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THE DISCOUNTED CASE

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ABSTRACT

STOCHASTIC EQUILIBRIUM AND TURNPIKE PROPERTY:

THE DISCOUNTED CASE

Ramon Marimon

The existence of the modified golden rule and the turnpike property are proved for a multisector stochastic growth model. The (exogenous) stochastic environment is represented by a stationary stochastic process that influences preferences, technology and resources. A social planner maximizes the expected sum of discounted utilities.

The conditions required in order to obtain these results, are the natural strengthening of the stability conditions of the deterministic case. As in the deterministic case, the discount factor must be close to one in order to guarantee the almost sure (and in the mean) convergence of optimal interior programs. It is proved that all optimal interior programs converge to each other. This fact is used to prove the existence of a unique optimal stationary program (*the modified golden rule*). These results imply that all optimal interior programs converge to the stationary program (*the turnpike property*).

I. INTRODUCTION

The rational expectations hypothesis is appropriate when the economic environment is stationary or fluctuates according to a stationary distribution. If equilibrium prices and allocations follow a stationary process, then agents can learn from experience.

A rational expectations model with intertemporal production may be turned into a stochastic growth model by introducing a planner who maximizes the equilibrium social welfare function and by decentralizing intertemporal decisions by means of Malinvaud prices. By this device, a stochastic growth model can be used in analyzing the properties of a rational expectations equilibrium when all agents share the same information.

I study the existence of a stationary equilibrium (the modified golden rule) and the convergence of equilibrium programs to the stationary equilibrium (the turnpike property). The model is a multisector growth model. An exogenous stationary stochastic process influences preferences, technology and resources. A representative consumer or planner maximizes the expected sum of discounted utilities.

In a deterministic context, these two properties have been studied extensively. (For an excellent survey of turnpike theory, see McKenzie [1980].) Bewley has used the idea that an Arrow-Debreu model can be transformed into a growth model (see Bewley [1982], and also Yano [1983] and Coles [1983]).

In a stochastic environment, the existence of a stationary equilibrium and the turnpike property have been proved only for two classes of economies: (1) the one-good, one-planner model of Brock and Mirman (Brock and Mirman [1972], [1973]), and (2) the multisector one-planner model when future utilities are not discounted (Radner [1973], Jeanjean [1974], Evstigneev [1974], Dana [1974]). For the multisector model with discounting, only the existence of a weak form of stationarity has been proved (Majumdar and Radner [1983]).

In a deterministic growth model, one proves the existence of the modified golden rule by means of a fixed point argument involving initial capital stocks. In the stochastic case the stationary optimal capital stocks may depend on the past history of the exogenous stochastic process. This means that these capital stocks belong to an infinite dimensional space. This infinite dimensionality seems to preclude the use of a fixed point argument.

My approach is to prove, loosely speaking, that all optimal programs converge to each other. This fact implies that there exist a unique optimal stationary program and that all optimal programs converge to it.

In order to prove the convergence result, I use the value loss approach which has a long tradition in turnpike theory. My method of proof is partially based on that of Bewley [1982]. I show that the value losses form a positive supermartingale. This proof is quite involved; it is necessary to find bounds on the value loss. In order

to obtain these bounds, I show that any optimal program can be reached in a finite number of periods from any capital stock in a certain compact set.* In order to prove this last fact, I use special assumptions on the desirability of labor and its role in production.

My work is closely related to that of Brock and Majumdar [1978] and Majumdar and Radner [1983]. Their work differs from mine in that they use dynamic programming techniques. In that approach, end of period capital stocks are functions of the exogenous stochastic process and the beginning of period capital stock. In my model, capital stocks depend only on the history of the exogenous process. Majumdar and Radner proved the existence of an invariant distribution of capital stocks in a non-linear activity analysis model.

Majumdar and Radner's concept of a stationary distribution is a weaker concept than the concept of stationarity used here. The difference between the two concepts of stationarity can be illustrated in a deterministic model. In such a model, an invariant distribution could be a probability that distributes mass uniformly over a periodic cycle. In the deterministic case, my stationary equilibrium can only be a steady state.

It is not difficult to introduce into my model a finite number of firms and consumers; one would then obtain a stationary state which would be an equilibrium in a truly general equilibrium context.

*Yano used a similar idea for capital stocks close to the optimal steady state. This argument stimulated the method of proof described above (Yano [1983a, b]).

The rest of the paper is structured as follows: Section 2 contains the description of the model, notation and definitions. Assumptions and theorems are listed in Section 3. I discuss the assumptions in Section 4. In Section 5 I relate my work to the literature. Lemmas are stated and proved in Section 6. Finally, the theorems are proved in Sections 7 and 8.

II. THE MODEL: DEFINITIONS AND NOTATIONS

My model is similar to the one used by Radner (1973) and Bewley (1981) in their study of the stationary model. The main difference is the introduction of impatience in the model. Whenever both models agree, I follow Bewley's notation.

The Stochastic Environment

There is an exogenous stochastic process $\{s_t\}_{t=-\infty}^{\infty}$ which influences the utility function, endowments and production possibilities. The random variables, s_t , take values in some finite set M . The sample space of the process is $S = \{(\dots, s_{-1}, s_0, s_1, \dots) \mid s_t \in M, \text{ for all } t\}$. The σ -field generated by S is denoted \mathcal{J} (i.e., $\sigma(S) = \mathcal{J}$). \mathcal{J} is the smallest complete σ -field such that all the random variables s_t are measurable with respect to \mathcal{J} . Similarly, if $S_t = \{(\dots, s_{t-1}, s_t) \mid s_n \in M \forall n \leq t\}$, then $\mathcal{J}_t = \sigma(S_t)$. \mathcal{J}_t is the smallest complete σ -field with respect to which all the random variables $s_n, n \leq t$, are measurable. Clearly, $\{\mathcal{J}_t\}_{t=0}^{\infty}$ is an increasing family of σ -fields (i.e., $\mathcal{J}_t \subset \mathcal{J}_{t+1} \subset \mathcal{J}$). (An increasing family of σ -fields is sometimes called a *filtration* on (S, \mathcal{J}) .) \mathcal{J}_t represents the information available at time t of the process $\{s_n\}_{n=-\infty}^{+\infty}$.

A probability P is defined on \mathcal{J} and (S, \mathcal{J}, P) is the underlying probability space of the model. Alternatively, one can define transition probabilities from (S, \mathcal{J}_t) to (S, \mathcal{J}_{t+1}) and derive the existence of P from them (I. Tolcea's theorem (Neveu [1970], Proposition V-I-1)). Transition probabilities do not play any role in this model and at t only the projection of (S, \mathcal{J}, P) denoted (S, \mathcal{J}_t, P) is considered.

An arbitrary stochastic process $\{z_t\}_{t=0}^{\infty}$ defined on (S, \mathcal{J}, P) is said to be *adapted* to $\{\mathcal{J}_t\}_{t=0}^{\infty}$ if z_t is \mathcal{J}_t -measurable for all $t \geq 0$. An event $A \in \mathcal{J}$ is said to occur almost surely (denoted a.s.) if $P(A) = 1$. Two random variables z and z' on (S, \mathcal{J}, P) are said to be equivalent (members of the same equivalence class) if $z(s) = z'(s)$ a.s.

E denotes the expectation operator corresponding to P . That is, if $x: S \rightarrow (-\infty, \infty)$ is integrable with respect to P , then $EX = \int x(s)P(ds)$. The conditional expectation operator is denoted by $E[\cdot | \mathcal{J}_t]$. That is, if X is integrable with respect to P , then $E[X | \mathcal{J}_t]$ is \mathcal{J}_t -measurable and integrable and for all $A \in \mathcal{J}_t$,

$$\int_A E[X | \mathcal{J}_t] P(ds) = \int_A X P(ds).$$

A generic element of S is simply denoted by s (i.e., $s \in S$; $s: (\dots, s_{-1}, s_0, s_1, \dots)$). The projection of s on t is denoted by $(s)_t = s_t$. The *shift* operator $\sigma: S \rightarrow S$ is defined by the formula $(\sigma s)_t = s_{t+1}$ (i.e., the value s_{t+1} appears now at t). $\{s_t\}$ is said to be *strictly stationary* if and only if σ is probability preserving (i.e., for all $A \in \mathcal{J}$, $P(A) = P(\sigma A)$). A set $A \in \mathcal{J}$ is said to be *invariant* if $P(A) = P(A \cap \sigma A)$. A stationary process is said to be *metrically transitive* (or alternatively, σ is said to be *ergodic*) if every invariant set is of probability 0 or 1. A stochastic process $\{z_t\}_{t=0}^{\infty}$ is said to be a *positive supermartingale* (resp. a *positive submartingale*) if, for $t \geq 0$, $z_t \geq 0$ a.s. and $E[z_{t+1} | \mathcal{J}_t] \leq z_t$ a.s. (resp. $z_t \geq 0$ a.s., $E[z_t] < +\infty$, and $E[z_{t+1} | \mathcal{J}_t] \geq z_t$ a.s.).

Commodity Space

I make the usual distinction between primary goods, produced goods and consumption goods. L is the set of different types of commodities; $L_c \subset L$ the set of consumption goods; $L_p \subset L$ the set of produced

goods, and $L_0 \subset L$ the set of primary goods (i.e., $L = L_p \cup L_0$ and $L_p \cap L_0 = \emptyset$). Pure intermediate goods are good in $L \setminus L_0 \cup L_c$.

Notation

\mathbb{R}^L is the L -dimensional Euclidian space. \mathbb{R}^{L_u} ($u = p, 0$ or c) is the corresponding subspace, that is $\mathbb{R}^{L_u} = \{x \in \mathbb{R}^L \mid x^i = 0 \text{ if } i \notin L_u\}$. (I always denote components of a vector by superindex).

The norm on \mathbb{R}^L is the maximum norm and is always denoted by $|\cdot|$. That is, if $x \in \mathbb{R}^L$, $|x| = \max \{\text{absolute value } (x^i), i \in L\}$.

$\mathcal{L}_{\infty, L}(S, \mathcal{J}_t, P)$ denotes the space of equivalence classes of \mathcal{J}_t -measurable functions from S to \mathbb{R}^L which are essentially bounded (i.e., if $x \in \mathcal{L}_{\infty, L}(S, \mathcal{J}_t, P)$, then exists $c > 0$ such that $P\{|x(s)| > c\} = 0$).

$\mathcal{L}_{1, L}(S, \mathcal{J}_t, P)$ denotes the space of equivalence classes of \mathcal{J}_t -measurable functions from S to \mathbb{R}^L that are integrable with respect to P .

For $v = p, 0$ or c , $\mathcal{L}_{\infty, L_v}(S, \mathcal{J}_y, P)$ (resp. $\mathcal{L}_{1, L_v}(S, \mathcal{J}_t, P)$) is the subspace of $\mathcal{L}_{\infty, L}(S, \mathcal{J}_t, P)$ (resp. $\mathcal{L}_{1, L}(S, \mathcal{J}_t, P)$) corresponding to L_v . That is, $\mathcal{L}_{\infty, L_v}(S, \mathcal{J}_t, P) = \{x \in \mathcal{L}_{\infty, L}(S, \mathcal{J}_t, P) \mid x^i(s) = 0 \text{ a.s. if } i \notin L_v\}$.

If $x \in \mathbb{R}^L$, then " $x \geq 0$ " means " $x^i \geq 0$, for all $i \in L$." " $x > 0$ " means " $x \geq 0$ and $x \not\leq 0$." " $x \gg 0$ " means " $x^i > 0$, for all $i \in L$." $\mathbb{R}_+^L \equiv \{x \in \mathbb{R}^L \mid x \geq 0\}$ and $\mathbb{R}_-^L \equiv \{x \in \mathbb{R}^L \mid x \leq 0\}$. $\mathbb{R}_{++}^L \equiv \{x \in \mathbb{R}^L \mid x \gg 0\}$.

If $x \in \mathcal{L}_{g, L}(S, \mathcal{J}_t, P)$, where $g = 1$ or ∞ , then " $x \geq 0$ " means " $x(s) \geq 0$ a.s." " $x > 0$ " means " $x \geq 0$ and $x \not\leq 0$ ". " $x \gg 0$ " means

" $x(s) \gg 0$ a.s." . Finally, " $x \gg \gg 0$ " means "there exists a positive real number r such that $x^i(s) \geq r$ a.s., for all $i \in L$." $\mathcal{L}_{g,L}^+(S, \mathcal{J}_t, P) \equiv \{x \in \mathcal{L}_{g,L}(S, \mathcal{J}_t, P) \mid x \geq 0\}$ and $\mathcal{L}_{g,L}^-(S, \mathcal{J}_t, P) \equiv \{x \in \mathcal{L}_{g,L}(S, \mathcal{J}_t, P) \mid x \leq 0\}$.

Finally, if $x \in \mathcal{L}_{g,L}(S, \mathcal{J}_t, P)$, where $g = 1$ or ∞ , then $\sigma^m x \equiv x(\sigma^m s)$, where σ is the shift operator. Clearly, $\sigma^m x \in \mathcal{L}_{g,L}(S, \mathcal{J}_{t+m}, P)$. x is said to be *invariant* if $\sigma x = x$ (i.e., if $x(\sigma s) = x(s)$ a.s.) . An adapted process $\{x_t\}_{t=0}^{\infty}$ where, for all t , $x_t \in \mathcal{L}_{g,L}(S, \mathcal{J}_t, P)$ is said to be *stationary* if, for all t , $x_t = \sigma^t x$ for some $x \in \mathcal{L}_{g,L}(S, \mathcal{J}_0, P)$.

At each date, the commodity space is \mathbb{R}^L , but the space in which economic decisions are made at time t (i.e., the action space at t) is $\mathcal{L}_{\infty,L}(S, \mathcal{J}_t, P)$.

Consumption

There is only one consumer or planner. The endowment of primary goods is determined by $w \in \mathcal{L}_{\infty,L_0}^+(S, \mathcal{J}_0, P)$, and at time t , $w_t = \sigma^t w$.

A *consumption program* is an adapted process $\{x_t\}_{t=0}^{\infty}$ such that for all $t \geq 0$, $x_t \in \mathcal{L}_{\infty,L_c}^+(S, \mathcal{J}_t, P)$.

Utility is additively separable with respect to time and satisfies the expected utility hypothesis. Future utility is discounted by a constant factor δ , where $\delta \in (0, 1)$.

The utility function for consumption in period zero is $U: \mathbb{R}_+^L \times S \rightarrow (-\infty, \infty)$. U is assumed to be $\sigma(\mathbb{R}_+^L) \otimes \mathcal{J}_0$ -measurable, where $\sigma(\mathbb{R}_+^L)$ is the σ -field generated by \mathbb{R}_+^L . Summing up, the total utility of the consumption program $\{x_t\}_{t=0}^{\infty}$ is

$$E \left[\sum_{t=0}^{\infty} \delta^t U(x_t(s), \sigma^t s) \right] .$$

(Note: To ease notation, the shift operator is usually dropped from the second argument of U , even if, in fact, it is present.)

Production

There is only one aggregate firm. The production set at time zero, is represented by means of an implicit production function $g: \mathbb{R}_-^L \times \mathbb{R}_+^P \times S \rightarrow (-\infty, \infty)$. I assume that g is $\sigma(\mathbb{R}_-^L \times \mathbb{R}_+^P) \otimes \mathcal{J}_1$ -measurable. An input-output vector $y = (y_0, y_1) \in \mathbb{R}_-^L \times \mathbb{R}_+^P$ is said to be *feasible* given that up to time 1, $(\dots, s_{-1}, s_0, s_1)$ has occurred if $g(y_0, y_1; s) \leq 0$. The interpretation is as follows: "Given that the history of the exogenous random shocks has been (\dots, s_{-1}, s_0) up to time zero, it is possible to obtain an output vector y_1 at the beginning of period 1 if s_1 occurs by using an input vector y_0 at time zero." The production set at time t is represented by $g(\cdot, \cdot; \sigma^t s)$. The production set Y is defined by

$$Y = \{(y_0, y_1) \in \mathcal{L}_{\infty, L}^-(S, \mathcal{J}_0, P) \times \mathcal{L}_{\infty, L}^+(S, \mathcal{J}_1, P) \\ | g(y_0(s), y_1(s); s) \leq 0 \text{ a.s.}\}$$

If $y \in \mathcal{L}_{\infty, L}^-(S, \mathcal{J}_t, P) \times \mathcal{L}_{\infty, L}^+(S, \mathcal{J}_{t+1}, P)$, then y is called a *production plan* at t . It is feasible if $y \in \sigma^t Y$.

Random capital stocks at time t are alternatively denoted by $y_{t-1,1}$ or by K_t (i.e., $y_{t-1,1} \equiv K_t$, for all $t \geq 0$). A *production program* is a sequence of production plans $\{(y_{y0}, y_{t1})\}_{t=0}^{\infty}$ together with an initial random capital stock K_0 .

A *production program* is said to be feasible if for all t , (y_{y0}, y_{t1}) is a feasible plan and $y_{t0} + y_{y-1,1} + \sigma^t w \geq 0$.

If $\{(y_{t0}, y_{t1})_{t=0}^{\infty}, y_{t-1,1}\}$ is a feasible production program, then $\{K_t\}_{t=0}^{\infty}$, where $K_t = y_{t-1,1}$ for every t , is said to be an *accumulation program*.

Programs

A *program* or allocation is of the form $\{(x_t)(y_t)\}_{t=0}^{\infty}$ where $\{x_t\}_{t=0}^{\infty}$ is a consumption program and $\{y_t\}_{t=0}^{\infty}$ is a feasible production program.

