(x)$ coincides with the usual space of normals at x.

(b) Let C be convex. Then $N_C(x)$ coincides with the cone of normals to C at x in the sense of convex analysis.

(c) Let C have convex cones of interior displacements $k(x,C)$, at each point x. Then $N_C(x)$ is identical to the polar of $k(x,C)$.

(d) $T_C(x)$ is non-empty and convex.

<u>Proof</u>: $T_C(x)$ is identical to the tangent cone originally and differently defined by Clarke [1975]; see the first paragraph of Rockafeller [1979]. Given this, (a), (b) and (c) follow from Proposition 3.3 in Clarke [1975]. (d) follows from Theorem 1 in Rockafeller [1979].

D.2 A **Quasi Marginal Cost Pricing (QMCP) Equilibrium** is a 4-tuple $((\bar{x}^i), (\bar{y}^j), \bar{y}^I, \bar{p})$ such that[4]

(i) For all i, $\bar{x}^i \in A^i(\bar{p},(\bar{y}^j),\bar{y}^I) = \{x^i \in X^i \mid \bar{p}x^i \leq a^i\bar{p}\ (\sum_j \bar{y}^i - \bar{y}^I + w)\}$ and $u(\bar{x}^i) \geq u(x^i)$ for all $x^i \in A^i(\bar{p},(\bar{y}^j),\bar{y}^I)$ unless $\bar{p}\ (\sum_j \bar{y}^j + \bar{y}^I + w) = 0$.

(ii) For all $j = 1, \ldots, \ell, I$, $\bar{y}^j \in \text{Bdry}\ (Y^j)$ and $\bar{p} \in N_{Y^j}(\bar{y}^j)$.

(iii) $\sum_{i=1}^{m} \bar{x}^i = \sum_{j=1}^{\ell} \bar{y}^i + \bar{y}^I + w.$

Our equilibrium concept is clear, $\bar{x}^i$ are quasi-demands as conventionally defined in the literature; by (b) of the proposition above, $\bar{y}^j$ are profit-maximizing bundles for all non-increasing returns to scale industries; $\bar{p}$ are the marginal cost prices corresponding to the production plan $\bar{y}^I$ for the increasing returns industry; and aggregate excess demands are zero.

For our main result we shall need one final assumption. This can be traced to Meade [1949, paragraph 12] and simply says that for any efficient production plan of the increasing returns industry, marginal cost pricing is **viable** in the sense that the aggregate revenue from production is not negative. More formally,

A4. For all $y^I \in \text{Bdry}(Y^I)$, $p \in N_{Y^I}(y^I)$, implies that

$$p\left(\sum_{j=1}^{\ell} y^j + y^I + w\right) \geq 0, \quad \text{where } y^j \in \operatorname{Arg\,Max}_{y \in Y^j} py \text{ for all } j = 1, \ldots, \ell.$$

Theorem 1: Under Assumptions A1 to A4, exists a QMCP equilibrium.

Corollary 1: If in addition to A1-A3, Y^I is assumed to be convex and consumers have endowments w^i, $\sum_i w^i = w$ and shares θ^{ij} in all firms $j = 1, \ldots, \ell, I$, then there exists a quasi-Walrasian equilibrium. In this case, A4 is automatically satisfied.

It is clear that if in A4 $p(\sum_{j=1}^{\ell} y^j + y^I + w) > 0$, the equilibrium referred to in Theorem 1 will be such that for all i, $\bar{x}^i$ belong to the demand correspondence rather than the quasi-demand correspondence.

The reader should take note of there being no presumption that the equilibria whose existence is established in Theorem 1 is productively efficient in the aggregate. A natural question arises as to the conditions under which there exist productively efficient marginal cost pricing equilibria. As mentioned in the introduction, the existence theorems presented by Beato, Brown-Heal and Cornet can be viewed as answers to this question. We present another sufficient condition here which appears not to have been noticed. We shall need

D3. An aggregate production plan $\bar{y} \in \hat{Y}$ is said to be **non-tight** if there exist $y^j \in Y^j$, $\sum_{j=I,1}^{\ell} y^j = \hat{y}$ such that for all $\varepsilon > 0$, for all $z \in B_\varepsilon(\hat{y}) \cap \hat{Y}$ there exist $z^j \in B_\varepsilon(y^j) \cap Y^j$ such that $z = \sum_{j=I,1}^{\ell} z^j$. If a production plan is not non-tight, it is said to be **tight.**

The above concepts can be given a very simple economic interpretation. An aggregate production plan $\hat{y}$ such that any other production plan "arbitrarily close" to $\hat{y}$ can be obtained by individual plans "arbitrarily close" to y^j. Put differently, the economy can accommodate marginal changes in aggregate production levels as a result of marginal changes in individual production levels. We shall given an example of a tight-production plan when we discuss the economy presented by Beato-Mas-Colell.

