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OPTIMAL SPENDING AND MONEY HOLDINGS IN THE PRESENCE
OF LIQUIDITY CONSTRAINTS AND RANDOM INCOME FLUCTUATIONS

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Optimal Spending and Money Holdings in the Presence
of Liquidity Constraints and Random Income Fluctuations

ABSTRACT

We examine the optimal spending behavior and money holdings of a risk averse individual who faces liquidity constraints and random fluctuations in his money income. Because of a cash-in-advance constraint, the individual has a well-defined transactions requirement for money balances. In addition, because money income is uncertain and money is - by assumption - the only available store of value, the risk averse individual also holds money balances as an inventory which can be drawn down in periods of unexpectedly low earnings. We establish the strict monotonicity properties of optimal expenditure and money demand decisions and show that the average propensity to consume is directly related to expected income. We show that, in the presence of random income fluctuations, the risk averse individual has a unique target level of money balances which depends directly on the dispersion of the probability distribution which governs these fluctuations. We also establish the existence, continuity, and monotonicity properties of the unique stationary probability distributions which characterize the behavior of optimal money holdings and beginning-of-period money balances in a stochastic steady-state. In particular, the stationary distribution which characterizes optimal money holdings is shown to be almost-everywhere continuous and strictly increasing with a single mass point at zero.

OPTIMAL SPENDING AND MONEY HOLDINGS IN THE PRESENCE
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Richard H. Clarida

In this paper, we examine the optimal spending behavior and money holdings of a risk averse individual who faces liquidity constraints and random fluctuations in his money income. The liquidity constraints are of two kinds: a prohibition against borrowing and a cash-in-advance requirement that goods must be purchased with and income received in the form of money. Because of the cash-in-advance constraint, the individual has a well-defined transactions requirement for money balances. In addition, because money income is uncertain and money is - by assumption - the only available store of value, the risk averse individual also holds money balances as an inventory which can be drawn down in periods of unexpectedly low earnings. The first objective of this paper is to establish the monotonicity properties of optimal spending and money demand decisions in such an environment. The second objective is to establish the existence, continuity, and monotonicity properties of the stationary probability distributions which characterize the behavior of optimal money holdings and beginning-of-period money balances in a stochastic steady-state.

Helpman (1981) studies a deterministic cash-in-advance model in which money income is constant. He shows that optimal spending is a continuous, strictly increasing function of beginning-of-period money balances and that the marginal propensity to spend out of money balances is strictly between zero and one (except when beginning balances are sufficiently small in which case the marginal propensity to spend is unity). Helpman's results also imply that the average propensity to spend out of beginning-of-period balances is directly related to anticipated income (again except when beginning balances are sufficiently small in which case the average propensity to spend out of

beginning balances is unity).¹ Finally, Helpman establishes that spending and money balances converge monotonically, and in a finite number of periods, to a stationary state in which they are constant and equal to the time independent income flow.

By contrast, very little is at present known about optimal spending and money demand decisions or about the limiting behavior of optimal money holdings in the presence of random money income and a cash-in-advance constraint. The results which are available are established in the important paper of Schechtman and Escudero (1977). Schechtman and Escudero show that optimal spending and money holdings are continuous, non-decreasing functions of beginning-of-period balances and that, if the probability distribution which governs random money income is discrete, there exists a limiting distribution for beginning-of-period money balances.

The contributions of this paper to the literature may now be summarized. We establish that virtually all of the properties of optimal spending and money demand decisions derived by Helpman (1981) for the deterministic, constant income case carry over to the decisions of a risk averse individual who confronts random fluctuations in his money income. Furthermore, we prove the intuitive result that, in the presence of random income fluctuations, the risk averse individual has a unique target level of money balances which depends directly on the dispersion of the probability distribution which governs his random wage earnings. We are able to sharpen substantially the weak-monotonicity results of Schechtman and Escudero (1977) by proving, along the lines of Lucas (1978), that the agent's indirect utility function (value function) inherits the strict concavity of his period utility function. Focusing our attention on the case in which the probability distribution which

governs random earnings is continuous, we establish the existence and examine the continuity and monotonicity properties of the unique stationary probability distributions which characterize the behavior of optimal money holdings and beginning-of-period money balances in a stochastic steady-state. We show that the limiting distribution which characterizes beginning-of-period money balances is continuous and strictly increasing over its compact support. By contrast, the stationary distribution which characterizes optimal money holdings is shown to be almost-everywhere continuous with a single mass point at zero. This latter result is shown to follow from the inventory-theoretic nature of the risk averse individual's intertemporal optimization problem and has been obtained by Lucas (1980) in a cash-in-advance model in which preferences, and not earnings, are uncertain.

The plan of the paper is as follows. In Section 1, we set up the model and prove the existence and properties of the indirect utility function which corresponds to the individual's intertemporal optimization problem. In Section 2, we derive the monotonicity properties of the optimal expenditure and money demand functions. In Section 3, we establish the existence, continuity, and monotonicity properties of the limiting distributions which characterize optimal money holdings and beginning-of-period money balances in a stochastic steady-state. In Section 4, we provide some concluding remarks.

