COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut 06520

COWLES FOUNDATION DISCUSSION PAPER NO. 706

Note: Cowles Foundation Discussion Papers are preliminary materials circulated to stimulate discussion and critical comment. Requests for single copies of a Paper will be filled by the Cowles Foundation within the limits of the supply. References in publications to Discussion Papers (other than mere acknowledgment by a writer that he has access to such unpublished material) should be cleared with the author to protect the tentative character of these papers.

THE COOPERATIVE FORM, THE VALUE AND THE ALLOCATION OF JOINT COSTS AND BENEFITS

M. Shubik

May 15, 1984

THE COOPERATIVE FORM, THE VALUE AND THE ALLOCATION OF JOINT COSTS AND BENEFITS*

by

M. Shubik

1. INTRODUCTION

In 1953 Lloyd Shapley published his elegant paper on the value solution to an n-person game in characteristic function form. In 1962 I suggested that Shapley's axioms could be reinterpreted in terms of accounting conventions and could be used to provide a means for devising incentive compatible cost assignments and internal pricing in a firm with decentralized decisionmaking.

The problem of the assignment of joint costs (and benefits) is one which has bedevilled accountants for many years. A reaction by a microeconomist oriented towards marginal analysis may be why bother to assign overheads or joint costs at all.[1] The reason for the different attitudes and perceived needs by accountants, economists, regulators, production managers, tax collectors, divisional vicepresidents and others is that they are all looking at the same institutional entity from different viewpoints.

*The research described in this paper was partially supported by a grant from ONR and was previously an IIASA Collaborative Paper.

[1]See Stigler (1966) for example in his textbook p. 165.

The accountant, among other things, wants the books to balance. He wants all costs allocated. Benefits, unless they can be translated directly into money, pose difficult problems and the convention of conservvatism more or less dictates that if you know that an item is of positive worth but that you cannot quantify its value, carry it out at zero or at a symbolic sum such as $1 for good will.

A tax accountant looks towards minimizing a specific evaluation, namely his clients' tax bill; a tax collector may try to maximize tax revenues collected. An economist advising on profit maximization from a given plant producing a joint product wants to make sure that the arbitrary assignment of joint costs or profits does not distort the profit maximization. The divisional vicepresident views this problem in terms of delegation. If there are joint products, can an internal control system be designed which enables him to delegate decisions to others who use only the information they are sent? Or is it desirable to have the decision centralized?

Even to this day microeconomic theory is disturbingly vague about what constitutes a long term or short term decision. In the corporation, marketing, pricing, production, product development, minor capacity change, major investment and innovation all have different time scales. One individual's decision variables are another individual's parameters.

Accountants must produce systems that are viable, acceptable and operational taking into account the pressures and problems of management, custom, law, economics and the tax collector. Generally Accepted Accounting Principles are not a rigid monolithic set of rules to be obeyed in the same way that chess players obey the rules of chess. But they are presented as a set of guidelines for the responsible businessman and others.

The problems in cost accounting are in many ways important and applied, but simpler versions of many of the problems encountered in the study of externalities and public goods. The accounting profession in its societal role is not attempting to solve all of the problems of welfare which may occur to the economist, but hopefully in trying to provide a reporting and allocation scheme of operational worth to the institutions being served.

This paper is being written from the viewpoint of economics and game theory rather than that of the professional accountant. It may be that various nuances of importance to the accounting profession are overlooked or treated in a somewhat different language. However the thrust of this investigation is in terms of cost and revenue allocations as control mechanisms of interest to the economist, accountant and business or public executive.

Any abstraction results in a distortion of reality from some point of view. Thus there is no universal all-purpose accounting scheme which can always satisfy the needs of a variety of individuals utilizing accounting schemes for different purposes. Cost and control accounting, stockholder financial reporting and tax accounting have different constituencies and purposes. Whitman and Shubik (1979) discuss the different motivations and problems which occur just at the level of financial accounting for stockholders, bondholders, other creditors and managers.

Once the full diversity of interested parties and their different purposes is recognized, even restricting ourselves to the allocation of costs and revenues for control leaving aside problems involving equity or taxes we would still need to differentiate different parties, purposes and problems. These call for special considerations in both modeling and in the selection of solutions.

In the subsequent sections we consider purpose and problems. In other words what are the questions and who asks them? We also consider modeling and problem representation. In particular our concern is with the uses and limitations of the characteristic function of an n-person game. This also involves an excursion into the problems posed by limits on information and by threats. Given a well defined problem and a model that is regarded as a satisfactory representation of the phenomenon being studies, a solution concept must be selected and a solution obtained. Among the candidates are the value, the core and nucleolus associated with models in characteristic function form and some varient of the noncooperative equilibrium associated with strategic form or extensive form representations of the problem.

2. FORMS OF MODELING

Three conceptually different forms of modeling have been suggested for interactive decisionmaking in situations which can be described as games of finite length. They are the extensive, strategic and cooperative forms of a game. Each can be used to describe the same game, but at a different level of detail. Thus each representation is best suited for a different class of questions and poses different levels of difficulty in mathematical analysis and in computation. If our purpose is to apply game theoretic analysis to answer questions concerning operational problems the ability to actually compute solutions becomes important.

2.1. The Extensive Form and Dynamics

The most realistic model of an organism is the organism itself. Any representation is an abstraction which removes or distorts information. If one is trying to answer a specific question concerning the behavior of the organism, a model judiciously selected may portray the features of the organism which are relevant and simplify the analysis by obliterating detail irrelevant to the question at hand.

Von Neumann and Morgenstern (1944) provide an extensive form description for games with each player having a finite set of strategies. The details of the game tree description are well known (for an exposition by Shapley and Shubik see Shubik (1982)) and are not gone into here. Instead our concern is with the modeling implications of what is included and excluded in the extensive form as usually utilized.

The extensive form provides a total contingent planning or historical process view of a game. The use of the game tree is explicitly historical. The same position on a chess board arrived at by different sequencing of moves will lead to a different node on a game tree for every sequence. The game tree is *process oriented*. Any path from the initial node or root of a game tree to a final or terminal node provides a move by move description of a play with information conditions indicated on the game tree.

A description of matching pennies as a game in extensive form is easy, the full game tree for tic-tac-toe is large but could be displayed. A full game tree for chess though logically feasible is technologically infeasible and operationally of little value. The difference in complexity between the detailed description of process in the play of a chess game and behavior in a corporation in a market is enormous both quantitatively and qualitatively. In particular it is easy to specify more or less unambiguously the rules of the game for chess and identify what is meant by a move. Tow person zero sum games are games of pure opposition. There is nothing to be gained by the palyers talking to or bargaining with each other. Although in fact chess players may try ploys and various forms of psychological warfare they are forbidden by the rules and as a reasonable first order approximation can be ignored in the description of most chess games.