We can now weaken A4 to A4' where

A4'. For all $\bar{y}$ Bdry $(\hat{Y})$, p $N_{\hat{Y}}(\bar{y})$ implies $p(\hat{y}+w) > 0$.

Theorem 2:

(a) **Under A1-A3 and A4', there exist** $((\bar{x}^i), \tilde{y}, \bar{p})$ **such that** $\tilde{y} \in$ Bdry $(\hat{Y})$, $\bar{p} \in N_{\hat{Y}}(\tilde{y})$ **and conditions (i) and (iii) of D2 are fulfilled.**

(b) **If** $\tilde{y}$ **is non-tight in** (a) **above, there exist** $\bar{y}^j$ **summing to** $\tilde{y}$ **such that (ii) of D2 is also fulfilled.**

Theorem 2(a) has already been presented by Cornet [1982] and constitutes his Theorem 1. Theorem 2(b) provides a sufficient condition under which an aggregate production plan can be decentralized through marginal cost prices. It also makes clear that such a decentralization is the principal source of difficulty in ensuring the existence of aggregate productive efficiency of MCPE. The reason for this is transparent once we observe that $\bar{y}^j$ constitute a solution of a nonlinear programming problem in which the output of a commodity produced by any one industry is maximized subject to material balance and to the requirement that the outputs of all other industries are fixed. The question then is whether there exist multipliers (marginal cost prices) such that this solution satisfies first order conditions. But it is well known that this is generally true only under a constraint qualification. Our non-tightness assumption can be viewed as constituting such a constraint qualification.

We end this section by applying Theorem 2 to the example presented by Beato and Mas-Colell [1982]. Let commodities 1 and 2 refer to x and y as defined by Beato and Mas-Colell. It is clear that in the sense of Theorem 2(a) there is an equilibrium in which $\tilde{y} = (-16,16)$, $\bar{p} = (3,2)$, $\bar{x}^1 = (0,42)$ and $\bar{x}^2 = (4,24)$. It can also be checked that there is no other equilibrium in which aggregate production efficiency is satisfied. The important observation, of course, is that this equilibrium cannot be sustained as a **bona fide** MCPE. The reason is that $\tilde{y} = (-16,16)$ is **tight.** Let $\bar{y}^1$ refer to the production plan of the constant returns to scale firm $\bar{y}^2$ to that of the increasing returns to scale firm. If $\tilde{\tilde{y}}$ is obtained as a sum of $\bar{y}^1 = (0,0)$ and $\bar{y}^2 = (-16,16)$, any perturbation of $\tilde{\tilde{y}}$ along the constant returns technology cannot be obtained as a perturbation of $\bar{y}^1$ and $\bar{y}^2$. An analogous statement is true if $\bar{y}^1$ and $\bar{y}^2$ are respectively chosen to be $(-16,16)$ and $(0,0)$.

This example also makes clear that by considering the boundary of the aggregate production set, we admit as feasible, marginal cost prices (as at the kink) which have no basis to be regarded as such at the individual firm level.

3. Proofs

We begin with the proof of Theorem 1. It relies on the fact that the attainable consumption and production sets are compact; and that a suitably truncated boundary of the production set of the increasing returns industry is homeomorphic to the simplex. We present these essentially known results first.

Let the set of attainable states be denoted by A where

$$A = \{(x^i),\ (y^j) \mid \sum_{i=1}^{m} x^i + \sum_{j=I,1}^{\ell} y^j = w, \text{ and } x^i \in X^i,\ y^j \in Y^j$$

for all $i,j\}$. Let the projection of A on X^i and Y^j be denoted respectively as $\hat{X}^i$ and $\hat{Y}^j$. We can now state

Lemma 1 (Hurwicz-Reiter): Under Assumption A1 to A3, $\hat{X}^i$ and $\hat{Y}^j$ are compact for all i,j.

Proof: See Lemma 3 in Brown-Heal [1982].

