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THE EXACT DISTRIBUTION OF THE WALD STATISTIC

by

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0. ABSTRACT

This paper derives the exact distribution of the Wald statistic for testing general linear restrictions on the coefficients in the multivariate linear model. This generalizes all previously known results including those for the standard F statistic in linear regression, for Hotelling's T^2 test and for Hotelling's generalized T_0^2 test. Conventional classical assumptions of normally distributed errors and nonrandom exogenous variables are employed.

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1. INTRODUCTION

One field in which econometric distribution theory would appear to be in a particularly well developed state is the linear model with normally distributed errors. In this model we have an established battery of exact statistical tests for practitioners, the most well known of which are the commonly used t and F ratio statistics, Hotelling's T^2 test and Hotelling's generalized T_0^2 test. Other test statistics for which distributional results have been obtained in this model are the likelihood ratio statistic, Pillai's statistic and Roy's largest latent root test. [6] contains a detailed review of analytical results in this field including both exact formulae and asymptotic expansions. All the above mentioned statistics are used for testing linear hypotheses about the coefficients in the multivariate linear model.

In spite of the extensive research in this field (exemplified by Chapter 10 of [6]) there are still major unsolved problems. One of these, which is particularly interesting to econometricians, is the distribution of the Wald statistic for testing general linear restrictions on the coefficients. In special cases where the coefficient matrix in the restrictions takes on a Kronecker product form (corresponding to the GMANOVA and classical MANOVA models in multivariate analysis) the Wald statistic reduces to the generalized T_0^2 statistic and known results for the distribution of this statistic apply. However, when the coefficient matrix is not of the Kronecker product form none of the presently available results are applicable. Important practical examples arise in applied demand analysis in econometrics where the Slutsky symmetry condition leads to simple linear across equation restrictions which cannot be formulated in a Kronecker product representation.

The purpose of the present paper is to provide a unified and comprehensive general solution to the distribution of the Wald statistic in this multivariate linear model setting. More specifically, the paper derives the exact probability density function (pdf) of the Wald statistic for testing quite general linear restrictions on the coefficients under the null hypothesis that the restrictions hold. The analytical methods employed here rely on the matrix fractional calculus that was developed and applied by the author in some other recent work [7], [8].

As special cases of our general solution we derive the distributions of the conventional regression F statistic, Hotelling's T^2 statistic and Hotelling's generalized T_0^2 statistic.

2. THE MODEL AND NOTATION

In the multivariate linear model

(1) $$y_t = Ax_t + u_t \ , \quad t = 1, \ldots, T$$

y_t is a vector of n dependent variables, A is an $n \times m$ matrix of parameters, x_t is a vector of nonrandom independent variables and the u_t are i.i. $N(0,\Omega)$ errors with Ω positive definite. The hypothesis under consideration takes the general form

(2) $$H_0 : \ D \operatorname{vec} A = d$$

where D is a $q \times nm$ matrix of known constants of rank q, d is a known vector and $\operatorname{vec}(A)$ stacks the rows of A.

From least squares estimation of (1) we have:

$$(3) \qquad A^* = Y'X(X'X)^{-1} \ , \quad \Omega^* = Y'(I - P_X)Y/N$$

where $Y' = [y_1, \ldots, y_T]$, $X' = [x_1, \ldots, x_T]$, $P_X = X(X'X)^{-1}X'$ and $N = T-m$. We take X to be a matrix of full rank $m \leq T$ and define $M = (X'X)^{-1}$.

The Wald statistic for testing the hypothesis (2) is

$$(4) \qquad \begin{aligned} W &= (D \text{ vec } A^* - d)'\{D(\Omega^* \otimes M)D'\}^{-1}(D \text{ vec } A^* - d) \\ &= N\ell' B\ell \end{aligned}$$

where $\ell = D \text{ vec } A^* - d$ is $N(0,V)$ under H_0 with $V = D(\Omega \otimes M)D'$, and $B = \{D(C \otimes M)D'\}^{-1}$ where $C = Y'(I - P_X)Y$ is central Wishart with covariance matrix Ω and N degrees of freedom.

We define $y = \ell' B \ell$ and write y in canonical form as

$$(5) \qquad y = g'Gg$$

where $g = V^{-1/2}\ell$ is $N(0, I_q)$ and $G = V^{1/2}\{D(C \otimes M)D'\}^{-1}V^{1/2}$.

In the MANOVA model (see [6] for example) $D = I \otimes R$ where R is $r \times m$ of rank $r \leq m$; and in the GMANOVA model we have $D = S \otimes R$ where S is $s \times n$ of rank $s \leq n$.