When we try to portray the relatively simple three or four person game of Poker or Monopoly even though the formal rules of the game are given, much of the dynamics of play hinges upon language and informal communication not specified in the rules of the game and yet not clearly or expressly ruled out by the formal rules. When are words merely words extraneous to deeps and when are words and informal communication a critical part of the process? The long history of bargaining, negotiation and contracting the giving of promises, the offering of one's word of honor are all examples of the importance of verbal communication as vital parts of the game. The perceptive book of Raiffa (1982) on negotiation serves as an important example of the difficulties in trying to match the formal decision structure of a game in extensive form with the squishy and poorly articulated realities of human communication systems in quasi cooperative and quasi competitive situations.

Any attempt to model much of the activities of the corporation as a game in extensive form must confront two new and critical sets of difficulties in contrast with trying to model formally the game of chess. Language and informal communication count and the rules of the game cannot be easily formulated. Implicit contract, implicit collusion, reputation, trust and social or institutional custom all play a role yet they are extremely difficult to formalize.

The existence of a vast body of law complete with intricate documents such as the commercial code and the law of contract testify both to the attempts and to the incompleteness of the attempts to formalize the rules for commercial behavior.

There is an old saying about the game of gold that "one should drive for show, but putt for dough." This in essence summarizes the difficulties faced by those trying to analyze institutions and trying to devize control systems or routines which provide appropriate incentives. We would like to be able to have a parsimonious description of the corporation akd discover broad general rules be they accounting measures, bonus systems or reporting routines. Yet experience teaches us that in the twilight of institutional complexity that characterizes large private corporations or public bureaucracies or state owned industries there are tax consultants, lawyers, fixers, millionaires and commissars beating the system by utilizing details of the mechanisms overlooked by their designers.[2]

We may summarize the relevance of our comments on the extensive form description of economic activity as follows. The extensive form places a laudable stress on process, but in general the modeling difficulties encountered in trying to provide a full process description are overwhelming. The sheer complexity of amount of detail combined with difficulties in characterizing rules limits the value of the extensive form as a satisfactory basis from which to start an analysis, except for highly stripped down and simplified representations.

---

[2]Or possibly left vague on purpose to provide loopholes for allies in a difficult game with a hidden agenda.

2.2. An Aside on Dynamics

Although almost all studies of the design and behavior of human systems involve some aspects of dyanmics and disequilibrium, most of our models and methods of analysis are static. The models, like a parlor game, have a definite beginning and end. Whereas societies and institutions do not have a neat starting point and doomsday. The finite game tree represents only a slice out of time suitable for the modeling of a parlor game, but less suitable for an ongoing process without a fixed date of termination.

There have been game theory methods designed to portray and analyze competitive structures of indefinite length. In general however virtually all models formulated for the study of cost assignment or of incentive systems are finite in length of time. Our modeling intuition and desire for simplification must serve as the justification for using a finite time slice model for the problem at hand.

2.3. The Strategic Form

Many game theory teaching examples are based upon two person two by two matrix games. The strategic form can be reinterpreted as an extensive form where each individual has a single move; he selects one among a set of strategies. But except where the underlying decision structure can be reasonably well approximated by a single move by all players a considerable amount of detail is lost. Although the game theoretic device of analyzing games by studying the choice of strategies is conceptually neat and clear, the modeling and computational problems to be faced in generating the strategy set for any reasonably complex multistage process are considerable.

In short, although both the extensive and strategic form descriptions of a game are process oriented, sensitive to institutional structure and reflect information conditions they are best employed for the development of theory and for didactic purposes in general. In application to specific problems with a certain amount of hand-tailoring on an *ad hoc* basis (such as studying some problems in pollution control, see Klevorick and Kramer (1973) for example) it may be fruitful to use the extensive or strategic form. Frequently the modeling is either too complicated or the simpler models present too great a distortion of the structure to be studied.

2.4. The Cooperative Form: One or Many?

Much of the work in the application of game theoretic methods to cost problems has been based upon a cooperative form of description of the game (see Shubik (1962), Littlechild and Owen (1973), Young, Okoda and Hashimoto (1980), Hamlen, Hamlen and Tschirhart (1977), Billera, Heath and Raanan (1977)). Yet, as is argued here, far more attention needs to be paid to the *ad hoc* aspects of modeling adequately in the cooperative form prior to the application of a solution concept.

Von Neumann and Morgenstern based their cooperative solution theory upon the characteristic function of an n-person game. This function v is a superadditive set function deferred on 2^n coalitions. *Pro forma* we may define the worth of the empty coalition Θ as zero or $v(\Theta) = 0$. Consider two coalitions S and T with no members in common. Superadditivity calls for

$$v(S \cap T) \geq v(s) + v(T) \quad \text{where} \quad S \cap T = \Theta .$$

This is merely an economic incentive condition where the assumption is that there are at least no overall losses from cooperation and there can in general be gains. Even this is not an innocent assumption. If the joining together of two coalitions has any concrete institutional meaning the mechanics of coalition formation may be expensive and superadditivity is not a foregone conclusion.

The casual reader of von Neumann's and Morgenstern's classic might be surprised to see the elaborate apparatus they erected in order to calculate and justify the characteristic function. They invented an $n+1$ person game with a fictitious player Nature who loses the amount that all the real players gain. This game is converted into a constant sum game by introducing Nature as a strategic dummy. In all constant sum games the strategic problem faced by a coalition S and its compliment $\bar{S}$ is one of pure opposition. A gain by S is reflected by an offsetting loss by $\bar{S}$.

The device of inventing the extra player was introduced to try to avoid the unpleasant modeling problems of describing the threat conditions that may exist when coalition S confronts $\bar{S}$ in a nonconstant sum game. Table 1a shows a simple 2×2 matrix game where if we calculate the characteristic function by assuming that the opposition to S plays in a way to minimize S's gain we obtain a symmetric function as shown in Table 1b.

TABLE 1a

		Player 2: 1	Player 2: 2
Player 1	1	10,-1	-1000,0
	2	0,10	0,10

TABLE 1b

$v(\bar{1}) = 0 \;,\quad v(\bar{2}) = 0$

$v(\overline{12}) = 10$

This completely masks the underlying nonsymmetry where in order for player 1 to hold player 2 to zero he must be willing to suffer a loss of -1,000 whereas if player 2 uses his second strategy it is in player 1's self interest to accept zero while player 2 obtains 10.

Shapley and Shubik (see Shubik (1982), Chapter 6) have suggested as a modeling concept, a c-game which is a game or strategic situation adequately represented by a characteristic function. A simple but important example of a c-game is the cooperative version of an exchange economy. Any set of traders S can trade among its members, but the coalition $\overline{S}$ has no threat beyond not trading. In the language of the economist the economic (or other) activity of S or $\overline{S}$ generates no externalities to the other.