1: The Individual's Optimization Problem

Consider an individual who plans his consumption for infinitely many periods $t = 1, 2, \dots$. He lives in an economy in which all payments are made in the form of money and in which money is the only asset. In particular, for purchases of goods he has to pay with money, and his wage income is received at the end of each period in money. The individual must decide at the beginning of each period how much to spend on consumption subject to the cash-in-advance constraint that such spending not exceed his beginning-of-period cash balances. That portion of the individual's beginning money balances which is not spent, which we shall refer to as money holdings, is carried over to the next period. Letting p_t denote the period t price level, c_t period t consumption, h_t period t money holdings, and m_t beginning of period t money balances, we can express the individual's cash-in-advance constraint as

$$p_t c_t + h_t = m_t. \quad (1)$$

At the end of period t , the individual receives a money wage payment, ε_t , for labor services which are inelastically supplied at the beginning of the period. The individual cannot use money income received at the end of period t to purchase goods in period t . Period t money holdings plus period t money income thus determine the money balances available at the beginning of period $t+1$:

$$h_t + \varepsilon_t = m_{t+1}. \quad (2)$$

Consumption of c_t goods in period t yields utility of $\beta^t u(c_t)$ where $\beta \equiv 1/1+\delta$, δ the positive rate of time preference. We shall make the standard assumptions that $u(c)$ is strictly concave, increasing, and differentiable and that the random variables ε_t are i.i.d. with a continuous, and continuously

differentiable c.d.f. $G(\epsilon)$. The range of ϵ is an interval $[\underline{\epsilon}, \bar{\epsilon}]$ which is a finite subset of R^+ .

We shall restrict our attention to the case in which the individual can rationally expect the money price of goods to remain constant. As has been shown by Lucas (1980) and Bewley (1980) in somewhat different contexts, this restriction is consistent with general equilibrium in an economy in which there are a continuum of agents, (outside) money is fixed in supply and is the only asset, and there is no aggregate uncertainty.² In what follows we shall normalize the constant price level to one so that $p_t = 1$, $t = 1, 2, \dots$.

The individual's intertemporal optimization problem may be expressed as:

$$\max E \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$\begin{aligned} c_t + h_t &= m_t; \\ h_t + \epsilon_t &= m_{t+1}; \\ c_1 + h_1 &= m; \\ c_t > 0; h_t > 0; m_t > 0. \end{aligned}$$

Let $v(m)$ be the value of the objective function of an individual who begins the period with money balances m and behaves optimally. This function must satisfy:

$$v(m) = \max_{c < m} \{ u(c) + \beta \int v(m - c + \epsilon) dG(\epsilon') \} \quad (3)$$

Following closely the arguments presented in Lucas (1978;1980), we now establish the existence and properties of the value function v .

Theorem 1: Under the assumptions on u and G , there exists a unique, bounded, continuous, strictly increasing, strictly concave, and once continuously differentiable function v such that

$$v(m) = \max_{c < m} \{ u(c) + \beta \int v(m - c + \varepsilon) dG(\varepsilon) \}. \quad (4)$$

Furthermore,

$$v'(w) = u'(c^*); \quad (5)$$

where c^* is the unique solution to (4).

Proof: We follow Lucas (1978;1980). Let L be the space of continuous, bounded functions $u: R^+ \rightarrow R$ normed by $\|u\| = \sup_m |u(m)|$. Define T as the operator on L such that (4) reads $v = Tv$. Using Berge (1963), $T: L \rightarrow L$. Using Blackwell (1965), T is a contraction so that $Tv = v$ has a unique solution $v^* \in L$ and $\|T^n u - v^*\| \rightarrow 0$ as $n \rightarrow \infty$ for all $u \in L$.

It is easy to verify that T takes nondecreasing functions of m into strictly increasing functions of m . We now show that T takes concave functions of m into strictly concave function of m . Let v be any element of L , let $m^0 \neq m^1$, let $\theta \in (0,1)$, and let $m^\theta = \theta m^0 + (1-\theta)m^1$. Let $c^0 < m^0$ attain $(Tv)(m^0)$, the right hand side of (4), and define c^1 similarly. Then $c^\theta = \theta c^0 + (1-\theta)c^1 < m^\theta$. It follows that

$$\begin{aligned} (Tv)(m^\theta) &> u(c^\theta) + \beta E v(m^\theta - c^\theta + \varepsilon) \\ &> \theta [u(c^0) + \beta E v(m^0 - c^0 + \varepsilon)] + (1-\theta) [u(c^1) + \beta E v(m^1 - c^1 + \varepsilon)] \\ &> \theta (Tv)(m^0) + (1-\theta) (Tv)(m^1); \end{aligned}$$

where the first, weak inequality, restates (4), the second strict inequality reflects the assumed concavity of v , the strict concavity of u , and the fact that $m^0 \neq m^1$ and $\theta \in (0,1)$, and the final inequality is the definition of m^0 and m^1 . Thus, we have shown that Tv is strictly concave if v is concave.

For all m such that $c^* < m$, the differentiability of v follows directly from Lucas (1978) proposition 2. For m such that $c^* = m$

$$v(m) = u(m) + \beta \int v(\epsilon) dG(\epsilon); \quad (6)$$

and (5) follows directly. Since the one-sided derivatives agree, continuous differentiability follows. Q.E.D.