3. THE NULL DISTRIBUTION OF W

It is convenient to work first with the canonical variate y . The conditional distribution of y given C is that of a positive definite quadratic form in normal variates. Many different series representations of this latter distribution are available and most of these are reviewed in [4] and [5]. One of the most elegant representations of this distribution is, in fact, not reported in either [4] or [5] but was given by James [3] in terms of a hypergeometric function with two matrix arguments. Explicitly, we have

$$\text{(6)} \qquad \mathrm{pdf}(y|C) = \frac{y^{q/2-1}}{2^{q/2}\Gamma\left(\frac{q}{2}\right)(\det G)^{1/2}}\ {}_0F_0\left(-\tfrac{1}{2}G^{-1},\ y\right)$$

where

$$\text{(7)} \qquad {}_0F_0\left(-\tfrac{1}{2}G^{-1},\ y\right) = \int_{O(q)} \mathrm{etr}\left(-\tfrac{y}{2}G^{-1}HE_{11}H'\right)(\underline{dH}) \ .$$

In (7) $O(q)$ is the orthogonal group of $q \times q$ matrices, $\mathrm{etr}(\ ) = \exp(\mathrm{tr}(\ ))$, $E_{11} = e_1e_1'$ where e_1 is the first unit vector and $(\underline{dH})$ represents the normalized Haar measure on $O(q)$. (7) also has the following series representation

$$\text{(8)} \qquad \sum_{j=0}^{\infty} \frac{1}{j!} \frac{C_{(j)}\left(-\frac{1}{2}G^{-1}\right)y^j}{C_{(j)}(I_q)}$$

in terms of top order zonal polynomials $C_{(j)}(\ )$ where (j) denotes the partition $(j, 0, \ldots, 0)$ of j with only one non zero part. Formulae for $C_{(j)}(\ )$ are known and are given in [3].

From the definition of G we deduce that:

$$\text{(9)} \qquad \mathrm{pdf}(y|C) = \frac{y^{q/2-1}\det(\overline{D}(C \otimes M)\overline{D}')^{1/2}}{2^{q/2}\Gamma\left(\frac{q}{2}\right)}\ {}_0F_0\left(-\tfrac{1}{2}\overline{D}(C \otimes M)\overline{D}',\ y\right)$$

where $\overline{D} = V^{-1/2}D$. Now

$$\mathrm{pdf}(C) = \frac{\mathrm{etr}\left(-\frac{1}{2}\Omega^{-1}C\right)(\det C)^{(N-n-1)/2}}{2^{nN/2}\Gamma_n\left(\frac{N}{2}\right)(\det \Omega)^{N/2}} .$$

Hence

$$\mathrm{pdf}(y) = \int_{C>0} \mathrm{pdf}(y|C)\,\mathrm{pdf}(C)\,dC$$

$$(10) = \frac{y^{q/2-1}}{2^{(q+nN)/2}\Gamma\left(\frac{q}{2}\right)\Gamma_n\left(\frac{N}{2}\right)(\det \Omega)^{N/2}} \int_{C>0} \mathrm{etr}\left(-\frac{1}{2}\Omega^{-1}C\right)\det\left(\overline{D}(C\otimes M)\overline{D}'\right)^{1/2}$$

$$\cdot(\det C)^{(N-n-1)/2}\,{}_0F_0\left(-\frac{1}{2}\overline{D}(C\otimes M)\overline{D}',\ y\right)dC .$$

Using the theory of matrix fractional calculus developed in [8] we now write

$$(11)\quad \det\left(\overline{D}(C\otimes M)\overline{D}'\right)^{1/2}{}_0F_0\left(-\frac{1}{2}\overline{D}(C\otimes M)\overline{D}',\ y\right)$$

$$= \det\left(\overline{D}(\partial Z\otimes M)\overline{D}'\right)^{1/2}\int_{O(q)} \mathrm{etr}\left(-\frac{y}{2}\overline{D}(\partial Z\otimes M)\overline{D}'HE_{11}H'\right)(\underline{dH})\cdot\mathrm{etr}(CZ)\Big|_{Z=0}$$

where Z is an auxiliary matrix of dimension $q\times q$ and ∂Z denotes the matrix differential operator $\partial/\partial Z$. The right side of (11) may be regarded as a linear (pseudodifferential) operator representation of this function. An absolutely convergent power series representation of the operator may also be used, instead of (11).