A program $\{(x_t)(y_t)\}_{t=0}^{\infty}$ is *feasible* for $K_0 \in \mathcal{L}_{\infty, L}^+(S, \mathcal{J}_0, P)$ if $y_{-1,1} = K_0$ and if for $t \geq 0$, $x_t \leq y_{t0} + y_{t-1,1} + \sigma^t w$. The set of feasible programs for K_0 is denoted $\mathcal{F}(K_0)$.

A \mathcal{J}_0 -*program* is a program defined using the information \mathcal{J}_0 . That is, once (\dots, s_{-1}, s_0) is known, the \mathcal{J}_0 -program specifies the program starting from that node, or in other words, the \mathcal{J}_0 -program is a conditional (or projected) program. A \mathcal{J}_0 -program is feasible if it is the conditional program of some feasible program. Given $K_0(s)$, the set of feasible \mathcal{J}_0 -programs for $K_0(s)$ is denoted $\mathcal{F}(K_0(s), s)$. Similarly, the set of feasible \mathcal{J}_t -programs for $K_t(s)$ is denoted $\mathcal{F}(K_t(s), \sigma^t s)$ (or simply $\mathcal{F}(K_t(s))$, when there is no confusion). It will follow from assumption A.10 and A.13 below, that \mathcal{F} is $\sigma(\mathbb{R}^c)^{\otimes \mathcal{J}_0}$ measurable, where $\sigma(\mathbb{R}^c)$ is the Borel σ -field generated by \mathbb{R}^c .

A *stationary program* is a program such that for all $t \geq 0$:

$$(1) \quad x_t = \sigma^t x, \text{ where } x \in \mathcal{L}_{\infty, L}^+(S, \mathcal{J}_0, P), \text{ and}$$

$$(2) \quad (y_{t0}, y_{t1}) = (\sigma^t y_0, \sigma^t y_1), \text{ where}$$

$$(y_0, y_1) \in \mathcal{L}_{\infty, L}^-(S, \mathcal{J}_0, P) \times \mathcal{L}_{\infty, L}^+(S, \mathcal{J}_1, P) \text{ and}$$

$$K_0 = y_{-1,1} = \sigma^{-1} y_1.$$

A program is said to be γ -interior if for all $t \geq 0$, and all $i \in L_p$, $K_t^1(s) \geq \gamma$ a.s. . When I refer to the class of interior programs, it is understood that there is some positive real γ for which all the programs in this class are γ -interior.

The good labelled "good 1" ($1 \in L_0 \cap L_c$) is interpreted as labor/leisure. A program is said to be ρ -leisured if for all $t \geq 0$, $x_t^1(s) \geq \rho$ a.s. . That is, if leisure is uniformly bounded below.

Optimality

A program $\{(x_t)(y_t)\}_{t=0}^{\infty} \in \mathcal{F}(K_0)$ is said to be *optimal* (for δ) if it is a solution to the problem:

$$(2.1) \quad \text{Max} \left\{ E \left[\sum_{t=0}^{\infty} \delta^t U(\tilde{x}_t(s), \sigma^t s) \right] \mid \{(\tilde{x}_t)(\tilde{y}_t)\}_{t=0}^{\infty} \in \mathcal{F}(K_0) \right\}$$

The *conditional valuation function* (or simply) the *valuation function*) of $K_0(s)$ is defined by

$$(2.2) \quad V(K_0(s), s; \delta) = \text{Max} \left\{ E \left[\sum_{t=0}^{\infty} \delta^t U(\tilde{x}_t(s), \sigma^t s) \mid \mathcal{Y}_0 \right] \mid \{(\tilde{x}_t(s)), (\tilde{y}_t(s))\}_{t=0}^{\infty} \in \mathcal{F}(K_0(s), s) \right\}$$

It follows from assumptions A.5, A.6, A.10 and A.13 below, that

$$V(\cdot, \cdot, ;) \text{ is } \sigma(\mathbb{R}^c) \otimes \mathcal{Y}_0\text{-measurable.}$$

Prices:

A system of *present value prices* is an adapted process $\{p_t\}_{t=0}^{\infty}$ such that for all $t \geq 0$, $p_t \in \mathcal{L}_{1,L}^+(S, \mathcal{J}_t, P)$, $p_t \neq 0$ for some t .

Given a system of current value prices, and a discount factor δ , the *current value price system* $\{q_t\}_{t=0}^{\infty}$ is defined by $q_t = \delta^{-t} p_t$.

A price system is said to be stationary if for all $t \geq 0$, $q_t = \sigma^t q$, where $q \in \mathcal{L}_{1,L}^+(S, \mathcal{J}_0, P)$ and $q \neq 0$.

Competitive Programs

A program $\{(x_t)(y_t)\}_{t=0}^{\infty}$ is said to be *competitive* if there exists a system of present value prices $\{p_t\}_{t=0}^{\infty}$, $p_t \gg 0$ for all t , such that

$$(2.3) \quad (i) \quad E[p_{t+1} \cdot y_{t1} + p_t \cdot y_{t0}] \geq E[p_{y+1} \tilde{y}_1 + p_t \tilde{y}_0], \text{ for all}$$

$$(y_0, y_1) \in \sigma^t Y, \text{ and } t \geq 0.$$

$$(2.4) \quad (ii) \quad E[\delta^t U(x_t(s), s) - p_t(s) \cdot x_t(s)]$$

$$\geq E[\delta^t U(\tilde{x}(s), s) - p'_t(s) \cdot \tilde{x}(s)],$$

$$\text{for all } \tilde{x} \in \mathcal{L}_{\infty, L_c}^+(S, \mathcal{J}_t, P) \text{ and } t \geq 0.$$

If, in addition, satisfies the transversality condition:

$$(2.5) \quad (iii) \quad \lim_{t \rightarrow \infty} E[p_t \cdot K_t] = 0 ,$$

then the program is said to be *supported* by a system of prices.

Remark 1:

Under the assumptions stated below, if conditions (i) to (iii) are satisfied, then the system of prices *supports* the value function in the sense that if $\{(x_t)(y_t)\}_{t=0}^{\infty} \in \mathcal{F}(K_0)$, then

$$(2.6) \quad (iv) \quad E[V(K_0(s), s; \delta) - p_0(s) \cdot K_0(s)] \\ \geq E[V(\tilde{K}_0(s), s; \delta) - p_0(s) \cdot \tilde{K}_0(s)] \\ \text{for all } \tilde{K}_0 \in \mathcal{L}_{\infty, L_p}^+(S, \mathcal{J}_0, P) .$$

Remark 2:

It can also be derived from (i), (ii) and (iv) the following properties (see Zilcha [1976], for these remarks):

$$(2.7) \quad (i') \quad E[\delta q_{t+1}(s) \cdot y_{t1}(s) | \mathcal{J}_t] + q_t(s) \cdot y_{t0}(s) \\ \geq E[\delta q_{t+1}(s) \cdot \tilde{y}_1(s) | \mathcal{J}_t] + g_t(s) \tilde{y}_0(s) , \text{ a.s.} \\ \text{for all } (\tilde{y}_0, \tilde{y}_1) \in \sigma^t Y , t \geq 0 .$$

$$(2.8) \quad (ii') \quad U(x_t(s), \sigma^t s) - q_t(s) \cdot x_t(s) \\ \geq U(c, \delta^t s) - q_t(s) \cdot c \quad \text{a.s.} \\ \text{for all } c \in \mathbb{R}_+^L , t \geq 0 .$$

$$\begin{aligned}
 (2.9) \quad & (iv') \quad V(K_0(s), s; \delta) - q_0(s) \cdot K_0(s) \\
 & \geq V(\tilde{K}_0, s; \delta) - q_0(s) \cdot \tilde{K}_0 \quad \text{a.s.} \\
 & \text{for all } \tilde{K}_0 \in \mathbb{R}_+^L, t \geq 0.
 \end{aligned}$$

Alternatively, conditions (i) and (ii) can be derived from the profit maximization problem and the utility maximization problem, respectively. One can think of the planner solving these two problems, then conditions (i) and (ii) simply say that intertemporal decentralization is possible. Condition (iii) says: prices are, in fact, Malinvaud prices or the transversality condition for efficiency is satisfied. It is a known result that conditions (i), (ii) and (iii) are sufficient for optimality and that alternatively, if a program is optimal, then there is a system of prices satisfying (i) to (iii). In order to obtain this result, some boundness and interiority assumptions have to be satisfied. I make enough assumptions below in order to guarantee this result. In Section 5 I comment on this point.

The Value Loss

In turnpike theory, the use of the value loss approach is a standard method to construct a Liapouov's process with which to study the convergence properties of optimal programs. I elaborate on this point in Section 5. Here I only give definitions and characterizations.

If $\{(x_t)(y_t)\}_{t=0}^{\infty} \in \mathcal{F}(K_0)$ is an optimal program supported by a system of current value prices $\{q_t\}_{t=0}^{\infty}$, and $\{(x_t)(y_t)\}_{t=0}^{\infty} \in \mathcal{F}(K_0)$ is a feasible program, then the *present value loss* is defined by

$$(2.10) \quad F_0(\tilde{K}_0(s), s; K_0, \delta) = q_0(s) \cdot (\tilde{K}_0(s) - K_0(s)) \\ - E \left[\sum_{t=0}^{\infty} \delta^t (U(\tilde{x}_t(s), s) - U(x_t(s), s)) \mid \mathcal{J}_0 \right].$$

It is easy to see (Lemma (6)) that

$$(2.11) \quad F(\tilde{K}_0(s), s; K_0, \delta) \\ \geq E \left[\sum_{t=0}^{\infty} \delta^t (U(x_t(s), s) - q_t(s) \cdot x_t(s)) \mid \mathcal{J}_0 \right] \\ - E \left[\sum_{t=0}^{\infty} \delta^t (U(\tilde{x}_t(s), s) - q_t(s) \cdot \tilde{x}_t(s)) \mid \mathcal{J}_0 \right] \\ + E \left[\sum_{t=0}^{\infty} \delta^t (\delta q_{t+1}(s) \cdot y_{t1}(s) + q_{t0}(s) \cdot y_{t0}(s)) \mid \mathcal{J}_0 \right] \\ - E \left[\sum_{t=0}^{\infty} \delta^t (\delta q_{t+1}(s) \tilde{y}_{t1}(s) + q_{t0}(s) \cdot \tilde{y}_{t0}(s)) \mid \mathcal{J}_0 \right] \\ = S_1(s) + S_2(s).$$

$E[S_1] \geq 0$ is the condition for utility maximization given that prices can be normalized by taking the marginal utility of income equal to one. Similarly, $E[S_2] \geq 0$ is the condition for profit maximization. In other words, $F_0(\tilde{K}_0(s), s; K_0, \delta)$ is no less than the conditionally expected losses in reduced utility and profits incurred in the program $\{(\tilde{x}_t) (\tilde{y}_t)\}_{t=0}^{\infty}$ with respect to the valuation of the program $\{(x_t) (y_t)\}_{t=0}^{\infty}$.

Notice that under the assumptions stated below, $F_0(\cdot, \cdot; K_0, \delta)$ is $\sigma(\mathbb{R}^c) \otimes \mathcal{J}_0$ -measurable.

The current value loss is simply

$$(2.12) \quad F_0(\tilde{K}_0(s), s; K_0, \delta) - \delta E[F_1(\tilde{K}_1(s), s; K_1, \delta) | \mathcal{J}_0]$$

and it is a direct computation to see that

$$(2.13) \quad \begin{aligned} & F_0(\tilde{K}_0(s), s; K_0, \delta) - \delta E[F_1(\tilde{K}_1(s), s; K_1, \delta) | \mathcal{J}_0] \\ & \geq U(x_0(s), s) - q_0(s) \cdot x_0(s) - (U(\tilde{x}_0(s), s) - q_0(s) \cdot \tilde{x}_0(s)) \\ & \quad + E[\delta q_1(s) \cdot y_{01}(s) | \mathcal{J}_0] + q_0(s) \cdot y_{00}(s) \\ & \quad - (E[\delta q_1(s) \cdot \tilde{y}_{01}(s) | \mathcal{J}_0] + q_0(s) \tilde{y}_{00}(s)) \\ & = M_1(s) + M_2(s) . \end{aligned}$$

But (i') and (ii') imply $M_1(s) \geq 0$ and $M_2(s) \geq 0$ a.s., respectively. In other words, the current value loss has also a direct interpretation in terms of the one period utility and expected profits maximization, respectively.

Finally, I define the "myopic"-value loss by

$$(2.14) \quad F_0(\tilde{K}_0(s), s; K_0, \delta) - E[F_1(\tilde{K}_1(s), s; K_1, \delta) | \mathcal{J}_0]$$

I introduce this term because the difficulties that appear in the discounted case lie in the difference between (2.12) and (2.14). I will explain this point in Sections (5) and (7). The myopic value loss can be characterized in two alternative ways:

$$\begin{aligned}
& F_0(\tilde{K}_0(s), s; K_0, \delta) - E[F_1(\tilde{K}_1(s), s; K_1, \delta) | \mathcal{J}_0] \\
(2.15) \quad & = F_0(\tilde{K}_0(s), s; K_0, \delta) - \delta E[F_1(\tilde{K}_1(s), s; K_1, \delta) | \mathcal{J}_0] \\
(2.16) \quad & - (1 - \delta) E[F_1(\tilde{K}_1(s), s; K_1, \delta) | \mathcal{J}_0] \\
& = U(x_0(s), s) - q_0(s) \cdot x_0(s) - (U(\tilde{x}_0(s), s) - q_0(s) \cdot \tilde{x}_0(s)) \\
& + E[q_1(s) y_{01}(s) | \mathcal{J}_0] + q_0(s) y_{00}(s) \\
& - (E[q_1(s) \tilde{y}_{01}(s) | \mathcal{J}_0] + q_0(s) \tilde{y}_{00}(s)) \\
& - (1 - \delta) E\left[\sum_{n=1}^{\infty} \delta^{n-1} (U(x_n(s), s) - U(\tilde{x}_n(s), s)) \mid \mathcal{J}_0 \right]
\end{aligned}$$

The myopic value loss reduces the losses incurred in the current period for not following the optimal program. Due to this "myopic effect," the myopic value loss might be negative (i.e., a gain!). In (2.15) current losses are reduced by expected future losses weighted by the "degree of impatience." In (2.16) the loss in reduced utility appears as in the current value loss but then expected losses in profits are undiscounted or accounted in current prices. Finally, this sum is reduced by the expected future losses in utility also weighted by the "degree of impatience."

Convergence Concepts

Finally, I define the different convergence concepts and some well known probability results.

Let $\{z_n\}_{n=0}^{\infty}$ be a stochastic process adapted to $\{\mathcal{J}_n\}_{n=0}^{\infty}$ with values in \mathbb{R}^L , and let $z: S \rightarrow \mathbb{R}^L$ be \mathcal{J} -measurable. Then it is said that

- z_n converges almost surely to z , written $z_n \xrightarrow{\text{a.s.}} z$, if

$$\liminf_{n \rightarrow \infty} z_n = z = \limsup_{n \rightarrow \infty} z_n \quad \text{a.s.}$$

- z_n converges in probability to z , written $z_n \xrightarrow{P} z$, if

$$\lim_{n \rightarrow \infty} P\{s: |z_n(s) - z(s)| \geq \epsilon\} = 0 \quad \text{for each positive } \epsilon$$

If for all n , $z_n \in \mathcal{L}_{1,L}(S, \mathcal{J}_n, P)$ and $z \in \mathcal{L}_{1,L}(S, \mathcal{J}, P)$, then it is said that

- z_n converges in the mean to z , written $z_n \xrightarrow{L_1} z$, if

$$\lim_{n \rightarrow \infty} E[|z_n - z|] = 0 \quad \text{and}$$

- z_n converges in the mean to z exponentially, if there exist $A > 0$ and $a \in (0, 1)$ such that

$$\forall n \geq 0, E[|z_n - z|] \leq A \cdot a^n$$

(Without the integrability assumption) it is said that

• z_n converges in distribution to z , written $z_n \Rightarrow z$, if

$$\lim_{n \rightarrow \infty} P\{s: z_n(s) \leq x\} = P\{s: z(s) \leq x\}$$

for every $x \in \mathbb{R}^L$ such that $P\{s: z(s) = x\} = 0$

For each vector valued random variable z_n , its *distribution* μ_n is the probability measure on $(\mathbb{R}^L, \sigma(\mathbb{R}^L))$ defined by

$$\mu_n(A) = P\{s: z_n(s) \in A\}, \text{ for every } A \in \sigma(\mathbb{R}^L),$$

then it is said that

• μ_n converges weakly to μ , written $\mu_n \Rightarrow \mu$, if $z_n \Rightarrow z$.

Let $z_n \in \mathcal{Z}_{1,L}(S, \mathcal{J}_n, P)$ for all $n \geq 0$, then the stochastic process $\{z_n\}_{n=0}^{\infty}$ is said to be *uniformly integrable* (or *equi-integrable*) if

$$\lim_{\alpha \rightarrow \infty} \sup_n \int_{\{|z_n| \geq \alpha\}} |z_n(s)| P(ds) = 0$$

Remark: It exists $h \in \mathcal{Z}_{1,L}^+(S, \mathcal{J}, P)$ such that, for all $u \geq 0$ $|z_n| \leq h$ a.s., then $\{z_n\}_{n=0}^{\infty}$ is uniformly integrable.

For the adapted process $\{z_n\}_{n=0}^{\infty}$ of a.s. finite vector valued random variables (in particular for the process $\{z_n\}_{n=0}^{\infty}$ such that $z_n \in \mathcal{Z}_{\infty,L}(S, \mathcal{J}_n, P)$, $\forall n \geq 0$), the following relations are satisfied:

(a) If $\sigma^{-n} z_n - \sigma^{-m} z_m \xrightarrow{P} 0$, then there exists a \mathcal{J}_0 -measurable vector valued random variable z , such that $\sigma^{-n} z_n \xrightarrow{P} z$.

