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CONSUMPTION, LIQUIDITY CONSTRAINTS, AND ASSET ACCUMULATION IN THE PRESENCE OF RANDOM INCOME FLUCTUATIONS

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by

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1. Introduction

Recent empirical research [Flavin (1981), Hagashi (1982)] has rejected the certainty-equivalent formulation of permanent income hypothesis [Hall (1978)]. These findings are often attributed to households' inability to borrow completely against expected future labor income. This paper is a theoretical investigation of optimal consumption behavior under risk aversion, random income fluctuations, and borrowing restrictions. Our principle objective is to establish the existence and to investigate the properties of the stationary probability distribution which characterizes the asymptotic behavior of consumption under these conditions.

Schechtman (1976) proves that optimal consumption converges almost surely to mean income if the rate of interest on savings is zero, the (infinitely lived) household does not discount future utility, borrowing is prohibited, and income is an i.i.d. random variable. Bewley (1976) generalizes Schechtman's result to the case in which income is a stationary stochastic process. In both analyses, asset holdings converge almost surely to infinity so that, asymptotically, the household is able to self-insure completely. Foley and Hellwig (1975) and Schechtman and Escudero (1977) generalize Schechtman (1976) by establishing the existence of a limiting distribution for wealth if future utility is discounted at a positive rate. Schechtman and Escudero also prove that the wealth accumulation process is bounded so long as the rate of time preference strictly exceeds the constant rate of interest on savings and the elasticity of marginal utility is itself bounded. Our contribution to this literature is to establish the existence and a number of properties of the stationary probability distribution which characterizes the asymptotic behavior of consumption in a version of Schechtman and Escudero's (1977) model which allows for lending as well as borrowing in amounts which can be repaid with probability one at a constant, positive rate of interest.

In Section 2, we establish the existence of a unique stationary probability distribution which characterizes the asymptotic behavior of consumption, relaxing Foley and Hellwig's (1975) and Schechtman and Escudero's (1977) assumptions that the rate of interest is zero and that borrowing is not allowed.

In Section 3, we derive some basic properties of this distribution. We show that in the stochastic steady-state, there is a strictly positive probability of being liquidity constrained, by which we mean that discounted expected marginal utilities of consumption are not equated across time. We also establish that if the rate of interest is zero — the case studied by Foley and Hellwig (1975) and Schechtman and Escudero (1977) — expected asymptotic consumption equals mean income independently of utility function curvature, the rate of time preference, and the probability distribution of productivity shocks. In the more general case of a positive interest rate, the relationship between expected asymptotic consumption and mean labor income will depend upon the rate of time preference, utility function curvature, and the probability distribution of labor income drawings. In particular, we show that if the rate of time preference is sufficiently large, consumption is an i.i.d. random variable in the stochastic steady-state with mean strictly less than expected labor income.

In Section 4, we begin by comparing the expected asymptotic consumption behavior of two individuals who differ in their pure rates of time preference. We prove that the expected asymptotic consumption of the low time preference individual is greater than or equal to that of the high time preference individual, even though the former consumes less at any given level of wealth. We conclude this section by comparing the asymptotic consumption and accumulation behavior of two individuals who confront probability distributions for labor income which differ by a location parameter. If the individuals are able to borrow completely against certain future labor income, we establish that the stationary probability distributions which characterize their consumption in the stochastic steady-state are identical. However, the stationary probability distributions which characterize the asset accumulation of two such individuals are shown to differ by a location parameter equal to the annuity value of the difference in the certain component of their labor incomes. This implies that differences in certain future labor income are, on average in the stochastic steady-state, offset by differences in asset accumulation. The variance and higher central moments of the stationary distribution for asset holdings are thus invariant to differences in certain future labor income.

2. An 'Income Fluctuations' Problem

We consider the following 'income fluctuations' problem. An individual with an infinite planning horizon must decide at the beginning of each period how much to consume and how many financial claims to purchase or sell. A financial claim costs one consumption good and entitles its owner to $\rho \equiv 1+r$ consumption goods next period, where r is the constant, non-negative rate of interest on consumption loans. Total resources available for consumption and saving in period t, w_t, are

$$(1) \qquad w_t = \rho a_t + s_t + \phi ,$$

where a_t is the stock of financial claims purchased at the beginning of period $t-1$, s_t is the wage received at the beginning of period t (the inelastic supply of labor being normalized to unity), and ϕ is a non-negative, finite limit on borrowing. Consumption of $c_t \leq w_t$ goods in period t yields utility of $\beta^t u(c_t)$ where $\beta \equiv 1/1+\delta$, δ the positive rate of time preference. The individual accumulates financial claims and wealth according to

$$(2) \qquad a_{t+1} = \rho a_t + s_t - c_t ,$$

$$(3) \qquad w_{t+1} = \rho(w_t - c_t) + s_{t+1} - r\phi .$$

Formally, the individual's problem is

$$\max E \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to $c_t + a_{t+1} = \rho a_t + \varepsilon_t$,

$$c_0 + a_1 = \varepsilon_0 ,$$

$$c_t \geq 0 , \quad a_t + \phi \geq 0 .$$

We shall find it useful to make the following standard assumptions:

(A1) $u(c)$ is a strictly increasing, strictly concave, bounded and twice continuously differentiable with $u'(0) = \infty$ and $u'(\infty) = 0$,

(A2) The income received in each period is an independent and identically distributed random variable. Furthermore, there exists finite $\underline{\varepsilon}$ and $\bar{\varepsilon}$ for which the stationary cumulative distribution function of ε , denoted G , satisfies:

$$G(\varepsilon) = 0 , \quad \varepsilon \leq \underline{\varepsilon} < 0 , \quad G(\varepsilon) = 1 , \quad \delta \geq \bar{\varepsilon} > 0 ,$$

$$dG(\varepsilon) \geq 0 , \quad \text{continuously for all} \quad \varepsilon - |\underline{\varepsilon}, \bar{\varepsilon}| ,$$

(A3) The limit on borrowing satisfies:

$$0 \leq \phi < \underline{\varepsilon}/r ,$$

a restriction which insures solvency with probability one.

Let $v(w)$ be the value of the objective function of an individual who begins the period with resources w and behaves optimally. This function must satisfy:

$$(4) \qquad v(w) = \max_{c \leq w}\{u(c) + \beta \int v(\rho(w-c) + \varepsilon' - r\phi)dG(\varepsilon')\} .$$

Denote the optimal consumption and asset accumulation decision rules by

c and g respectively. The following proposition establishes the existence and properties of optimal consumption and asset accumulation decisions in the presence of random income fluctuations and a borrowing limit which satisfies (A3).

Proposition 2.1: *By Assumptions* (A1), (A2), *and* (A3), *there exists a unique solution to the agent's problem. There is a unique, bounded, continuous, strictly increasing, strictly concave, and once continuously differentiable function* v *such that*

$$(5) \qquad v(w) = \max_{c \leq w}\{u(c) + \beta\int v(\rho(w-c) + \varepsilon' - r\phi)dG(\varepsilon')\} .$$

There also exists a unique, continuous, and strictly increasing function c *such that*

$$(7) \qquad v'(w) = u'(c(w)) .$$

If $r < \delta$, *there exists a unique* $\hat{w}(\delta)$ *which solves*

$$(8) \qquad u'(\hat{w}) = \beta\rho\int v'(\varepsilon' - r\phi)dG(\varepsilon') ,$$

and for all $\varepsilon - r\phi \leq w \leq \hat{w}$, *optimal consumption is given by*

$$(9) \qquad c(w) = w .$$

For all $w > \hat{w}$, *the optimal consumption decision rule satisfies* $\hat{w} < c(w) < w$ *and is uniquely defined by*

$$(10) \qquad u'(c(w)) = \beta\rho\int v'(\rho(w - c(w)) + \varepsilon' - r\phi)dG(\varepsilon') .$$

Proof. With the exception of the strict concavity of v and the strict monotonicity of c, all results are proved in Schechtman and Escudero (1977). Following Lucas (1980), let L be the space of continuous, bounded functions $u : R^+ \to R$ normed by

$$(11) \qquad ||u|| = \sup_w |u(w)| \; .$$

Define T as the operator on L such that (5) reads $v = Tv$. Using Berge (1963), $T : L \to L$. Using Blackwell (1965), T is a contraction so that $Tv = v$ has a unique solution $v^* \in L$ and $||T^n u - v^*|| \to 0$ as $n \to \infty$ for all $u \in L$. Lucas, Prescott and Stokey (1983) prove that T takes concave functions of w into strictly concave functions of w so that v^* is strictly concave. Q.E.D.