Since all attainable consumption and production sets are compact, we can assume that there exists a cube centered at zero and with length k such that all $\hat{X}^i$ and $\hat{Y}^j$ are in its interior. Now let[5]

$$K = ((1+\ell k)e+w) \in R^n_{++}$$

and $E_K(Y^I) = \text{Bdry } (Y^I + \{K\}) \cap R^n_+$. We shall focus on the following subset of $E_K(Y^I)$

$$\hat{E}_K(Y^I) = \{z \in E_K(Y^I) \mid \nexists\ \bar{z} \in E_K(Y^I),\ \ \bar{z} < z \ \text{ with strict inequality for all } i \text{ with } z_i > 0\}.$$

The reason for this apparently arbitrary truncation will become clear in the proof of Theorem 1. For the moment, we can state

Lemma 2 (Moore): If $Y^I - R^n_+ \subseteq Y^I$, then $\hat{E}_K(Y^I)$ is homeomorphic to the simplex Δ, i.e., there exists a continuous mapping $\nu\colon \Delta \to E_K(Y^I)$ which is bijective with a continuous inverse and such that $\delta >> 0$ implies $\nu(\delta) >> 0$.

Proof: See Appendix.

Proof of Theorem 1: Let $N(z)$ denote the normal to the set $\hat{E}_K(Y^I)$ at a point z in the set and define the correspondence $\phi\colon\ \hat{E}_K(Y^I) \to \Delta$ such that

$$\begin{aligned} \phi(z) &= N(z) \cap \Delta \quad \text{for all } z \in R^n_{++} \\ &= \Delta \quad \text{otherwise.} \end{aligned}$$

Next, we develop our concepts of pseudo-demands and supplies. For any $p \in \Delta$ let

$$s^j(p) = \underset{y \in Y^j}{\text{Arg Max}}\ py,\quad \pi^j(p) = \underset{y \in Y^j}{\text{Max}}\ py \qquad j = 1, \ldots, \ell.$$

For any $p \in \Delta$, for any $\delta \in \Delta$ and $\nu(\delta)$ as in Lemma 2, for all $i = 1, \ldots, m$, let

$$M^i(p,\delta) = a^i(pw + \sum_{j=1}^{\ell} \pi^j(p) + p(\nu(\delta) - K))$$

$$\hat{A}^i(p,\delta) = \{x \in \hat{X}^i \mid px < M^i(p,\delta)\}$$

$$d^i(p,\delta) = \underset{x^i \in \hat{A}^i(p,\delta)}{\text{Arg Max}}\ u^i(x^i) \quad \text{if} \quad M^i(p,\delta) > 0$$

$$= \hat{A}^i(p,\delta) \quad \text{if} \quad M^i(p,\delta) = 0$$

$$= \{0\} \qquad \text{if} \quad M^i(p,\delta) < 0.$$

The above concepts are clear once we recognize the fact that in a private-ownership economy with an increasing returns industry, incomes may well be negative.[6] In this case we let each consumer obtain a zero commodity bundle. Note also, that it is through the incomes $M^i(p,\delta)$ that the production plan of the increasing returns industry, as parameterized by δ, affects demands.

Our proof of existence of QMCP equilibria will revolve around the following mappings in which $(X \times Y)$ denote the compact set $(\sum_{i=1}^{m} \hat{X}^i) \times (\sum_{j=1}^{\ell} \hat{Y}^j)$.

$\mu: \Delta \to \Delta$ where $\mu(\delta) = \phi(\nu(\delta))$, $\nu(\cdot)$ as in Lemma 2.

$$\xi: \Delta \times \Delta \to X \times Y \quad \text{where} \quad \xi(p,\delta) = \left(\sum_{i=1}^{m} d^i(p,\delta)\right) \times \left(\sum_{j=1}^{\ell} s^j(p)\right)$$

$$\pi: X \times Y \to \Delta \quad \text{where} \quad \pi_1(x,y) = \frac{(x_i - y_i - w_i + K_i)}{\sum_{i=1}^{n} (x_i - y_i - w_i + K_i)}$$

The economics behind our mappings are clear once the reader takes note of the fact that the (n-1) dimensional simplex Δ is representing two different concepts, viz., the prices p and the production plans δ of the increasing returns industry. The fact that the latter are equivalently represented by a point on the simplex is precisely the content of Lemma 2. Given any δ, μ determines the corresponding normalized marginal rates of transformation for the increasing returns industry. Given any (p,δ), ξ determines the consumer demands and competitive supplies. Finally, π projects an arbitrary non-negative vector in $R_+^{\ell n}$ on to the (n-1) dimensional simplex. Note, that given our truncation,

$$x - y - w + K \geq e \gg 0 \qquad (*)$$

and hence the mapping $\pi(\cdot)$ is well defined.