The existence and basic properties of v are established in Schechtman and Escudero (1977), although our application of the contraction mapping results to this "income fluctuations" problem would appear to be new. The contribution of Theorem 1 is to show that v inherits the strict concavity of u . Earlier papers by Schechtman and Escudero (1977) and Mendelson and Amihud (1982) have established only the concavity of v . We shall now use the strict concavity of v to investigate the monotonicity properties of optimal spending and money demand decisions in the presence of random income fluctuations and a cash-in-advance constraint.

2: Properties of Optimal Expenditure and Money Demand Functions

In this section, we establish the monotonicity properties of the optimal expenditure and money demand functions. We begin by proving the existence, uniqueness, and continuity of these functions.

Lemma 1(Schechtman and Escudero (1977)): Under the assumptions on u and G , there exists a unique and continuous function $c(m;G)$ such that

$$v(m) = u(c(m;G)) + \beta \int v(m - c(m;G) + \epsilon) dG(\epsilon); \quad (7)$$

that is, c is the optimal expenditure function in the sense of obtaining maximum expected utility. Furthermore,

$$v'(m) = u'(c(m;G)). \quad (8)$$

There also exists a unique a continuous function $h(m;G)$ such that

$$v(m) = u(m - h(m;G)) + \beta \int v(h(m;G) + \epsilon) dG(\epsilon); \quad (9)$$

that is, h is the optimal money demand function in the sense of obtaining maximum expected utility.

Proof: The maximization problem (4) involves maximizing a continuous, strictly concave function over a compact, convex set. Hence $c(m;G)$ is uniquely defined and, from Berge (1963), p. 116, continuous. The existence, uniqueness, and continuity of $h(m;G)$ follows immediately from these results and the cash-in-advance constraint (1). Q.E.D.

We now establish that optimal expenditure is a strictly increasing function of beginning-of-period money balances. We also show that the marginal

propensity to spend out of money balances is strictly between zero and one, except when beginning balances fall below a uniquely defined critical level in which case the marginal propensity to spend out of money balances is unity.

Theorem 2: Under the assumptions on u and G , the optimal expenditure function $c(m;G)$ is strictly increasing in beginning-of-period money balances (see Figure 1). For all $0 < m < \bar{m}$, where \bar{m} is the unique solution to

$$u'(\bar{m}) = \beta \int v'(\varepsilon) dG(\varepsilon); \quad (10)$$

the marginal propensity to spend out of money balances is unity and the optimal expenditure function is given by

$$c(m;G) = m. \quad (11)$$

For all $m > \bar{m}$, the marginal propensity to spend out of money balances is strictly between zero and one and the optimal expenditure function is uniquely defined by

$$u'(c(m;G)) = \beta \int v'(m - c(m;G) + \varepsilon) dG(\varepsilon). \quad (12)$$

Proof: Equations (10), (11), and (12) follow directly from (7), the definition of v given in (4), the assumed properties of u and the differentiability and strict concavity of v established in Theorem 1. Noting that for all $\lambda > 0$

$$\int v'(m + \lambda - c(m;G) + \varepsilon) dG(\varepsilon) < \int v'(m - c(m;G) + \varepsilon) dG(\varepsilon); \quad (13)$$

and using (10) and (11), we see that c is strictly increasing in m . To show that the marginal propensity to spend out of money balances is less than one

for all $m > \bar{m}$, we just observe that (12) implies

$$u'(c(m;G) + \lambda) < \beta \int v'(m + \lambda - (c(m;G) + \lambda) + \varepsilon) dG(\varepsilon). \quad (14)$$

Thus $c(m + \lambda;G) - c(m;G) < \lambda$. Q.E.D.

Corollary: For all $0 < m < \bar{m}$, optimal money holdings are zero (see Figure 2):

$$h(m;G) = 0; \quad m < \bar{m}. \quad (15)$$

For all $m > \bar{m}$, optimal money holdings are strictly increasing in m and are uniquely defined by

$$u'(m - h(m)) = \beta \int v'(h(m) + \varepsilon) dG(\varepsilon); \quad m > \bar{m}. \quad (16)$$

Proof: Follows immediately from the theorem and the cash-in-advance constraint $c + h = m$.

We next establish that the average propensity to spend out of money balances is directly related to expected income $\mu = E\varepsilon$ (except when such balances fall below the critical level \bar{m} in which case the this propensity is equal to one).

Theorem 3: The average propensity to spend out of money balances is directly related to expected income. In particular for all $\lambda > 0$

$$c(m;\mu)/m = c(m;\mu+\lambda)/m = 1; \quad 0 < m < \bar{m}; \quad (17)$$

$$c(m;\mu)/m < c(m;\mu+\lambda)/m < 1; \quad m > \bar{m}. \quad (18)$$

Proof: From Theorem 2 we know that a necessary and sufficient characterization of the optimal expenditure function is given by

$$u'(c(m;\mu+\lambda)) = \beta \int v'(m - c(m;\mu+\lambda) + \varepsilon + \lambda; \mu+\lambda) dG(\varepsilon), \quad (19)$$

for $m > \hat{m}$; otherwise $c(m) = m$. We must show that $v'(m;\mu) > v'(m;\mu+\lambda)$. We work recursively with the fact that v can be viewed as the limit of a sequence of valuation problems. The agent's problem in the next to last period is given by

$$v(m;\mu+\lambda;1) = \max_{c \leq m} \{ u(c) + \beta E u(m - c + \varepsilon + \lambda) \} \quad (20)$$