From (10) and (11) we deduce that

$$(12)\qquad \mathrm{pdf}(y) = \frac{y^{q/2-1}}{2^{(q+nN)/2}\Gamma\left(\frac{q}{2}\right)\Gamma_n\left(\frac{N}{2}\right)(\det \Omega)^{N/2}} \int_{C>0}\Bigg[\det\left(\bar{D}(\partial Z \otimes M)\bar{D}'\right)^{1/2}$$

$$\cdot {}_0F_0\left(-\tfrac{1}{2}\bar{D}(\partial Z \otimes M)\bar{D}',\ y\right)\cdot \mathrm{etr}\left\{-\left(\tfrac{1}{2}\Omega^{-1} - Z\right)C\right\}\Bigg]_{Z=0} (\det C)^{(N-n-1)/2}dC\ .$$

The integral over C in (12) is absolutely and uniformly convergent for all Z satisfying $\mathrm{Re}(Z) \leq \varepsilon I$ where ε is any positive quantity less than the smallest latent root of $\Omega^{-1}/2$. We may, therefore, take both the operator and the evaluation at $Z = 0$ outside the integration, yielding:

$$\mathrm{pdf}(y) = \frac{y^{q/2-1}}{2^{(q+nN)/2}\Gamma\left(\frac{q}{2}\right)(\det \Omega)^{N/2}}\Bigg[\det\left(\bar{D}(\partial Z \otimes M)\bar{D}'\right)^{1/2} {}_0F_0\left(-\tfrac{1}{2}\bar{D}(\partial Z \otimes M)\bar{D}',\ y\right)$$

$$\cdot\left[\det\left(\tfrac{1}{2}\Omega^{-1} - Z\right)\right]^{-N/2}\Bigg]_{Z=0}$$

$$(13)\qquad = \frac{y^{q/2-1}}{2^{q/2}\Gamma\left(\frac{q}{2}\right)}\Bigg[\det\left(\bar{D}(\partial Z \otimes M)\bar{D}'\right)^{1/2}\cdot {}_0F_0\left(-\tfrac{1}{2}\bar{D}(\partial Z \otimes M)\bar{D}',\ y\right)\det(I - 2\Omega Z)^{-N/2}\Bigg]_{Z=0}\ .$$

We now employ the simply proven result that if T is a nonsingular matrix and $X = TZT'$ then

$$(14)\qquad \partial Z = T'\partial X T\ .$$

In particular, writing $X = TZT'$ with $T = (2\Omega)^{1/2}$ in (13) we obtain

$$(14)\qquad \mathrm{pdf}(y) = \frac{y^{q/2-1}}{\Gamma\left(\frac{q}{2}\right)}\Bigg[\det\left(L(\partial X \otimes I)L'\right)^{1/2} {}_0F_0\left(-L(\partial X \otimes I)L',\ y\right)\det(I-X)^{-N/2}\Bigg]_{X=0}$$

where

$$(15)\qquad L = V^{-1/2}D(\Omega^{1/2} \otimes M^{1/2}) = [D(\Omega \otimes M)D']^{-1/2}D(\Omega^{1/2} \otimes M^{1/2})\ .$$

The distribution of the Wald statistic W is therefore:

(16) $$\mathrm{pdf}(w) = \frac{w^{q/2-1}}{N^{q/2}\Gamma\left(\frac{q}{2}\right)}\left[\det\left(L(\partial X \otimes I)L'\right)^{1/2} {}_0F_0\left(-L(\partial X \otimes I)L', \; w/N\right)\det(I-X)^{-N/2}\right]_{X=0} \; .$$

This is a simple and very general expression for the exact density which is very useful in analytic work. As later sections of the paper show, all presently known results for this statistic, including asymptotic results and special case distributions such as those for Hotelling's T^2 and generalized T_0^2, can be deduced quite simply from (14) and (16).

4. THE ASYMPTOTIC DISTRIBUTION OF W

In the neighborhood of $X = 0$ and to the first order of approximation

(17) $$\det(I-X)^{-N/2} \sim \mathrm{etr}(NX/2) \; .$$

Making this replacement in (16) we deduce the following first order asymptotic approximation to the distribution of W :

(18) $$\mathrm{pdf}(w) = \frac{w^{q/2-1}(N/2)^{q/2}}{N^{q/2}\Gamma\left(\frac{q}{2}\right)} \, {}_0F_0\left(-(N/2)I_q, \; w/N\right)$$

$$= \frac{w^{q/2-1} \, e^{-w/2}}{2^{q/2}\Gamma\left(\frac{q}{2}\right)}$$

that is, the density of the χ_q^2 distribution.

