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A MEAN-VARIANCE APPROACH TO FUNDAMENTAL VALUATIONS

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A Mean-Variance Approach to Fundamental Valuations

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Here I consider a risk asset to be an equity title to the stream of earnings from a unit of physical capital. The "fundamental valuation" of such an asset is the valuation of those earnings. It excludes speculative capital gains and losses, arising from variations of market prices of the equities. The fundamental valuation takes account of earnings whether or not distributed as dividends. Reinvestment of retained earnings may be reflected, of course, in appreciation of actual equity issues, but such retention is regarded as equivalent to the issue of new shares. That is, a share here is always title to one unit of capital at replacement cost. Fundamental valuations will depend on the means, variances, and covariances of the joint probability distributions of the earnings per shares of the various risk assets; on their supplies relative to each other and to the safe asset; and on the return available from the safe asset.

I certainly would be the last person to assert that asset markets in fact generate fundamental valuations. The speculative content of market prices is all too apparent in their excessive volatility.¹ Keynes's classic description of equity markets as casinos where assessments of long-term investment prospects are overwhelmed by frantic short-term guesses about what average opinion will think average opinion will think — and so on, to

the nth degree -- rings as true today as when he wrote it.²

Indeed, it is a decisive reason to be skeptical of the accepted Capital Asset Pricing Model as it is generally implemented. The empirical joint probability distributions of asset returns, inclusive of capital gains, contain so much speculative noise that the betas and other parameters estimated from them cannot be expected to continue to hold in the future. CAPM, it seems to me, is a "bootstrap" explanation of asset prices, wherein prices are supposed to be derived from movements in the very same prices.

The exercise of fundamental valuation focuses on the basic returns that are available to portfolio owners and the basic risks that they must somehow assume, and asks at what prices these supplies will be matched by demand. Everything else is a zero-sum game, in which some investors win and others lose. The speculative game itself doubtless adds, from the standpoint of any individual investor, to the inescapable underlying risks. A theory of fundamental valuation, therefore, is a beginning, but only a beginning, to a story of the determination of market prices.

The key to the simple approach to be discussed in this paper is a very primitive observation. Expected earnings and risks on equities, as defined above, are given to the economy as a whole, independently of the asset prices. But the expected returns and risks of any individual investor depend on those prices. If R_i is the stochastic earning on an equity and its price is q_i then the return to an investor is $r_i = R_i/q_i$. The higher is q_i , the lower will be the expected value of r_i and its variance, and the absolute value of its covariance with any r_j .

I begin with a single risk asset and consider the determination of its value in terms of a single safe asset. The relevance of this model is justified in the subsequent section, where the familiar separation theorem, reinterpreted, is shown to hold. The market portfolio of numerous risk assets takes the place of the single risk asset in the simpler model. In the process, the q_i for the individual risk assets are related to the q for the market portfolio.

Valuation of a single risk asset

A safe asset is available to the market in amount s and bears a return per unit of a . The supply of the risk asset, capital at replacement cost normalized to be 1 in terms of the safe asset, is z . Without loss of generality, the supplies s and z add to 1. The risk asset will earn R per unit, with expected value \bar{R} and variance v^2 . The price (fundamental valuation here) of the risk asset is q . The representative individual investor has wealth of $s+qz$. She is free to determine the proportions $(1-x, x)$ in which to divide her wealth between the safe and risk assets, respectively. Her return on the safe asset is a , and on the risk asset $r = R/q$. Her expectation of return on the risk asset is $\bar{r} = \bar{R}/q$, and its standard deviation is $\sigma_r = v/q$. Thus her portfolio return is $e = a+x(r-a)$, with expected value $\bar{e} = a+x(\bar{r}-a)$, and its standard deviation is $\sigma_e = x\sigma_r$. In supply-demand equilibrium, her choice of x must result in her holding of the basic supplies, s

and z , of the two assets. The following conditions must be met:

$$(1a) \quad \bar{e} = x\bar{r} + (1-x)a = x\bar{R}/q + (1-x)a$$

$$(1b) \quad \sigma_e = x\sigma_r = xv/q$$

$$(2) \quad \bar{e} = a + (\sigma_e/\sigma_r)(\bar{r}-a) = a + (q/v)(\bar{R}/q - a)$$

This is the budget line for the individual investor. As her

choice x varies from 0 to 1, (\bar{e}, σ_e) goes from $(a, 0)$ to

$(\bar{r}, \sigma_r) = (\bar{R}/q, v/q)$. Its slope is $(\bar{r}-a)/\sigma_r$ or $(\bar{R}-aq)/v$.