Define the maximand as $\xi(m;c;\mu;1)$. Then, $\xi_m(m;c;\mu;1) > \xi_m(m;c;\mu+\lambda;1)$. Since $v(m;\mu;1) = \xi(m;c^*; \mu;1)$ for some c^* , we have

$$v'(m;\mu;1) > v'(m;\mu+\lambda;1) \quad (21)$$

Proceeding recursively, we can show that

$$v'(m;\mu;i) > v'(m;\mu+\lambda;i); \quad (22)$$

provided $v'(m;\mu;i-1) > v'(m;\mu+\lambda;i-1)$. It follows directly from Mendelson and Amihud (1982) Lemma p. 696, that $v'(m;i)$ converges to $v'(m)$ uniformly on every closed interval so long as $u'(0) < \infty$. It follows that

$$v'(m;\mu) > v'(m;\mu+\lambda). \quad (23)$$

From (19) and (23), we obtain, for $m > \hat{m}$

$$\begin{aligned} u'(c(m;\mu+\lambda)) &= \beta E v'(m - c(m;\mu+\lambda) + \varepsilon + \lambda; \mu+\lambda) \\ &< \beta E v'(m - c(m;\mu) + \varepsilon; \mu) = u'(c(m;\mu)). \end{aligned} \quad (24)$$

The theorem follows immediately from (24). Note also that $\hat{m}(\mu+\lambda) > \hat{m}(\mu)$. Q.E.D.

Theorems 1 and 2 establish that, with one exception to be discussed now, the properties of optimal spending and money demand decisions derived by Helpman (1981) for the deterministic, constant income case carry over to the decisions of a risk averse individual who confronts random fluctuations in his money income. A feature of the deterministic, constant income cash-in-advance model which does not appear to be shared by its stochastic counterpart is that the former predicts that the marginal propensity to spend out of expected income is exactly T times larger than the marginal propensity to spend out of money balances, where T is the (finite) number of periods it takes the individual to optimally reach steady-state spending and money balances equal to the time independent income flow.³

For completeness, we now state without proof the result, due to Miller (1976), that the average propensity to spend out of money balances is inversely related to the dispersion of the probability distribution which governs random money income.⁴

Lemma 2 (Miller (1976)): The average propensity to spend out of cash balances is inversely related to a mean preserving spread of G :

$$c(m;G(\mu;\sigma))/m < c(m;G(\mu))/m; \quad (25)$$

where $G(\mu)$ second-degree stochastically dominates $G(\mu;\sigma)$; i.e.

$$\int u(\epsilon) dG(\epsilon;\mu;\sigma) < \int u(\epsilon) dG(\epsilon;\mu). \quad (26)$$

for all increasing, strictly concave u .

Corollary: The average propensity to hold money balances is directly related to the dispersion of the probability distribution which governs money income. In particular,

$$h(m;G(\mu;\sigma))/m = h(m;G(\mu)) = 0; \quad 0 < m < \hat{m}(\sigma) < \hat{m}. \quad (27)$$

$$h(m;G(\mu;\sigma))/m < h(m;G(\mu)) < 1; \quad m > \hat{m}(\sigma). \quad (28)$$

This obvious corollary to Miller's (1976) result highlights the inventory-theoretic, precautionary motive for holding money balances in the presence of random money income fluctuations. Confronting more uncertainty about his end-of-period money income receipt, the risk averse individual spends less and holds more of his beginning-of-period balances so as to at least partially self-insure against an unanticipated income shortfall.⁵

Define a desired or target level of money balances to be any stock of beginning-of-period money balances such that planned hoarding

$$\bar{s}(m) \equiv \mu - c(m); \quad (29)$$

is zero. Formally, a stock of target money balances is just a fixed point to the equation

$$m^{\tau} = h(m^{\tau}) + \mu. \quad (30)$$

We conclude this section by establishing that there in fact exists a unique target level of money balances which depends directly on the dispersion of the probability distribution which governs the random income fluctuations.

Proposition: There exists a unique level of target money balances. That is, the equation

$$m^T = h(m^T) + \mu \quad (31)$$

has exactly one solution (see Figure 3) which is given by

$$m^T = c^{-1}(\mu) > \mu. \quad (32)$$

Furthermore, the level of target money balances is directly related to the dispersion of the probability distribution which governs the random income fluctuations. That is, for all $G(\mu)$ which second-degree stochastically dominate $G(\mu; \sigma)$

$$m^T(G(\mu; \sigma)) > m^T(G(\mu)) \quad (33)$$

Proof: Existence and uniqueness follow directly from the Theorem 2 result that

$$0 < h(m+\lambda) - h(m) < \lambda \quad (34)$$

and the fact that c is continuous and strictly increasing over a compact set and is thus invertible. The relationship between σ and m^T follows directly from (31) and the Corollary to Lemma 2. Q.E.D.