When, as in the example in Table 1a the cost of carrying out a threat is important we would like this fact reflected in the cooperative representation of the game that is used. Harsanyi (1959) in connection with the development of his value solution suggested a way to evaluate threats which for situations involving monetary sidepayments can be described as

$$h(S) + h(\overline{S}) = v(N) \ , \quad \text{when} \quad S \cup \overline{S} = N$$

$$h(S) - h(\overline{S}) = \text{maxmin}[\text{payoff to } S - \text{payoff to } \overline{S}] \ .$$

The first of these two linear equations states that the coalitions S and $\overline{S}$ when cooperating will obtain everything. When threatening each other they will try to maximize the difference between their scores. This defines a damage exchange rate where both the damage to the other and the cost of inflicting the damage are taken into account.

The distinction between the characteristic function $v(S)$ and the Harsanyi function $h(S)$ is in the modeling argument concerning how threats are treated. One could use other possibly *ad hoc* arguments to decide upon the joint product obtainable by a group of S firms, individuals, departments or machines. A different way of looking at the characteristic function is to regard it merely as a production function with values defined only on sets of resources.

In essence the joint cost, revenue and externality problems involve a finite set of profit centers whose activities influence each other. When interests are either independent (no externalities) or completely opposed it is clear how to calculate $v(S)$. ($v(S)$ and $h(S)$ will coincide.) When this is not so we may need to consider the special properties of the problem at hand.

Although for a large n $(n = |N|)$ the number 2^n becomes considerable (for $n = 20$, $2^n = 1,048,576$); in actual application many combinations can be ruled out quickly. Even though the characteristic function provides many degrees of freedom in modeling the economic environment it may still not provide enough. There are at least two larger representations which merit consideration. They are the characteristic function supplemented by a list of weights indicating the relative importance of the players; or the game represented in partitition form.[3]

Instead of describing the game merely by the set N of players and the characteristic function $v(\cdot)$ we add a vector of weights to the description, one for each player or $w_1, w_2, \ldots, w_n$. Shapley (1981) has suggested an application of the weighted game to the payment of expenses; this is discussed further in Section 4. Shubik and Weber (1981) have considered the adding of weights to players in the allocation of costs to a defence system.

The characteristic function and its variants are based upon considering

[3]We could go one step further by considering a weighting vector on the partition function form thus attributing different importance to the various partitions. No work has been done on this probably because it is too complex and messy and as yet no important set of phenomena depending on this structure has been identified.

a coalition S and a counter coalition $\bar{S}$. It is possible however that the amount a coalition S can obtain depends upon the configuration of subgroups formed by the remaining players. Thrall and Lucas (1963) provided the formulation of games in partition function form and investigated an extension of the von Neumann and Morgenstern solution. Let $N = \{1, 2, \ldots, n\}$ be the set of players and $P = \{P_1, P_2, \ldots, P_r\}$ be a partition of N into r coalitions. Let Π denote the set of all patterns and R^1 the real numbers. Then for each pattern P there is an outcome function $F_p : P \to R^1$, i.e. it assigns an outcome $F_p(P_i)$ to coalition P_i given the partition P. The function F that assigns to each partition its outcome function is called the partition function of the game. An n-person game in partition function form is characterized by (N,F).

Given any partition function we can obtain an upper bounding and lower bounding characteristic function $\bar{v}$ and v^1 where for a subset s of N

$$\bar{v}(S) = \max_{\{P|S\in P\}} F_p(S)$$

and

$$v^1(S) = \min_{\{P|S\in P\}} F_p(S) .$$

In the first instance a coalition S is given the maximum worth it can obtain as a coset in any partition to which it belongs. In the second instance it is awarded the minimum. These functions will not necessarily be superadditive.

The partition function picks up the possibility that the yield to a group depends not merely upon the set of the remaining players but on the specifics of its organization.

It is easy to see that without externalities, as in an exchange economy the partition function and characteristic function forms coincide.

In summary we note that the cooperative representation of an n-person game or multidivision corporation or other institutions is in general far more parsimonious than either the extensive or strategic form. The cooperative representation surpresses institutional and process detail. The danger in utilizing the characteristic function is that the threat structure present in some situations may not be adequately represented. For many cost allocation and revenue assignment problems the cooperative representation appears to be the simplest and easiest to work with if it can be established that it provides an adequate model for the problem at hand.

3. WHAT ARE THE QUESTIONS?

There are many different questions concerning the assignment of joint costs and revenues and the possibility of answering them depends heavily upon *ad hoc* institutional and technological facts which determine how well the game can be formulated in coalitional or other forms.

3.1. Information, Agency and Auditing

Among the more important questions are how to design a system where it does not pay individuals to lie to the central office, the tax collector and to whoever else to whom they submit their reports. In general the cooperative form is not adequate to study problems of auditing, enforcement and agency relationships under incomplete information. There is a bargaining literature based heavily upon strategic or simple extensive form models.[4] Items such as the cost of spot checks and random audits call for strategic analysis. Our hope is that there are worthwhile problems where at least to a good first approximation truth revelation is incentive compatible with the accounting scheme. Thus we will not need to worry about the strategic structure in detail as the appropriate incentive design has removed the need for information distortion as part of planned strategy.

3.2. Incentives, Power and Fair Division

In a large corporation the choice is not between centralization or decentralization, but the degree of decentralization needed for viability and the level of decentralization that results in optimal performance. The virtues of the price system as an efficient decentralizing device are well known. But when externalities are present the price system in general is not efficient.

A corporate central office has considerable power in deciding upon the nature of the structure of the firm. Among the factors determining the nature of decentralization are geographical location, nature of products, differentiation of functions and frequency with which some functions are needed; jointness or separability of production processes, interlinkage of marketing of products and sharing of common facilities. Given that the central management has decided upon a structure for the firm it decides upon the freedom of decisionmaking for its executives, the management information system and the incentive system under which they will operate.

In my previous paper (Shubik (1962), p. 331) I suggested a partial

---

[4]See for example Groves and Ledyard (1977), Shubik (1970).

list of relevant decisions which might be aided by an appropriate allocation procedure. They were:

1) decision on major investment,
2) liquidation of a department,
3) abolition of a product line,
4) introduction of a new product,
5) other innovations (such as a change in distribution),
6) the merger of several departments,
7) the splitting of a department into several entities,
8) pricing, purchase of raw materials and sales of final products.

All of these can be described as internal corporate decisions (or inner directed decisions) in as much as the decisions are all internal to one bureaucratic structure. There are other classes of decisions where the emphasis among incentives, power and fair division is somewhat different, but which are also amenable to game theoretic methods. Four classes of problems are suggested. They vary considerably in terms of differences in modeling required to arrive at an adequate description of the game.

Internal corporate incentive systems

Internal to the corporation there is an intermix of geography, technology, economics, politics, accounting, legal and cultural conventions which limit the divisional structure of the firm.