There is a family of budget lines, fanning out from $(a, 0)$; they

depend on q , with greater slope for lower q . Members of this

family are depicted in Figure 1. An upper limit on q is \bar{R}/a ,

for which the line is horizontal. The budget line points for $x=1$

-- let us call them "termini" even though it is conceivable for

investors to go short in the safe asset -- all lie on a ray from

the origin, $\bar{e} = \sigma_e \bar{R}/v$, also shown in Figure 1. As q declines,

the points on this terminal locus are farther from the origin;

indeed this distance is proportional to q^{-1} . The lower limit of

q is zero, at which the slope of the budget line is the same as

that of the terminal locus ray.

In equilibrium of demand and supply, the representative

investor's choice $(x, 1-x)$ must correspond to the supplies of

the two assets (z, s) . Since $x = qz/(s+qz)$ and $s = 1-z$, the

following relationship holds: $x/(1-x) = qz/(1-z)$, implying

$$q = z(1-x)/x(1-z)$$

This locates the point on each member of

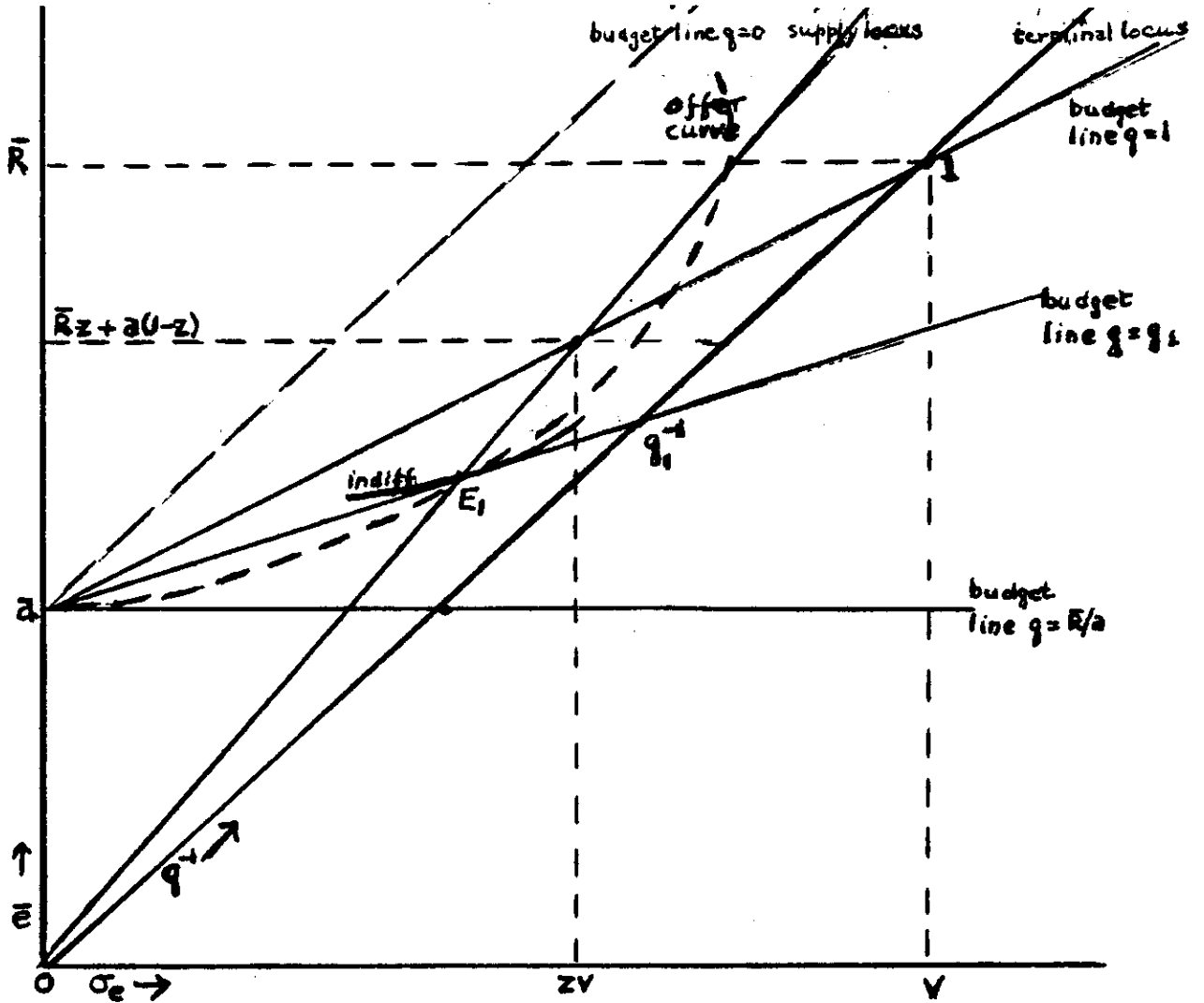
the family of budget lines at which the two assets are held in