- (b) If $z_n \xrightarrow{\text{a.s.}} z$, then $z_n \xrightarrow{P} z$.
- (c) If $z_n \xrightarrow{P} z$, then there exists a subsequence $\{z_{n_i}\}$ such that $z_{n_i} \xrightarrow{\text{a.s.}} z$.
- (d) If $\{z_n\}_{n=0}^{\infty}$ is uniformly integrable and $z_n \xrightarrow{P} z$, then $z \in \mathcal{L}_{1,L}(S, \mathcal{Y}, P)$ and $z_n \xrightarrow{L_1} z$.
- (e) Let $z_n \in \mathcal{L}_{1,L}(S, \mathcal{Y}_n, P)$ and $z \in \mathcal{L}_{1,L}(S, \mathcal{Y}, P)$. If $z_n \xrightarrow{L_1} z$, then $\{z_n\}_{n=0}^{\infty}$ is uniformly integrable and $z_n \xrightarrow{P} z$.
- (f) If $z_n \xrightarrow{P} z$, then $z_n \Rightarrow z$ (and $\mu_n \Rightarrow \mu$).
- (g) If $\{z_n\}_{n=0}^{\infty}$ is uniformly integrable and $z_n \Rightarrow z$ (or equivalently $\mu_n \Rightarrow \mu$), then $z \in \mathcal{L}_{1,L}(S, \mathcal{Y}, P)$ and $z_n \xrightarrow{L_1} z$.

Remark: It follows from (b), (d) and (f) that if $\{z_n\}_{n=0}^{\infty}$ is uniformly integrable and $z_n \xrightarrow{\text{a.s.}} z$, then $z_n \xrightarrow{P} z$, $z_n \xrightarrow{L_1} z$, $z_n \Rightarrow z$ and $\mu_n \Rightarrow \mu$. Conversely, it follows from (c), (e) and (g) that if $\{z_n\}_{n=0}^{\infty}$ is uniformly integrable and $z_n \Rightarrow z$ (i.e., $\mu_n \Rightarrow \mu$), then $z_n \xrightarrow{L_1} z$, $z_n \xrightarrow{P} z$, and there exist a subsequence $\{z_{n_i}\}$ such that $z_{n_i} \xrightarrow{\text{a.s.}} z$, but, in general, is not true that $z_n \xrightarrow{\text{a.s.}} z$. (Note: For the proofs of (a) to (g), see Neveu [1970], p. 46-51, and Billingsley [1979, p. 284-292.]

III. ASSUMPTIONS AND THEOREMS

3.1 Assumptions*Commodity Space*

A.1 (Nontriviality) $L_c \cap L_p \neq \emptyset$, $L_c \cap L_0 \neq \emptyset$ ($1 \in L_c \cap L_0$).

Environment

A.2 M is a finite set

A.3 The probability space (S, \mathcal{J}_0, P) is complete and nonatomic.

A.4 (Stationarity and ergodicity) The stochastic process

$\{s_t\}_{t=-\infty}^{+\infty}$ is strictly stationary and metrically transitive.

Preferences

A.5 (Measurability) $U: \mathbb{R}^L_c \times S \rightarrow (-\infty, \infty)$ is measurable with respect to $\sigma(\mathbb{R}^L_c) \otimes \mathcal{J}_0$, where $\sigma(\mathbb{R}^L_c)$ is the Borel σ -field on \mathbb{R}^L_c .

A.6 (Differentiability) For each $s \in S$, $U(\cdot, s): \mathbb{R}^L_c \rightarrow (-\infty, \infty)$ is \mathcal{C}^2 (twice continuously differentiable). For each $s \in S$, let $Df(\cdot, s)$ and $D^2f(\cdot, s)$ denote the first and second derivatives, respectively, of the function $f(\cdot, s)$.

A.7 (Strict Monotonicity) $DU(x, s) \gg 0$ a.s. for all $x \in \mathbb{R}^L_c$.

A.8 (Strict Concavity) For each $x \in \mathbb{R}^L_c$ and each s , $D^2U(x, s)$ is negative definite; uniformly in s is a set of probability one.

A.9 (Desirability of Leisure) $D_1U(x, s) \rightarrow +\infty$ a.s. as $x^1 \rightarrow 0$.

For all $i \in L_c$, $i \neq 1$, $\lim_{x^1 \rightarrow 0} D_iU(x, s) < +\infty$ a.s.

Endowments

A.10 (Stationary Availability of Primary Resources)

$w \in \mathcal{L}^+_{\infty, L_0}(S, \mathcal{J}_0, P)$, $w \gg 0$ and $w_t(s) = w(\sigma^t s)$.

A.11 (Uniform Boundness of the Initial Capital Stocks) There exist

$B > 0$ such that $\|K_0\|_{\infty} \leq B$, where $K_0 \in \mathcal{L}^+_{\infty, L_p}(S, \mathcal{J}_0, P)$ is an initial random capital stock.

- A.12 (*Interiority*) There exist $\gamma \geq 0$ such that for any initial random capital stock K_0 , and for all $i \in I_p$, $K_-^i(s) \geq \gamma$ a.s. .

Technology

The production set $Y \subset \mathcal{Z}_{\infty, L}^-(S, \mathcal{J}_0, P) \times \mathcal{Z}_{\infty, L_p}^+(S, \mathcal{J}_1, P)$ is described by means of an implicit production function, as follows:

$$Y = \{(y_0, y_1) \in \mathcal{Z}_{\infty, L}^-(S, \mathcal{J}_0, P) \times \mathcal{Z}_{\infty, L_p}^+(S, \mathcal{J}_1, P) \mid$$

$$g(y_0(s), y_1(s); s) \leq 0 \text{ a.s.}\}$$

- A.13 (*Measurability*) $g: \mathbb{R}_-^L \times \mathbb{R}_+^P \times S \rightarrow (-\infty, \infty)$ is measurable with respect to $\sigma(\mathbb{R}_-^L \times \mathbb{R}_+^P) \otimes \mathcal{J}_1$.

- A.14 (*Differentiability*) For each $s \in S$, $g(\cdot, \cdot; s): \mathbb{R}_-^L \times \mathbb{R}_+^P \rightarrow (-\infty, \infty)$ is \mathbf{L}^2 .

- A.15 (*Free Disposability*) $Dg(y_0, y_1; s) \gg 0$ a.s. for all $(y_0, y_1) \in \mathbb{R}_-^L \times \mathbb{R}_+^P$.

- A.16 (*Strict Convexity*) $D^2g(y_0, y_1; s)$ is positive definite on the subspace of $\mathbb{R}_-^L \times \mathbb{R}_+^P$ orthogonal to $Dg(y_0, y_1; s)$; with respect to s this property is satisfied uniformly in a set of probability one.

- A.17 (*Possibility of Zero Production*) $g(0, 0; s) = 0$ for every s .

- A.18 (*Necessity of Primary Inputs*) For $(y_0, y_1) \in \mathbb{R}_-^L \times \mathbb{R}_+^P$ if $y_1 > 0$ and $y_0^i = 0$ for all $i \in I_0$, then $g(y_0, y_1; s) > 0$; for every s .

- A.19 (*Uniformly Bounded Marginal Productivity of Capital*) There exists a positive constant η such that if (y_0, y_1) and $(\tilde{y}_0, \tilde{y}_1)$ are two production plans and for some s $g(y_0(s), y_1(s); s) = 0$ and $g(\tilde{y}_0(s), \tilde{y}_1(s); s) = 0$, then $|y_1(s) - \tilde{y}_1(s)| \leq \eta |y_0(s) - \tilde{y}_0(s)|$.

A.20 (*Finite Different Productions at Period Zero*) There is a finite partition A_i of s , such that, for all $A_i \in \mathcal{J}_0$, the function $g(y_0, y_1; \sigma^{-1}s)$ is constant on A_i ; for every $(y_0, y_1) \in \mathbb{R}_-^L \times \mathbb{R}_+^L$.

A.21 (*Existence of an Expansible Plan*) There exists $\xi \in \mathcal{L}_{\infty, L}^+$ (S, \mathcal{J}_0, P) $\xi \gg 0$, such that $g((- \xi(s), -w(s)), \xi(\sigma s)) < 0$ a.s. . In particular, for some $\rho > 0$, the expansible capital stock can be reproduced with a ρ -leisured program.

Recall that a program $\{(x_t)(y_t)\}_{t=0}^{\infty}$ is said to be γ -interior if $K_t^i \geq \gamma$ for all $i \in L_p$ and all $t \geq 0$. It is said to be ρ -leisured if for all $t \geq 0$, $x_t^1(s) \geq \rho$ a.s. .

A.22 (*Monotone Productivity of Labor in Production*)

There exist a fixed positive integer $m (m \geq \#|L_p|)$ and for each pair $\delta > 0$, $\rho > 0$, a positive real ρ' , such that if $\{(x_t)(y_t)\}_{t=0}^{\infty} \in \mathcal{F}(K_0)$ is a γ -interior ρ -leisured program, then there exists a program $\{(\tilde{x}_t)(\tilde{y}_t)\}_{t=0}^{\infty} \in \mathcal{F}(K_0)$ satisfying

- (i) for all $t, \tilde{K}_t \geq K_t$; a.s. ;
- (ii) If $y_{t0}^i(s) = 0$, then $\tilde{y}_{t0}^i(s) = 0$, and
- (iii) $\tilde{K}_m \geq K_m + \rho' \cdot e$, a.s., where e is the unit vector on \mathbb{R}_+^L .

3.2 Theorems

Assume A.1 to A.21

Theorem 1 (*Stochastic Convergence of Optimal Interior Programs*)

There exist $\delta^* \in (0, 1)$, $A > 0$, $a \in (0, 1)$ and $\epsilon \in (0, 1)$ such that if $\{(x_t)(y_t)\}_{t=0}^{\infty}$ and $\{(\tilde{x}_t)(\tilde{y}_t)\}_{t=0}^{\infty}$ are two optimal interior programs for $\delta \in (\delta^*, 1)$, then

$$P\{|x_t - \tilde{x}_t, y_t - \tilde{y}_t| > \epsilon^t\} \leq A \cdot a^t$$

It follows that $|x_t - \tilde{x}_t, y_t - \tilde{y}_t|$ converges to zero a.s. and exponentially in the mean.

Theorem 2 (*Existence of the Modified Stochastic Golden Rule*) There

exists $\delta^* \in (0, 1)$ such that if $\delta \in (\delta^*, 1)$, there exists an optimal interior stationary program

$\{((\sigma^t \bar{x}), (\sigma^t \bar{y}))^{(\delta)}\}_{t=0}^{\infty} \in \mathcal{F}(\bar{K}^{(\delta)})$. This program is unique up to equivalence.

Theorem 3 (*Stochastic Turnpike Property*)

There exists $\delta^* \in (0, 1)$ such that if $\{(x_t)(y_t)\}_{t=0}^{\infty}$ is an optimal interior program for $\delta \in (\delta^*, 1)$, then $((x_t)(y_t))$ converges to $((\bar{x})(\bar{y}))^{(\delta)}$ a.s. and exponentially in the mean.

IV. DISCUSSION OF THE ASSUMPTIONS

Most of the assumptions are the natural extension to a stochastic environment of the assumptions commonly made in turnpike theory and general equilibrium theory. Except for assumptions A.9 and A.22, the rest of the assumptions are a combination of the assumptions made in Radner [1973] or the stochastic version of the assumptions made in Bewley [1982]. I will only comment on the most conflictive ones.

Separable Utility and the Fixed Discount Factor

It has been repeatedly argued that intertemporal complementarity in consumption should not be ruled out in intertemporal models. In the current trade interpretation of the model, this criticism is particularly valid. An alternative formulation, that still preserves the necessary stationarity of the model, would be to use a stationarity utility function of the type introduced by Koopmans. This type of utility function has been already introduced in growth models. In a deterministic model, Lucas and Stokey [1982] have used stationary utility functions in order to obtain nontrivial asymptotic solutions for the case of different "degrees of impatience" between consumers that trade over time. Epstein [1983] has generalized the Brock and Mirman [1972] stochastic growth model in a model where the preferences of the planner are represented by a stationary cardinal utility. A possible line of further research is to study a multisector model with stationary cardinal utility. There are, however, several problems in using this formulation: (1) The class of utility functions that satisfies Koopmans' axioms and the axioms of

choice under uncertainty is not "too much larger" than the class of functions that are additively separable with respect to time (see Epstein [1983]); (2) in order to obtain optimal stationary programs, it is necessary to introduce further assumptions that still narrow more the admissible class (for example, that the degree of impatience increases with accumulated wealth; (3) with impatiently heterogeneous consumers, the problem of trivial asymptotic solutions (i.e., the most patient consumer receives the aggregate equilibrium allocation in the limit (see Bewley [1982] and Lucas and Stokey [1982]), is not avoided if a specific functional form, different from the additively separable, is not defined. But, finally, has been the major tractability of the additively separable class, and the need to solve first this case, that has determined my choice.

There is another issue that, to my account, has not been pointed out. It is the introduction of a state dependent discount factor. That is, to make $\delta: S \rightarrow (0, 1]$ a \mathcal{J}_0 -measurable function, and then to define the total utility of a consumption program $\{x_t\}_{t=0}^{\infty}$ by

$$E\left[\sum_{t=0}^{\infty} \delta(\sigma^t s) \cdot U(x_t(s), \sigma^t s)\right]$$

It seems reasonable to think that in a situation of war or fatal illness, not only our current preferences are affected, but also the way that we see the future. State dependent discount factors had been studied in dynamic programming, but little is known about the properties of optimal programs (existence of prices, etc.) and, as I stated previously, first one must study the fixed discount case.

The Triviality of the von Neumann Facet

As in Bewley [1982], I assume strict convexity of the production set. The standard interpretation is that there are fixed entrepreneurial inputs or resources. Yano has pointed out that: "since we are concerned with a very long-run economy, it is not realistic to give up adjustability in the level of factors" (Yano [1983a], p. 1). Yano partially generalizes Bewley's model by assuming that there is an aggregate production set which is *laterally strictly convex* (i.e., the aggregate production set is a cone that is strictly convex along rays other than those starting at the origin of the cone (Coles [1983], p. 4)), or more explicitly, by assuming that the von Neumann facet at the optimal steady state is a ray and lateral strict convexity is preserved. I have not made Yano's extension; first, for technical reasons, and second, for "strategic" reasons. Technically, I have to prove the turnpike property with respect to an arbitrary optimal program, not necessarily stationary, and even if Yano's technique can be applied, I have decided, at this point, not to complicate more some of the proofs. Strategically, there are two reasons: First, with the current trade interpretation of the model Yano's criticism is less important and the model can be extended to a general equilibrium framework with a finite number of firms, not just an aggregate production set (this approach seems more adequate for an integration with the theory of finance), and by aggregation of laterally strictly convex sets one does not necessarily obtain an aggregate laterally strictly convex set. Second, when uncertainty is introduced in production, the problem of the von Neumann facet deserves a specific treatment. (Bibliographic note: The work of Bewley and Yano

has been recently extended by Coles who studies facets that might be flats and the turnpike theorem when consumers have different discount rates (Coles [1983].)

Remark: There is another strong assumption in technology that is not a general equilibrium assumption: A.19 (Uniformly Bounded Marginal Productivity of Capital). This assumption rules out technologies of the Cobb-Douglas type. I use this assumption in the proof of Lemma 3.

The Role of Labor and Leisure

As I said in the introduction, the only peculiar assumptions of my model are A.9 (Desirability of Leisure) and A.22 (Monotone Productivity of Labor in Production). A.9 is an Inada's assumption that, loosely speaking, says that the "representative consumer" must have some relax. A.22 is more complex. Basically, it says: (a) labor is used directly or indirectly in the production of all producible goods; (b) if there is a program such that, in each period, all the inputs (i.e., $(L_0 \cup L_p)$) are available in a uniform positive (negative) amount and the labor input is not fully used (uniformly), then there is an alternative program that (with probability one) increases the level of production uniformly by using more labor at each period (possibly all the labor endowment) and without using the inputs that were not used in the original program; (c) the required number of periods (m) for this increase in production is no less than the number of producible goods and the uniform increment in production depends on the uniform lower bound in capital stocks and the uniform lower bound in leisure in the original program. This assumption is introduced for convenience,

but it seems reasonable if producible goods are perishable. If one wants to introduce storage in the model, then uniform rates of depreciation must be assumed and, with further complications, all the same results can also be obtained.

Assumption A.9 is used in Lemma 4 to prove that optimal programs are ρ -leisured.

Assumption A.22 is used in Lemma 2 and 3. The role of Lemma 2, together with Lemma 4, is to prove that any optimal program can be reached in a finite number of periods from any interior capital stock. This result is used in Lemma 7 to prove that the differences in utility between two optimal programs are uniformly bounded. If one only has to prove that an optimal program converges to a stationary program, then the above result can be obtained by using a technique developed by Gale (Gale [1976]), i.e., from an interior capital stock, a program is constructed that converges exponentially to the stationary program. (Bibliographic note: In the deterministic model with discounting turnpike theorems equivalent to my Theorem 1, have been proved by assuming reachability (see, for example, McKenzie [1976]).

The Interiority of Optimal Programs

As previously stated, my results are valid for interior optimal programs. This is an important limitation since I have not introduced assumptions that guarantee the existence of interior programs. To show that a stationary program must be interior is not difficult if one assumes that initial capital stocks are interior (A.9), but, again, this is not my starting point. There are, at least, two forms to

guarantee the interiority of the optimal programs; one is to introduce enough Inada conditions, the other is to assume enough interdependency in production (e.g., all producible goods are used in the production of each producible good). The problems with these assumptions are that in one case the utility surface has flats and in the other case that pure consumption goods might have to be excluded; in other words, that the assumption might be too strong. A possible compromise can be a careful combination of both types of assumptions. I still have not studied this problem with enough detail.