Items which may appear to be trivial to the academic economist may be of paramount importance to those concerned with corporate control. For example, should one keep at least three sets of books and should one openly admit to keeping the three sets of books! One may want one set for the tax collector where the operational consideration is to minimize the tax bill. Another set may be required for stockholders and creditors and a third set for internal control and incentives.

In some countries both legal and societal pressures may leave the corporation open to political and populist attack when large differences are found in different sets of books. A decentralized multinational corporation such as an oil company is faced with designing cost and revenue allocations which minimize taxes over dozens of countries, yet which do not destroy the morale of regional directors. For example some years ago it would have been difficult for even the most incompetent manager of a Venezuelan branch of an oil company to fail to report enormous profits whereas a competent manager in Great Britain would have been derelict in his duties had he not reported losses.

If incentive pay or bonuses depend upon local performance and must be justified to the stockholders then the corporation must explain why one set of books is used for one purpose and another for the other.

If the firm is modeled as an n person game in coalitional form, it is the general management which has the opportunity to decide upon the number of players and the constraints on their strategy sets. For example what limits are placed upon the amount of money that a general manager can invest without having to see his divisional or group vicepresident? Who is permitted to generate the suggestion to merge two departments, or to split a department into two?

The frequency of the need for special services, the costs of record-keeping, accounting, calculating, auditing and communicating all enter into the decisions to choose among markets, divisions and hierarchies. Williamson's (1975) perceptive book spells out many of the detailed factors. These and other factors must already have been adequately reflected in the characteristic function description of the corporation. Furthermore any merger of departments or institutional change unlike the costless coalition formation in much theory may have important administrative costs attached to it. This needs to be accounted for. Shapley and Shubik (1966) have suggested an approximate way to charge all coalitions an organizational cost.

In short, prior to even discussing what solution concept to employ in any serious application much of the work involves providing a sufficiently relevant description of the firm in cooperative form. Possibly one of the major contributions of game theory is to provide a conceptual framework and a strategic audit (see Shubik (1983)) which provides a guideline to problem formulation and data gathering rather than a rigid accounting system for the design of incentive compatible allocation schemes.

In my 1962 article I did not stress sufficiently the importance of tailoring the cooperative form for any serious application. I suggested the reinterpretation of Shapley's axioms as cost accounting maxims as a reasonable way to try to operationalize the question of what are the desired properties of an allocation system having specified its purposes.

The use of the axiomatic approach by several authors (Shubik (1962), Billera, Heath and Raanan (1978), Roth and Verrechia (1979)) I believe illustrates an important difference between the physical sciences and management sciences. In the physical sciences the production of axiom systems may symbolize the postscript to application rather than the prescript. Because accounting and corporate structure are completely human artifacts axioms may be regarded as a way to clarify broad precepts such as the properties of an incentive system or of fair division; this may be the way to start rather than end application.

External corporate incentive systems

All of the problems of measurement noted for internal corporate control systems hold for external corporate problems which can be considered as a cooperative game. These include:

1) mergers and acquisitions,

2) cartel arrangements (where legal), and

3) the splitting of costs and revenues in joint ventures.

The differences here are that the players and their strategy sets are more naturally identified with independent decisionmaking groups. Furthermore the number of players is usually below ten and often two or three.

Politico-economic externalities and public goods

The literature on public utilities, public goods in general and externalities is enormous and has been in existence for some time (see for example the collection of Musgrave and Peacock (1958)). The treatment of these topics here is not intended except to note the key modeling differences among public goods problems, externalities and corporate allocation problems. A major aspect of public goods provision is that at some part of the process direct or indirect political bargaining and voting is involved. A major aspect of the structure of most situations involving externalities is that they involve a high component of legal as well as political process. In contrast to both of these most corporate problems have a far higher economic content combined with much better defined recording and reporting procedures. Furthermore for-profit operations tend to have fewer measurement problems than public services.

The work of Klevorick and Kramer (1973) on the Genossenschaften and the work of Young, Okada and Hashimoto (1980) provide examples of public goods and externality problems at the level of municipal finance.

Technological, taxation and legal fair division problems (TTLFD)

There is a class of problems where the technological component may be high and the cooperative form natural and relatively easy to define. The purpose at hand is well defined and the players and their strategy sets are reasonably easy to define parsimonously. This class includes the telephone system, time sharing computers, aircraft loading, the sharing of joint services, peak load pricing and expense account allocation.

The problems have in common the features that moral hazard and reporting distortion are clearly defined, minimal or non existent. In many of the network or joint service problems the strategy of the users amounts to using or not using the facility. The individuals need not be looked at as players or can be regarded as extremely small without individual power. The characteristic function can be looked at more as a production function than as a game of strategy.

As a first order approximation in economic applications it is useful to completely separate out the question of how to price systems, or tax or cost for the use of an existing facility from the questions concerning whether or not capacity is to be changed. Possibly one of the great psychological barriers that exists between many microeconomic theorists and corporate economists and accountants is the mystical belief that somehow or other demand curves have an objective reality beyond a small region of current sales and that the model of the single product firm with a continuous differentiable marginal cost function provides the general paradigm for understanding optimal pricing and production. It is noted in Section 4 that marginal cost pricing can be deduced as a special instance of the value but not the other way round.

If we take a corporation or a public good as given, then a host of worthwhile operational questions concerning pricing, taxation and assignment of costs for equity considerations can be formulated and I believe can be usefully answered provided that our initial model is adequate.

Table 2 provides an informal check list for the four types of problems discussed here. Most of the comments should be reasonably clear but in some instances a few words of explanation are called for.

For corporate incentive systems the form of the internal hierarchy is important even in deciding who is to be modeled as a strategic player. Moral hazard, organizational slack and revelation problems may all be important. Depending upon the nature of the problem and size of the firm anywhere from two to several hundred divisions, plants or profit and cost centers could be involved.

Although I believe that on an *ad hoc* basis joint cost and profit allocations can be developed based upon a cooperative game description of the firm, many incentive problems appear to require detail best described by more complex modeling. In my 1962 article I suggested several examples which although highly simplified were based on actual experience, illustrating what happened when the then extant methods of costing were used for decisionmaking.

It is easy for the academic to state why bother to allocate joint costs or overheads. This is not a valid criticism unless it can be demonstrated that for the purpose at hand a better alternative is available. Furthermore the alternative must be operationally feasible. It was with this criticism in mind that I suggested trying to specify desirable properties for an accounting system in axiomatic form.