5. HOTELLING'S T^2 STATISTIC

When $D = F \otimes g'$ for some m-vector g and $q \times n$ matrix F of full rank $q < n$ the null hypothesis has the form

$$H_0 : FAg = d \tag{19}$$

so that the restrictions apply to the same linear combination of the columns of A. In this case the Wald statistic becomes

$$W = N\left[\frac{FA^*g - d}{(g'Mg)^{1/2}}\right]' (FCF')^{-1}\left[\frac{FA^*g - d}{(g'Mg)^{1/2}}\right] \tag{20}$$

$$= Nx'S^{-1}x$$

where $x = (FA^*g - d)/(g'Mg)^{1/2}$ and $S = FCF'$ is $W_q(N, F\Omega F')$. Hence, W Hotelling's T^2 statistic and, in particular,

$$F = \frac{N-q+1}{q}\left(\frac{W}{N}\right) \equiv F_{q,N-q+1} \ .$$

Specializing (14) to this case, we note that

$$L = (F\Omega F'g'Mg)^{-1/2}(F\Omega^{1/2} \otimes g'M^{1/2})$$

so that (14) becomes

$$\mathrm{pdf}(y) = \left[\Gamma\left(\frac{q}{2}\right)\right]^{-1} y^{q/2-1}\left[\det(E\partial XE')^{1/2}\,{}_0F_0(-E\partial XE',\ y)\det(I-X)^{-N/2}\right]_{X=0} \tag{21}$$

where $E = (F\Omega F')^{-1/2}F\Omega^{1/2}$. Notice that $EE' = I_q$. We therefore introduce an $(n-q) \times n$ matrix K such that $P' = [E' \,\vdots\, K']$ is orthogonal. Transforming $X \to PXP' = Z$ so that $\partial X = P'\partial ZP$ we find

$$(22)\qquad \mathrm{pdf}(y) = \left[\Gamma\left(\frac{q}{2}\right)\right]^{-1} y^{q/2-1}\left[(\det \partial Z_{11})^{1/2}\,{}_0F_0(-\partial Z_{11},\ y)\det(I - Z_{11})^{-N/2}\right]_{Z_{11}=0}$$

where Z_{11} is the leading $q \times q$ submatrix of Z. Using the rules of fractional differentiation proved in an earlier article [8], we have

$$(23)\qquad (\det \partial Z_{11})^{1/2}\det(I - Z_{11})^{-N/2} = \frac{\Gamma_q\left(\frac{N+1}{2}\right)}{\Gamma_q\left(\frac{N}{2}\right)}\det(I - Z_{11})^{-(N+1)/2}$$

and

$$(24)\qquad {}_0F_0(-\partial Z_{11},\ y)\det(I - Z_{11})^{-(N+1)/2}\Big|_{Z_{11}=0}$$

$$= \int_{O(q)} \det(I + yHe_1e_1'H')^{-(N+1)/2}(\underline{dH})$$

$$= (1+y)^{-(N+1)/2} .$$

Hence from (22), (23) and (24) we obtain

$$\mathrm{pdf}(y) = \frac{y^{q/2-1}\Gamma_q\left(\frac{N+1}{2}\right)}{\Gamma\left(\frac{q}{2}\right)\Gamma_q\left(\frac{N}{2}\right)(1+y)^{(N+1)/2}}$$

$$= \frac{\Gamma\left(\frac{N+1}{2}\right)y^{q/2-1}}{\Gamma\left(\frac{q}{2}\right)\Gamma\left(\frac{N-q+1}{2}\right)(1+y)^{(N+1)/2}}$$

$$(25)\qquad = \frac{y^{q/2-1}}{B\left(\frac{q}{2},\ \frac{N-q+1}{2}\right)(1+y)^{(N+1)/2}} .$$

Upon transformation, we deduce that

$$F = \frac{N-q+1}{q}\,y \equiv F_{q,N-q+1} .$$

6. CONSTANTINE'S DISTRIBUTION FOR THE GENERALIZED T_0^2 STATISTIC

Let $D = F \otimes G'$ where F is an $f \times n$ matrix of rank $f \leq n$ and G is an $m \times g$ matrix of rank $g \leq m$. Suppose the null hypothesis has the form

$$(26) \qquad H_0 : FAG = 0 .$$

The Wald statistic may in this case be written as:

$$(27) \qquad W = \mathrm{tr}\{FA^*G(G'MG)^{-1}(FA^*G)'(F\Omega F')^{-1}\}$$
$$= N\ \mathrm{tr}(S_1 S_2^{-1})$$

where $S_1 = FA^*G(G'MG)^{-1}(FA^*G)'$, $S_2 = FCF'$. Writing $S_1 = KK'$ where $K = FA^*G(G'MG)^{-1/2}$ has columns which are i.i. $N(0, F\Omega F')$ under H_0 we see that S_1 has a (possibly singular) central Wishart distribution with covariance matrix $F\Omega F'$ and g degrees of freedom. S_2 is independent Wishart with the same covariance matrix and N degrees of freedom. Thus W in (27) is proportional to Hotelling's generalized T_0^2 statistic. We will consider the distribution of $y = W/N$, corresponding to Constantine's [1] statistic T.