proportions that absorb their given supplies. The locus of these

points in (\bar{e}, σ_e) space may be found by substituting the above

Figure 1

Valuation of the risk asset in supply=demand equilibrium.



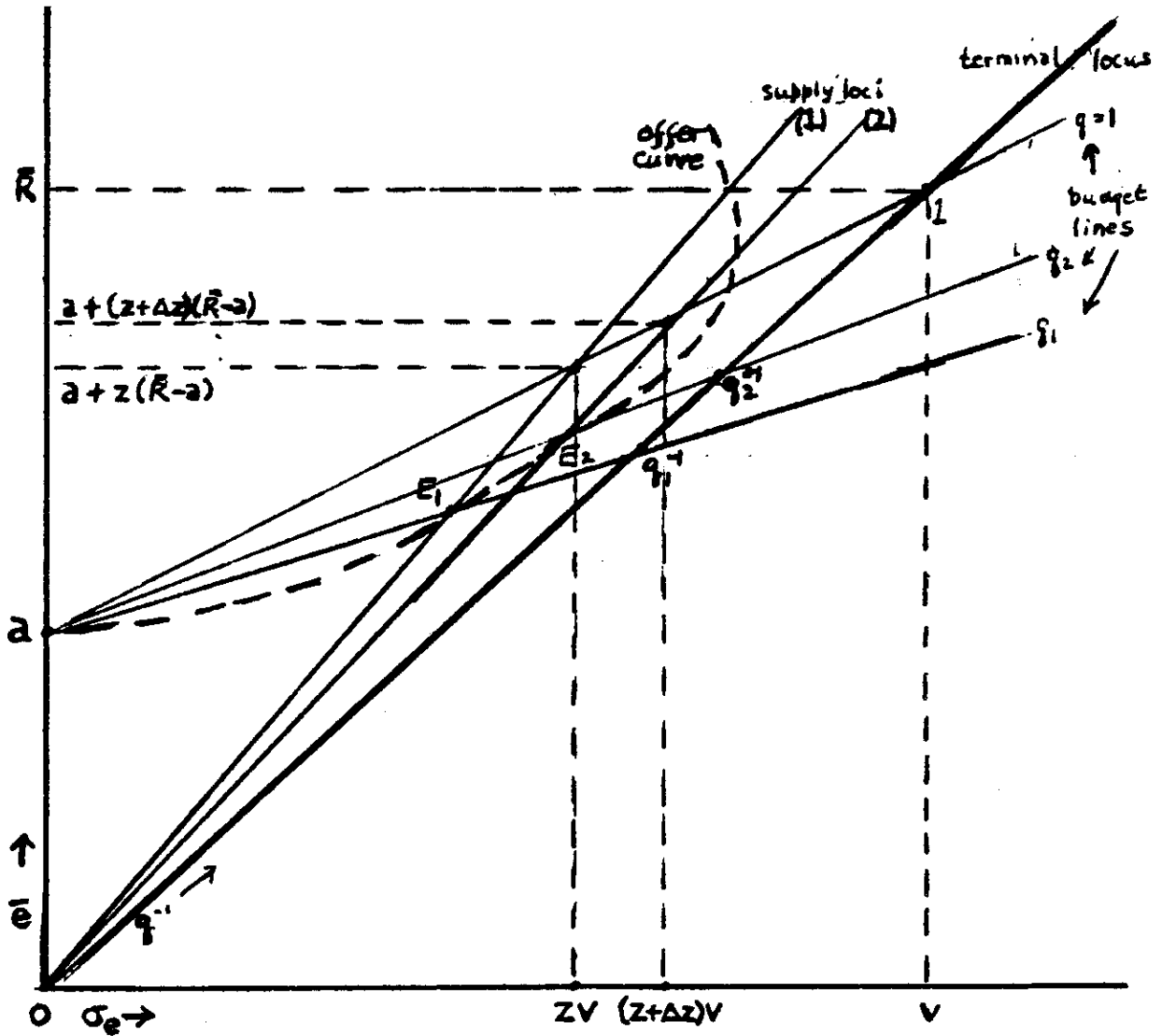
The stable equilibrium is at E_1 , the lower intersection of the offer curve and the supply locus. There an indifference curve is tangent to the budget line corresponding to $q=q_1^{-1}$. The value of q_1^{-1} is proportional to distance from the origin along the terminal locus. E_1 divides the budget line distance from a to the terminal locus (the point marked q_1^{-1}) in the proportions $x, 1-x$, such that $x/(1-x) = qz/(1-z)$.