V. REVIEW OF THE LITERATURE

Bibliographic Comment

My model can be seen as a generalization of the one good model of Brock and Mirman [1972]. The Brock and Mirman [1972] model has been extensively studied and generalized; see, for example, Mirman and Zilcha [1975] and [1976], Razin and Yahar [1979], Blume and Easley [1982], and Epstein (1983). The hypothesis that the exogenous stochastic process is a sequence of identically distributed independent variables is maintained in all these works. Brock has also generalized this model in his integration of Stochastic Growth Theory and the Theory of Finance (Brock [1979], [1982]).

As far as I know, the only stochastic multisector models with discount that study similar problems to the ones I study are: Brock and Majumdar [1978] and Majumdar and Radner [1983]. They all use a dynamic programming approach. In Majumdar and Zilcha [1979] an attempt is made to solve the existence problem. These works are discussed below.

With respect to the deterministic multisector model with discount, my work can be seen as a stochastic version of Scheinkman [1976] and McKenzie [1976]. However, in its formulation and in the methods of proof, my work is more closely related to Bewley [1982] and Yano [1983b]. My model differs from theirs in that there is a unique representative consumer (Bewley has a finite number of firms, while in Yano's model

and in mine, there is an aggregate firm).

For the stochastic multisector model without discounting, the existence of a stationary optimal program supported by prices was proved first, I believe, in Radner [1973] and also in Jeanjean [1974], Dana [1974], Dynkin [1974] and Evstigneev [1974]. I believe the turnpike property was proved simultaneously, in Evstigneev [1974] and Jeanjean [1974]. It has also been proved in Dana [1974], Evstigneev and Katyshev [1979] and Föllmer and Majumdar [1978]. In general, these models can be seen as a stochastic version of the Gale's model (Gale [1967]) and the turnpike results as the stochastic version of Brock [1970]. Finally, in the undiscounted case, Bewley has proved the existence of a stationary equilibrium in a general equilibrium model (Bewley [1981]).

The existence of an optimal program from any interior random capital stock has been repeatedly proved for both the discounted and the undiscounted case (see, for example, Evstigneev [1974] and Dynkin [1974]).

The existence of supporting prices for optimal programs in the discounted case has been proved in Tacsar [1974] and also Zilcha [1976a].

The stochastic growth model presents two main differences with respect to the deterministic growth model. (1) The commodity space is an infinite dimensional space, and (2) the convergence concepts are not uniquely defined. Different approaches to stochastic growth theory are based on the use of different, but related, concepts of convergence. On the other hand, (1) rules out simple generalizations of the methods used in the deterministic model for the proof of existence of the modified golden rule. Now, I want to discuss these issues and relate my work with the existent literature.

The Distributional Approach

The existence of the modified golden rule and the turnpike property was originally proved for the one sector stochastic growth model with discounting by Brock and Mirman (Brock and Mirman [1972]), using a dynamic programming approach. More precisely, they show that there exists a continuous stationary optimal policy (function) of investment and consumption. By deriving further properties of this policy function, they finally prove the existence of *a unique invariant distribution* of optimal capital stocks and that any optimal accumulation program *converges in distribution* to the stationary optimal program defined by the invariant distribution.

Brock and Majumdar (Brock and Majumdar [1978]) have used a similar approach for the multisector growth model. The main limitation of their work is that they make strong *ad hoc* assumptions. In particular, they assume (1) the exogenous stochastic process is a sequence of identically distributed independent random variables (denoted an i.i.d. process); (2) optimal random capital stocks belong (a.s.) to a compact set, (3) there exists a continuous stationary optimal policy function, and (4) a strong value loss assumption is satisfied. The meaning of (4), as well as the turnpike part of the paper, are discussed below.

Brock and Majumdar show that under assumptions (1) to (3), there exist at least one invariant distribution of optimal capital stocks.

In a nonlinear activity analysis model, Majumdar and Radner offer a rigorous proof of this result in a more general stochastic environment (they only assume that the exogenous process is stationary, takes values

in a compact set and satisfies Feller's property.) They only place assumptions on preferences and technology. Since the exogenous stochastic process is not i.i.d., the process of optimal capital stocks is not necessarily Markovian, and the invariant distribution is defined on the (state) space defined by the cartesian product of the capital space and the sample space of the exogenous process. In other words, the invariant distribution does not define a (strictly) stationary optimal accumulation program.

In all these models, the invariant distribution is obtained from a stochastic kernel defined by the policy function and the stochastic law of the exogenous process. This invariant distribution is a linear combination of distributions, each one having as a support an ergodic set of the Markov process (Doob [1953], Theorem 5.7, p. 214). The proof of the turnpike property in the Brock and Mirman model is basically to show that given the monotonicity properties of the optimal policy function there exists a unique ergodic set. As in the deterministic case, these monotonicity properties of the optimal policy function are, in general, not satisfied in the multisector model. It is at this point that Brock and Majumdar introduce the value loss approach.

The uniqueness of the invariant distribution and the convergence of distributions of optimal capital stocks are linked to properties of the stochastic kernel (see, for example, Futia [1983]). The main difficulty is to prove that such properties are satisfied.

The convergence in distribution of the optimal accumulation programs is, in fact, a form of *stochastic neighborhood turnpike theorem*. In other words, even if there is a unique ergodic set, this might be composed by *cyclically moving subsets* (Doob [1953], p. 211). If the exogenous stochastic process is i.i.d., this corresponds to optimal accumulation programs with *a.s. periodic cycles*. If the exogenous stochastic process is not i.i.d., then the optimal accumulation programs presents *stochastic cycles* "i.e., fluctuations of varying magnitude and duration" (Majumdar and Radner [1983] p. 1822). With positive probability, the realizations of different optimal programs can diverge systematically. I prove a.s. convergence which precludes this type of behavior.

In summary, it should be possible to prove a *stochastic neighborhood turnpike theorem* in terms of *convergence in distribution* with weaker concavity conditions than the ones required for the *stochastic asymptotic turnpike theorem* in terms of *a.s. convergence*.^{*} Since in growth models feasible allocations are usually uniformly bounded, all different concepts of convergence can be reduced to these two types (see p. 20).

^{*} José Scheinkman stated a similar remark to me.

The Existence of the Modified Golden Rule Problem

I have defined a stationary equilibrium as a strictly stationary stochastic process of optimal allocations and prices. As we have seen, Majumdar and Radner (Majumdar and Radner [1983]) prove the existence of an optimal program which is stationary in a weaker sense. Similar to the deterministic model, their existence result is satisfied for any positive discount factor. In contrast with the deterministic model, their invariant distribution may include optimal programs that in a deterministic model will not qualify as steady states. Furthermore, even if in Majumdar and Radner model there is only one planner the possible nonuniqueness of the invariant distribution is a phenomenon which does not appear in the deterministic model under similar concavity/convexity assumptions. As I stated before, in order to derive the existence of a stationary equilibrium (stochastic version of the deterministic steady state) from an invariant distribution, additional information is needed on the optimal policy function and the exogenous stochastic process.

Another approach to the existence problem is to try to construct an argument similar to the one used in the deterministic case. The difficulty is that the commodity space is \mathcal{L}_∞ . One might think that with an adequate choice of a topology for \mathcal{L}_∞ , this problem can be solved in a similar way that the general equilibrium existence problem is solved for \mathcal{L}_∞ (Bewley [1972]). Unfortunately, this is not the case. This is a technical problem and I only want to mention it briefly (the reader not interested can pass to the next point).

The proofs of existence of the modified golden rule in the deterministic case (Peleg and Ryder [1974] and Bewley [1982]) basically consist of a fixed point argument for a two periods economy. The same type of argument can be used in the stochastic model without discounting (Bewley [1981]). The reason why the same line of proof does not work when there is discount can be summarized in three facts: (1) With discount there is a transfer (or tax) payment $((1-\delta) \cdot E[q_1 \cdot y_{01} | \mathcal{J}_0])$. This factor does not appear in the proof of existence of equilibrium (not necessarily stationary). (2) In order to obtain compactness a weak topology must be chosen (say, the weak^{*}-topology), and (3) the product operation is not continuous in that topology. Majumdar and Zilcha have tried this approach using a fixed point argument but, unfortunately, their proof is not correct since they incur in the problem of inconsistency of (1), (2) and (3) (Majumdar and Zilcha [1979], p. 7; the proof of the super-semicontinuity of the F correspondence).*

*There are other problems with their proof, but I think this is the crucial one. I am not aware of any other proof that overcomes this problem.

The Supermartingale Property of the Value Loss

In order to prove the turnpike theorem using the value loss approach, two properties have to be proved: (1) the sum of current value losses is a convergent series, and (2) the value loss assumption is satisfied. Loosely speaking, the value loss assumption says that whenever optimal capital stocks diverge, the current value loss is strictly positive. In probabilistic terms, property (1) can be defined in terms of martingale theory. Föllmer and Majumdar (Föllmer and Majumdar [1978]) have used this approach in their proof of the turnpike property for the nondiscounted case. They use a *two sided current value loss* (i.e., the current value losses defined in (2.12) are added for two optimal programs) and they show that the resulting process is a submartingale. I use a similar approach, in particular I show that the one sided *present value loss* is a positive supermartingale.

The problem with the discounted case is that the supermartingale property does not follow from the characterization of the value loss. As I stated in Section 2, the *myopic value loss* is not necessarily positive. Furthermore, in order to prove the value loss assumption it is necessary to show that the *myopic value loss* is strictly positive whenever optimal capital stocks diverge. At this point of the argument, Brock and Majumdar assume that this condition is satisfied. They justify the assumption by saying that "it is the natural stochastic version of the deterministic curvature conditions" (Brock and Majumdar [1978], p. 234). As I show in my proof, this conjecture is correct. However, a central part of the proof is precisely to show

that this property is satisfied.

Further Research

There are three possible lines for further research: (1) the relaxation of some of the assumptions of the model and its extension to a general equilibrium model; (2) to study weaker turnpike properties and its relation to the different concept of stationarity; and (3) the application of the results in a rational expectations interpretation of the model. What I mean by (1) and (2) has been already covered. In general, my results can be applied to any general equilibrium model with intertemporal production. One particular application is to extend Brocks' work on integration of stochastic growth theory and the theory of finance (Brock [1979][1982]). In fact, I believe, it was Brock who introduced the rational expectations interpretation of the stochastic growth model (Brock [1974]).*

*In the introduction, I have simply translated Brock's words in a backwards direction (Brock [1982], p. 33). But this has been the direction in which I arrived at the turnpike. It is at this point that I must thank Robert Lucas, Jr. for his early encouragement.

VI. LEMMAS

In this section I state and prove the lemmas that will be used in the proof of Theorem 1. The boundness of feasible allocations is stated in Lemma 1. Lemmas 2 and 3 establish the reachability conditions of γ -interior ρ -leisured programs. In Lemma 4 I prove that optimal programs are ρ -leisured. The uniform boundness of current value prices supporting optimal interior programs is proved in Lemma 5. Finally, the main properties of the present value loss function are proved in Lemmas 6 to 9. In this, and in the next two sections, I assume A.1 - A.22.

Lemma 1: (Radner): There exist $B > 0$ such that if $(x)(y)$ is a feasible allocation, then $\|x, y\|_{\infty} \leq B$.

Proof: The proof follows from assumptions A.2, A.3, A.4, A.10, A.13, A.18 (Necessity of Primary Inputs) and A.20 (Finite Different Productions at period zero). It is a routine adequation of Radner's proof (Radner [1973], Theorem 3.1) and is omitted.

Lemma 2: For every pair of numbers $\rho > 0$ and $\gamma > 0$, there exist a positive integer N such that if $\{(\hat{x}_t)(\hat{y}_t)\}_{t=0}^{\infty} \in \mathcal{F}(\hat{K}_0)$ and $\{(\tilde{x}_t)(\tilde{y}_t)\}_{t=0}^{\infty} \in \mathcal{F}(\tilde{K}_0)$ are two γ -interior ρ -leisured programs then there are two accumulation programs $\{\hat{K}_n\}_{n=0}^N \in \mathcal{F}(\hat{K}_0)$ and $\{\tilde{K}_n\}_{n=0}^N \in \mathcal{F}(\tilde{K}_0)$ such that $\hat{K}_N \geq \tilde{K}_N$ and $\tilde{K}_N \geq \hat{K}_N$.

Proof: Since the problem is symmetric, it is enough to prove the existence of $\{\tilde{K}_n\}_{n=0}^N$. By assumption there exists $\rho' > 0$ and $m \geq L_p$ such that $\tilde{K}_m(s) \geq \hat{K}_m(s) + \rho' \cdot e$ a.s., where e is the unit vector in \mathbb{R}^L , and $\{\tilde{K}_n\}_{n=0}^m \in \mathcal{F}(\tilde{K}_0)$ is the alternative program of A.22.

Define the \mathcal{J}_m -measurable map $\alpha_m: S \rightarrow [0,1]$ by

$$\alpha_m(s) = \operatorname{argmax}_{\alpha \in [0,1]} \{ \alpha \cdot \hat{K}_m(s) + (1-\alpha) \tilde{K}_m(s) \leq \tilde{K}_m(s) \}$$

By convexity of Y , the finite accumulation program

$$\{ \alpha_m \cdot \hat{K}_n + (1-\alpha_m) \cdot \tilde{K}_n \}_{n=m}^{2m} \in \mathcal{F}(\tilde{K}_m)$$

is feasible. It is also γ -interior and ρ -leisured, and, again by assumption, it is possible to construct the program $\{\tilde{K}_n\}_{n=0}^{2m}$ in such a way that

$$\tilde{K}_{2m}(s) \geq \alpha_m(s) \cdot \hat{K}_{2m}(s) + (1-\alpha_m(s)) \tilde{K}_{2m} + \rho' \cdot e \quad \text{a.s.}$$

Define the \mathcal{J}_{2m} -measurable map $\alpha_{2m}: S \rightarrow (0,1)$ by

$$\alpha_{2m}(s) = \operatorname{argmax}_{\alpha \in [0,1]} \{ \alpha \cdot \hat{K}_{2m}(s) + (1-\alpha) \cdot \tilde{K}_{2m}(s) \leq \tilde{K}_{2m}(s) \} .$$

The process of defining the program $\{\tilde{K}_n\}$ and the α_n maps is recursively repeated until for some \bar{n} $\alpha_{\bar{n}}(s) = 1$ a.s. . I only have to prove that this process actually stops in a finite number of iterations.

Let \bar{n} be the smallest positive integer such that $\bar{n} \geq (B - \gamma)/\rho'$, where B is the upper bound of Lemma (1). I claim that $\alpha_{\bar{n}}(s) = 1$ a.s.

By the definition of α_m , for a given s , there exist an $i \in L_p$ such that

$$\alpha_m(s) \cdot \hat{K}_m^i(s) + (1 - \alpha_m(s)) \tilde{K}_m^i(s) = \tilde{K}_m^i(s) \geq \tilde{K}_m^i(s) + \rho'$$

i.e.,
$$\alpha_m(s) \cdot (\hat{K}_m^i(s) - \tilde{K}_m^i(s)) \geq \rho'$$

and
$$\alpha_m(s) \geq \rho' / (B - \gamma) .$$

Similarly,

$$\begin{aligned} \alpha_{2m}(s) \cdot \hat{K}_{2m}^j(s) + (1 - \alpha_{2m}(s)) \tilde{K}_{2m}^j(s) &= \tilde{K}_{2m}^j(s) \\ &\geq \alpha_m(s) + \hat{K}_{2m}^j(s) + (1 - \alpha_{2m}(s)) \tilde{K}_{2m}^j(s) + \rho' \end{aligned}$$

for some $j \in L_p$. Hence,

$$(\alpha_{2m}(s) - \alpha_m(s)) (\hat{K}_{2m}^j(s) - \tilde{K}_{2m}^j(s)) > \rho'$$

and
$$\alpha_{2m}(s) \geq \alpha_m(s) + \rho' / (B - \gamma) \geq 2 \cdot \rho' / (B - \gamma) .$$

In general, we have

$$1 \geq \alpha_{nm}(s) \geq n \frac{\rho'}{(B - \gamma)}$$

and, in particular,

$$1 \geq \alpha_{nm}(s) \geq \bar{n} \frac{\rho'}{(B - \gamma)} \geq 1 \quad \text{a.s.}$$

Let $N = m \cdot \bar{n}$ and the lemma is proved.

Q. E. D.

I have proved that ρ -leisured γ -interior programs can be reached from any interior capital stock in a finite number of periods. In the proof of the turnpike property, Lemma 2 (together with Lemma 4) plays the same role that *good programs* have traditionally played in turnpike theory (see Gale [1967]), i.e., guarantees that the difference in utility between optimal programs is uniformly bounded.

Next lemma strenghtens the reachability condition. For capital stocks sufficiently close to the original capital stock of the ρ -leisured γ -interior program, I prove that the approximating program can be constructed in a way that consumptions and productions are not "too different" from the original program. I use some ideas that were first developed by Yano (Yano [1983a], Lemma I.8.8). My result is stronger than Yano's result since I prove that not only consumptions are close but also that productions are efficient.