Concerning the analysis for corporate external behavior such as joint ventures, mergers and acquisitions, modeling difficulties are in

TABLE 2

	Corporate Internal	Corporate External	Public Goods and Externalities	TTLFD (Tech.Tax.Legal Fair Division)
Strategic Hierarchy	yes	no	no	not relevant
Political Context	low	low	often high	often not relevant
Legal Context	low	variable	often high	conforming to rules of game
Voting	no	no	often yes	no
Truth Revelation, Moral Hazard & Incentive Problems	often	sometimes	sometimes	often not relevant
Number of Players	anywhere from two to several hundred	two to ten	a few municipalities or many consumer voters	often many customers
Conceptual Problems in Selecting & Defining Cooperative Form	high	middling	middling	low
Informational & Computational Problems in Calculating Cooperative Form	high	high but easily justified	middling to high	middling, primarily technological
Major Purposes	incentive systems (see list)	bargaining "fair division"	efficient supply & appropriate taxation "fair division"	"fair division" legal settlement
Modeling Selection Problems	extensive or strategic form called for but may quickly be too complex; cooperative form for limited results	cooperative form probably adequate; Harsanyi function or partition function may be called for	extensive form for some; for others cooperative form probably adequate;Harsanyi function or partition function may be called for	cooperative form characteristic function
Solution Concepts	noncooperative equilibrium with variants value Nucleolus	core value Nucleolus	core value Nucleolus noncooperative equilibrium	value Nucleolus

general not too bad. Furthermore the coalitional form is the natural form to use to decide if a deal is worthwhile. When there are only two parties (as is often the case) this is done as a matter of course.

In investment banking due diligence studies must be made to clean up legal, tax and many other fine point features which could destroy a deal. But in essence as many hundreds of millions of dollars may be at stake the expenditure of a few million in doing what is in essence the calculation of the game in cooperative form appears to be reasonable.

The game theoretic solution of the core might be used as a lower bound for group demands and the value (especially if it lies with the core) suggests a "fair" way of dividing the proceeds.[5]

It is suggested here that for many, but not all, problems concerning public goods and the control of externalities the institutional, legal and political setting is considerably different than for the corporation. Political and legal factors are often important. Furthermore voting may be involved and especially on the revenue side in spite of the interest some years ago in the development of social accounting the evaluation of the worth of justice or smog control or many other social services poses many unsolved conceptual and technological problems.

There are some nice problems such as water resource allocations where the structure is reasonably well definable and a cooperative game analysis is reasonable.

The last class of problems involving setting time sharing prices or setting the division of tax bills or splitting expenses appear to me to be the clearest candidates for game theoretic methods at a directly

[5]Roth and Verrecchia (1979) raise questions about risk neutrality which are not discussed here.

applied level. The problems do not involve in a direct way an inordinate amount of politics; legal considerations are restricted to either conforming to a specified code or challenging it explicitly (for example Verrecchia (1981) discusses Cost Accounting Standard 403 with regard to the allocation of taxes). In manh instances if there are a large number of customers an individual is easy to model strategically. He either is captive hence cannot threaten to go elsewhere for his services or he has a fixed threat which amounts to buying his services elsewhere. In either case this is relatively easy to model in coalitional form. The data required are in essence technological qualified by legal requirements for the form of reporting and the problems are concerned with efficient use and equitable charges of a given facility or possibly with the addition of capacity.

4. MODELS AND SOLUTIONS

Associated with the extensive and strategic form description of a game are the various forms of noncooperative equilibrium solution. Associated with the cooperative form are the core, value, nucleolus, kernel, bargaining set, stable set and several other solutions. It is suggested here that the three solution concepts most appropriate to the study of allocation and incentive problems are the noncooperative equilibrium, the value and the nucleolus. Of these the last two apply to games in cooperative form.

The kernel, bargaining set and stable set (see Shubik (1982) for definitions and discussion) are more appropriate for bargaining and sociological analysis than for allocation problems.

4.1. The Core

When there is the appropriate economic structure to the problem at hand the core can be defined for either a characteristic function or partition function description. A considerable body of literature exists on the relationship between the core of an economic system modeled as a game and the efficient price system (see Shubik (1959); Debreu and Scarf (1962), and for a survey Shubik (1984), Part III). Yet when the somewhat special conditions concerning technological independence of production and lack of various externalities do not hold there is no guarantee that the core will exist. Although Hamlen *et al.* (1977) suggest the use of core theory in evaluating joint cost allocation, of the four types of problems noted the only one for which the core will probably exist is in mergers, joint ventures and cartel arrangements. The existence of the core when joint production, joint revenues, externalities and voting procedures are present is not guaranteed. Thus, for example, the relationship between the core and Lindahl prices (see Musgrave and Peacock (1958)) at least must be tenuous as the core depends on the characteristic function which in turn depends upon technological and institutional restrictions on the behavior of groups, whereas the Lindahl prices do not depend upon coalition structure.

A key feature in determining the possibility for decentralization by prices is whether the economic entity being studied can be represented in cooperative form by a totally balanced game (see Shapley and Shubik (1968), or Shubik (1982), Chapter 6). Thus one way of considering limitations on individual strategic independence is to see if it is consistent with finding a characteristic function that defines a totally balanced game.

An interesting possibility which does not appear to have been investigated in any detail is to use a partition function representation thereby permitting a completely *ad hoc* evaluation of threats and then deriving an upper and lower bounding characteristic function based upon giving the coalition S its largest and smallest payoff in any partition of which it is a coset. The values of $\overline{v}(S) \geq v^1(S)$ for all S where $\overline{v}(S)$ is the upper and $v^1(S)$ the lower characteristic function. The lower characteristic function will have the larger core. There is no guarantee that either will be superadditive. However if the upper characteristic function game is totally balanced then the lower characteristic function game will also be balanced. Thus there will be at least one and possibly two sets of shadow prices which permit decentralized decision-making depending upon whether the corporate rules limit the payoffs of S to $\overline{v}(S)$ or $v^1(S)$.

It is possible that the upper characteristic function game has no core, yet the lower characteristic function game is totally balanced in which case a unique set of shadow prices will permit decentralized decisionmaking. The point of these observations is to stress that the design of decentralized systems has two components--the laws of physics and the laws of organizations. If the technological facts of life are sufficiently bad it may not be possible to maintain a given degree of decentralization.

For the three person game however if the lower game is balanced so is the upper game. Suppose

$$P(1|2,3) = a,\ P(2|1,3) = b,\ P(3|1,2) = c,\ P(1|23) = 0,\ P(2|13) = 0$$

$$P(3|12) = 0,\ P(12|3) = d,\ P(13|2) = e,\ P(23|1) = f,\ P(123) = 1$$

then for the lower game

$$\bar{v}(\Theta) = 0$$

$$v^1(1) = v^1(2) = v^1(3) = 0$$

$$v^1(12) = d,\ v^1(13) = e,\ v^1(23) = f\ ,\quad d, e, f \geq 0$$

$$v^1(123) = 1$$

Total balance requires that $\frac{1}{2}(d + e + f) < 1$. For the upper game

$$\bar{v}(\Theta) = 0$$

$$\bar{v}(1) = a,\ \bar{v}(2) = b,\ \bar{v}(3) = c$$

$$\bar{v}(12) = d,\ \bar{v}(13) = e,\ \bar{v}(23) = f$$

$$\bar{v}(123) = 1$$

renormalizing so that all one person coalitions have a value of zero then checking the total balance we require

$$\frac{1}{2}(d + e + f - 2(a + b + c)) < 1 - a - d - c\ .$$

The possibility for differences in the existence of total balance starts with $n = 4$.