expression for q into (1a) and (1b). The substitution eliminates x and determines a locus which is a line from the origin through the point $(z\bar{R}+(1-z)a, zv)$ which corresponds to $q=1$. This "supply locus" is shown in Figure 1. Geometrically z and $1-z$ are always proportional to the distances zv and $v-zv$ along the horizontal axis. But portfolio shares x and $1-x$ depend on q ; they are proportional to the distances along a budget line from the vertical axis to the supply locus and from the supply locus to the terminal locus. The equilibrium value of q corresponds to an intersection of the supply locus and a budget line where the allocation $(x, 1-x)$ is desired by a representative portfolio manager. Graphically, as exemplified in Figure 1, such an intersection will be a point where a return-risk (\bar{e}, σ_e) indifference curve is tangent to the budget line.

An investor's offer curve, a locus of such tangencies, may be derived in a standard way, showing essentially a curve of demand for risk e at various values of q . The offer curve depends on the sure return a . At the upper limit $q=\bar{R}/a$, the curve starts at $(a, 0)$. As q declines and the price of risk increases from zero, the curve is upward sloping. At some point, as the income effects of an increase in the price of risk become important, the curve may bend backwards, as illustrated in Figure 1. In any case, there may be two intersections with the supply locus in its relevant range. The lower intersection, E_1 in the diagram, is stable in the following sense: If q is lower than q_1 then demand for the risk asset exceeds supply; excess demand raises q . The symmetric argument applies for a positive

Figure 2

Comparative statics: effect of increase in supply of risk asset.



The supply locus is rotated clockwise by an increase Δz in the supply of the risk asset relative to the safe asset. The equilibrium moves from E_1 to E_2 . The new tangency is on the budget line corresponding to q_2 , which is smaller than q_1 , as indicated by its greater distance up the unchanged terminal locus ray.