Lemma 3: Let $\{(x_t)(y_t)\}_{t=0}^{\infty} \in \mathcal{F}(K_0)$ be a ρ -leisured γ -interior program. Assume for all $t \geq 0$, $x_t(s) = y_{t0}(s) + y_{t-1,1}(s) + w(\sigma^t s)$. Then there is a positive integer m and positive constants $\bar{\epsilon}$ and \bar{A} (depending on ρ and γ), satisfying:

For any $\epsilon \in (0, \bar{\epsilon})$: If $\tilde{K}_0^{(i)}(s) = \max\{0, K_0^{(i)}(s) - \epsilon\}$, $i \in L_p$, then there is a \mathcal{J}_0 -program $\{(\tilde{x}_t(s))(\tilde{y}_t(s))\}_{t=0}^{\infty} \in \mathcal{F}(K_0(s))$ with the following properties:

$$(i) \{(\tilde{\alpha}_t(s))\tilde{y}_t(s)\}_{t=m}^{\infty} \equiv \{(x_t(s))(y_t(s))\}_{t=m}^{\infty} \text{ and}$$

$$\tilde{K}_m(s) \equiv K_m(s).$$

(ii) For all $t \geq 0$ and for all $i \in L$: if $\tilde{x}_t^{(i)}(s) = 0$ then $\tilde{x}_t^{(i)}(s) = 0$, if $y_{t1}^{(i)}(s) = 0$ then $\tilde{y}_{t0}^{(i)}(s) = 0$, and if $y_{t1}^{(i)}(s) = 0$ then $y_{t1}^{(i)}(s) = 0$.

(iii) For $0 \leq t \leq m-1$, $g(\tilde{y}_{t0}(s), \tilde{y}_{t1}(s); s) = 0$ a.s.

(iv) For all $t \geq 0$, $|x_t(s) - \tilde{x}_t(s)| \leq \bar{A} \cdot \epsilon$

(v) For all $t \geq 0$, $\tilde{x}_t(s) = \tilde{y}_{t0}(s) + \tilde{y}_{t-1,1}(s) + w(\sigma^t s)$.

Proof: By assumption there is an m and a \mathcal{J}_0 -program $\{('x_t(s)), ('y_t(s))\}_{t=0}^{m-1} \in \mathcal{F}(K_0(s))$ such that $'y_{t0}^{(1)}(s) = 0$ if $y_{t0}^{(1)}(s) = 0$, $'K_t(s) \geq K_t(s)$ and

$'K_m(s) \geq K_m(s) + \rho' \cdot \epsilon$. I can also assume $'x_t(s) \leq x_t(s)$

and $'x_t(s) = 'y_{t0}(s) + 'y_{t-1,1}(s) + w(\sigma^t s)$, since the alter-

native program is more intensively accumulative.

In order to satisfy (i) to (v) simultaneously, I have to keep track of many different elements; the cost of doing this is the introduction of a quite tedious notation. For $z \in \mathbb{R}^L$ I define the characteristic functions $I_{\{z\}}$ and $I_{\{z\}}^c$ by

$$I_{\{z\}} = \begin{cases} 1 & \text{if } z^{(i)} > 0 \\ 0 & \text{if } z^{(i)} = 0 \end{cases} ; \quad I_{\{z\}}^c = e - I_{\{z\}}$$

where, as usual, e is the unit vector in \mathbb{R}^L . The $L \times L$ characteristic matrix is defined by

$$\bar{I}_{\{z\}} = \begin{bmatrix} I_{\{z\}}^{(1)} & 0 & \dots & 0 \\ 0 & I_{\{z\}}^{(2)} & \dots & 0 \\ \vdots & & \ddots & \\ 0 & & & I_{\{z\}}^{(L)} \end{bmatrix}$$

and similarly for $\bar{I}_{\{z\}}^c$.

Finally, $I_{\{z\}} \wedge I_{\{z'\}}$ denotes the componentwise minimum of both vectors. For any \mathcal{L}_0 -measurable map $\alpha: s \rightarrow (0,1)$, I can define the convex combination program

$$(x_t(\alpha(s), s), y_t(\alpha(s), s)) \equiv (\alpha(s)'x_t(s) + (1-\alpha(s))x_t(s),$$

$$\alpha(s)'y_t(s) + (1-\alpha(s))y_t(s)), \text{ for } t = 0, \dots, m-1,$$

$$\text{and } K_0(\alpha(s), s) = \alpha(s)'x_0(s) + (1-\alpha(s))K_0(s) = K_0(s).$$

I now define a sequence of maps, $\{\beta_n(\alpha(s), s)\}_{n=0}^{m-1}$, as follows:

$$\beta_{m-1}(\alpha(s), s) = \max\{\beta \in \mathbb{R}_t \mid g((y_{m-1,0}(\alpha(s), s) + \bar{I}_{\{K_{m-1}(s)\}}^c) \cdot K_{m-1}(\alpha(s), s) + \beta \cdot I_{\{K_{m-1}(s)\}} \wedge I_{\{y_{m-1,0}(s)\}}) , \\ K_m(s) ; s \leq 0\} ,$$

$$\beta_{m-2}(\alpha(s), s) = \max\{\beta \in \mathbb{R}_t \mid g((y_{m-2,0}(\alpha(s), s) + \bar{I}_{\{K_{m-2}(s)\}}^c) \cdot K_{m-2}(\alpha(s), s) + \beta \cdot I_{\{K_{m-2}(s)\}} \wedge I_{\{y_{m-2,0}(s)\}}) , \\ (\bar{I}_{\{K_{m-1}(s)\}} \cdot K_{m-1}(\alpha(s), s) - \beta_{m-1}(\alpha(s), s) \cdot I_{\{K_{m-1}(s)\}} \wedge I_{\{y_{m-1,0}(s)\}}) ; s \leq 0\}$$

and recursively, I define in a similar way

$\beta_{m-r}(\alpha(s), s)$ for $r = 3, \dots, m$, i.e.,

$$\beta_0(\alpha(s), s) = \max\{\beta \in \mathbb{R}_t \mid g((y_{0,0}(\alpha(s), s) + \beta \cdot I_{\{K_0(s)\}} \wedge I_{\{y_{0,0}(s)\}}) , \\ (\bar{I}_{\{K_1(s)\}} \cdot K_1(\alpha(s), s) - \beta_1(\alpha(s), s) \cdot I_{\{K_1(s)\}} \wedge I_{\{y_{1,0}(s)\}}) ; s \leq 0\}$$

Notice that the definition of $\beta_0(\alpha(s), s)$ is consistent with the definition of the β_n 's function since $\bar{I}_{\{K_0(s)\}}^c K_0(\alpha(s), s) = 0$.

Claim 1: The function $\beta_0(\cdot, s) : (0, 1) \rightarrow \mathbb{R}_t$ is continuous.

In order to see this, I apply recursively the maximum principle.

$$\begin{aligned} \text{Let } \psi_{m-1}(\alpha, s) = & \{ \beta \in \mathbb{R}_t^L \mid g((y_{m-1,0}(\alpha, s) \\ & + \bar{I}_{\{K_{m-1}(s)\}}^c \cdot K_{m-1}(\alpha(s), s) \\ & + \beta \cdot I_{\{K_{m-1}(s)\}} \wedge I_{\{y_{m-1,0}(s)\}}), K_m(s); s) \leq 0 \} \end{aligned}$$

By assumption $g(\cdot, \cdot; s) : \mathbb{R}_t^L \times \mathbb{R}_t^P \rightarrow \mathbb{R}$ is a continuous function this implies that ψ_{m-1} is closed valued. By lemma 1 (uniform boundness of feasible allocations) ψ_{m-1} is also compact valued. The map $y_{m-1}(\cdot, s) : (0, 1) \rightarrow \mathbb{R}^L$ is clearly continuous and similarly for $K_{m-1}(\cdot, s)$. Using this fact and the continuity of g it is easy to prove that ψ_{m-1} is a continuous correspondence. Let $\psi_{m-1}(\alpha^q, s) \neq \emptyset$, $\alpha^q \rightarrow \alpha^0$ and $\beta^q \in \psi_{m-1}(\alpha^q, s)$. By compactness there exist a subsequence $\beta^{q_i} \rightarrow \beta^0$, From $\{\alpha^{q_i}\}$, I can extract another subsequence that it is either increasing or decreasing. (I maintain the same notation for this subsequence.) If $\{\alpha^{q_i}\}$ is increasing, then $\{\psi_{m-1}(\alpha^{q_i})\}$ is an increasing sequence of closed sets and, hence, $\beta^0 \in \psi_{m-1}(\alpha^0)$; if $\{\alpha^{q_i}\}$ is decreasing, then for all q_i , $\beta^0 \in \psi_{m-1}(\alpha^{q_i})$ the continuity of y_{m-1} and q imply that $\beta^0 \in \psi_{m-1}(\alpha^0)$. Thus, ψ_{m-1} is upperhemicontinuous. A similar argument can be used to prove that ψ_{m-1} is lowerhemicontinuous (i.e., if $\alpha^{q_i} \rightarrow \alpha^0$ and $\beta^0 \in \psi_{m-1}(\alpha^0)$, then there is a $\{\beta^q\}$ such that $\beta^q \in \psi_{m-1}(\alpha^q)$ and $\beta^q \rightarrow \beta^0$), for the decreasing subsequence $\{\alpha^{q_i}\}$ choose $\beta = \beta^0$ and for the increasing subsequence use the continuity of y_{m-1} and q .

Let $h(\cdot, \cdot, s) : (0, 1) \times \mathbb{R}_+ \rightarrow \mathbb{R}$ be defined by

$$h(\alpha, \beta; s) = g((y_{m-1,0}(\alpha, s) + \bar{I}^c_{\{K_{m-1}(s)\}} \cdot K_{m-1}(\alpha, s) \\ + \beta \cdot I_{\{K_{m-1}(s)\}} \wedge I_{\{y_{m-1,0}(s)\}} \cdot K_m(s) : s)$$

by continuity at $y_{m-1}(\cdot, s)$, $K_{m-1}(\cdot, s)$ and $g(\cdot, \cdot; s)$ it follows that $h(\cdot, \cdot; s)$ is also continuous. Now,

$$\beta_{m-1}(\alpha; s) \equiv \max \{h(\alpha, \beta; s) \mid h(\alpha, \beta; s) \in \psi_{m-1}(\alpha; s)\} .$$

ψ_{m-1} is compact valued and continuous and h is continuous applying the Maximum Principle theorem, it follows that $\beta_{m-1}(\cdot; s)$ is continuous. The same argument can be applied to prove the continuity of $\beta_{m-2}(\cdot; s)$ and so on.

The claim has been proved and I now return to the proof of the lemma.

$$\text{Recall that } |K_m(\alpha(s), s) - K_m(s)| = |\alpha(s) \cdot K_m(s) \\ + (1 - \alpha(s))K_m(s) - K_m(s)| = \alpha(s) \cdot |K_m(s) \\ - K_m(s)| \geq \alpha(s) \cdot \rho' .$$

By assumption A.19 (uniformly bounded marginal productivity of capital) we have

$$\begin{aligned}
\alpha(s) \cdot \rho &\leq |K_m(\alpha(s), s) - K_m(s)| \leq \eta \cdot |\bar{I}_{\{K_{m-1}(s)\}}^c \cdot K_{m-1}(\alpha(s), s) \\
&\quad + \beta_{m-1}(\alpha(s), s) \cdot I_{\{K_{m-1}(s)\}} \wedge I_{\{y_{m-1,0}(s)\}}| \\
&\leq \eta^2 \cdot |\bar{I}_{\{K_{m-2}(s)\}}^c \cdot K_{m-2}(\alpha(s), s) \\
&\quad + \beta_{m-2}(\alpha(s), s) \cdot I_{\{K_{m-2}(s)\}} \wedge I_{\{y_{m-2,0}(s)\}}| \\
&\leq \dots \leq \eta^m |\beta_0(\alpha(s), s) \cdot I_{\{K_0(s)\}} \wedge I_{\{y_{0,0}(s)\}}| \\
&= \eta_m \cdot \rho_0(\alpha(s), s) .
\end{aligned}$$

Let $\bar{\varepsilon} = \rho' \cdot \eta^{-m}$, then $\bar{\varepsilon} \leq \lim_{\alpha \rightarrow 1} \beta_0(\alpha, s)$. Furthermore,

$\lim_{\alpha \rightarrow 0} \beta_0(\alpha, s) = 0$. For any $\varepsilon \in (0, \bar{\varepsilon})$, using the continuity of $\beta_0(\cdot, s)$ and the intermediate value theorem, there exist a \mathcal{J}_0 -measurable map $\alpha: s \rightarrow (0, 1)$ such that $\alpha(s) \cdot \rho' \leq \eta^m \cdot \varepsilon$. Given $\alpha(s)$ and defining the corresponding program

$$\{(x_t(\alpha(s), s), (y_t(\alpha(s), s)))_{t=0}^{m-1}, K_0(\alpha(s), s) \equiv K_0(s),$$

it follows, from the above construction, that $K_m(s)$ can be reached from $K_0(s) - \varepsilon \cdot I_{\{K_0(s)\}} \wedge I_{\{y_{0,0}(s)\}} \geq 0$.

It only remains to define the program $\{(x_t(s), (y_t(s)))_{t=0}^{\infty}$ and check the conditions.

Remark: Given $\varepsilon \in (0, \bar{\varepsilon})$, $\alpha(s)$ is changed using the continuity of $\beta_0(\alpha, s)$ as we have shown before. $\alpha(s)$, in turn, defines a sequence of maps $\beta_u(\alpha(s), s)$, $n = 0, \dots, m-1$, using the above construction. Notice, however, that $\beta_n(\alpha(s), \cdot)$

only depends on the finite history (s_1, \dots, s_u) and, in particular, $\beta_0(\alpha(s), s) = \varepsilon$ (the past history $(\dots s_0)$ is registered through $\alpha(s)$).

Let

$$\{(\tilde{x}_n(s)), (\tilde{v}_n(s))\}_{k=m}^{\infty} \equiv (x_n(s)), (y_n(s))\}_{n=m}^{\infty},$$

$$y_{m-1}(s) \equiv \tilde{K}_m(s) \equiv K_m(s) \quad \text{for } n=1, \dots, m-1,$$

$$\tilde{y}_{n,0}(s) = y_{n,0}(\alpha(s), s) + \bar{I}_{\{K_n(s)\}}^c = K_n(\alpha(s), s)$$

$$+ \beta_n(\alpha(s), s) \cdot I_{\{K_n(s)\}} \wedge I_{\{y_{n,0}(s)\}}$$

$$\tilde{y}_{n-1,1}(s) \equiv \tilde{K}_n(s) = \bar{I}_{\{K_n(s)\}} \cdot K_n(\alpha(s), s)$$

$$- \beta_n(\alpha(s), s) \cdot I_{\{K_n(s)\}} \wedge I_{\{y_{n,0}(s)\}}$$

$$\tilde{x}_n(s) = x_n(\alpha(s), s) = \alpha(s) \cdot x_n(s) + (1 - \alpha(s))x_n(s)$$

and for $n = 0$

$$\tilde{y}_{0,0}(s) = y_{0,0}(\alpha(s), s) + \varepsilon \cdot I_{\{K_0(s)\}} \wedge I_{\{y_{0,0}(s)\}}$$

$$\tilde{y}_{-1,1}^{(i)}(s) \equiv \tilde{K}_0^{(i)}(s) = \max \{0, K_0^{(i)}(s) - \varepsilon\} \quad i \in L_p$$

$$\tilde{x}_0^{(i)}(s) = \begin{cases} x^{(i)}(\alpha(s), s) = \alpha(s) x^{(i)}(s) + (1-\alpha(s))x^{(i)}(s) & \text{if } y_{00}^{(i)} < 0 \\ \max\{0, K_0^{(i)}(s) - \epsilon\} & \text{if } y_{00}^{(i)} = 0 \text{ and } i \in L_p \\ w^{(i)}(s) & \text{if } y_{00}^{(i)} = 0 \text{ and } i \in L_0 \end{cases}$$

Now I claim that the program $\{(\tilde{x}_t(s), \tilde{y}_t(s))\}_{t=0}^{\infty}$ satisfies (i) to (v) .

(i) Is satisfied by definition.

(ii) $y_{n0}^{(i)}(\alpha(s), s) = 0$ whenever $y_{n0}^{(i)}(s) = 0$ and

$x_n^{(i)}(\alpha(s), s) = 0$ whenever $x_n^{(i)}(s)$ by assumption.

For $n \geq 1$, we have $\tilde{x}_n^{(i)}(s) = 0$ if $x_n^{(i)}(s) = 0$. For $n = 0$ if $y_{00}^{(i)} < 0$ it is also true that $\tilde{x}_0^{(i)}(s) = 0$

if $x_0^{(i)}(s) = 0$; suppose $y_{00}^{(i)} = 0$ for $i \in L_p$.

By assumption $x_0^{(i)}(s) = y_{00}^{(i)}(s) + K_0^{(i)}(s) + w^{(i)}(s) = K_0^{(i)}(s)$, hence, if $x_0^{(i)} = 0$ (i.e., $K_0^{(i)}(s) = 0$), then $\tilde{x}_0^{(i)}(s) = 0$ and the same argument applies if

$y_{00}^{(i)}(s) = 0$ and $i \in L_0$ (i.e., $x_0^{(i)}(s) = w^{(i)}(s)$).

The role played by the characteristic functions is precisely to guarantee this condition for

$(\tilde{y}_{n0}(s), \tilde{y}_{n1}(s)) \forall n \geq 0$. For $\tilde{y}_{-1,1}(s)$ is true by definition.