Corollary 2.2: There exists a unique, continuous, and non-decreasing function g such that

$$(13) \qquad v(w) = u(w - g(w) - \phi) + \beta\int v(\rho(g(w) + \phi) + s' - r\phi)dG(s') \; ,$$

that is, g is the optimal asset accumulation decision rule. For all $s - r\phi \le w \le \hat{w}$, optimal asset accumulation is given by

$$(14) \qquad g(w) = -\phi \; .$$

For all $w > \hat{w}$, g is strictly increasing and is uniquely defined by

$$(15) \qquad u'(w - g(w) - \phi) = \beta\int v'(\rho(g(w) + \phi) + s' - r\phi)dG(s') \; .$$

The function c and the cumulative distribution function G of s together define a Markov process

$$(16) \qquad w_{t+1} = \rho(w_t - c(w_t)) + s_{t+1} - r\phi$$

with state space R^+. That is, given an initial distribution for total wealth $F_0(w)$, the distribution $G(s)$ and the difference equation (16) together determine the sequence of distributions $F_1(w)$, $F_2(\ )$, ... which prevail at dates $t = 1, 2, \ldots$. If this sequence converges, its limit is the stationary distribution for wealth. Schechtman and Escudero (1977) provide the following restrictions on preferences which insure that the wealth accumulation process is bounded:

(A4) The elasticity of $u'(c)$ is uniformly bounded; i.e., there exists a $\bar{c}$ s.t. for all $c > \bar{c}$:

$$-cu''(c)/u'(c) \leq M < \infty ,$$

(A5) The rate of time preference strictly exceeds the real rate of interest on consumption loans; i.e.,

$$0 \leq r < \delta .$$

Schechtman and Escudero (1977) go on to show that (A1), (A4), and the additional assumptions that the income received in each period is a countable random variable, that borrowing is prohibited and that the rate of interest is zero are sufficient to prove the existence of a limiting distribution for total wealth.

Our first task shall be to relax these latter three assumptions and to prove the existence of a unique, continuous limiting distribution for total wealth under assumptions (A1) through (A5). Armed with these results and the properties of c and g, we than establish the existence of unique, stationary cumulative distribution functions which characterize asymptotic consumption and asset accumulation behavior in a stochastic steady-state.

To say that a sequence $w_0, w_1, \ldots$ is subject to the transition probabilities K means that $K(w, w')$ is the conditional probability of the event $\{w_1 \leq w'\}$ given that $w_0 = w$. Equation (16) and the definition of G imply that

$$(17) \qquad K(w, w') = G(w' - h(w) + r\phi) ,$$

where $h(w) \equiv w - c(w)$. If the probability distribution of w_0 is F_0, the distribution of w_1 is given by

$$(18) \qquad F_1(w') = \int_\Omega F_0\{dw\}K\{w, w'\} .$$

<u>Definition</u>: The distribution F is a stationary distribution for K if $HF = F$, where the operator H is defined by

$$(19) \qquad HF(w') = \int_\Omega F\{dw\}K\{w, w'\} .$$

Following the approach suggested by Mendelssohn and Sobel (1980) as adapted from Feller (1971), Donaldson and Mehra (1983), and Rosenblatt (1967), we now prove the following theorem.

<u>THEOREM 2.1</u>: <u>By Assumptions</u> (A1) <u>through</u> (A5), <u>there exists a continuous and increasing stationary distribution function for wealth, denoted</u> $F(w)$, <u>which is the unique solution to the functional equation</u> $HF = F$:

$$(20) \qquad F(w') = \int_\Omega G(w' - \rho h(w) + r\phi)dF(w) .$$

<u>For any</u> F_0,

(21) $\lim_{n\to\infty} H^n F = F$.

__Furthermore__, F __possesses a continuous density function which is positive on the compact subset__ $\Omega \equiv [\underline{s} - r\phi, \bar{w}]$ __where__

(22) $\bar{w} = \min[w : \rho h(w) + \bar{s} - r\phi = w]$.

__Proof__. By Assumption (A2), the stochastic kernel

(23) $K(w, w') \equiv G(w' - \rho h(w) + r\phi)$

has a continuous density, denoted $k(w; w')$. We first show that K is regular in the sense of Feller (1971), p. 272. That is, we must show that the family of transforms $\mu_t(\ )$ defined by

(24) $\mu_t(w') = \int k(w, w')\mu_{t-1}(w)dw$, $\mu_0 = 0$

for μ continuous and bounded is equicontinuous whenever μ_0 is uniformly continuous on any closed interval $[\underline{w},\bar{w}]$.

Let $M = \max_{w\in[\underline{w},\bar{w}]} |\mu_0(w)|$. By the recursive definition of $\mu_t(\ )$, $\forall t$ $|\mu\ (w)| \leq M$. Then

(25) $$|\mu_t(w) - \mu_t(w')| = |\int k(v,w)\mu_{t-1}(v)dv - \int k(v,w')\mu_{t-1}(v)dv|$$
$$\leq \int|k(v,w) - k(v,w')||\mu_{t-1}(v)|dv$$
$$< e$$

for some δ sufficiently small to ensure $|w-w'| < \delta \rightarrow |k(v,w)-k(v,w')| < e/M$, $\forall v$. Such a δ is possible as k is uniformly continuous on $[\underline{w},\bar{w}]$.