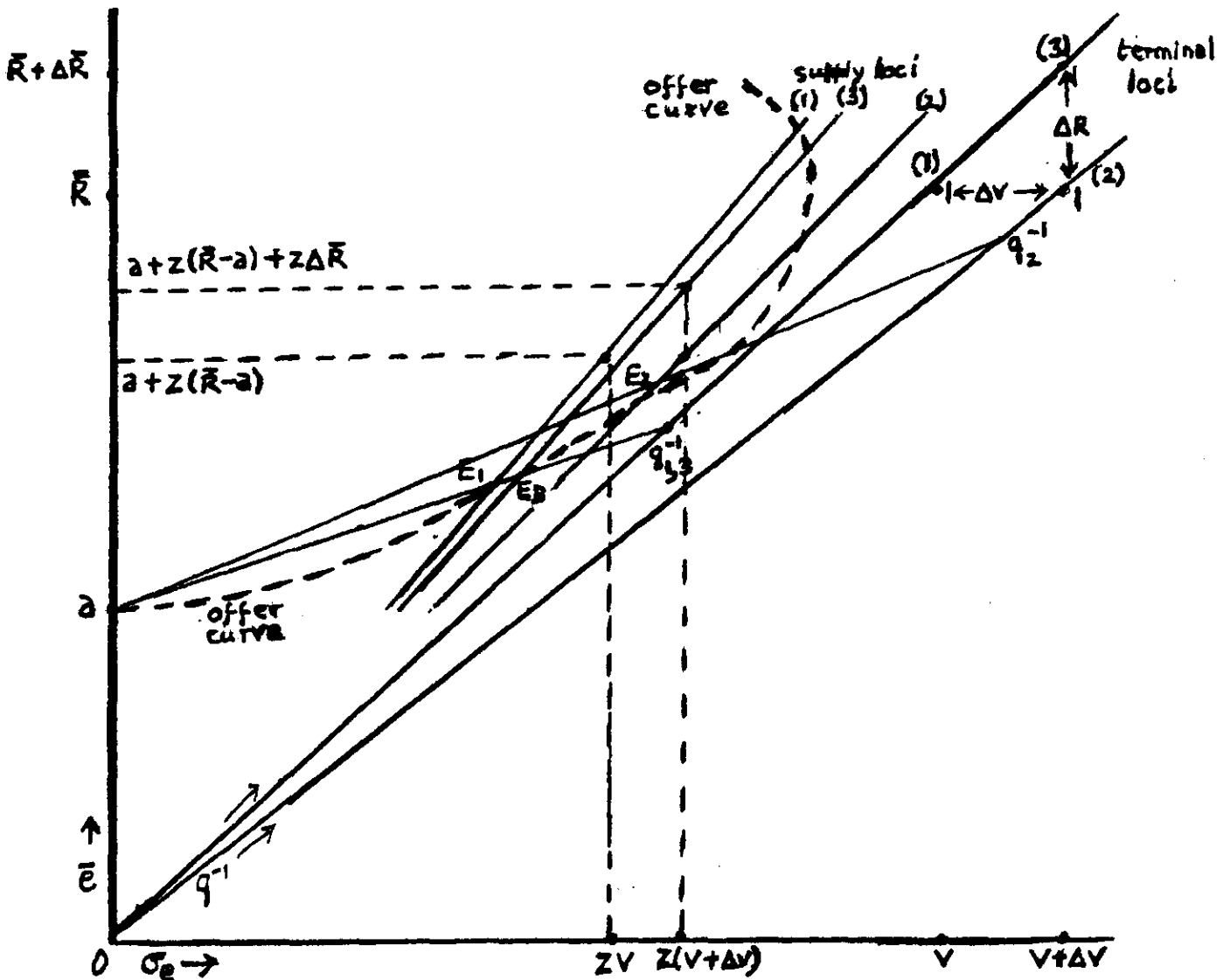
deviation from q_1 . By the same argument, the second intersection is not stable.