3: The Limiting Distribution for Optimal Money Holdings

The function c and the cumulative distribution function G of ϵ together define a Markov process

$$m_{t+1} = m_t - c(m_t) + \epsilon_t \quad (35)$$

with state space R^+ . That is, given an initial distribution for money balances, $F_1(m)$, the distribution $G(\epsilon)$ and the difference equation (37) together determine the sequence of distributions $F_1(m), F_2(m), \dots$ which prevail at dates $t = 1, 2, \dots$. The stationary distribution for beginning-of-period money balances, if it exists, is the limit of this sequence. Schechtman and Escudero (1977) establish that, if the probability distribution which governs the random income fluctuations is discrete, there exists a limiting distribution for beginning-of-period money balances.⁶

Our first task shall be to relax this assumptions and to prove the existence of a unique, continuous limiting distribution for beginning-of-period money balances. Armed with these results and the properties of h established in Section 2, we then establish the existence of a unique, almost-everywhere continuous stationary cumulative distribution function which characterizes optimal money holdings in a stochastic steady-state.

To say that a sequence $m_1, m_2 \dots$ is subject to the transition probabilities K means that $K(m, m')$ is the conditional probability of the event $\{m_2 < m'\}$ given that $m_1 = m$. Equation (2) and the definition of G imply that

$$K(m, m') = G(m' - h(m)). \quad (36)$$

If the probability distribution of m_t is $F_t(m)$, then the distribution of m_{t+1} is given by

$$F_{t+1}(m') = \int_{\Omega} F_t(dm) K(m, m'). \quad (37)$$

Definition: The distribution F is a stationary distribution for K if $TF = F$, where the operator T is defined by

$$TF(m') = \int_{\Omega} F(dm) K(m, m'). \quad (38)$$

Feller's (1971) ergodic theorems for Markov chains establish conditions on K and Ω which guarantee the existence of a unique stationary distribution F .

Furthermore, F represents the asymptotic distribution of m_n under any initial distribution. That is, the influence of the initial state fades away and the system tends to a steady state governed by the stationary solution.

Mendelson and Sobel (1980) have applied the Feller (1971) ergodic theorems to study capital accumulation in the context of a general renewable resource model.

Following closely the arguments used by Danthine and Donaldson (1981) in their examination of the limiting behavior of capital in the Brock - Mirman (1972) stochastic optimal growth model, we now prove the following theorem.

Theorem 4: Under the assumptions on u and G , there exists a continuous and increasing stationary distribution function for beginning-of-period money balances, denoted $F(m)$, which is the unique solution to the functional equation $TF = F$:

$$F(w') = \int_{\Omega} G(m' - h(m)) dF(w). \quad (39)$$

For any F_1 ,

$$\lim_{n \rightarrow \infty} T^n F_1 = F. \tag{40}$$

Furthermore, F possesses a continuous density function which is positive on the compact subset $\Omega \equiv [\underline{\epsilon}, \bar{m}]$ of R^+ where \bar{m} is the unique solution to

$$\bar{m} = h(\bar{m}) + \bar{\epsilon}. \tag{41}$$

Proof: By assumption, the stochastic kernel

$$K(m, m') \equiv G(n' - h(m)) \tag{42}$$

has a continuous density, denoted $k(m, m')$. It follows from Feller (1971), p. 272 that $K(m, m')$ is regular. That is, the family of transforms $\mu_t(\cdot)$ defined by

$$\mu_t(m') = \int k(m; m') \mu_{t-1}(m) dm; \quad \mu_0 = \delta_0; \tag{43}$$

for μ continuous and bounded is equicontinuous whenever μ_0 is uniformly continuous.

We must show that the compact set $\Omega \equiv [\underline{\epsilon}, \bar{m}]$ is the ergodic set of the Markov process (35) with transition probabilities $K(m, m')$, and that the complement of Ω is the transient set. This will insure that all of the probability weight of the stationary distribution lies in Ω , and that any point of Ω can be reached from any other point in that interval in a finite number of steps.

We first show that, once the process (35) has entered Ω , there is zero probability that it will depart from it (see Figure 4). It follows immediately from the corollary to Theorem 2 that there exists exactly one \bar{m} such that

$$h(m) + \bar{\epsilon} < m; \quad m > \bar{m}. \quad (44)$$

This implies that $h(m) + \bar{\epsilon} < \bar{m}$ for $0 < m < \bar{m}$. Now observe that for $m > \hat{m}$, $u'(c(m)) < \beta pu'(c(h(m) + \underline{\epsilon}))$ which implies that $m > h(m) + \underline{\epsilon}$. It follows that

$$\underline{\epsilon} < h(m) + \epsilon < \bar{m}. \quad (45)$$

That is, for M not in Ω and $m \in \Omega$

$$k(m, M) = dG(M - h(m)) = 0. \quad (46)$$

We next show that there is no transient subset of Ω . This follows from the fact that, for all $m \in \Omega$, $h(m) + \bar{\epsilon} > m$, $h(m) + \underline{\epsilon} < m$, and that c and dG are continuous. Hence, for all $m \in \Omega$, there exists an interval of positive length surrounding m , $I(m)$, such that $k(m, m') > 0$. These intervals cover Ω .