We next show that, once the process (16) has entered Ω , there is zero probability that it will depart from it (see Figure 3). Schechtman and Escudero (1977), Theorem 3.9, p. 162 show that Assumptions (A4) and (A5) are sufficient to ensure that there exists a $\tilde{w} < \infty$ such that

$$(26) \qquad \rho(w - c(w)) + \bar{\varepsilon} - r\phi < \tilde{w} , \quad w > \tilde{w} .$$

This implies that there exists a unique $\bar{w}$ which solves (22) and that

$$(27) \qquad \rho(w - c(w)) + \bar{\varepsilon} - r0 \leq \bar{w} , \quad w \leq \bar{w} .$$

Observe that for $w > \hat{w}$, $u'(c(w)) = \beta\rho Eu'(c(\rho h(w) + \varepsilon' - r\phi))$ by the envelop theorem. This implies that $u'(c(w)) < \beta\rho u'(c(\rho h(w) + \underline{\varepsilon} - r\phi))$ so that $w > \rho h(w) + \underline{\varepsilon} - r\phi$. It follows that, for $w \in \Omega$

$$(28) \qquad \underline{\varepsilon} - r\phi \leq \rho h(w) + \varepsilon' - r\phi \leq \bar{w} .$$

That is, for W not in Ω and $w - \Omega$

$$(29) \qquad k(w,W) = dG(W - \rho h(w) + r\phi) = 0 .$$

We now show that any interval disjoint from Ω is a transient set. There are two types of intervals to consider, depicted as $[\bar{w}, w_1]$ and $[w_1, w_2]$ in Figure 3. Suppose first that, for some time t, $w_t = w_0 \in [\bar{w}, w_1]$; we must show that the process passes out of $[\bar{w}, w_1]$. Consider any sequence $\{\varepsilon_{t+n}\}$, $n = 1, 2, \ldots$ and form the sequences w_{t+n} and $w_{t+n}(\bar{\varepsilon})$ defined, respectively, by

$$(30) \qquad w_{t+n} = \rho h(w_{t+n-1}) + \varepsilon_{t+n} ,$$

$$(31) \qquad w_{t+n}(\bar{\varepsilon}) = \rho h(w_{t+n-1}(\bar{\varepsilon})) + \bar{\varepsilon} , \quad w_t = w_t(\bar{\varepsilon}) = w_0 .$$

Without loss of generality, assume $\varepsilon_{t+1} < \bar{\varepsilon}$, then $\forall n$, $w_{t+n} < w_{t+n}(\bar{\varepsilon})$. By construction $w_{t+n}(\bar{\varepsilon}) \to \bar{w}$ so that for some N, $w_{t+n} < \bar{w}$ with probability one. Thus $[\bar{w}, w_1]$ is a transient set. Suppose now that

$w_t = w_0 \in [w_1, w_2]$. Since $w > \rho h(w) + \underline{\varepsilon} - r\phi$, there exists by continuity an $\hat{\varepsilon} > \underline{\varepsilon}$ such that, for all $w \in [w_1, w_2]$, $\rho h(w) + \hat{\varepsilon} < w$. Consider any sequence $\{\varepsilon_{t+n} \leq \hat{\varepsilon}\}$, $n = 1, 2 \ldots$. By construction $w(\hat{\varepsilon})$ converges to some point to the left of w_1 , so that for some N , $w_{t+n} < w_1$ so long as each $\varepsilon_{t+n} \leq \hat{\varepsilon}$. The probability of this occurrence is at least ξ^N where

$$(32) \qquad \xi = \int_{\underline{\varepsilon}}^{\hat{\varepsilon}} dG(\varepsilon) > 0 .$$

Thus with probability at least ξ^N , the process leaves $[w_1, w_2]$ never to return. The expected number of visits to $[w_1, w_2]$ is less than

$$(33) \qquad \sum_{j=1}^{\infty} (1 - \xi^N)^j < \infty .$$

Hence, $[w_1, w_2]$ is transient.