(iii) The fact that the β_n^i 's are chosen to be maximal guarantees this condition for $n \geq 1$. The particular choice of $\alpha(s)$, through ϵ , guarantees this property for $n = 0$. In fact, all the argument concerning the

continuity of $\beta_0(\cdot, s)$ has been constructed in order to satisfy (ii) at $n = 0$.

$$\begin{aligned}
 \text{(iv) For } n \geq 1, \quad & |\tilde{x}_n(s) - x_n(s)| = |\alpha(s)'x_n(s) \\
 & + (1-\alpha(s))x_n(s) - x_n(s)| \\
 & \leq |(1-\alpha(s))x_n(s) - x_n(s)| \\
 & = \alpha(s) \cdot |x_n(s)|.
 \end{aligned}$$

$$\begin{aligned}
 \text{For } n = 0, \quad & |\tilde{x}_0(s) - x_0(s)| = \max\{\varepsilon, |\tilde{x}_0^{(i)}(s) - x_0^{(i)}(s)| \\
 & \forall i \in L \text{ s.t. } y_{00}^{(i)}(s) < 0\},
 \end{aligned}$$

$$\text{hence, either } |\tilde{x}_0(s) - x_0(s)| = \varepsilon$$

$$\text{or } |\tilde{x}_0(s) - x_0(s)| \leq \alpha(s) \cdot |x_0(s)|.$$

Given the choice of $\alpha(s)$, we have that $\alpha(s) \leq \eta^m \cdot \varepsilon / \rho'$.

Let $\bar{A} = \eta^m \cdot B / \rho'$ where B is the upper bound on feasible allocations (there is no loss of generality in assuming $\bar{A} \geq 1$), and (iv) is satisfied.

$$\begin{aligned}
 \text{(v) By assumption } & x_n(s) = y_{n0}(s) + y_{n-1,1}(s) + w(\sigma^n s) \\
 \text{and } 'x_n(s) = & 'y_{n0}(s) + 'y_{n-1,1}(s) + w(\sigma^n s) \text{ for } n \geq 0.
 \end{aligned}$$

$$\begin{aligned}
\text{If } n \geq 1, \text{ then } \tilde{x}_n(s) &= y_{n0}(\alpha(s), s) + y_{n-1,1}(\alpha(s), s) \\
&+ w(\sigma^n s) = y_{n0}(\alpha(s), s) \\
&+ \bar{I}_{\{K_n(s)\}}^c \cdot K_{n-1,1}(\alpha(s), s) \\
&+ \beta_n(\alpha(s), s) \cdot I_{\{K_n(s)\}} \wedge I_{\{K_{n,0}(s)\}} \\
&+ \bar{I}_{\{K_n(s)\}} \cdot y_{n-1,1}(\alpha(s), s) \\
&- \beta_n(\alpha(s), s) \cdot I_{\{K_n(s)\}} \wedge I_{\{y_{n,0}(s)\}} \\
&+ w(\sigma^n s) = \tilde{y}_{n0}(s) + \tilde{y}_{n-1,1}(s) + w(\sigma^n s) .
\end{aligned}$$

The same equality is satisfied for $\tilde{x}_0^{(i)}(s)$ if $y_{00}^{(i)} < 0$. Suppose that $y_{00}^{(i)} = 0$ and $i \in L_p$, then by definition $\tilde{x}_0^{(i)}(s) = \max\{0, K_0^{(i)}(s) - \epsilon\} = \tilde{K}_0^{(i)}$.

If $y_{00}^{(i)} = 0$ and $i \in L_0$, then also by definition

$\tilde{x}_0^{(i)} = w^{(i)}(s)$. Which proves (v). Notice that

given the particular choice of $\tilde{x}_0^{(i)}(s)$ it is possible

to decrease $K_0(s)$ without violating (ii).

Q.E.D.

Corollary 3: For any pair of positive numbers (ρ, γ) , there exist $\bar{\epsilon} > 0$ and $E > 0$, such that if $\{(x_t)(y_t)\}_{t=0}^{\infty} \in \mathfrak{F}(K_0)$ is a ρ -leisured γ -interior program and, for $\epsilon \in (0, \bar{\epsilon})$, \tilde{K} is defined by $\tilde{K}^1(s) = \max \{K^1(s) - \epsilon, \gamma\}$, then there exist a program $\{(x_t)(y_t)\}_{t=0}^{\infty} \in \mathfrak{F}(\tilde{K})$ such that

$$E \left[\sum_{t=0}^{\infty} \delta^t (U(x_t(s), s) - U(\tilde{x}_t(s), s)) \right] \leq \epsilon \cdot E, \quad \forall \delta \in (0, 1].$$

Proof: Let $\{(\tilde{x}_t)(\tilde{y}_t)\}_{t=0}^{\infty} \in \mathfrak{F}(\tilde{K})$ be the program defined in Lemma 3, then

$$\begin{aligned} E \left[\sum_{t=0}^{\infty} (U(x_t(s), s) - U(\tilde{x}_t(s), s)) \right] \\ = E \left[\sum_{t=0}^m (U(x_t(s), s) - U(x_t(s), s)) \right] \end{aligned}$$

by condition (iv) of Lemma 3 and continuity of $U(\cdot, s)$ it follows that there exists $E > 0$ such that

$$E \left[\sum_{t=0}^m (U(x_t(s), s) - U(X_t(s), s)) \right] \leq \epsilon \cdot E$$

Q.E.D.

In the next two lemmas, the first order conditions of the one period maximization problems are used to derive bounds on labor supplies and current value prices. Before I state these lemmas, I characterize the one period maximization problems (see Rockafellar and Watts [1975]).

The one period profit maximization problem is:

$$(6.1) \quad \text{Max} \{E[\delta q_{t+1}(s) \cdot y_{t1}(s) | \mathcal{J}_t] + q_t(s) y_{t0}(s)\}$$

$$\text{subject to} \quad g(y_{t0}(s), y_{t1}(s); s) \leq 0 \quad \text{a.s.}$$

the Lagrangean for this problem is

$$\begin{aligned} & E[\delta q_{t+1}(s) y_{t1}(s) | \mathcal{J}_t] + q_t(s) y_{t0}(s) \\ & - E[\lambda_{t+1}(s) g(y_{t0}(s), y_{t1}(s); s) | \mathcal{J}_t] \end{aligned}$$

where $\lambda_{t+1} \in \mathcal{L}_1(S, \mathcal{J}_{t+1}, P)$ and the Kuhn-Tucker conditions for this problem are

$$(6.2a) \quad q_t^i(s) - E[\lambda_{t+1}(s) \frac{\partial g(y_{t0}(s), y_{t1}(s); s)}{\partial y_{t0}^i(s)} | \mathcal{J}_t] \geq 0 \quad \text{a.s.}$$

$$(6.2b) \quad y_{t0}^i(s) [q_t^i(s) - E[\lambda_{t+1}(s) \frac{\partial g(y_{t0}(s), y_{t1}(s); s)}{\partial y_{t0}^i(s)} | \mathcal{J}_t]] = 0 \quad \text{a.s.}$$

$$(6.3a) \quad \delta q_{t+1}^i(s) - \lambda_{t+1}(s) \frac{\partial g(y_{t0}(s), y_{t1}(s); s)}{\partial y_{t1}^i(s)} \leq 0 \quad \text{a.s.}$$

$$(6.3b) \quad y_{t1}^i(s) [\delta q_{t+1}^i(s) - \lambda_{t+1}(s) \frac{\partial g(y_{t0}(s), y_{t1}(s); s)}{\partial y_{t1}^i(s)}] = 0 \quad \text{a.s.}$$

Similarly, for the one period utility maximization problem we have:

$$(6.4a) \quad D_i U(x_t(s), s) - q_t^i(s) \leq 0 \quad \text{a.s.}$$

$$(6.4b) \quad x_t^i(s) [D_i U(x_t(s), s) - q_t^i(s)] = 0 \quad \text{a.s.}$$

Lemma 4: There exists $\rho > 0$ such that if

$\{(x_t)(y_t)\}_{t=0}^{\infty}$ is an optimal program, then

$\{(x_t)(y_t)\}_{t=0}^{\infty}$ is a ρ -leisured program, provided $\delta > 0$.

Proof: Suppose the lemma is not true, then for any $\rho > 0$ there exists a set $A_\rho \in \mathcal{T}_0$ of positive probability and an optimal allocation $((^\rho x_0)(^\rho y_0))$ such that $0 \leq {}^\rho x_0^1(s) < \rho$ for $s \in A_\rho$. As $\rho \rightarrow 0$, ${}^\rho x_0^1(s) \rightarrow 0$ for $s \in A_\rho$. By assumption A.9, $D_1 U({}^\rho x_0(s), s) \rightarrow +\infty$. Then, by condition (6.4), ${}^\rho q_0^1(s) \rightarrow +\infty$ a.s. if $s \in A_\rho$.

For small ρ the supply of labor is positive. Without loss of generality I can assume that labor is used to produce the good $j \in L_p \cap L_c$ and that $x_1^j(s) > 0$ if $s \in A_\rho$. In general, the choice of j depends on (\dots, s_0, s_1) and consumption may be delayed for a finite number of periods, but the argument is the same. So I assume that if $s \in A_\rho$, $x_1^j(\dots, s_0, s_1) > 0$ a.s. It follows from (6.2), (6.3) and (6.4) that

$$(6.5) \quad \rho_{q_0^1}(s) = E \left[\delta D_j(x_1(s), s) \left(\frac{g(y_{00}(s), y_{01}(s); s)}{y_{01}^j(s)} \right)^{-1} \right. \\ \left. \cdot \frac{\partial g(y_{00}(s), y_{01}(s); s)}{\partial y_{00}^1(s)} \Big| \mathcal{J}_0 \right](s) \text{ a.s. for } s \in A_\rho$$

But by assumptions A.8 and A.16 the right hand side of 6.5 is uniformly bounded which contradicts the unboundness of $\rho_{q_0^1}(s)$ on A_ρ .
Q.E.D.

Lemma 5: There exist $\underline{\delta} \in (0, 1)$ and positive numbers \underline{q} , \bar{q} , such that if $\{q_t\}_{t=0}^\infty$ is a system of current value prices supporting an optimal γ -interior program, then $\underline{q} \leq q_t^i(s) \leq \bar{q}$ a.s. for all $i \in L_p$ and all $t \geq 0$.

Proof: I first compute the lower bound \underline{q} . Let $i \in L_p \cap L_c$, then for any $j \in L$,

$$(6.6) \quad q_0^j(s) \geq E \left[\delta D_i U(x_1(s), s) \cdot \left(\frac{\partial g(y_{00}(s), y_{01}(s); s)}{\partial y_{01}^i(s)} \right)^{-1} \right. \\ \left. \frac{\partial g(y_{00}(s), y_{01}(s); s)}{\partial y_{00}^i(s)} \Big| \mathcal{J}_t \right] \geq \underline{\delta} \cdot \underline{d} \cdot \underline{a} \equiv \underline{q}$$

where \underline{d} is the uniform lower bound of $D_i U(\cdot, s)$ on the set $\{x \in \mathbb{R}^c \mid |x| \leq B\}$ and \underline{a} is the uniform lower bound of the marginal rates of transformation on the set $\{y_0 \in \mathbb{R}_-^L \times \mathbb{R}^p \mid |y| \leq B\}$.

An upper bound can be computed using the fact that marginal utilities and marginal rules of transformation are bounded from above.

An alternative argument can be constructed using Lemma 3.

Let $\{(x_t)(y_t)\}_{t=0}^{\infty} \in \mathcal{J}(K_0)$ be an optimal γ -interior program.
 Let $\epsilon = \frac{1}{2} \min(\bar{\epsilon}, \gamma)$ where $\bar{\epsilon}$ is defined in Lemma 3. Define \tilde{K}_0 by

$$\tilde{K}_0^i(s) = K_0^i(s) - \epsilon$$

and

$$\tilde{K}_0^j(s) = K_0^j(s) \quad \text{for } j \neq i .$$

Let $\{(\tilde{x}_t)(\tilde{y}_t)\}_{t=0}^{\infty} \in \mathcal{X}(\tilde{K}_0)$

be the program reaching $\{(x_t)(y_t)\}_{t=0}^{\infty}$ defined in Lemma 3. Then, by the same argument used in the proof of Corollary 3, there exists \bar{q} such that

$$\begin{aligned} \epsilon \cdot \bar{q} &\geq E \left[\sum_{t=0}^m (U(x_t(s), s) - U(\tilde{x}_t(s), s)) \mid \mathcal{J}_t \right] \\ &\geq E \left[\sum_{t=0}^{\infty} \delta (U(x_t(s), s) - U(\tilde{x}_t(s), s)) \mid \mathcal{J}_t \right] \\ &\geq V(K_0(s), s; \delta) - V(\tilde{K}_0(s), s; \delta) \\ &\geq q_0(s)(K_0(s) - \tilde{K}_0(s)) = q_0^i(s) \cdot \epsilon \end{aligned}$$

Q.E.D.

Remark: In fact prices are uniformly bounded for all $i \in L$ but I only need for $i \in L_p$. Mirman and Zilcha have shown that prices may be unbounded (Mirman and Zilcha [1976]). In my model this case is

excluded, since the Inada condition is only satisfied for the good leisure/labor and labor is used directly or indirectly in the production of all producible goods. Notice that if the upper bound is computed using Kuhn-Tucker conditions, the γ -interiority is not required.

Next, I prove some properties of the Liapunov process defined by the present value loss. The proofs of these properties are, in general, the stochastic version of Bewley's proofs (Bewley [1982]). The main difference is that I use Lemmas 2 and 3 and I can simplify some of his arguments (in particular his Lemma 12.15, which corresponds to my Lemmas 3 and 9).

Let $\{(x_t)(y_t)\}_{t=0}^{\infty} \in \mathcal{X}(K_0)$ be an optimal interior program supported by a system of current value prices $\{q_t\}_{t=0}^{\infty}$ and $\{(\tilde{x}_t)(\tilde{y}_t)\}_{t=0}^{\infty} \in \mathcal{X}(K_0)$ be another optimal interior program. The present value loss was defined in (2.10) by

$$F_L(\tilde{K}_t(s), s; K_t, \delta) = q_t(s) \cdot (\tilde{K}_t(s) - K_t(s)) \\ - (V(\tilde{K}_t(s), s; \delta) - V(K_t(s), s; \delta))$$

Note: From now on, in order to simplify notation, I will denote the value loss simply by $F_t^\delta(s)$. Recall that I assume all optimal interior programs are uniformly interior, i.e., there exists $\gamma > 0$ for which all optimal interior programs are, in fact, ρ -leisured and γ -interior optimal programs (Lemma 4).

Lemma 6: For all $t \geq 0$, $F_t^\delta \geq 0$ a.s., provided that F_0^δ is well defined.

Proof: Is a direct computation (see 2.11)

Lemma 7: There exist $C > 0$, such that for all $t \geq 0$ and $\delta \in [\underline{\delta}, 1]$, $F_t^\delta \leq C$ a.s., where $\underline{\delta}$ is defined in Lemma 5.

Proof: Let $\{(\tilde{x}_n)(\tilde{y}_n)\}_{n=0}^\infty \in \mathcal{G}(\tilde{K}_t(s))$ be the program defined in Lemma 2, i.e., the program that reaches $\{(x_n)(y_n)\}_{n=t}^\infty$.

$$\begin{aligned} & V(K_y(s), s; \delta) - V(\tilde{K}_t(s), s; \delta) \\ & \leq E \left[\sum_{n=0}^{\infty} \delta^n (U(x_{t+1}(s), s) - U(\tilde{x}_n(s), s)) | \mathcal{Y}_t \right] \\ & = E \left[\sum_{n=0}^N \delta^n (U(x_{t+n}(s), s) - U(\tilde{x}_n(s), s)) | \mathcal{Y}_t \right] \leq \bar{c} \text{ a.s.} \end{aligned}$$

The two last inequalities follow from Lemma 2 and the uniform boundedness of $U(\cdot, s)$, respectively. Then it follows that

$$F_t^\delta(s) \leq q_t(s) (\tilde{K}_t(s) - K_t(s) + \bar{c}) \leq \bar{q} \cdot B + \bar{c}$$

where \bar{q} is the upper bound of Lemma 5 and B the upper bound of Lemma 1. Let $C = \bar{q} \cdot B + \bar{c}$.