The comparative statics are shown graphically in Figures 2-3. First, an increase in z , i.e. in the supply of the risk asset relative to that of the safe asset, rotates the supply curve to the right and lowers q . Second, an increase in v , the riskiness of equity, would likewise lower q , as implied by movement from E1 to E2 in Figure 2. Third, an increase in \bar{R} will have the opposite effect. It will raise q ; the equilibrium moves to E3 in Figure 3. Indeed, as shown, proportionate increases in \bar{R} and v will raise q . Fourth, but not shown diagrammatically, an increase in the sure return a will lower the demand for risk and reduce q . Both the offer curve and the supply locus shift up; the budget line through their intersection hits the terminal locus, unchanged, farther from the origin.

All the comparative static results are straightforward and consistent with intuition.

A situation in which q differs from 1 is a short run equilibrium, though the short run may be long in real time. Deviations of fundamental value from replacement cost, if

Figure 3
Comparative statics: effects of changes in risk and return.



The figure diagrams two exercises. The initial situation (1) has equilibrium E_1 and $q = q_1$. In situation (2) the risk on the risk asset is increased by Δv . The terminal locus and the supply locus are both rotated clockwise, and in the new equilibrium E_2 q is lowered to q_2 . In situation (3)

\bar{R} has also been increased, in the same proportion as v . Terminal locus (3) coincides with that of (1), but the point $q=1$ has moved out in proportion to the increases in R and v . Supply locus (3) is between the other two. In E_3 q is higher than in the other two equilibria. (The budget lines for (1) and (3) coincide in the diagram, but this is not a geometric necessity.)

realized in the market or even expected to be realized, will be incentives for investment in the underlying real capital assets, or for disinvestment. By raising relative supply z , lowering expected earnings \bar{R} , and possibly increasing risk v , positive net investment would over time correct a positive premium in q . Another avenue of adjustment would be monetary policy lowering s or raising a or both.

Multiple risk assets

Extension to the case of one safe asset and many risk assets is straightforward, because "separation" applies in this framework as well as in the familiar CAPM. Here the supply of risk asset i ($i = 1, 2, \dots, n$) is z_i at replacement cost. Its earnings per unit are R_i with mean \bar{R}_i and with variances and covariances v_{ij} ($j = 1, 2, \dots, n$). The total supply of risk assets $\sum_i z_i$ is denoted z as before; likewise the safe asset is in supply s with return a ; and $s + z = 1$. The total return on the market supplies of risk assets is $R = \sum_i z_i R_i / z$. Its expected value is \bar{R} , and its variance is $v^2 = \sum_{ij} z_i z_j v_{ij} / z^2$.

Let q_i be the valuation of asset i in terms of the safe asset. To the individual investor the rate of return is $r_i = R_i / q_i$, the mean return is $\bar{r}_i = \bar{R}_i / q_i$, and the variance-covariance matrix is $\sigma_{ij} = v_{ij} / q_i q_j$. The investor's problem is to determine, given the q_i , the portfolio shares x_i . Let the sum of the x_i be x , as before. Let q be the weighted average of the q_i , such that

$qz = \sum_i q_i z_i$. Thus $x_i = q_i z_i / (s + qz)$ and $x = qz / (s + qz)$ as before.

The solution to the investor's problem can be separated into two parts: (1) the determination of the risk portfolio x_i/x , and (2) the determination of x , the fraction of the total portfolio at market value to be placed in the risk portfolio. The risk portfolio depends on the r_i , σ_{ij} , and on a . The second part of the problem was discussed in the preceding section of the paper.

In considering the first problem it is convenient to define \hat{x}_i as x_i/x , \hat{z}_i as z_i/z , and \hat{q}_i as q_i/q . Then we have the following identities:

$$\begin{aligned}
 \hat{x}_i &= \hat{q}_i \hat{z}_i \\
 \bar{r}_i &= \bar{R}_i / \hat{q}_i q \\
 \bar{r} &= \sum_i \hat{x}_i \bar{r}_i = \sum_i \hat{z}_i \bar{R}_i / q = \bar{R} / q \\
 (3) \quad \sigma_{ij} &= v_{ij} / \hat{q}_i \hat{q}_j q^2 \\
 \sigma_r^2 &= \sum_{ij} \hat{x}_i \hat{x}_j \sigma_{ij} = \sum_{ij} \hat{z}_i \hat{z}_j v_{ij} / q^2 = v^2 / q^2 \\
 \bar{e} &= a + x(\bar{r} - a) = a + \sum_i x_i (\bar{r}_i - a) \\
 \sigma_e^2 &= x^2 \sigma_r^2 = \sum_{ij} x_i x_j \sigma_{ij}
 \end{aligned}$$