We finally show that the set $[\underline{\varepsilon} - r\phi, \bar{w}]$ is irreducible [Rosenblatt (1967)] by which we mean that any open set $A \subseteq [\underline{\varepsilon} - r\phi, \bar{w}]$ can be reduced from any point $w \subseteq [\underline{\varepsilon} - r\phi, \bar{w}]$ in a finite number of steps with positive probability. Pick an open subset $A = (w_3, w_4)$ and a point $w_0 \notin A$ in Ω . Since A is open, for some w^* A , we can find a neighborhood $J(w^*, \eta) = (w^* - \eta, w^* + \eta)$ such that $J(w^*, \eta) \subseteq A$ and thus $w^* - \eta > w_1$. Consider the following set

$$J^{\varepsilon}(w^*, \eta) = \{\varepsilon : \rho h(w) + \varepsilon = w \text{ for some } w \in J\} .$$

We consider first the case in which J^{ε} has positive measure. Since G is continuous

$$(34) \qquad \int_{J^{\varepsilon}} dG(\varepsilon) = \omega > 0 .$$

Suppose that at some time t, $w_t = w_0$ and for each $s - J^s$ consider the sequence $\{w^s_{t+n}\}$, $n = 1, 2, 3, \ldots$ defined by $w^s = \rho h(w^s_{t+n-1}) + s - r\phi$. For each s there is a $T(s)$ such that for $n \geq T(s)$, $w^s_{t+n} > w_3$ and let $T^* - \sup_{s \in J^s} \{T(s)\}$. By construction $T^* < \infty$. Thus

$\text{Prof}(w_{t+T^*} \in A | w_t = w_0) \geq \omega^{T^*+1} > 0$.

Consider next the case in which J^s has only one element. Using the arguments of the previous paragraph, there is a positive probability of reaching some interval to the right of A, say $B = (w_5, w_4)$, in a finite number of steps. Now let $w_t \in B$. The process $w_{t+n} = \rho h(w_{t+n-1}) + \underline{s} - r\phi$ converges to $0 < w$ and by continuity, there exists $\hat{\hat{s}} > \underline{s}$ such that $w_{t+n} = \rho h(w_{t+n-1}) + \hat{\hat{s}} - r\phi$ converges to the left of (w_3, w_4). Thus, there is a positive probability that, starting from $w_t \in B$, the process enters A in a finite number of steps. Q.E.D.

Consumption and asset holdings also evolve according to Markov processes which are given by

$$(35) \qquad c_{t+1} = c(\rho(c^{-1}(c_t) - c_t) + s_{t+1} - r\phi) ,$$

$$(36) \qquad a_{t+1} = \rho a_t + s_t - c(\rho a_t + s_t + \phi) .$$

The existence of limiting distributions which characterize the asymptotic behavior of consumption and asset accumulation in the stochastic-steady state follow immediately from the theorem.

Corollary 2.3: *There exists a unique, continuous, increasing stationary distribution, denoted* $J(c)$, *which characterizes the behavior of consumption in the stochastic steady-state. The support of* $J(c)$ *is the compact interval* $[s - r\phi, c(\bar{w})]$. $J(c)$ *is given by*

$$(37) \qquad J(c') \equiv \text{Prob}(c_t \leq c') = F(c^{-1}(c')) .$$

Corollary 2.4: *There exists a unique, almost-everywhere continuous, increasing stationary distribution, denoted* $X(a)$, *which characterizes asset accumulation in the stochastic steady-state. The support of* $X(a)$ *is the compact interval* $[-\phi, g(\bar{w})]$ *of* R . X *has a single mass point at* $a = -\phi$ *such that*

$$(38) \qquad X(-\phi) \equiv \text{Prob}(a = -\phi) = F(\hat{w}) .$$

For $a > -\phi$, $X(a)$ *is defined by*

$$(39) \qquad X(a') \equiv \text{Prob}(a \leq a') = F(g^{-1}(a')) .$$

Proof. $g(w) = -\phi$ for $0 \leq w \leq w$ which implies (38). $g(w)$ is continuous and strictly increasing over the compact interval $[\hat{w},\bar{w}]$ so that the inverse $g^{-1}(a')$ exists and is right continuous at $a' = -\phi$ which implies (39) and the right continuity of X at $-\phi$. Q.E.D.

3. Consumption in the Stochastic Steady-State: Basic Properties

In this section, we derive some basic properties of the stochastic steady-state behavior of consumption. We begin with the following result.

Theorem 3.1: There is a strictly positive probability of being liquidity constrained in the stochastic steady-state this probability is given by

$$\text{(40)} \qquad \text{Prob}\left[u'(c_t) > \frac{1+r}{1+\delta}Eu'(c_{t+1})\right] = F(\hat{w}) > 0 \ .$$

Proof. Follows directly from Proposition 2.1, equation (8) and the result that the stationary distribution for wealth is strictly increasing over the ergodic set $[\underline{s} - r\phi, \bar{w}]$. Q.E.D.