Let us now consider the composite map $\alpha(\cdot)$ which takes $\Delta \times \Delta \times X \times Y$ into itself such that

$$\alpha(p,\delta,x,y) = \mu(\delta) \times \xi(p,\delta) \times \pi(x,y).$$

We now show that α satisfies all the conditions of Kakutani's fixed point theorem. It is clear that given continuity of $\nu(\cdot)$, all arguments are standard except for the fact that ϕ is a non-empty, upper-hemi-continuous, convex-valued map. The fact that $N(z)$ is non-empty and convex follows from the proposition in Section 2. To show that $N(z) \cap \Delta \neq \phi$, we first observe that, as a consequence of free disposal,[7] $(-R^n_+) \subseteq T(z)$. For any $y \in (-R^n_+)$, simply consider the sequence $y^k = y$ for all k, corresponding to arbitrary sequences $z_k \to z$, $z_k \in \hat{E}_k(Y^I)$ and $t_k \downarrow 0$. This implies that $N(z) \subset R^n_+$. It is also clear that $T(z) \neq R^n$ so that $N(z) \neq \{0\}$. Finally, upper-hemi-continuity follows from Rockafeller [1972, Corollary 2, Section 2] given that Int $T(z) \neq \phi$ for any z. Hence $\alpha(\cdot)$ has a fixed point $(\bar{p}, \bar{\delta}, \bar{x}, \bar{y})$.

It follows from (*) that $\bar{\delta} = \pi(\bar{x},\bar{y}) >> 0$ so that, by Lemma 2, $\nu(\bar{\delta}) >> 0$. Hence, from the mapping $\mu(\cdot)$, $\bar{p}$ are indeed marginal cost prices corresponding to the production plan $\bar{y}^I = (\nu(\bar{\delta}) - K)$ of the increasing returns industry. Since, from the mapping ξ, we know that there exist $\bar{y}^j \in Y^i$ such that $\sum_{j=1} \bar{y}^j = \bar{y}$ and $\bar{y}^j \in s^j(\bar{p})$ $j = 1, \ldots, \ell$, we can appeal to assumption A4 to assert that $M^i(\bar{p},\bar{\delta}) > 0$ for all i. From ξ we know that for all i, there exist $\bar{x}^i \in d^i(\bar{p},\bar{\delta})$ such that $\sum_{i=1}^{m} \bar{x}^i = \bar{x}$. Clearly, $((\bar{x}^i), (\bar{y}^j), \bar{y}^I, \bar{p})$ satisfy conditions (i) and (ii) of QMCP equilibrium. It remains to be verified that $\bar{x} = \bar{y} + \bar{y}^I + w$. Since (i) is satisfied, given local non-satiation, we can write

$$\overline{px} = \sum_{i=1}^{m} M^i(\bar{p},\bar{\delta}) = \overline{pw} + \overline{py} + \overline{py}^I. \qquad (W)$$

From the construction of $\pi(\cdot)$ we know that $(\bar{x} - \bar{y} - w + K) = \lambda(\bar{y}^I + K)$, where λ is a positive real number. By appealing to (W) we can write this as $\bar{p}(\bar{y}^I + K) = \lambda\bar{p}(\bar{y}^I + K)$. Since $\bar{y}^I + K = \nu(\delta) >> 0$ by Lemma 2, we have

$\bar{p}(\bar{y}^I + K) > 0$. Thus, $\lambda = 1$ and the proof is complete.