The exact distribution of y may be deduced from our general expression (14) with the matrix L given in this case by:

$$(28) \qquad L = (F\Omega F')^{-1/2}F\Omega^{1/2} \otimes (G'MG)^{-1/2}G'M^{1/2} .$$

This leads to

$$(29) \qquad \mathrm{pdf}(y) = \frac{y^{q/2-1}}{\Gamma\left(\frac{q}{2}\right)}\left[\det(E\partial XE')^{g/2}\,{}_0F_0(-E\partial XE'\otimes I,\ y)\det(I-X)^{-N/2}\right]_{X=0} .$$

Transforming auxiliary variates as in the argument following (21) we obtain

$$(30) \qquad \mathrm{pdf}(y) = \left[\Gamma\left(\frac{q}{2}\right)\right]^{-1} y^{q/2-1}\left[(\det\ \partial Z_{11})^{g/2}\,{}_0F_0(-\partial Z_{11}\otimes I,\ y)\det(I - Z_{11})^{-N/2}\right]_{Z_{11}=0} .$$

This provides a natural generalization of the density (22). Applying the operator $(\det\ \partial Z_{11})^{g/2}$ we have

$$(31) \qquad \mathrm{pdf}(y) = \frac{y^{q/2-1}\Gamma_f\left(\frac{N+g}{2}\right)}{\Gamma\left(\frac{q}{2}\right)\Gamma_f\left(\frac{N}{2}\right)}\left[{}_0F_0(-\partial Z_{11}\otimes I,\ y)\det(I - Z_{11})^{-(N+g)/2}\right]_{Z_{11}=0}$$

$$= \frac{y^{q/2-1}\Gamma_f\left(\frac{N+g}{2}\right)}{\Gamma\left(\frac{q}{2}\right)\Gamma_f\left(\frac{N}{2}\right)}\left[\int_{V_{1,fg}} \mathrm{etr}\left(-y(\partial Z_{11}\otimes I)hh'\right)\underline{(dh)}\det(I - Z_{11})^{-(N+g)/2}\right]_{Z_{11}=0}$$

$$(32) \qquad = \frac{y^{q/2-1}\Gamma_f\left(\frac{N+g}{2}\right)}{\Gamma\left(\frac{q}{2}\right)\Gamma_f\left(\frac{N}{2}\right)}\int_{V_{1,fg}} \det(I + yQ)^{-(N+g)/2}\underline{(dh)}$$

where $V_{1,fg}$ denotes the Stiefel manifold $h'h = 1$ where h is $fg\times 1$, $\underline{(dh)}$ denotes the normalized Haar measure on this manifold and

$$(33) \qquad Q = \sum_{i=1}^{g} h_i h_i'$$

where the h_i are $f\times 1$ vectors taken from the partition of $h' = (h_1', h_2', \ldots, h_g')$ into g component vectors.

The integral in (32) may be evaluated as follows

$$(34) \qquad \frac{1}{2}\Gamma\left(\frac{fg}{2}\right)\int_{V_{1,fg}} \det(I + yQ)^{-(N+g)/2}(\underline{dh})$$

$$= \frac{1}{2}\left(\frac{2\pi^{fg/2}}{\Gamma\left(\frac{fg}{2}\right)}\right)^{-1}\int_{V_{1,fg}} \det(I + yQ)^{-(N+g)/2}(dh)\int_0^\infty e^{-r}\, r^{(fg-2)/2}dr$$

where (dh) represents the unnormalized Haar measure on $V_{1,fg}$. Now note that if x is any $fg \times 1$ vector we may write $x = hr^{1/2}$ where $r = x'x$ and h lies in $V_{1,fg}$. The measure changes according to

$$dx = \frac{1}{2}r^{(fg-2)/2}dr(dh) \ .$$

Under this transformation we may write (34) as

$$(35) \qquad \frac{\Gamma\left(\frac{fg}{2}\right)}{2\pi^{fg/2}}\int_{\mathbb{R}^{fg}} \det\left(I + yR/tr(R)\right)^{-(N+g)/2}e^{-x'x}dx$$

where $R = \sum_{i=1}^{g} x_i x_i'$ and the partition of $x' = (x_1', x_2', \ldots, x_g')$ corresponds to that of h given earlier.