Finally, we show that any interval disjoint from Ω is a transient set. Any interval to the right of \bar{m} , say $[\bar{m}, m_1]$ as depicted in Figure 4, is characterized by the fact that, for all ϵ , $h(m) + \epsilon < m$. The process (35) will clearly leave such intervals in a finite number of steps with probability one. That is, there exists an N such that, for all $n > N$,

$$T^n[h(m) + \epsilon] < \bar{m} \quad (47)$$

for all possible realization of ϵ . Hence, $[\bar{m}, m_1]$ is transient (probability one of leaving, zero probability of re-entering). Exactly the same reasoning applies to intervals to the left of Ω . Q.E.D.

Corollary: There exists a unique, almost-everywhere continuous, increasing stationary distribution, denoted $X(h)$, which characterizes optimal money holdings. The support of $X(h)$ is the compact interval $[0, h(\bar{w})]$ of R . X has a single mass point at $h = 0$ such that

$$X(0) \equiv \text{Prob}(h = 0) = F(\hat{m}). \quad (48)$$

In general, $X(h')$ is given by

$$X(h') \equiv F(h^{-1}(h')). \quad (49)$$

Proof: From the Corollary to Theorem 2, $h(m) = 0$ for $0 < m < \hat{m}$ which implies (48). $h(m)$ is continuous and strictly increasing over the compact interval $[\hat{m}, \bar{m}]$ so that the inverse $h^{-1}(h')$ exists and is right continuous at $h' = 0$, which implies (49) and the right continuity of X at 0. Q.E.D.

The shape of the probability distribution which characterizes optimal money holdings is shown in Figure 5. There is a mass point at zero and then a smooth distribution on $(0, h(\bar{m})]$. The intuition is straightforward. Money is an inventory which is held to self-insure against adverse income fluctuations. Since the rate of time preference is positive, the optimal inventory policy cannot be bounded away from zero. In other words, if the individual were setting money demand so that, with probability one, money balances were never exhausted, he could increase his expected discounted utility by holding less money and spending more on consumption. This is the result obtained by Lucas (1980) in a cash-in-advance model in which preferences, not income, are random.

4: Concluding Remarks

In this paper, we have examined the optimal spending behavior and money holdings of a risk averse individual who faces liquidity constraints and random fluctuations in his money income. Because of a cash-in-advance constraint, the individual has a well-defined transactions requirement for money balances. In addition, because money income is uncertain and money is - by assumption - the only available store of value, the risk averse individual also holds money balances as an inventory which can be drawn down in periods of unexpectedly low earnings. We have shown that virtually all of the properties of optimal expenditure and money demand decisions derived by Helpman (1981) for the deterministic, constant income carry over to the decisions of a risk averse individual who confronts random fluctuations in his money income. We also showed that, in the presence of random income fluctuations, the risk averse individual's desired or target level of money balances is unique and depends directly on the dispersion of the probability distribution which govern these fluctuations. Finally, we established the existence, continuity, and monotonicity properties of the unique stationary probability distributions which characterize the behavior of optimal money holdings and beginning-of-period money balances in a stochastic steady-state. We showed that the limiting distribution which characterizes beginning-of-period money balances is continuous and strictly increasing over its compact support. By contrast, the stationary distribution which characterizes optimal money holdings was shown to be almost-everywhere continuous with a single mass point at zero.

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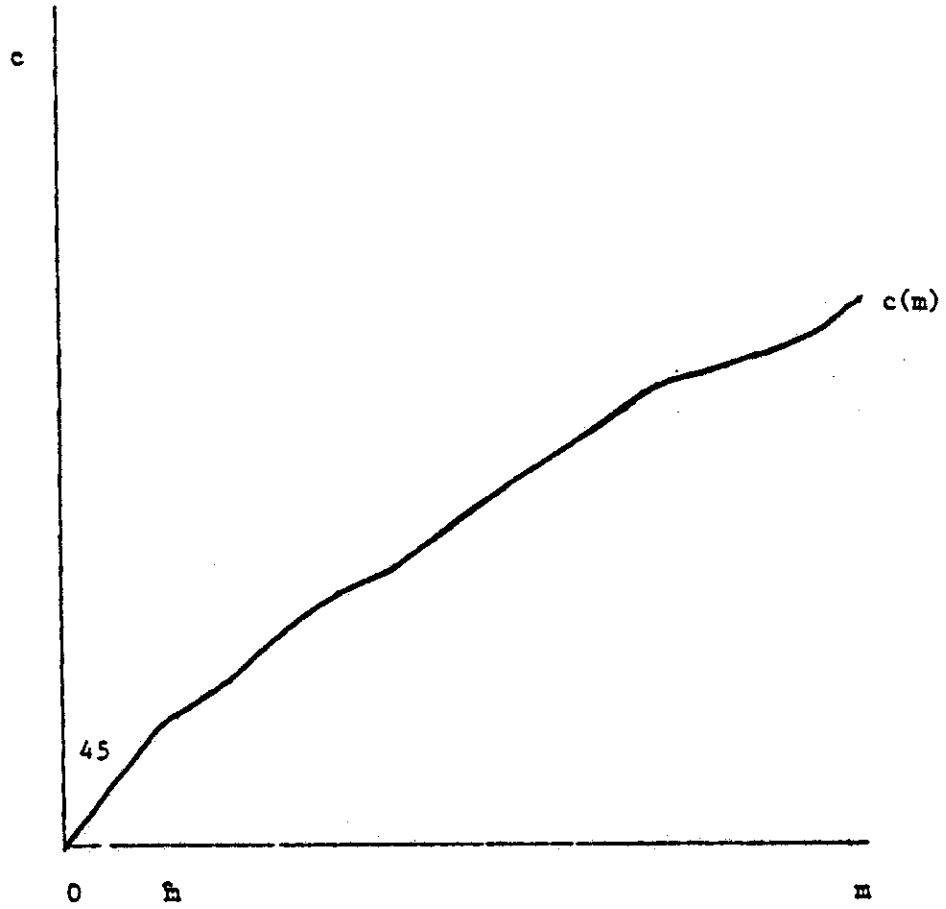
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NOTES