The investor's problem is to choose x_i to minimize σ_r^2 subject to achieving an arbitrary value of $\bar{e} - a$, or equivalently an arbitrary value of $x(\bar{r} - a)$.

The determining equations are:

$$\begin{aligned}
 \sum_j x_j \sigma_{ij} - \lambda (\bar{r}_i - a) &= 0 \quad (i=1, 2, \dots, n) \\
 (4) \quad \sum_i x_i (\bar{r}_i - a) &= \bar{e} - a = x(\bar{r} - a)
 \end{aligned}$$

Their solution yields the x_i and λ , which is equal to $\sigma_e^2 / (\bar{e}-a)$. That all the ratios x_i/x_j are independent of $\bar{e}-a$ is a familiar result. For the present purpose, it is convenient to exploit this result by dividing every equation by x , giving:

$$(5) \quad \begin{aligned} \sum_j \hat{x}_j \sigma_{ij} - \lambda(\bar{r}_i - a)/x &= 0 \quad (i=1,2,\dots,n) \\ \sum_i \hat{x}_i (\bar{r}_i - a) &= (\bar{e}-a)/x = \bar{r}-a \end{aligned}$$

These $n+1$ equations may be solved for the \hat{x}_i . Moreover, multiplying each of the first n equations by its x_i gives

$$(6) \quad \sum_j \hat{x}_i \hat{x}_j \sigma_{ij} - \lambda \hat{x}_i (\bar{r}_i - a)/x = 0 \quad (i=1,2,\dots,n)$$

Summing these equations and using the last equation of (5) gives

$$\lambda/x = \sigma_r^2 / (\bar{r}-a).$$

We can now introduce the supply side of the model with the help of the identities (3), transforming equations (5) into:

$$(7) \quad \sum_j z_j v_{ij} / q^2 = (\lambda/qz) (\bar{R}_i / \hat{q}_i q - a) \quad (i=1,2,\dots,n)$$

Likewise, substitutions into equations (6) and their summation give $\lambda/qz = v^2/q^2 (\bar{R}/q - a)$. Define β_i to be $\sum_j \hat{x}_j v_{ij} / v^2$. Then:

$$(8) \quad (\bar{R}_i/a - q_i) = \beta_i (\bar{R}/a - q)$$

Equation (8) says that the gap between q_i and \bar{R}_i/a is proportional to the gap between overall q and its maximum value

\bar{R}/a . The factor of proportionality is β_i . The specific gap is larger or smaller than the overall gap as β_i is larger or smaller than 1. Given the overall gap $\bar{R}/a - q$, and given \bar{R}_i , q_i will be larger the smaller is β_i . Note also that if $\bar{R}_i = \bar{R}$ and if $\beta_i = 1$ then q_i will equal q .

Here, in distinction to the betas of CAPM as usually interpreted and applied, the β_i depend solely on fundamentals. They are independent of asset market prices and market rates of return. The numerator is the covariance of the earnings on replacement cost of the specific asset with the earnings of the economy-wide portfolio. The economy-wide portfolio is the set of supplies of risk assets actually available, quantities not values. These must be held in the typical investor's portfolio, and that fact constrains the aggregate risk and return of the portfolio. The asset prices q_i are those that induce the typical investor to hold the given supplies.

It is important to be careful about what "separation" means and does not mean. The \hat{q}_i are not independent of q . However, equation (8) says that the size of above-defined "gap" for a specific asset, relative to the overall gap, depends only on the asset's beta. That, in turn, depends only on the fundamentals; relative supplies of risk assets and their variances and covariances. It does not depend on the total supply of risk assets z relative to the safe asset s . An exogenous reduction of s would leave equation (8) intact, although it would of course lower q . Of course, neither q nor the \hat{q}_i are independent of the return s on the safe asset.