Foley and Hellwig (1975) and Schechtman and Escudero (1977) study the special case in which $r = 0$. We now prove that in this special case, expected asymptotic consumption equals mean labor income independently of risk aversion, time preference, and the probability distribution of labor income drawings. This result is the analogue to Schechtman's (1976) result in the no-discounting case that consumption itself converges almost surely to mean income.

Theorem 3.2: If the rate of interest is zero, expected asymptotic consumption equals mean labor income for all u satisfying (A1), G satisfying (A2), and $\delta > 0$.

Proof. Asset holdings evolve according to the Markov process

$$\text{(41)} \qquad a_{t+1} = a_t + s_t - c(\rho a_t + s_t + \phi) \ .$$

By Corollary 2.4, there exists a unique limiting distribution for assets

$X(a)$ such that $\lim_{t\to\infty} X(a, t, a_0) = X(a)$ where $X(a', t, a_0)$ $\equiv \text{Prob}(a_t \leq a', a_0)$. By Feller (1971), Theorem 1, p. 249

(42) $$\lim_{t\to\infty} E(a, X(a, t, a_0)) = E(a, X(a)) ,$$

which, along with (41) implies

(43) $$\lim_{t\to\infty} E(c, J(c, t, c_0)) = E\varepsilon . \quad \text{Q.E.D.}$$

In general, the relationship between expected asymptotic consumption and mean labor income depends upon u , G , and δ . We now establish that for any given u , G , r , there exists a finite δ such that consumption is an i.i.d. random variable in the stochastic steady-state with mean strictly less than expected labor income.

<u>Theorem 3.3</u>: <u>There exists a finite</u> $\bar{\delta}$ <u>such that</u> if $\delta > \bar{\delta}$

(44) $$E(c, J(c,\delta,u,G,r) = E\varepsilon - r\phi$$

<u>for all</u> u <u>satisfying</u> (A1), G <u>satisfying</u> (A2), <u>and finite</u> r . <u>Furthermore</u>,

(45) $$J(c', \delta, u, G, r) = G(c' + r\phi) .$$

<u>Proof</u>. The ergodic set for wealth is given by $[\underline{\varepsilon} - r\phi, \bar{w}]$. The theorem will be true if there exists a $\bar{\delta}$ such that $\hat{w}_{\bar{\delta}} > \bar{\varepsilon}$ since this implies that wealth evolves into the interval $[\underline{\varepsilon} - r\phi, \bar{\varepsilon}]$. In this interval $c(w_t) = w_t$, $a_t = -\phi$ so that $w_t = \varepsilon - r\phi$. Now

(46) $$u'(\hat{w}) = \beta\rho\int v'(\varepsilon' - r\phi)dG(\varepsilon')$$

(47) $$\leq \beta\rho v'(\underline{\varepsilon} - r\phi)$$

(48) $$\leq \beta\rho u'\left[\frac{r}{1+r}(\underline{\varepsilon} - r\phi)\right]$$

where the last inequality follows from Schechtman and Escudero (1977), Theorem 1.1 and Lemma 2.2, pp. 153–155. Let w solve

$$(49) \qquad u'(\hat{\hat{w}}) = \frac{1+r}{1+\delta}u'\left[\frac{r}{1+r}\left(\underline{s} - r\phi\right)\right] .$$

$\hat{\hat{w}}$ is a continuous, strictly increasing function of δ with $\lim_{\delta\to\infty} \hat{\hat{w}}_{\delta} = \infty$. Also $\hat{w} \geq \hat{\hat{w}}$. So there exists a finite $\bar{\delta}$ such that $w \geq \hat{w}_{\delta} > \bar{s}$ for $\delta > \bar{\delta}$. In the stochastic steady-state, consumption is i.i.d. with $E(c, J(c,\delta)) = Es - r\phi$ and $J(c',\delta,r) = G(c' + r\phi)$. Q.E.D.

4. Consumption in the Stochastic Steady-State: The Role of Time Preference and Certain Labor Income

We now compare the expected asymptotic consumption of two individuals who differ only in their pure rates of time preferences. Clarida (1984) shows that the average propensity to consume is directly related to the rate of time preference. This result, along with Clarida's (1984), Theorem 4, p. 19, implies that the expected asymptotic asset holdings of low time preference individuals are no less than those of high time preference individuals. However, this reasoning does not necessarily apply to the probability distribution of consumption, for, although expected asset holdings are inversely related to time preference and consumption is a strictly increasing function of assets, less is consumed at each level of asset holdings given a lower rate of time preference. We now establish that, in the income fluctuations problem studied here, expected asymptotic consumption is itself inversely related to the rate of time preference.