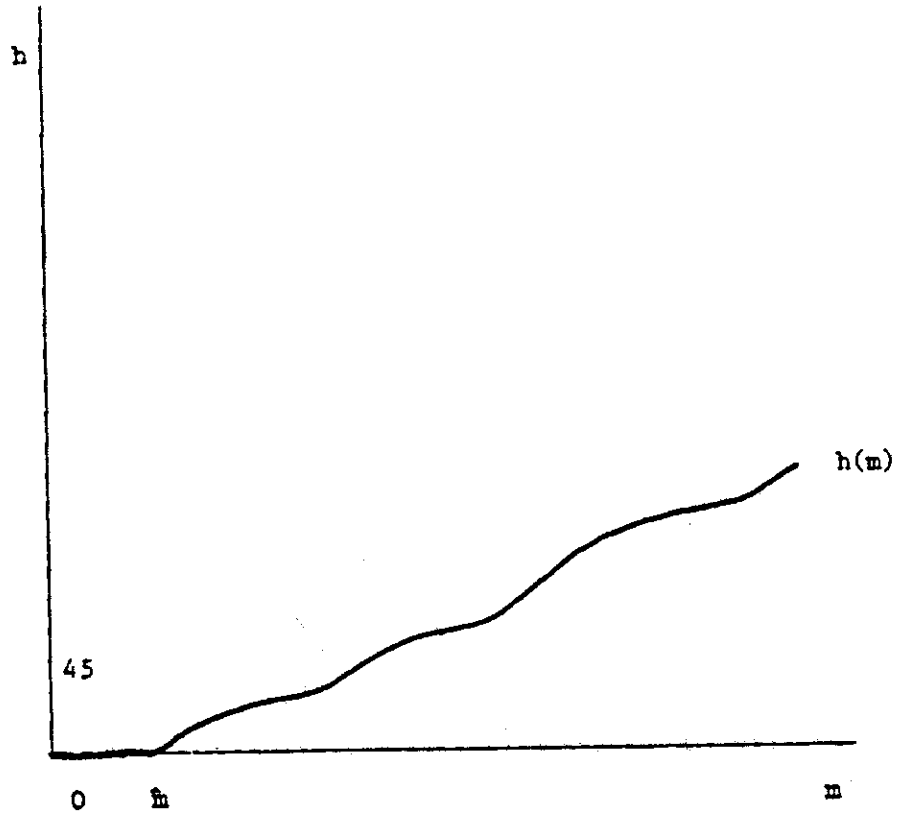
- (1) Helpman (1981) also shows that the marginal propensity to spend out of anticipated earnings exceeds the marginal propensity to spend out of money balances. It is not clear that this is even a meaningful experiment in a stochastic context since the agent is assumed to know the stationary probability distribution which governs his random income.
- (2) In Lucas (1980), the agent-specific disturbances are to preferences while in Bewley (1980) each agent's endowment of labor is random. Although we do not do so here, it is possible to extend the Lucas (1980) proof of existence in a cash-in-advance economy to the income fluctuations case studied in this paper.
- (3) However, using (23) and (24) it is possible to show that the marginal propensity to spend out of expected earnings does exceed the marginal propensity to spend out of cash balances. See note (1).
- (4) The proof can be found in Miller (1976), Theorem 2, pp. 163 - 165.
- (5) See Bewley (1980) for an in-depth analysis of the self-insurance motive for holding money.
- (6) In particular, under the assumptions outlined in the text, Schechtman and Escudero (1977) show that money holdings evolve according to a delayed renewal process.