4.2. The Value

In my estimation the value and possibly the nucleolus are the two most important solution concepts for the allocation of joint costs and revenues. Depending upon the problem at hand we already know how to apply the value to the characteristic function, the player weighted characteristic function and the partition function. It is my opinion that the outer reaches of generality in the application of value theory to a coalitional structure would be to a partition function form with weighted importance to players, but there still remains much to be done before this extra complication is explored.

The Shapley value was originally based on the von Neumann-Morgenstern characteristic function. Shapley (1951, 1953) gave two versions of an axiom system for the value. Shubik (1962) used the first to suggest accounting desiderata. Since then, supported by the work of Aumann and Shapley (1974) various formulations and modifications of the axioms for the value with a continuum of players have been offered. Myerson (1977) has extended the axiom system for the value to games in partition function form and Shapley (1981) has extended his axiom system to include players with different weights. Every one of these systems merits consideration in terms of accounting desiderata.

The value is the natural extension of the type of thinking in economics that made the use of marginal analysis so fruitful. In essence the value is the combinatoric version of marginal analysis. Instead of evaluating a margin at a single point the marginal contribution is evaluated over all combinations. In the unweighted value all combinations are deemed to be generated by a selection of all players with equal probability. The weighted value treats them nonsymmetrically.

Although, as was illustrated by Billera *et al.* (1978) for many small players it is possible to calculate the value by assuming a continuum of players; when numbers are few but bigger than five or six the calculation of the value is laborious unless use can be made of special properties. Littlechild and Owen (1973) provide an example of a simple calculation (see Shapley and Shubik (1969) and Mann and Shapley (1964) for some relatively large calculations).

4.3. The Nucleolus

We define the excess of a coalition S at a particular payoff vector by

$$e(S,a) = v(S) - \sum_{c \in S} i \ .$$

It provides a measure for how much more (or less) a coalition obtains in an imputation than it could obtain by acting alone. Any imputation in the core has an excess less than or equal to zero.

The *nucleolus* (Schmeidler (1969)) is a single point solution which always exists which minimizes the dissatisfaction of the most dissatisfied coalition. More formally we construct an ε-*core* and vary ε until we find the smallest nonempty ε-core. This is known as the *near core*. This is the set of imputations at which maximum excess has been minimized. If we were to continue to vary ε for all coalitions we would wipe out the near core. Instead we consider only those coalitions whose set is not constant through the near core and using the ε minimize their maximum excess. We repeat this procedure until only a single point remains. Littlechild (1974) and Littlechild and Thompson (1977) have used the nucleolus as a costing allocation device.

Sobolev (1975) has produced a set of axioms from which the nucleolus can be derived. These axioms have not yet been published in English. I expect that they will be usefully interpretable in terms of costing principles but this has not yet been done. The attractive feature of the nucleolus is that even without the axioms the ε adjustment has a direct interpretation in terms of taxes or subsidies and the minimization of claims of inequality.

4.4. The Noncooperative Equilibrium

The contrast between cooperative and noncooperative solution theories can be misleading. It appears to be clear for games of finite length. But when the problem at hand is considered in a fully dynamic setting the distinction between cooperative and noncooperative blurs. In essence a good institutional design is one that provides a self policing system; efficiency and individual self interest are compatible. But for games with a finite number of players Dubey and Rogawski (1982) have shown that noncooperative equilibria are generally inefficient. This tells us that we must have a special structure if we expect to find efficient noncooperative equilibria.

In contrast with games of finite length, in games of indefinite length it is often possible to enforce an efficient outcome by threat strategies where the outcome appears to be cooperative but the enforcement mechanism is noncooperative.

Apart from the complexities encountered in modeling the extensive or strqtegic forms already noted, new conceptual and modeling difficulties appear when one tries to reconcile the extensive form with a full dynamics. In terms of application these difficulties are manifested in the limitations of short term economic measures or accounting control systems to reflect adequately the variety of goal structures to be found in control groups of any major institutions. As the time span is increased it becomes more difficult to sort out or to ignore sociopsychological, social, cultural and political factors. An active area of current research is the study of repeated games considering reputations and how to describe and categorize threats.

My caveat here is that useful application of the noncooperative equilibrium solution calls for difficult *ad hoc* modeling. This can be done as Klevorick and Kramer (1973) have shown. But in general both difficulties in constructing and justifying the usefulness of extensive or strategic form models and the justification of the selection procedure among the equilibrium points (see Harsanyi (1959)) has limited the application of the noncooperative equilibrium solution.

5. SOME PROBLEMS AND PROSPECTS

5.1. What Is an Application?

An unkind caricature of much of operations research and management science is that they consist of a set of techniques looking for a problem. The manager undoubtedly will be somewhat institutionally oriented. He has a specific organization to run. The economist, management scientist, game theorist or programmer often think in terms of their specialized techniques and are willing to treat institutional reality to their specialized bed of Procrustes. Thus a manager and his accountants may view large calculations of the value or nucleolus more as an exercise in the employment of surplus Ph.Ds than a serious new way to allocate costs and revenues.

I do not subscribe to either a belief in the caricature or that game theory models provide full answers to many of the problems in cost accounting. My view is mildly optimistic and the remainder of this paper is devoted to two topics. They are what constitutes a worthwhile application and a sketch of several different applications which I believe illustrate the worth of game theoretic methods in the assignment of costs and revenues.

I suggest that there are at least five levels of meaning as to what constitutes an application of a methodology to an applied problem. In particular concerning game theoretic analysis as applied to cost and revenue allocation its uses are as follows:

1) to stop errors and challenge the basis for practices;
2) to suggest good questions, formalized operationally;
3) to suggest simple, better alternatives;
4) to provide new, formal accounting control systems;
5) to provide the calculations for specific answers to specific questions.

The first three applications are more at the level of high level advice and criticism rather than an explicit formal program for accounting. Experience as a consultant, professional knowledge and enough technological and institutional background can lead to identifying bad practices and to raising important questions concerning current procedures. On occasion an immediate *ad hoc* improvement may be spotted which requires little institutional adjustment.

When the climate is right, the use of modeling and computers have become more or less accepted and enough managers and accountants are receptive, there is a possibility for a conceptual and institutional reorganization of accounting practices.

It is my belief, that at least in economics, a true conceptual breakthrough shows its ultimate importance in application when it serves as the basis for a new accounting scheme. The three accounting schemes of signal importance to the development of the modern economy were

1) double-entry bookkeeping which vastly increased the possibility for individual trade and enterprise;
2) national income accounting which provided a conceptual basis for much of the economic accounting control structure of the modern nation state; and

3) the input-output system accounting schemes which provided major links in accounting for production, derived demand and final demand.