We now assume that $g \geq f$ and will show later how this condition may be relaxed. Writing $X = [x_1, x_2, \ldots, x_g]$ we transform $X \to (R,V)$ by using the decomposition $X' = VR^{1/2}$, where $R = XX'$ and V lies in the Stiefel manifold $V_{f,g}$, the collection of all f frames of orthonormal g-vectors. The measure changes according to

$$(36) \qquad dX = 1/2^f(\det\ R)^{(g-f-1)/2}dR(dV)$$

where (dV) denotes the invariant measure on $V_{f,g}$.

(35) then takes the form

$$\frac{\Gamma\left(\frac{fg}{2}\right)}{2^{f+1}\pi^{fg/2}} \int_{R>0} \det\left(I + yR/tr(R)\right)^{-(N+g)/2} etr(-R)(\det R)^{(g-f-1)/2} dR \int_{V_{f,g}} (dV)$$

$$(37) \quad = \frac{\Gamma\left(\frac{fg}{2}\right)}{2\Gamma_f\left(\frac{g}{2}\right)} \int_{R>0} etr(-R)(\det R)^{(g-f-1)/2} \det\left(I + yR/tr(R)\right)^{-(N+g)/2} dR \ .$$

Taking $0 \leq y < 1$ we may expand

$$\det(I + yR/tr\ R)^{-(N+g)/2} = \sum_{k=0}^{\infty} \frac{(-y)^k}{k!} \sum_{\kappa} \left(\frac{N+g}{2}\right)_{\kappa} C_{\kappa}(R)(tr\ R)^{-k}$$

and integrating term by term in (37) we obtain

$$\frac{\Gamma\left(\frac{fg}{2}\right)}{2\Gamma_f\left(\frac{g}{2}\right)} \sum_{k=0}^{\infty} \frac{(-y)^k}{k!} \sum_{\kappa} \left(\frac{N+g}{2}\right)_{\kappa} \int_{R>0} etr(-R)(\det R)^{(g-f-1)/2} (tr\ R)^{-k} C_{\kappa}(R) dR$$

$$= \frac{\Gamma\left(\frac{fg}{2}\right)}{2\Gamma_f\left(\frac{g}{2}\right)} \sum_{k=0}^{\infty} \frac{(-y)^k}{k!} \sum_{\kappa} \left(\frac{N+g}{2}\right)_{\kappa} [\Gamma(k)]^{-1} \int_0^{\infty} s^{k-1} \int_{R>0} etr(-(1+s)R)$$

$$\cdot (\det R)^{(g-f-1)/2} C_{\kappa}(R) dR ds$$

$$= \frac{\Gamma\left(\frac{fg}{2}\right)}{\Gamma_f\left(\frac{g}{2}\right)} \sum_{k=0}^{\infty} \frac{(-y)^k}{k!} \sum_{\kappa} \left(\frac{N+g}{2}\right)_{\kappa} \Gamma_f\left(\frac{g}{2}, \kappa\right) [\Gamma(k)^{-1} \int_0^{\infty} s^{k-1} C_{\kappa}\left((1+s)^{-1} I_f\right)\left(\det((1+s)I_f)\right)^{-g/2} ds$$

$$= \frac{1}{2}\Gamma\left(\frac{fg}{2}\right) \sum_{k=0}^{\infty} \frac{(-y)^k}{k!} \sum_{\kappa} \left(\frac{N+g}{2}\right)_{\kappa} \left(\frac{g}{2}\right)_{\kappa} C_{\kappa}(I_f) [\Gamma(k)]^{-1} \int_0^{\infty} s^{k-1} (1+s)^{-fg/2-k} ds$$

$$(38) \quad = \frac{1}{2}\Gamma\left(\frac{fg}{2}\right) \sum_{k=0}^{\infty} \frac{(-y)^k}{k!(fg/2)_k} \sum_{\kappa} \left(\frac{N+g}{2}\right)_{\kappa} \left(\frac{g}{2}\right)_{\kappa} C_{\kappa}(I_f) \ .$$

It follows from (34) and (38) that

$$(39) \qquad \int_{V_{1,fg}} \det(I + yQ)^{-(N+g)/2}(\underline{dh})$$

$$= \sum_{k=0}^{\infty} \frac{(-y)^k}{k!(fg/2)_k} \sum_{\kappa} \left(\frac{N+g}{2}\right)_{\kappa} \left(\frac{g}{2}\right)_{\kappa} C_{\kappa}(I_f)$$

for $0 \leq y < 1$.