Figure 1



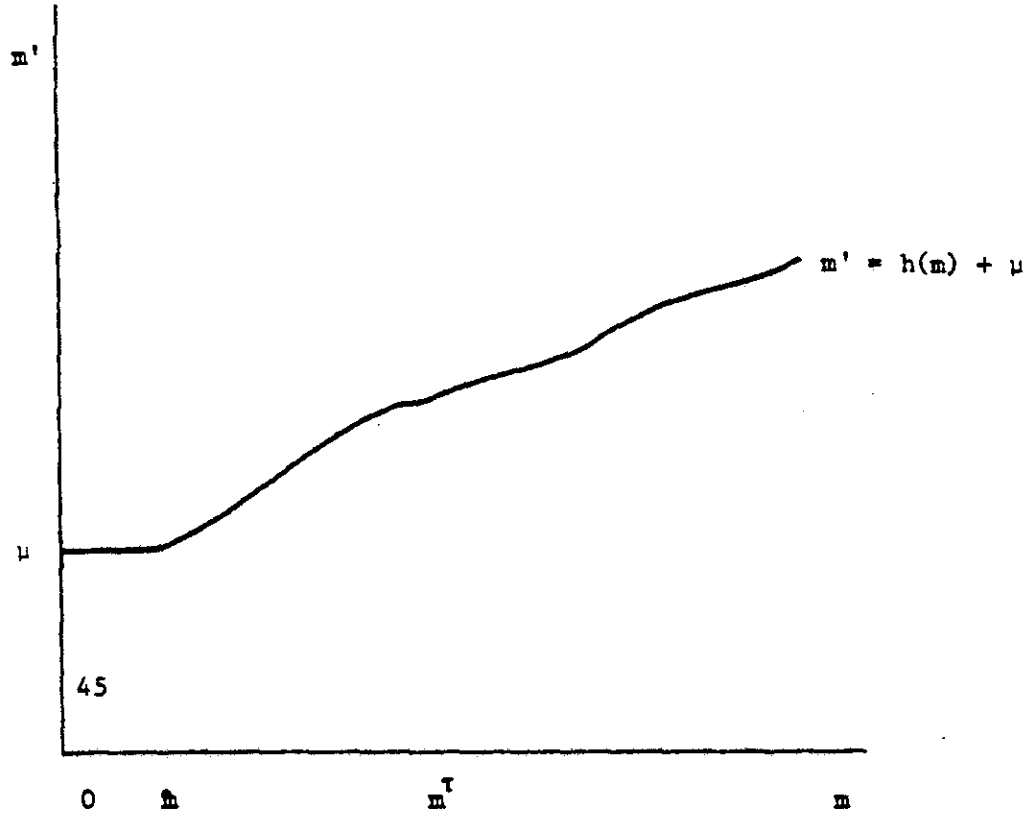
The optimal expenditure function $c(m)$.

Figure 2



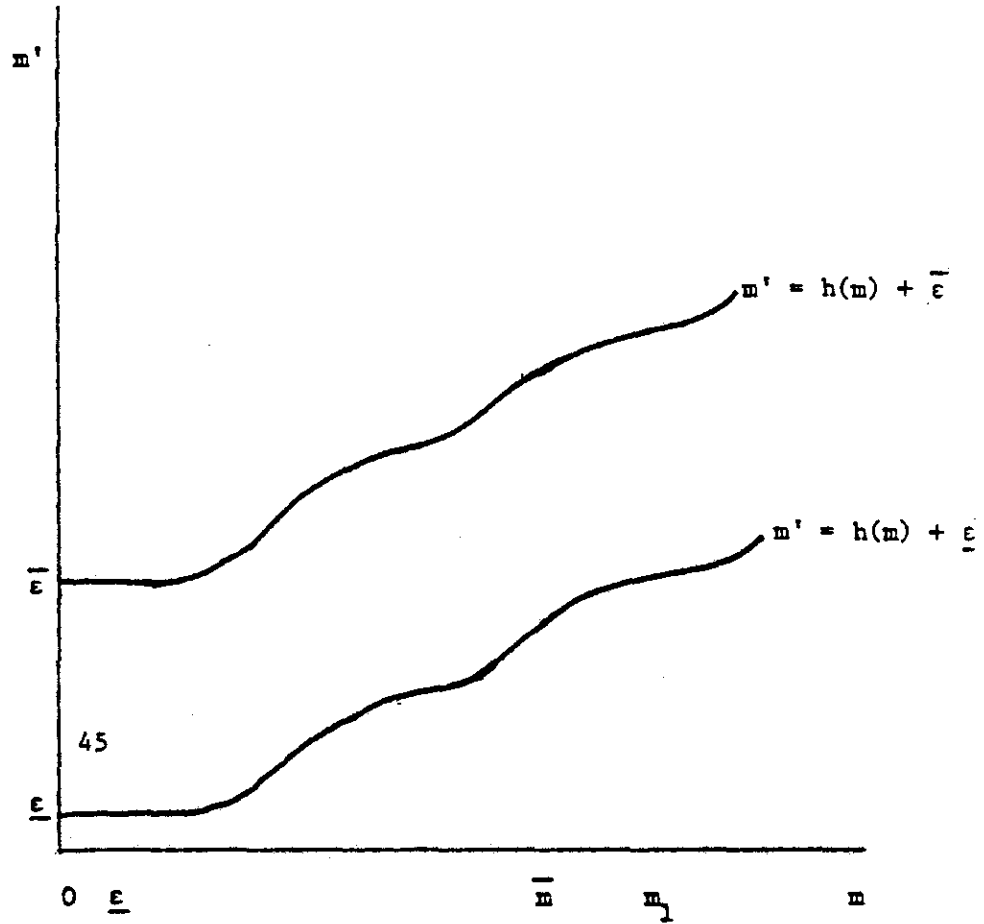
The optimal money demand function $h(m)$.

Figure 3



The target level of money balances m^T .

Figure 4



The ergodic set $\Omega = [\underline{\epsilon}, \bar{m}]$.