In each instance the accounting system generated data that were not previously available. In the process of devising routines and having many professionals consider the new accounting schemes, new problems, conceptual difficulties and gaps in information gathering were discovered. The interplay between economic theory and practice has to be sufficiently two way if benefits are to be derived.

The next accounting revolutions eventually will be in the accounting for the combinatorics of joint costs and revenues. This has already begun and is the prime topic of this paper. Tied in with this but different from it is the vast field of the development of social indicators and the socio-economic and technological development of cost and revenue criteria for public goods and services and for many externalities and other hard to measure manifestations of economic activity such as the worth of advertising or the costs and worth or research and development. In the last twenty years there was much activity and enthusiasm generated concerning measures of the quality of life and social indicators but the developments have proved to be far more difficult than initially expected. Even so current trends point both to the combinatorics of joint costs and revenues and to development of our understanding of both private and public off balance sheet items.

5.2. Some Examples

The papers in the collection of Moriarity (1981) and the extensive set of references provided by Biddle and Steinberg (1983) provide an array of examples to the growing literature of theory and applications. No attempt is made here to summarize all of them. Instead I wish to select a few examples to illustrate concretely the different types of applications.

In Section 3, four categories of problems were suggested and illustrated in Table 2. The applications discussed cover categories 1, 3 and 4. A number in parentheses after each example notes its category. The examples are:

Corporate incentive systems (1)

Water resources (3)

Runaway costs (3)

Telephone time sharing (4)

Expense account sharing (4)

Tax bill sharing (4)

Costs and revenues allocation of a brokerage house communication system (4)

My initial interest in the joint cost problem came about in the context of consulting work with Harlan Mills for a chemical company. The first example in my paper (Shubik (1962), p. 336) was based upon the experience of what can happen if fixed overheads are allocated by several acceptable accounting methods, yet there are independent profit centers who can take action based on this information. The application was by example and was tutorial aimed at preventing error, making clear the dangers in the assignment of overhead costs and pointing out that if you wanted to have tidy accounts there was a way which could avoid the error for overhead assignment. The same level of advice giving was also applied to a major oil company in terms of understanding why underestimation of demand was prevalent in the reporting of independent divisions.

Game theoretic type of reasoning applied to water resources predates publication of the formal theory of cooperative games. Ransmeier's work on the Tennessee Valley Authority provides the example. The work of Young, Okada and Hashimoto (1980), Klevorick and Kramer (1973) and Suzuki and Nakayama (1976) most of whom have been previously noted provide a set of sufficiently concrete examples midway between tutorials, normative suggestions and concrete recommendations. They use cooperative and strategic models and consider the core, value and noncooperative equilibrium solutions. At the least they help to raise and clarify relevant questions concerning an important class of public goods.

The paper of Littlechild and Thompson (1977) provides an in-depth example of the difficulties encountered in trying to find out who is trying to maximize what in even as apparently simple a problem as the solution of aircraft landing fees. It provides a different example of the same type of problem encountered in the water resource studies. The application is a mixture of advice giving, question clarification and pointing out that it is even possible to carry out computations with the suggested new schemes.

The remaining four examples all belong to the category of technological, taxation and fair division problems where the greatest level of formalization appears to be feasible and the greatest opportunity for useful answers to relatively specific questions can be had.

The first example is that of Billera, Heath and Raanan (1978) which describes in detail the method applied in allocating the costs of Cornell University telephone system. Even here the *ad hoc* modeling problems were considerable.

The next example is the scheme proposed by Shapley (1981) for the apportioning of an expense account involving trips to several different locations. Suppose an individual visits several divisions of his firm for consulting. He could have taken several separate trips or could arrange the trip in various sequences. Furthermore assume that he spends w_j days of consulting at the j^{th} location and that can be billed to it. How should the travel expenses be split? Shapley suggests, according to the weighted value. This appears to me to be sufficiently straightforward and feasible that it could provide a way to apportion the overheads generated by internal consulting groups.

Shapley used his scheme to split his expenses on a trip that involved a visit to a joint cost allocation conference and a trip to Yale. Although both groups accepted his expenses as billed this might not work in general among different institutions as there is no enforcement mechanism which prevents double billings or at least shading the savings in the consultant's favor.

The next two examples involve an intermix of cost accounting data, tax laws, questions of equity and legal settlement. Verrecchia (1981) describes a case study of a dispute between McDonnell Douglas Corporation and the United States Government concerning the interpretation of Cost Accounting Standard 403 dealing with the allocation of state, local income and franchise taxes. He proposes the use of the value. Bentz comments that Verrecchia does not establish that the value is more equitable than other methods, but he fails to consider the axioms which are satisfied by the value.

The last example was considered in connection with a law case involving a large brokerage company with both a brokerage business and a large money market fund. The basis of the case was that the outside directors had fiduciary responsibility to the holders of the money market shares. In particular if, as was the case, the advisory fee paid to the brokerage house far exceeded the cost involved in buying the services by hiring an inhouse investment group then it could be argued that it was the duty of the outside directors to vote against the advisory contract. It was argued however that the firm supplied to the customers at no cost to them but at great cost to itself a communications and computer system which enabled customers to switch with ease directly from stock to a money market fund or vice versa. The brokerage firm thereby claimed that the expenses were grossly understated and hence the fee was justified. This argument failed to take into account the fact that the provision of both services resulted in joint costs and revenues. How is one to jointly allocate the costs and benefits to different users of the firm's communication network?

A simple three person example illustrates some of the features of the problem. Suppose that there were only three customers, one holding only a stock account, one a money market account, and one both. Suppose the first two generate the same amount of communication and computer use, while the third generates twice as much. Suppose there is a fixed cost of 100 for the network. The net profit per customer (leaving out communication) is 100, 50 and 200 respectively. How should communication costs be allocated? The characteristic function is

$$v(1) = 0, \quad v(2) = -50, \quad v(3) = 100$$

$$v(12) = 50 \quad v(13) = 200, \quad v(23) = 150$$

$$v(123) = 250$$

this gives a value of (66-2/3, 16-2/3, 166-2/3) with the cost assignment being split evenly among the three players.

In order to have applied the value to the actual law case it would have been necessary to consider to a reasonable degree of accuracy the different classes of users, the variable costs of the communication net and an attribution of joint revenues. As all classes of traders were numerous a nonatomic game calculation was called for. The actual case was terminated before the costing questions came into evidence.

The feature worth noting with this example is that had not the case been terminated for other legal reasons the key consideration would have been a fair assessment of costs and revenues to be attributed to joint economic activity. The assessment of shared costs together with the consideration of extra revenues due to the provision of joint services was critical in determining if the directors had failed in their fiduciary duty by approving a management contract rather than buying the services elsewhere.*[6] The value is not necessarily the only solution possible, but in this instance the need for the allocation and for the specification of fair division criteria is operationally clear.