Proof of Corollary 1: Define all the mappings as in the proof of Theorem 1 but with consumer incomes $M^i(\cdot)$ modified as follows:

$$M^i(p,\delta) = pw^i + \sum_{j=1}^{\ell} \theta^{ij}\pi^j(p) + \theta^{iI}p(\nu(\delta) - K).$$

As in the proof of Theorem 1 we can obtain a fixed point $(\bar{p}, \bar{\delta}, \bar{x}, \bar{y})$. Let $\bar{y}^I = \nu(\bar{\delta}) - K$. Then, it is clear that $\bar{y}^I \in s^I(\bar{p})$ and there exist $\bar{y}^j \in s^i(\bar{p})$, $j = 1, \ldots, \ell$ such that $\sum_{j=1}^{\ell} \bar{y}^j = \bar{y}$. Also, there exist $\bar{x}^i \in \hat{A}^i(\bar{p},\bar{\delta})$ such that, if $M^i(\bar{p},\bar{\delta}) > 0$, then $\bar{x}^i \in \underset{x^i \in \hat{A}^i(\bar{p},\bar{\delta})}{\text{Arg Max}}\ u^i(x^i)$, for all i. The mapping π ensures, as before, that $\bar{x} = \bar{y} + \bar{y}^I + w$. Thus, $((\bar{x}^i), (\bar{y}^j), \bar{y}^I, \bar{p})$ is a quasi-Walrasian equilibrium. It is clear that A4 is automatically satisfied since Y^I is convex.

We now turn to a

Proof of Theorem 2: The proof of Theorem 2(a) follows as a straightforward modification of the proof of Theorem 1. We simply consider $\hat{E}_K(\hat{Y})$ instead of $\hat{E}_K(Y^I)$. The homeomorphism between $\hat{E}_K(\hat{Y})$ and the simplex is established as before but now the mapping ξ maps from prices to demands only and the projection map π projects from X to the simplex. Given our assumption A4', we can show the existence of a fixed point. This is an equilibrium in the sense required by Theorem 2(a).

In order to show that this equilibrium is a MCPE, we use Lemma A1 and A2 in the Appendix.

APPENDIX

<u>Proof of Lemma 2</u>: We begin by establishing compactness of $\hat{E}_K(Y^I)$. Define attainable sets with respect to K by

$$\hat{Y}^j_K = \{y^j \in Y^j | y^h \in Y^h,\ h \neq j,\ x^i \in X^i \text{ such that}$$

$$x - y - y^I - K = 0\},\quad j = 1, \ldots, I$$

$$\hat{X}^i_K = \{x^i \in X^i | x^h \in X^h,\ h \neq i,\ y^j \in Y^j \text{ such that}$$

$$x - y - y^I - K = 0\},\quad i = 1, \ldots, m.$$

By the same argument as in Lemma 3 of Brown-Heal [1982] it can be shown that these sets are compact. Moreover, given free disposal and the assumption that $0 \in X^i$, we have $(Y^I + \{K\}) \cap R^n_+ \subseteq \hat{Y}^I_K + \{K\}$. But,

$$E_K(Y^I) = \text{Bdry } (Y^I + \{K\}) \cap R^n_+ \subseteq \hat{Y}^I_K + \{K\}$$

and is therefore compact. Recall that $\hat{E}_K(Y^I) \subseteq E_K(Y^I)$ and is therefore bounded. To show that $\hat{E}_K(Y^I)$ is closed, consider a sequence $z^\nu \to z$, where $z^\nu \in \hat{E}_K(Y^I)$. Since $\hat{E}_K(Y^I)$ is compact, $z \in E_K(Y^I)$. Suppose $z \notin \hat{E}_K(Y^I)$, i.e., $\exists\ \bar{z} \in E_K(Y^I)$ such that $\bar{z}_i < z_i$ for all $i \in I_1 = (\{1, \ldots, n\} | z_i > 0)$ and $\bar{z}_i = 0$ for $i \in I_2 = (\{1, \ldots, n\} | z_i = 0)$. Then, there exists a ν large enough such that $\bar{z}_i < z^\nu_i$ for all $i \in I_1$ and $\bar{z}_i = 0$ for all $i \in I_2$, i.e., $\bar{z} < z^\nu$ and $\bar{z}_i < z^\nu_i$ for all i such that $z^\nu_i > 0$. This contradicts the fact that $z^\nu \in \hat{E}_K(Y^I)$ completing the proof of our claim.