Theorem 4.1: Expected asymptotic consumption is inversely related to the rate of time preference.

Proof. Consider two otherwise identical individuals with rates of time preference δ_1 and δ_2 such that $\delta_1 < \delta_2$. Then, from Clarida (1985), Theorem 3, p. 17

(50) $c(w, \delta_2) = c(w, \delta_1) = w$, for $w \leq \hat{w}(\delta_2)$,

(51) $c(w, \delta_2) < c(w, \delta_1) \leq w$, for $w > \hat{w}(\delta_2)$.

From Clarida (1985), Corollary 4.2, p. 231

(52) $E(a, X(a, \delta_2)) \geq E(a, X(a, \delta_1))$.

Using Theorem 3.2

(53) $E(a, X(a, \delta_i)) = (1+r)E(a, X(a, \delta_i)) + Es - E(c, J(c, \delta_i))$,

for $\delta_i = \delta_1, \delta_2$. Thus

(54) $E(c, J(c, \delta_2)) = rE(a, X(a, \delta_2)) + Es$

$\geq rE(a, X(a, \delta_1)) + Es = E(c, J(c, \delta_1))$. Q.E.D.

We conclude with a comparison of the asymptotic consumption and accumulation behavior of two individuals who confront probability distributions for labor income which differ by a location parameter Ψ. We shall assume that the individuals can borrow completely against that part of their future labor income which is known with certainty. We shall also assume that $u'(0) < \infty$. The location parameter is thus equal to the difference in the individuals' certain labor income, the lower support of G.

We begin by showing that c, the optimal consumption function, is invariant to additive differences in the probability distribution governing income fluctuations.

<u>Theorem 4.2</u>: <u>Consider two otherwise identical individuals whose incomes are governed by the probability distributions</u> $G(s_t)$ <u>and</u> $G(s_t, \Psi)$ <u>respectively,</u> $\Psi > 0$, <u>such that</u> $G(s_t, \Psi) = G(s_t + \Psi)$. <u>The two individuals have identical consumption functions. That is,</u>

$$(55) \qquad c(w) = c(w,\Psi) \ .$$

<u>Proof</u>: From Theorem 1 we know that a necessary and sufficient characteristization of optimal consumption is given by

$$(56) \qquad u'(c(w)) = \beta\rho\int v'(\rho(w - c(w)) + s' - r\phi)dG(s') \ ,$$

for $w > \hat{w}$, $c = w$ otherwise. The theorem follows directly from the assumption that $\phi(\Psi) = \phi + \Psi/r$ and definition of $G(s,\Psi)$. Together these imply that the random variables $s' - r\phi$ and $s' + \Psi - r\phi(\Psi)$ are identical. Q.E.D.

<u>Corollary 4.3</u>: <u>At any given level of wealth, the agent with the greater certain income purchases fewer financial assets. Furthermore,</u>

$$(57) \qquad g(w,\Psi) - g(w) = -\Psi/r \ .$$

<u>Proof</u>. Follows directly from the theorem and the budget constraint $c + g = w - \phi(\Psi)$.

These results make sense. So long as the individual with the greater certain income is allowed to borrow against it — to a limit of Ψ/r which can be repaid with probability one — the stationary statistical decision problems faced by him and another individual with lesser certain income are identical except for initial conditions. Thus, the optimal consumption functions coincide. However, it is the case that the individual with the greater certain income does consume more initially, accomplishing this by borrowing against his greater human capital.

The next theorem establishes the stochastic steady-state implications of these results.

<u>Theorem 4.4</u>: <u>Consider two otherwise identical individuals whose income fluctuations are governed by the probability distributions</u> $G(s_t)$ <u>and</u> $G(s_t, \Psi)$ <u>respectively. The stationary probability distributions which characterize these individuals' consumption in stochastic steady-state are identical,</u> $J(c) = J(c,\Psi)$. <u>The stationary probability distributions which characterize these individuals' asset accumulation differ by the location parameter</u> Ψ/r . <u>In particular,</u>

(58) $$X(a,\Psi) = X(a + \Psi/r) .$$

<u>Proof</u>. The stationary probability distribution which governs total wealth is the unique solution to the functional equation

(59) $$F(w') = \int G(w' - \rho h(w) + r\phi)dF(w) .$$

The solution to equation (58) depends only on G and h , which by Theorem 4.2, is independent of Ψ . Thus $J(c') = F(c^{-1}(c'))$ is invariant with respect Ψ so long as the individual can borrow completely against the lower support of G . Using the fact that $g(w,|) = h(w) - \phi(\Psi)$, we obtain,