---

*The directors' problem was to compare the overall assignment to their class of shareholders with what could be earned by setting up business as an independent. Thus the directors had to perform a cooperative analysis between the three person game with all customers in and the two person game with their customers alone.

REFERENCES

Aumann, R. J. and L. S. Shapley (1974). *Values of Monotonic Games.* Princeton: Princeton University Press.

Biddle, G.C. & R. Steinberg (1984). "Allocations of Joint and Common Costs," Journal of Accounting Literature, 3 to appear.

Billera, L. J., D. C. Heath and J. Raanan (1978). "Internal Telephone Billing Rates: A Novel Application of Monotonic Game Theory," *Operations Research.*

Debreu, G. and H. E. Scarf (1963). "A Limit Theorem on the Core of an Economy," *International Economic Review*, 4: 235-246.

Dubey, P. and J. Rogawski (1982). "Inefficiency of Nash Equilibria in a Private Goods Economy," Cowles Foundation Discussion Paper No. 631.

Groves, T. and J. Ledyard (1977). "Optimal Allocation of Public Goods. A Solution to the Free Rider Problem," *Econometrica*, 45: 785-809.

Hamlen, S. S., W. A. Hamlen and J. T. Tschirhart (1977). "The Use of Core Theory in Evaluating Joint Cost Allocation Schemes," *Accounting Review*, 52: 616-627.

Harsanyi, J. C. (1959). "A Bargaining Model for the Cooperative n-Person Game," in A. W. Tucker and R. D. Luce (eds.), *Contributions to the Theory of Games 4.* Princeton: Princeton University Press, pp. 324-356.

Klevorick, A. K. and G. H. Kramer (1973). "Social Choice on Pollution Management: The Genossenschaften," *Journal of Public Economics*, 2: 101-146.

Littlechild, S. C. (1974). "A Simple Expression for the Nucleolus in a Special Case," *International Journal of Game Theory*, 3: 21-29.

Littlechild, S. C. and G. Owen (1973). "A Simple Expression for the Shapley Value in a Special Case," *Management Science*, 20: 370-372.

Littlechild, S. C. and G. F. Thompson (1977). "Aircraft Landing Fees: A Game Theory Approach," *Bell Journal of Economics*, 8: 186-204.

Mann, I. and L. S. Shapley (1964). "The a priori Voting Strength of the Electoral College," in M. Shubik (ed.), *Game Theory and Related Approaches to Social Behavior*. New York: John Wiley, pp. 151-164.

Moriarity, S. (1981). *Joint Cost Allocations*. Norman, OK: University of Oklahoma.

Musgrave, R. A. and P. B. Peacock, eds. (1958). *Classics in the Theory of Public Finance*. New York: McGraw Hill.

Myerson, R. (1977). "Values of Games in Partition Function Form," *International Journal of Game Theory*, 6: 23-31.

Raiffa, H. (1982). *The Art and Science of Negotiation*. Cambridge, MA: Harvard University Press.

Roth, A. E. and R. E. Verrecchia (1979). "The Shapley Value as Applied to Cost Allocation: A Reinterpretation," *Journal of Accounting Research*.

Schmeidler, D. (1969). "The Nucleolus of a Characteristic Function Game," *SIAM Journal of Applied Maths.*, 17: 1163-1170.

Shapley, L. S. (1951). "Notes on the n-Person Game, II: The Value of an n-Person Game," *RAND RM-641*.

Shapley, L. S. (1953). "A Value for n-Person Games," in H. Kuhn and A. W. Tucker (eds.), *Contributions to the Theory of Games*. Princeton: Princeton University Press, pp. 307-317.

Shapley, L. S. (1981). "Comments on P. D. Banker's 'Equity Considerations in Traditional Full Cost Allocation Practices': An Axiomatic Perspective," in S. Moriarity (ed.), *Joint Cost Allocation*. University of Oklahoma.

Shapley, L. S. and M. Shubik. (1969). "Pure Competition, Coalitional Power, and Fair Division," *International Economic Review*, 10: 337-362.

Shapley, L. S. and M. Shubik, (1968). "On Market Games," *Journal of Economic Theory*, 1: 9-25.

Shapley, L. S. and M. Shubik (1966). "Quasi-cores in a Monetary Economy with Non-convex Preferences," *Econometrica*, 39: 805-827.

Shubik, M. (1959). "Edgeworth Market Games," in R. D. Luce and A. W. Tucker (eds.), *Contributions to the Theory of Games, Vol. 4*. Princeton: Princeton University Press.

Shubik, M. (1962). "Incentives, Decentralized Control, the Assignment of Joint Costs and Internal Pricing," *Management Science*, 325-343.

Shubik, M. (1970). "Budgets in a Decentralized Organization with Incomplete Information," RAND Publication.

Shubik, M. (1982). *Game Theory in the Social Sciences, Vol. 1.* Cambridge, MA: MIT Press.

Shubik, M. (1983). "The Strategic Audit; A Game Theoretic Approach to Corporate Competitive Strategy", Managerial and Decision Economics 4, 3.

Shubik, M. and R. Weber (1981). "Systems Defense Games: Colonel Blotto Command and Control," *Naval Research Logistics Quarterly*, 28.

Shubik, M. (1984). *Game Theory in the Social Sciences, Vol. II.* Cambridge, MA: MIT Press.

Sobolev, A. I. (1975). "The Characterization of Optimality Principles in Cooperative Games by Functional Equations. (In Russian with English summary.) Mathematical Methods in the Social Sciences, Vilma: 150-165.

Stigler, G. (1966). *The Theory of Price,* 3rd Edition. New York: MacMillan.

Suzuki, M. and M. Nakayama (1976). "The Cost Assignment of Cooperative Water Resource Development: A Game Theoretic Approach", Management Science 22, pp. 1081-1088.

Thrall, R. M. and W. F. Lucas (1963). "N-person Games in Partition Function Form," *Naval Research Logistics Quarterly*, 10: 281-298.

Verrecchia, R. (1981). "A Question of Equity: Use of the Shapley Value to Allocate State and Local Income and Franchise Taxes," in S. Moriarity (ed.), *Joint Cost Allocations.* Norman, OK: University of Oklahoma, pp. 14-40.

Von Neumann, S. and O. Morgenstern (1944). *Theory of Games and Economic Behavior.* Princeton: Princeton University Press.

Whitman, M. J. and M. Shubik (1979). *The Aggressive Conservative Inventor.* New York: Random House.

Williamson, O. E. (1975). *Markets and Hierarchies: Analysis and Anti-trust Implications.* New York: Free Press.

Young, H. P., M. Okada, and T. Hashimoto (1980). *Cost Allocation in Water Research Development: A Case Study of Sweden.* Laxenburg: International Institute for Applied Systems Analysis.