The remainder of this proof is based on Moore's [1971] proof of Theorem 3.7. Let $A = \{a \in R^n_+ | \exists\, z \in \hat{E}_K(Y^I)$ such that $a < z\}$, and $\Phi\colon \Delta \to A$ be defined by $\Phi(\delta) = \{a \in A | \exists\, \lambda \in R_+$ such that $a = \lambda\delta\}$. Let $\nu(\delta) = \underset{a \in \Phi(\delta)}{\text{Arg Max}} \sum_{i=1}^{n} a_i$. Since $\Phi(\delta)$ is non-empty, because $0 \in A$, and compact, because A is compact, $\nu(\delta)$ is a well defined function. From the definition of $\hat{E}_K(Y^I)$ it also follows that $\nu(\delta) \in \hat{E}_K(Y^I)$. Let $g\colon \hat{E}_K(Y^I) \to \Delta$ be defined by $g(z) = \dfrac{z}{\sum_{i=1}^{n} z_i}$. Given free disposal and $K >> 0$, $\sum_i z_i > 0$ so that $g(z)$ is well-defined and continuous. We now show that $g(z)$ is one-to-one on $\hat{E}_K(Y^I)$. Suppose $z,\ z' \in \hat{E}_K(Y^I)$, $z \neq z'$ and $g(z) = g(z')$.

Then $\dfrac{z}{\sum_i z_i} = \dfrac{z'}{\sum_i z_i'}$, or $z = \left|\dfrac{\sum_i z_i}{\sum_i z_i'}\right| z'$, which, given the definition of $\hat{E}_K(Y^I)$, is not possible. It also follows that $\lambda = g^{-1}$ and is therefore onto Δ. Thus, g is a continuous function one-to-one on $\hat{E}_K(Y^I)$ and onto Δ. Since $\hat{E}_K(Y)$ is compact and $\nu = g^{-1}$ it follows from Nikaido [1968, Theorem 1.4] that ν is continuous, one-to-one on Δ and onto $\hat{E}_K(Y^I)$. Thus $\hat{E}_K(Y^I)$ and Δ are homeomorphic. It is also clear that $\delta >> 0$ implies $\nu(\delta) >> 0$.

<u>Lemma Al</u>: Let y $\hat{Y} \equiv (Y^I + \sum_{j=1}^{\ell} Y^j)$ be nontight. Then there exist $y^j \in Y^j$ such that $T_{Y^I}(y^I) + \sum_{j=1}^{\ell} T_{Y^j}(y^j) \subseteq T_{\hat{Y}}(y)$.

Proof: Since y is nontight, there exist $y^j \in Y^j$, $j = 1, \ldots, \ell, I$ such that $\sum_j y^j = y$ and for any $y_k \overset{\hat{Y}}{\to} y$. there exist $y_k^j \in Y^j$, $j = 1, \ldots, \ell, I$ such that $y_k^j \to y^j$ and $\sum_j y_k^j = y$.

We shall show that for any $z^j \in T_{Y^j}(y^j)$, $j = 1, \ldots, \ell, I$, $\sum_j z^j = z \in T_{\hat{Y}}(y)$. Since z^j are in the individual tangent cones, there exist $z_k^j \to z^j$ such that for $t_k \downarrow 0$, for all k large enough, $t_k z_k^j + y_k^j \in Y^j$ for all j. Let $z_k = \sum_j z_k^j$. Certainly $z_k \to z$ and for all k large enough $y_k + t_k z_k = \sum_j (y_k^j + t_k z_k^j) \in \sum_j Y^j = \hat{Y}$ and the proof is complete.

Lemma A2: Under the notation and assumptions of Lemma A1,

$$N_{\hat{Y}}(y) \subseteq \bigcap_{j=I,1}^{\ell} N_{Y^j}(y^j).$$

Proof: See, for example, Property 6 in Beato [1982, p. 684].

FOOTNOTES

[1]This follows from the fact that the income hypothesis would now relate to the production plans on the boundaries of the truncated production sets and some of these production plans may be in the interior of their respective production sets.

[2]R^n is n-dimensional Euclidean space whose non-negative orthant is denoted by R^n_+ and the strictly positive orthant by R^n_{++}. We shall use the following convention for ordering the elements in R^n: $\gg, >, \geq$.

[3]Compare, for example, his proof and his Lemma 4(i) with that of ours on page 13 below.

[4]Bdry (A) denotes the boundary of the set A, $\text{Arg}\max_{x \in A} f(x)$ denotes $\bar{x}$ such that $f(\bar{x}) \geq f(x)$ for all x in A; $B_\varepsilon(x)$ is the open ball of radius ε centered at x.

[5]e is an element of R^n_{++}, all of whose components are unity.

[6]This is assuming, of course, that the increasing returns industry is being constrained to operate under marginal cost pricing.

[7]$T(z)$ is the tangent space corresponding to $N(z)$, i.e., $N(z)$ is the polar cone of $T(z)$.

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