(60) $$X(a') = F[h^{-1}(a' + \phi)] ,$$

(61) $$X(a', \Psi) = F[h^{-1}(a' + \phi + \Psi/r)] .$$

It follows directly that

(62) $$X(a', \Psi) = X(a' + \Psi/r) . \quad \text{Q.E.D.}$$

<u>Corollary 4.5</u>: <u>Differences in certain labor income, on average, in the stochastic steady-state offset by asset accumulation of</u>

$$E[a,\phi] - E[a,\ \phi(\Psi)] = \Psi/r\ . \tag{63}$$

<u>Corollary 4.6</u>: <u>The central moments of the stationary probability distribution which governs the accumulation of assets are invariant to differences in certain labor income</u>:

$$E[a - E(a,\phi),\ \phi]^{j} = E[a - E(a,\ \phi(\Psi)),\ \phi(\Psi)]^{j}. \tag{64}$$

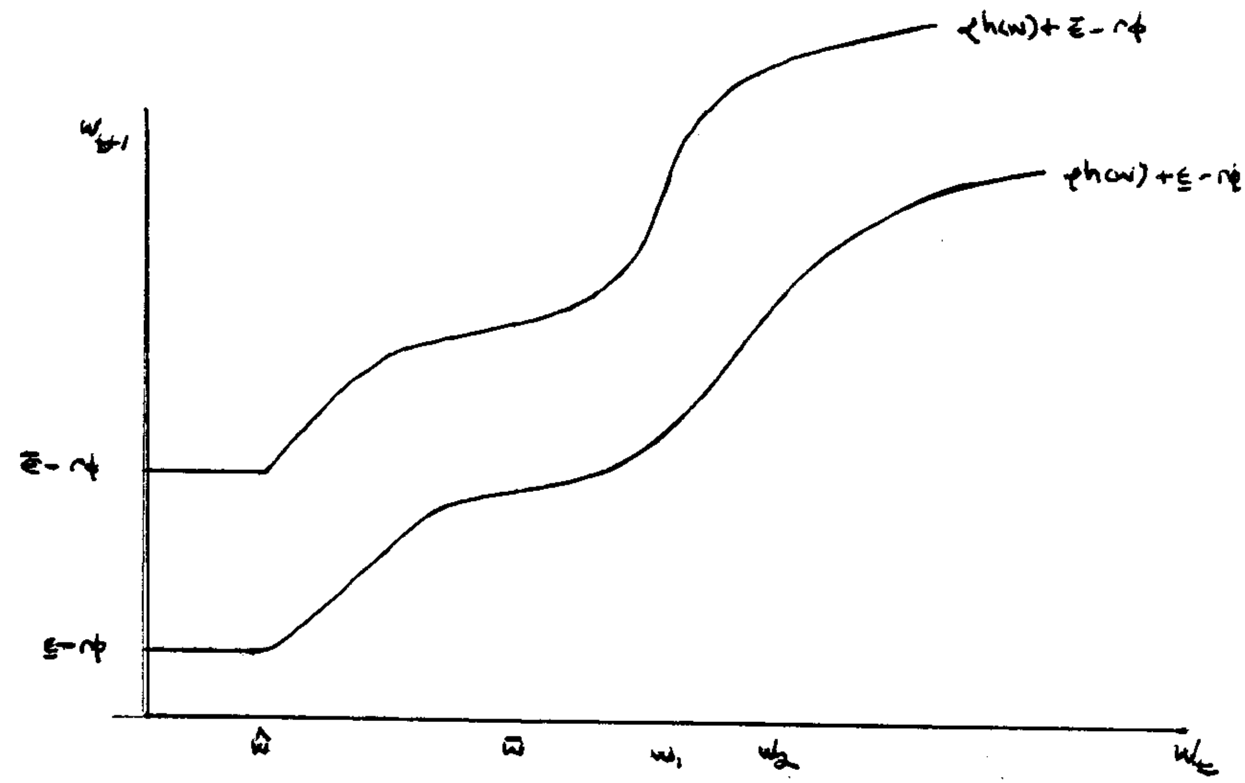
$\varphi h(w) + \bar{\varepsilon} - r\phi$
$\varphi h(w) + \underline{\varepsilon} - r\phi$
w_{t+1}
$\bar{\varepsilon} - r\phi$
$\underline{\varepsilon} - r\phi$
$\hat{w}$
$\bar{w}$
w_1
w_2
w_t

Figure 1

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