Using (39) in (32) and noting that $q = fg$, we find that

$$(40) \qquad \mathrm{pdf}(y) = \frac{y^{fg/2-1}\Gamma_f\left(\frac{N+g}{2}\right)}{\Gamma\left(\frac{fg}{2}\right)\Gamma_f\left(\frac{N}{2}\right)} \sum_{k=0}^{\infty} \frac{(-y)^k}{k!(fg/2)_k} \sum_{\kappa} \left(\frac{N+g}{2}\right)_{\kappa} \left(\frac{g}{2}\right)_{\kappa} C_{\kappa}(I_f) \ , \quad 0 \leq y < 1 \ .$$

Upon notational translation this is precisely the result obtained by Constantine [1]. This series converges as stated in the interval $0 \leq y < 1$. An alternative everywhere convergent series representation of the density is derived in a companion paper that the author has currently under preparation.

When $g < f$ the argument following (35) is reworked as before using $\det(I_g + yX'X/x'x)^{-(N+g)/2}$ in place of $\det(I_f + yXX'/x'x)^{-(N+g)/2}$. We also note that in this case

$$\frac{\Gamma_f\left(\frac{N+g}{2}\right)}{\Gamma_f\left(\frac{N}{2}\right)} = \frac{\Gamma_g\left(\frac{N+g-f+f}{2}\right)}{\Gamma_g\left(\frac{N+g-f}{2}\right)} \ .$$

The density of y when $g < f$ is then given by (40) after the notational changes

$$f \to g \ , \quad g \to f \ , \quad N \to N+g-f \ .$$

This also corresponds with Constantine's result in [1].

7. HOTELLING'S DISTRIBUTION FOR THE GENERALIZED T_0^2 STATISTIC WHEN $f = 2$

For the special case in which $f = 2$ Hotelling [2] found an everywhere convergent power series representation of the density of y. This representation may also be deduced from our general expression (32).

In fact, taking $f = 2$, $q = 2g$ and $g \geq 2$ (otherwise the results of Section 5 apply) we have

$$\mathrm{pdf}(y) = \frac{y^{g-1}\Gamma_2\left(\frac{N+g}{2}\right)}{\Gamma(g)\Gamma_2\left(\frac{N}{2}\right)} \int_{V_{1,2g}} \det(I + yQ)^{-(N+g)/2}(\underline{dh}) \ .$$

We expand $\det(I + yQ) = (1 + y + y^2 \det Q)$ and transform variates as in Section 6 to obtain

$$\mathrm{pdf}(y) = \frac{\Gamma_2\left(\frac{N+g}{2}\right)y^{g-1}(1+y)^{-(N+g)/2}}{\Gamma(g)\Gamma_2\left(\frac{N}{2}\right)4\pi^g} \int_{V_{2,g}} (dV)$$

$$\cdot \int_{R>0} \mathrm{etr}(-R)(\det R)^{(g-f-1)/2}\left\{1 + \frac{y^2}{1+y}\,\frac{\det R}{(\mathrm{tr}\ R)^2}\right\}^{-(N+g)/2} dR$$

$$= \frac{\Gamma_2\left(\frac{N+g}{2}\right)y^{g-1}(1+y)^{-(N+g)/2}}{\Gamma(g)\Gamma_2\left(\frac{N}{2}\right)\Gamma_2\left(\frac{g}{2}\right)\Gamma\left(\frac{N+g}{2}\right)} \int_{R>0} \mathrm{etr}(-R)$$

$$\cdot(\det R)^{(g-f-1)/2}\int_0^\infty e^{-t} \sum_{j=0}^\infty \frac{(-1)^j}{j!}\left(\frac{y^2}{1+y}\right)^j \frac{(\det R)^j}{(\mathrm{tr}\ R)^{2j}}\, t^{(N+g)/2+j-1}dt\ dR \ .$$

Term by term integration is justified by the absolute and uniform convergence of the series. We find

$$\mathrm{pdf}(y) = \frac{\Gamma_2\left(\frac{N+g}{2}\right) y^{g-1}(1+y)^{-(N+g)/2}}{\Gamma(g)\Gamma_2\left(\frac{N}{2}\right)\Gamma_2\left(\frac{g}{2}\right)}$$

$$\cdot \sum_{j=0}^{\infty} \frac{(-1)^j}{j!}\left(\frac{y^2}{1+y}\right)^j \left(\frac{N+g}{2}\right)_j \int_{R>0} \mathrm{etr}(-R)(\mathrm{tr}\ R)^{-2j}(\det\ R)^{(g-3)/2+j} dR$$