Although the betas here are fundamental, they are not immutable. For example, an increase in the supply z_1 relative to supplies of other risk assets, or an increase in the variance of its return, will generally raise β_1 and lower q_1 and \hat{q}_1 . The supply-determined economy-wide portfolio varies over time. One systematic source of variation, mentioned in the previous section, is the response of investment to deviations of the q_1 from 1, and from each other. It is tempting to seek estimates of the β_1 in time series regressions using equation (8) or some variant, but this procedure may give misleading estimates if structural changes in the "fundamentals" have altered betas over the period of observation. Even so, the regression procedure is probably less unreliable for these fundamental β 's than for those of CAPM.

Problems not addressed

In this simple exposition I have obviously finessed many important and vexing problems. At the very beginning, I ruled out speculation, capital gains and losses. I will mention just a few others.

The holding period: I was careful not to specify it, preferring not to tackle sequential portfolio strategies, transactions and decision costs, expectations other than those involved in one-period joint probability distributions, and other messy complications.

Aggregation: I do not really believe in that convenient abstraction, the representative agent, beloved of theorists in

economics and finance. I do not really believe in the anthropomorphic image of "the market." Yet I used it in this paper in order to make some particular observations on asset price modeling. In practice investors differ in their estimates of expected returns, variances, and covariances, and in their attitudes towards risk. The supply constraints emphasized above apply in aggregate but not to each investor individually. One could in principle regard the mean-variance model as a description of the derivation of individual asset demand functions relating desired stocks of assets to their price vector. Then the aggregate demand functions would confront the economy-wide supplies. This procedure is analogous to our theory of competitive pricing of commodities. Single-agent models yield neater results, while leaving unexplained the formidable volumes of financial transactions.

Quantity constraints: Aggregation problems are especially acute once we drop the unrealistic assumption that portfolio managers can buy or sell all assets, safe or risky, in any desired quantity at prevailing prices. Short positions are often impossible, almost always constrained in size, and always costly. There are many reasons, including the prevalence of risk-seekers who, unafraid of risk themselves, are quite willing to share it with anyone who will lend to them.

Volatility of market prices: According to the model of this paper, fundamental valuations do not change unless fundamentals change. Earnings are stochastic, but current realizations should not affect values unless they contain information leading to revision of estimates of the relevant joint probability

distribution. If those estimates are based on a large past sample, and on long run projections, the latest daily, monthly, or annual data should move the priors very little. Actual volatility of market prices appears to be wildly disproportionate to the information content of current data. This is a puzzle for the conventional CAPM as well as for the model of this paper. It is another way of making the point at the beginning of the paper, that economists and finance theorists are unable to explain speculation and the transactions volumes associated with gambling.

1. Robert J. Shiller, "Do Stock Prices Move Too Much to be Justified by Subsequent Changes in Dividends?" American Economic Review, vol. 71, 1981, 421-436.

2. J.M. Keynes, The General Theory of Employment, Interest, and Money, New York: Harcourt Brace, 1936, Chapter XII, especially pp. 156-60.

3. If $a=0$ all budget lines, for whatever value of q , coincide with the terminal locus. The value of x determines only the location of the point on that ray for which $q=1$. Likewise the supply locus coincides with the terminal ray. An "offer curve" can be constructed as the locus of tangencies of indifference curves to rays from the origin, even though only one of these rays is a possible budget line. This curve will cross the supply locus from below at a point where an indifference curve is

tangent to that ray. If (\bar{e}, σ) are the coordinates of that

tangency, then $q = 1 - 1/z + v/\sigma e$, or equivalently

$q = 1 - 1/z + \bar{R}/\bar{e}$. This point, however, is an unstable equilibrium by the criterion in the text. The stable equilibrium is at the origin where, like the safe asset, the risk asset has zero return and risk--because q is infinite!