Lemma 8: There exist $\alpha' > 0$, $\varepsilon' > 0$ such that

$$\begin{aligned} & F_t^\delta(s) - \delta E[F_{t+1}^\delta(s) | \mathcal{Y}_t] \\ & \geq \alpha' \cdot E[\min\{\varepsilon'^2, |x_t(s) - \tilde{x}_t(s), y_{t0}(s) \\ & \quad - y_{t0}(s), \tilde{y}_{t1}(s) - y_{t1}(s)|^2 | \mathcal{Y}_t\}] \end{aligned}$$

Proof:

$$\begin{aligned}
F_t^\delta(s) - \delta E[F_{t+1}^\delta(s) | \mathcal{J}_t] \\
&= U(x_t(s), s) - q_t(s) \cdot x_t(s) - (U(\tilde{x}_t(s), s) \\
&\quad - q_t(s) \cdot \tilde{x}_t(s)) \\
&\quad + E[\delta q_{t+1}(s) y_{t1}(s) | \mathcal{J}_t] + q_t(s) \cdot y_{t0}(s) \\
&\quad - \left(E[\delta q_{t+1}(s) \tilde{y}_{t1}(s) | \mathcal{J}_t] + q_t(s) \tilde{y}_{t0}(s) \right)
\end{aligned}$$

By assumption A.8, there exist $\alpha_c > 0$, $\epsilon_c > 0$, such that if $|x| \leq B$ and $|x - \hat{x}| \leq \epsilon_c$, then

$$-\frac{1}{2}(\hat{x} - x)^T \cdot D^2 U(x, s)(\hat{x} - x) \geq \alpha_c \cdot |x - \hat{x}|^2.$$

Assume $|x_t(s) - \tilde{x}_t(s)| \leq \epsilon_c$, then

$$\begin{aligned}
&U(x_t(s), s) - U(\tilde{x}_t(s), s) - q_t(s)(x_t(s) - \tilde{x}_t(s)) \\
&\geq U(x_t(s), s) - U(\tilde{x}_t(s), s) - DU(x_t(s), s)(x_t(s) - \tilde{x}_t(s)) \\
&\geq \alpha_c \cdot |x_t(s) - \tilde{x}_t(s)|^2
\end{aligned}$$

By concavity of $U(\cdot, s)$, if $|x_t(s) - \tilde{x}_t(s)| > \epsilon_c$, then

$$U(x_t(s), s) - U(\tilde{x}_t(s), s) - q_t(s)(x_t(s) - \tilde{x}_t(s)) \geq \alpha_c \cdot \epsilon_c^2.$$

Similarly, by assumption A.16, there exist $\alpha_p > 0$, $\epsilon_p > 0$ such that if $g(y_0, y_1; s) = 0$, $|y| \leq B$, and $|y - \hat{y}| \leq \epsilon_p$, then

$$\frac{1}{2} (y - \hat{y})^1 D^2 g(y_0, y_1; s) (y - \hat{y}) \geq \alpha_p \cdot |y - \hat{y}|^2 .$$

By (6.2) and (6.3),

$$\begin{aligned} & E[\delta q_{t+1}(s)(y_{t1}(s) - \tilde{y}_{t1}(s)) + q_t(s)(y_{t0}(s) - \tilde{y}_{t0}(s)) | \mathcal{J}_t] \\ & \geq E[\lambda_{t+1}(s) \cdot Dg(y_{t0}(s), y_{t1}(s); s) \cdot (y_t(s) - \tilde{y}_t(s)) | \mathcal{J}_t] \\ & E[\lambda_{t+1}(s) \cdot \min(\varepsilon_1^2, |y_t(s) - \tilde{y}_t(s)|^2) | \mathcal{J}_t] \end{aligned}$$

It follows from Lemma 5 that there exist positive numbers $\underline{\lambda}$, $\bar{\lambda}$ such that $\underline{\lambda} \leq \lambda_{t+1}(s) \leq \bar{\lambda}$ a.s. Let

$$\alpha' = \min(\alpha_c, \underline{\lambda} \cdot \alpha_p) , \varepsilon' = \min(\varepsilon_c, \varepsilon_p) .$$

Q.E.D.

Corollary 8: Let $\delta \in (\underline{\delta}, 1)$. There exist $\alpha > 0$, $\tilde{\varepsilon} > 0$, such that

$$\delta^{-1} F_t^\delta(s) - \underline{F}[F_{t+1}^\delta(s) | \mathcal{J}_t] \geq 2\alpha\{\tilde{\varepsilon}^2, |K_t(s) - \tilde{K}_t(s)|^2\} \quad \text{a.s.}$$

Proof: Let $\tilde{\varepsilon} = 2\varepsilon'$ and $\alpha = 1/8 \alpha'$.

Lemma 9: Let $\delta \in (\underline{\delta}, 1)$. There exist $A > 0$, $\hat{\varepsilon} > 0$, such that if $|K_t(s) - \tilde{K}_t(s)| < \hat{\varepsilon}$, then $F_t^\delta(s) \leq A \cdot |K_t(s) - \tilde{K}_t(s)|^2$.

Proof: By assumption A.8, there exist $\varepsilon_c > 0$ and $\bar{\alpha}_c > 0$, such that if $|x - \hat{x}| \leq \varepsilon_c$, then

$$U(x, s) - U(\hat{x}, s) - DU(x, s)(x - \hat{x}) \leq \bar{\alpha}_c \cdot |x - \hat{x}|^2 .$$

Similarly, by assumption A.16, there exist $\varepsilon_p > 0$ and $\bar{\alpha}_p > 0$, such that if $g(y; s) = 0$ and $|y - \hat{y}| \leq \varepsilon_p$, then

$$Dg(y; s)(y - \hat{y}) \leq \bar{\alpha}_p |y - \hat{y}|^2.$$

Define $\hat{\varepsilon}$ by $\hat{\varepsilon} = \min(\bar{\varepsilon}, \varepsilon_c/\bar{A}, \varepsilon_p/\bar{A} \cdot \bar{\eta})$, where

$$(1 + \bar{\eta}) = \sum_{t=0}^{m-1} \eta^t \text{ and } \eta \text{ is as in A.19 and } \bar{\varepsilon} \text{ is defined}$$

in Lemma 3. Let $\{(\tilde{x}_t, \tilde{y}_t)\} \in \mathcal{F}(\tilde{K}_0(s))$, be the alternative program of Lemma 3. Notice that without loss of generality I can assume $t = 0$.

$$\begin{aligned} F_0^\delta(s) &= q_0(s)(\tilde{K}_0(s) - K_0(s)) - (V(\tilde{K}_0(s), s; \delta) - V(K_0(s), s; \delta)) \\ &\leq q_0(s)(\tilde{K}_0(s) - K_0(s)) - E\left[\sum_{t=0}^{\infty} \delta^t (U(\tilde{x}_t(s), s) - U(x_t(s), s)) | \mathcal{J}_0\right] \\ &= E\left[\sum_{t=0}^{m-1} \delta^t (U(x_t(s), s) - U(\tilde{x}_t(s), s) - q_t(s)(x_t(s) - \tilde{x}_t(s)) | \mathcal{J}_0\right] \\ &\quad + E\left[\sum_{t=0}^{m-1} \delta^t (\delta q_{t+1}(s)(y_{t+1}(s) - \tilde{y}_{t+1}(s)) + q_{t0}(s)(y_{t0}(s) - \tilde{y}_{t0}(s)) | \mathcal{J}_0\right]. \end{aligned}$$

The last equality follows from 2.11 and conditions (i) to (v) of Lemma 3. Now, by conditions (ii) to (iv) of Lemma 3, it follows that for $t = 0, \dots, m-1$,

$$\begin{aligned}
& U(x_t(s), s) - U(\tilde{x}_t(s), s) - q_t(s)(x_t(s) - \tilde{x}_t(s)) \\
&= U(x_t(s), s) - U(\tilde{x}_t(s), s) - DU(x_t(s), s)(x_t(s) - \tilde{x}_t(s)) \\
&\leq \bar{\alpha}_c \cdot |\alpha_t(s) - \tilde{x}_t(s)|^2 \leq \bar{\alpha}_c \cdot \bar{A}^2 \cdot |K_0(s) - \tilde{K}_0(s)|^2 .
\end{aligned}$$

By assumption A.19 ,

$$\begin{aligned}
|y_{01}(s) - \tilde{y}_{01}(s)| &\leq \eta |y_{00}(s) - \tilde{y}_{00}(s)| \leq \eta \cdot |x_0(s) - \tilde{x}_0(s)| \leq \eta \cdot \bar{A} \cdot \varepsilon \\
|y_{10}(s) - \tilde{y}_{10}(s)| &\leq |y_{01}(s) - \tilde{y}_{01}(s)| + |x_{10}(s) - \tilde{x}_{10}(s)| \\
&\leq (\eta + 1) \cdot \bar{A} \cdot \varepsilon
\end{aligned}$$

Similarly,

$$|y_{11}(s) - \tilde{y}_{11}(s)| \leq \eta(\eta + 1)\bar{A} \cdot \varepsilon$$

and

$$|y_{m-1,0}(s) - \tilde{y}_{m-1,0}(s)| \leq \bar{A} \cdot \varepsilon \cdot \left(\sum_{t=0}^{m-1} \eta^t - 1 \right)$$

i.e., for $t = 0, \dots, m-1$, $|y_t(s) - \tilde{y}_t(s)| \leq \bar{\eta} \cdot \bar{A} \cdot \varepsilon$

Now, for $t = 0, \dots, m-1$,

$$\begin{aligned}
& E[\delta q_{t+1}(s) \cdot (y_{t1}(s) - \tilde{y}_{t1}(s)) + q_t(s)(y_{t0}(s) - \tilde{y}_{t0}(s)) | \mathcal{I}_0] \\
&= E[\lambda_{t+1}(s) \cdot Dg(y_{t0}(s), y_{t1}(s); s)(y_t(s) - \tilde{y}_t(s)) | \mathcal{I}_0] \\
&\leq \bar{\alpha}_p \cdot \bar{\lambda} \cdot E[|y_t(s) - \tilde{y}_t(s)|^2 | \mathcal{I}_0] \leq \bar{\alpha}_p \cdot \bar{\lambda} \cdot \bar{\eta}^2 \cdot \bar{A}^2 \cdot |K_0(s) - \tilde{K}_0(s)|^2 .
\end{aligned}$$

Let $A = m \cdot \bar{A}^2 \cdot \max(1, \bar{\eta}^2)$ and the lemma is proved.

VII. PROOF OF THEOREM 1

As I have explained in Section 5, with the value loss approach, the proof of the stochastic turnpike theorem is, in essence, the proof of the following two properties:

- (i) "The present value loss process is a finite positive supermartingale."
- (ii) "The value loss property" i.e., if (i) is satisfied, then optimal programs converge a.s.

I have made strong differentiability assumptions and it is possible to prove directly that the present value loss process converges in the mean exponentially to zero. Then, as a direct application of the Borel - Cantelli lemma (Doob [19532, p. 104] and of lemma 8, the a.s. convergence of optimal plans can be derived. I offer both proofs. The first approach (Proposition 1 and corollary, and a.s. convergence in theorem 1) has the advantage that states in probabilistic terms the main ideas of turnpike theory. (As I said, a similar argument was used by Föllmer and Majumdar [1978] in the nondiscounted case.) The second approach (exponential convergence of theorem 1) is, basically, the stochastic version of the method of proof used by Bewley [1982] (in fact, part of his argument is used in the proof of Proposition 1). The main advantage is that it gives a stronger result: with respect to the initial conditions, optimal programs converge uniformly in probability.

Now I turn to the proofs. To ease the reading, I first restate the main inequalities that will be used.

For all $t \geq 0$, the following a.s. inequalities are satisfied:

$$(7.1) \quad 0 \leq F_t^\delta(s) \leq C \quad (\text{Lemmas 6 and 7})$$

$$(7.2) \quad F_t^\delta(s) - \delta E[F_{t+1}^\delta(s) | \mathcal{J}_t] \geq \alpha \cdot E[\min\{(\frac{1}{2}\bar{\varepsilon})^2, \\ |x_t(s) - \tilde{x}_t(s), y_{t0}(s) - \tilde{y}_{t0}(s), y_{t1}(s) \\ - \tilde{y}_{t1}(s)|^2 | \mathcal{J}_t\}] \quad (\text{Lemma 8})$$

$$(7.3) \quad \delta^{-1} F_t^\delta(s) - E[F_{t+1}^\delta(s) | \mathcal{J}_t] \\ \geq 2\alpha \min\{\bar{\varepsilon}^2, |K_t(s) - \tilde{K}_t(s)|^2\} \quad (\text{Cor. to Lemma 8})$$

$$(7.4) \quad \text{If } |K_t(s) - \tilde{K}_t(s)| < \bar{\varepsilon}, \text{ then}$$

$$F_t^\delta(s) \leq A \cdot |K_t(s) - \tilde{K}_t(s)|^2 \quad (\text{Lemma 9})$$

where $\bar{\varepsilon} = \min(\tilde{\varepsilon}, \hat{\varepsilon})$ and $\tilde{\varepsilon}$ and $\hat{\varepsilon}$ where defined in the corollary to Lemma 8 and in Lemma 9, respectively. I first prove the supermartingale property.

Proposition 1: Assume A.1 - A.22. there exist $\delta_1 \in (0, 1)$ such that if $\{(x_t)(y_t)\}_{t=0}^\infty$ and $\{(\tilde{x}_t)(\tilde{y}_t)\}_{t=0}^\infty$ are two optimal interior programs for $\delta \in (\delta_1, 1]$, then the present value loss process $\{F_t(\tilde{K}_y; K_y, \delta)\}_{t=0}^\infty$ (resp. $\{\tilde{F}_t(K_t; K_y, \delta)\}_{t=0}^\infty$) is a finite positive supermartingale.

Proof: Let

$$(7.5) \quad \delta_1 = \max\left\{\frac{\delta}{2}, \frac{A}{2\alpha+A}, \frac{A}{2\alpha\bar{\varepsilon}^2+C}\right\}.$$

The random variables F_t are positive and finite (indeed, uniformly bounded!) by (7.1). The process $\{F_t, \mathcal{J}_t\}$ is adapted. It only remains

to prove that, for all t , $E[F_{t+1}^\delta | \mathcal{J}_t] \leq F_t^\delta$ a.s. .

Assume that $\delta \in (\delta_1, 1]$

$$(7.6) \quad F_t^\delta - E[F_{t+1}^\delta | \mathcal{J}_t] = \delta^{-1} F_t^\delta - E[F_{t+1}^\delta | \mathcal{J}_t] - (\delta^{-1} - 1) F_t^\delta$$

If $|K_t(s) - \tilde{K}_t(s)| < \bar{\epsilon}$, then by (7.3) and (7.4),

$$(7.7) \quad F_t^\delta(s) - E[F_{t+1}^\delta(s) | \mathcal{J}_t] \geq (2\alpha + (1 - \delta^{-1})A) |K_t(s) - \tilde{K}_t(s)|^2 \text{ a.s.}$$

and by (7.5)

$$2\alpha + (1 - \delta^{-1})A \geq 0 .$$

If $|K_t(s) - \tilde{K}_t(s)| \geq \bar{\epsilon}$, then by (7.1) and (7.3)

$$(7.8) \quad F_t^\delta(s) = E[F_{t+1}^\delta(s) | \mathcal{J}_t] \geq 2\alpha\bar{\epsilon}^2 + (1 - \delta^{-1})C \quad \text{a.s.}$$

and by (7.5)

$$2\alpha\bar{\epsilon}^2 + (1 - \delta^{-1})C \geq 0$$

Therefore, $\{F_t^\delta\}$ is a positive supermartingale. Q. E. D.

Remark: There exists a r.v. F_∞^δ such that $F_t^\delta \rightarrow F_\infty^\delta$ a.s. (since every positive supermartingale is convergent (see Neveu [1972] theorem II-2-9)).

For our purposes, however, this result does not have much interest.

What we need is, in fact, $F_\infty = 0$ a.s. .

In the discussion of Section 5, I have called $F_t^\delta - E[F_{t+1}^\delta | \mathcal{J}_t]$ the "myopic value loss," to distinguish it from $F_t^\delta - \delta E[F_{t+1}^\delta | \mathcal{J}_t]$, the "current expected value loss." I now prove a corollary to Proposition 1 that when $\delta = 1$ has a clear interpretation and it would stand by itself (Föllmer and Majumdar [1978], Theorem 3.1), but when $\delta < 1$ has interest,

only, as an intermediate step for the proof of Theorem 1. For any $\varepsilon > 0$, let

$$\Lambda_{\varepsilon}^{\delta} \equiv \{(t, s) | F_t^{\delta} - E[F_{t+1}^{\delta} | \mathcal{J}_t] \geq \varepsilon\}$$

i.e., $\Lambda_{\varepsilon}^{\delta}$ is the ε -myopic value loss region and

$$N_{\varepsilon t}^{\delta}(s) = \sum_{n=t}^{\infty} I_{\Lambda_{\varepsilon}^{\delta}}(n, s),$$

where I is the characteristic function, i.e., $N_{\varepsilon t}^{\delta}$ is the time spent after t in the ε -myopic value loss region.

Corollary: For any $\varepsilon > 0$, the following a.s. inequalities are satisfied:

$$\varepsilon \cdot E[N_{\varepsilon t}^{\delta} | \mathcal{J}_t] \leq F_t^{\delta} \leq C,$$

for some constant C . In particular, at time zero the conditional expected time spent in the ε -myopic value loss region is finite. (C is defined in Lemma (7)).

I introduce some additional probabilistic terminology that will be used in the proof of this corollary. (For a more complete discussion of these concepts, see Neveu [1972], Chapter VII). A sequence $\{Z_n\}_{n=0}^{\infty}$ is said to be *predictable* with respect to the filter $\{\mathcal{J}_n\}_{n=0}^{\infty}$ if the r.v. Z_0 is \mathcal{J}_0 -measurable and if for all $n \geq 0$ the r.v. Z_{n+1} is \mathcal{J}_n -measurable. An *increasing process* is a predictable sequence $\{A_n\}_{n=0}^{\infty}$ of finite r.r.v.'s such that

$$0 = A_0 \leq A_1 \leq \dots \quad \text{a.s.}$$

Fact 1: If $\{A_n\}_{n=0}^{\infty}$ is an increasing process such that $E[A_{\infty} | \mathcal{J}_0] < \infty$ a.s., then the process $\{E[A_{\infty} - A_n | \mathcal{J}_n]\}_{n=0}^{\infty}$ is a finite positive supermartingale called the *potential* of the increasing process (Neveu [1972], Prop. VIII-1-2).

Fact 2: (*Riesz decomposition* of the supermartingale). Every finite positive supermartingale can be uniquely written as the sum of a finite positive martingale and the potential of an increasing process.

Proof: I first define the Riesz decomposition of the finite positive supermartingale $\{F_t\}_{t=0}^{\infty}$. Let

$$A_0 = 0 ; A_{n+1} - A_n = F_n - E[F_{n+1} | \mathcal{J}_n] \forall n=0 .$$

Then,
$$A_{\infty} = \sum_{n=0}^{\infty} (F_n - E[F_{n+1} | \mathcal{J}_n])$$

and

$$\begin{aligned} E[A_{\infty} - A_n | \mathcal{J}_n] &= E\left[\sum_{m=n}^{\infty} (F_m - E[F_{m+1} | \mathcal{J}_m]) \middle| \mathcal{J}_n\right] \\ &= E\left[\sum_{m=n}^{\infty} (F_m - F_{m+1}) \middle| \mathcal{J}_n\right] = E[F_n - F_{\infty} | \mathcal{J}_n] . \end{aligned}$$

Let
$$M_n = E[F_{\infty} | \mathcal{J}_n] .$$

It is clear that

$$F_n = M_n + E[A_{\infty} - A_n | \mathcal{J}_n]$$

is the unique Riesz decomposition.