$$= \frac{\Gamma_2\left(\frac{N+g}{2}\right) y^{g-1}(1+y)^{-(N+g)/2}}{\Gamma(g)\Gamma_2\left(\frac{N}{2}\right)\Gamma_2\left(\frac{g}{2}\right)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!}\left(\frac{y^2}{1+y}\right)^j \left(\frac{N+g}{2}\right)_j$$

$$\cdot \int_{R>0} \frac{1}{\Gamma(2j)} \int_0^{\infty} \mathrm{etr}\Big(-(1+s)R\Big) s^{2j-1} ds (\det\ R)^{(g-3)/2+j} dR$$

$$= \frac{\Gamma_2\left(\frac{N+g}{2}\right) y^{g-1}(1+y)^{-(N+g)/2}}{\Gamma(g)\Gamma_2\left(\frac{N}{2}\right)\Gamma_2\left(\frac{g}{2}\right)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!}\left(\frac{y^2}{1+y}\right)^j$$

$$\cdot \frac{\left(\frac{N+g}{2}\right)_j \Gamma_2\left(\frac{g}{2}+j\right)}{\Gamma(2j)} \int_0^{\infty} s^{2j-1}(1+s)^{-g+2j} ds$$

$$= \frac{\Gamma_2\left(\frac{N+g}{2}\right) y^{g-1}(1+y)^{-(N+g)/2}}{\Gamma_2\left(\frac{N}{2}\right)\Gamma_2\left(\frac{g}{2}\right)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!}\left(\frac{y^2}{1+y}\right)^j \frac{\left(\frac{N+g}{2}\right)_j \Gamma_2\left(\frac{g}{2}+j\right)}{\Gamma(g+2j)}$$

$$= \frac{\Gamma_2\left(\frac{N+g}{2}\right) y^{g-1}(1+y)^{-(N+g)/2} \pi\Gamma\left(\frac{g-1}{2}\right)}{2^{g-1}\Gamma_2\left(\frac{N}{2}\right)\Gamma_2\left(\frac{g}{2}\right)\Gamma\left(\frac{g+1}{2}\right)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!}\left(\frac{y^2}{4(1+y)}\right)^j \frac{\left(\frac{N+g}{2}\right)_j \left(\frac{g-1}{2}\right)_j}{\left(\frac{g+1}{2}\right)_j}$$

$$= \frac{\Gamma(N+g-1) y^{g-1}(1+y)^{-(N+g)/2}}{2^g\Gamma(g)\Gamma(N-1)}\ {}_2F_1\left(\frac{g-1}{2}, \frac{N+g}{2}; \frac{g+1}{2}; \frac{-y^2}{4(1+y)}\right)$$

$$= \frac{(N+g-1) y^{g-1}(1+y)^{-(N+g)/2}}{2^g\Gamma(g)\Gamma(N-1)}\left[\frac{(2+y)^2}{4(1+y)}\right]^{-(N+g)/2} {}_2F_1\left(1, \frac{N+g}{2}; \frac{g+1}{2}; \left(\frac{y}{2+y}\right)^2\right)$$

$$= \frac{2^N\Gamma(N+g-1) y^{g-1}}{\Gamma(g)\Gamma(N-1)(2+y)^{N+g}}\ {}_2F_1\left(1, \frac{N+g}{2}; \frac{g+1}{2}; \left(\frac{y}{2+y}\right)^2\right)$$

$$(41) \qquad = \frac{\Gamma(N+g-1)(y/2)^{g-1}}{2\Gamma(g)\Gamma(N-1)(1+y/2)^{N+g}} \cdot {}_2F_1\left(1, \frac{N+g}{2}; \frac{g+1}{2}; \left(\frac{y}{2+y}\right)^2\right) .$$

This differs from Constantine's [1] formula on page 223 by a factor of 1/2. In fact, Constantine's expression is in error as can be verified by integrating (41) to unity.

8. THE REGRESSION F STATISTIC

When $f = 1$ the hypothesis (26) becomes the linear hypothesis in multiple regression. In particular let $F = e_1'$, $G = R'$ and a' be the first row of A . Then (26) is simply $H_0 : Ra = 0$. The distribution of y given by (32) reduces immediately to

$$\text{pdf}(y) = \left[B\left(\frac{g}{2}, \frac{N}{2}\right)\right]^{-1} y^{g/2-1}(1+y)^{-(N+g)/2}$$

so that

$$F = Ny/g \equiv F_{g,N}$$

as in standard regression theory.

9. CONCLUSION

This paper provides a comprehensive and unified treatment of the distribution of the Wald statistic for testing general linear hypotheses in the multivariate linear model. All presently known null distributions may be obtained from the general formulae (14) and (32) given here. The noncentral distributions are left for subsequent work.

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