Now, for any $\epsilon > 0$ let $\Lambda_{\epsilon}^{\delta}$ be ϵ -myopic value loss region. If $(t, s) \in \Lambda_{\epsilon}^{\delta}$

$$\varepsilon \cdot I_{\Lambda_{\varepsilon}^{\delta}}(t, s) = \varepsilon \leq A_{t+1} - A_t$$

and if

$$(t, s) \notin \Lambda_{\varepsilon}^{\delta}$$

$$\varepsilon \cdot I_{\Lambda_{\varepsilon}^{\delta}}(t, s) = 0 \leq A_{t+1} - A_t$$

since $\{A_t\}$ is an increasing process. Thus,

$$\begin{aligned} \varepsilon \cdot E[N_{\varepsilon t}^{\delta}(s) | \mathcal{J}_t] &= \varepsilon \cdot E\left[\sum_{n=t}^{\infty} I_{\Lambda_{\varepsilon}^{\delta}}(n, \cdot s) | \mathcal{J}_t\right] \\ &\leq E\left[\sum_{n=t}^{\infty} (A_{n+1} - A_n) | \mathcal{J}_t\right] \\ &= E[A_{\infty} - A_t | \mathcal{J}_t]. \end{aligned}$$

Since $\{M_t\}_{t=0}^{\infty}$ is a finite positive martingale, it follows that

$$\varepsilon \cdot E[N_{\varepsilon t}^{\delta} | \mathcal{J}_t] \leq F_t \quad \text{a.s.}$$

and by Lemma 7

$$F_t \leq C \quad \text{a.s.}$$

Q. E. D.

Proof of Theorem 1:

Step 1: (a.s. convergence)

Let

$$(7.9) \quad \delta^* = \max \left\{ \delta, \frac{A}{\alpha+A}, \frac{C}{\alpha\varepsilon^2 + C} \right\} \geq \delta_1$$

then from (7.7) and (7.8) and assuming $\delta \in (\delta^*, 1]$, we obtain

$$(7.10) \quad F_t^\delta(s) - E[F_{t+1}^\delta(s) | \mathcal{J}_t] \geq \alpha \cdot \min\{\bar{\varepsilon}^{-2}, |K_t(s) - \tilde{K}_t(s)|^2\}$$

Suppose that there is a set $A \in \mathcal{J}$ and a positive constant $\rho > 0$ such that

$$\limsup_{t \rightarrow \infty} |K_t(s) - \tilde{K}_t(s)| \geq \rho \quad \text{for } s \in A$$

then on A ,

$$F_t^\delta(s) - E[F_{t+1}^\delta(s) | \mathcal{J}_t] \geq \alpha \cdot \min\{\bar{\varepsilon}^{-2}, \rho^2\}$$

infinitely often. By the above corollary,

$$\alpha \cdot \min\{\bar{\varepsilon}^{-2}, \rho^2\} \cdot E[N_{\varepsilon 0}^\delta] \leq C.$$

This implies that $P(A) = 0$, i.e.,

$$\lim_{t \rightarrow \infty} |K_t(s) - \tilde{K}_t(s)| = 0 \quad \text{a.s.}$$

Step 2: (Exponential convergence)

Let

$$D = \max\{A, C / \bar{\varepsilon}^2\}$$

(note that without loss of generality $D = C / \bar{\varepsilon}^2$). By (7.4) if

$$|K_t(s) - \tilde{K}_t(s)| < \bar{\varepsilon},$$

then $F_t^\delta(s) \leq A \cdot |K_t(s) - \tilde{K}_t(s)|^2 \leq D \cdot |K_t(s) - \tilde{K}_t(s)|^2$.

From (7.10) we have

$$(7.11) \quad E[F_{t+1}^\delta(s) | \mathcal{J}_t] \leq \max\{F_t(s) - \alpha \bar{\epsilon}^2, (1 - \alpha D^{-1}) F_t^\delta(s)\}$$

By (7.1), $F_t^\delta(s) \leq C \leq D \bar{\epsilon}^2$ a.s., which implies that $-\alpha \bar{\epsilon}^2 \leq -\alpha D^{-1} F_t^\delta(s)$ a.s. Hence,

$$E[F_{t+1}^\delta(s) | \mathcal{J}_t] \leq (1 - \alpha D^{-1}) F_t^\delta(s) \quad \text{a.s.}$$

by taking expectations.

$$E[F_{t+1}^\delta] \leq (1 - \alpha D^{-1}) E[F_t^\delta]$$

i.e.,

$$(7.12) \quad E[F_t^\delta] \leq (1 - \alpha D^{-1})^t \cdot C.$$

By (7.2) and taking $\epsilon' = \frac{1}{2} \bar{\epsilon}$, we have

$$(7.13) \quad F_t^\delta(s) \geq \alpha' \cdot E[\min\{\epsilon'^2, |x_t(s) - \tilde{x}_t(s), y_t(s) - \tilde{y}_t(s)|^2 | \mathcal{J}_t\}]$$

Using Markov's inequality, it follows that for all $\epsilon > 0$,

$$(7.14) \quad \begin{aligned} P\{|x_t(s) - \tilde{x}_t(s), y_t(s) - \tilde{y}_t(s)| > \epsilon\} \\ \leq P\{F_t^\delta(s) > \alpha' \cdot \epsilon^2\} \leq \frac{1}{\alpha' \cdot \epsilon^2} \cdot E[F_t^\delta] \\ \leq \frac{1}{\alpha' \cdot \epsilon^2} \cdot (1 - \alpha D^{-1})^t \cdot C \end{aligned}$$

which proves that the convergence in probability is exponential. Finally, by Lemma (1) feasible programs are uniformly bounded. By Lebesgue's bounded convergence theorem, we have:

$$E[|x_t - \tilde{x}_t, y_t - \tilde{y}_t|] \rightarrow 0.$$

As I said, one can also obtain a.s. convergence of optimal interior programs using this second approach. This is a direct consequence of (7.14). In fact, the result is stronger since it also gives a lower bound on the speed of convergence.

Let $b = (1 - \alpha D^{-1})$, $e = b^{1/4}$, $a = b^{1/2}$ and $A = C/\alpha'$, then (7.14) can be rewritten as

$$(7.15) \quad P\{|x_t - \tilde{x}_t, y_t - \tilde{y}_t| > e^t\} \leq A \cdot a^t$$

Let $E_t = \{s: |x_t(s) - \tilde{x}_t(s), y_t(s) - \tilde{y}_t(s)| > e^t\}$, then

$$\sum_{t=0}^{\infty} P(E_t) \leq A \cdot \sum_{t=0}^{\infty} a^t = \frac{A}{1-a} < +\infty.$$

As an application of the Borel-Cantelli lemma (Neveu [1970], Proposition I-4-4), it follows that:

$$\limsup_{t \rightarrow \infty} E_t = \emptyset \quad \text{a.s.}$$

which, in turn, implies

$$(7.16) \quad \lim_{t \rightarrow \infty} |x_t - \tilde{x}_t, y_t - \tilde{y}_t| = 0 \quad \text{a.s.}$$

Remark: Recall that a sequence of measurable functions is said to converge *almost uniformly* if for every $\varepsilon > 0$, exists a measurable set M such that $P(M) < \varepsilon$ and the sequence converges uniformly on M^c . Furthermore, in a probability space almost uniform convergence is equivalent to a.s. convergence (Egoroff's theorem).

Let $M_t = \bigcup_{n=t}^{\infty} E_n$,

then if $s \in M_t^c$, we have that for $n \geq t$,

$$|x_n(s) - \tilde{x}_n(s), y_n - \tilde{y}_n(s)| \leq e^n \leq e^t$$

and

$$P(M_t) \leq \sum_{n=t}^{\infty} P(E_n) \leq \frac{A \cdot a^t}{1-a}$$

This says that the uniform convergence is exponential and that the probability of the sets in which the uniform convergence is not necessarily satisfied also converges exponentially to zero. In fact, this is a form of a.s. exponential convergence.

Finally, it is possible to derive a similar bound on the speed of convergence respect to convergence in the mean.

$$\begin{aligned} E[|x_t - \tilde{x}_t, y_t - \tilde{y}_t|] &= \\ &= \int_{E_t^c} |x_t - \tilde{x}_t, y_t - \tilde{y}_t| P(ds) + \int_{E_t} |x_t - \tilde{x}_t, y_t - \tilde{y}_t| P(ds) \\ &\leq e^t \cdot P(E_t^c) + B \cdot P(E_t) \\ &\leq e^t + A \cdot B \cdot a^t = e^t(1 + A \cdot B \cdot e^{-t}) \rightarrow 0 \end{aligned}$$

exponentially.

B is the upper bound on feasible allocations of lemma (1) Q. E. D.

VIII. PROOF OF THEOREM 2

I do not use a fixed point argument in order to prove the existence of a stationary optimal program. Instead, I use Theorem (1) and the fact that the topology of convergence in probability defines a complete metric space. I first define this well known topology and prove that production sets are closed with respect to this topology.

Let

$$z, z' \in \mathcal{L}_{\infty, L}(S, \mathcal{J}, P)$$

and define

$$d(z, z') = \int \frac{|z - z'|}{1 + |z - z'|} dP$$

then the following fact can be easily derived:

Fact: (i) d is a metric on $\mathcal{L}_{\infty, L}(S, \mathcal{J}, P)$;

(ii) $d(z_n, z) \rightarrow 0$ if and only if $z_n \xrightarrow{P} z$, and

(iii) $(\mathcal{L}_{\infty, L}(S, \mathcal{J}, P), d)$ is a complete metric space (Neveu [1970], problem II-4-3.)

By (ii) the topology of convergence in probability is the topology on $\mathcal{L}_{\infty, L}(S, \mathcal{J}, P)$ induced by d . Let

$$\mathcal{X}_0 = \{K \in \mathcal{L}_{\infty, Lp}(S, \mathcal{J}_0, P) \mid \forall i \in L_p, \gamma \leq K^i(s) \leq B \text{ a.s.}\}$$

where B is the upper bound of Lemma (1) and γ the uniform lower bound for interior programs.

Recall that

$$Y = \{(y_0, y_1) \in \mathcal{L}_{\infty, L}^-(S, \mathcal{J}_0, P) + \mathcal{L}_{\infty, Lp}^+(S, \mathcal{J}_1, P) \mid$$

$$g(y_0(s), y_1(s); s) \leq 0 \text{ a.s.}\}$$

Lemma 10: \mathcal{X}_0 and Y are closed on the topology of convergence in probability.

Proof: I prove the closedness of Y . The same argument can be used to prove the closedness of \mathcal{X}_0 . Let $\{y^{(n)}\}_{n \in \mathbb{N}}$ be a sequence on

$$(\mathcal{L}_{\infty, L}^-(S, \mathcal{J}_0, P) \times \mathcal{L}_{\infty, L_t}^+(S, \mathcal{J}_0, P), d).$$

Assume

$$y^{(n)} \equiv (y_0^{(n)}, y_1^{(n)}) \in Y \text{ and } d(y^{(n)}, y) \rightarrow 0.$$

Using the previous fact $y^{(n)} \xrightarrow{P} y$, this, in turn, implies that there is a subsequence $\{y^{(n_i)}\}$ such that $y_{n_i} \rightarrow y$ a.s. (Neveu [1970], Proposition II-9-3). By assumption A.14, for every

$$s, g(\cdot, \cdot; s) : \mathbb{R}_-^L \times \mathbb{R}_+^L \rightarrow \mathbb{R}$$

is continuous. Hence, $g(y_0(s), y_1(s); s) \leq 0$ a.s. Q. E. D.

Notation: If $\{(x_t)(y_t)\}_{t=0}^{\infty}$ solves the problem

$$\text{Max} \left\{ E \left[\sum_{t=0}^{\infty} \delta^t u(\tilde{x}_t(s), s) \right] \mid \{(\tilde{x}_t)(\tilde{y}_t)\}_{t=0}^{\infty} \in \mathcal{J}(K_0) \right\}$$

then let

$$K_t(K_0, \delta) \equiv \sigma^{-t} y_{t-1,1}; x_t(K_0, \delta) \equiv \sigma^{-t} x_t,$$

and $y_{t0}(K_0, \delta) \equiv \sigma^{-t} y_{t0}.$

Remark: Let $\{(x_t)(y_t)\}_{t=0}^{\infty} \in \mathcal{F}(K_0)$ be an optimal interior program, then the following two facts are satisfied:

- (i) For all $t \geq 0$, $K_t(K_0, \delta) \in K_0$.
- (ii) For all $m \geq 0$, $\{K_{t+m}(K_m(K_0, \delta), \delta)\}_{t=0}^{\infty}$ is an optimal interior accumulation program (Bellman's optimality equation).

Proof of Theorem 2: Let $\{(x_t)(y_t)\}_{t=0}^{\infty}$ be an optimal interior program. By the previous remark and Theorem (1), we have that, for all $m \geq 0$,

$$d(K_{t+m}(K_m(K_0, \delta), \delta), K_t(K_0, \delta)) \rightarrow 0.$$

Since the metric space $(\mathcal{L}_{\infty, Lp}^+(S, \mathcal{J}_0, P), d)$ is complete, there exists $\bar{K}(K_0, \delta) \in \mathcal{L}_{\infty, Lp}^+(S, \mathcal{J}_0, P)$ such that

$$(8.1) \quad d(K_t(K_0, \delta), \bar{K}(K_0, \delta)) \rightarrow 0$$

Step 1: (Uniqueness of $\bar{K}^{(\delta)}$)

In the last part of the proof of Theorem (1), I have obtained the a.s. convergence between optimal interior programs as a consequence of the exponential convergence in probability. I use here this uniform convergence to derive the uniqueness of $\bar{K}^{(\delta)}$.

Consider another optimal interior program

$$\{(\tilde{x}_t)(\tilde{y}_t)\}_{t=0}^{\infty} \in \mathcal{F}(\tilde{K}_0).$$

Using the above argument, there exists

$$\bar{K}(\tilde{K}_0, \delta) \in \mathcal{L}_{\infty, Lp}^+(S, \mathcal{J}_0, P)$$

such that

$$d(K_t(\tilde{K}_0, \delta), \bar{K}(\tilde{K}_0, \delta)) \rightarrow 0 .$$

The limiting r.v.'s satisfy:

$$(8.2) \quad \begin{aligned} \bar{K}(K_0, \delta) &= \lim_{t \rightarrow \infty} K_t(K_0, \delta) = K_0(K_0, \delta) \\ &+ \sum_{t=0}^{\infty} (K_{t+1}(K_1(K_0, \delta), \delta) - K_t(K_0, \delta)) \quad \text{a.s.} \end{aligned}$$

$$\begin{aligned} \bar{K}(\tilde{K}_0, \delta) &= \lim_{t \rightarrow \infty} K_t(\tilde{K}_0, \delta) = K_0(\tilde{K}_0, \delta) \\ &+ \sum_{t=0}^{\infty} (K_{t+1}(K_1(\tilde{K}_0, \delta), \delta) - K_t(\tilde{K}_0, \delta)) \quad \text{a.s.} \end{aligned}$$

(In fact, this is how they are derived (Neveu [1970], Proposition II-4-2)).

Let $\{\varepsilon_t\}_{t=0}^{\infty}$ be the decreasing sequence of real numbers defined in Theorem (1); i.e., $\varepsilon_{2n} = a^n$, $\varepsilon_{2n+1} = a^{(n+1)}$; $n \geq 0$).

Define:

$$C_t = \{s: |K_{t+1}(s; K_1(\tilde{K}_0, \delta), \delta) - K_t(s; \tilde{K}_0, \delta)| > \varepsilon_t\}$$

$$D_t = \{s: |K_{t+1}(s; K_1(K_0, \delta), \delta) - K_t(s; K_0, \delta)| > \varepsilon_t\}$$

$$E_t = \{s: |K_t(s; K_0, \delta) - K_t(s; \tilde{K}_0, \delta)| > \varepsilon_t\}$$

Suppose that $d(\bar{K}(K_0, \delta), \bar{K}(\tilde{K}_0, \delta)) \neq 0$, then there exist $\rho > 0$ and $\rho' > 0$ such that

$$(8.3) \quad P\{|\bar{K}(s; K_0, \delta) - \bar{K}(s; \tilde{K}_0, \delta)| > \rho\} \geq \rho'$$

Choose an even integer t such that

$$\rho > \frac{a^t(1+a)}{(1-a)} = \sum_{n=t}^{\infty} \epsilon_n > \epsilon_t.$$

$$\text{Then} \quad P\{|\bar{K}(s; K_0, \delta) - \bar{K}(s; \tilde{K}_0, \delta)| > \rho\}$$

$$\leq P\{|\bar{K}(s; K_0, \delta) - K_t(s; K_0, \delta)| > \sum_{n=t}^{\infty} \epsilon_n\}$$

$$+ P\{|K_t(s; K_0, \delta) - K_t(s; \tilde{K}_0, \delta)| > \epsilon_t\}$$

$$+ P\{|K_t(s; \tilde{K}_0, \delta) - \bar{K}(s; \tilde{K}_0, \delta)| > \sum_{n=t}^{\infty} \epsilon_n\}$$

$$(8.4) \quad \leq P(\overset{\infty}{\cup}_{n=t} D_n) + P(\overset{\infty}{\cup}_{n=t} E_n) + P(\overset{\infty}{\cup}_{n=t} C_n) \leq 6A \frac{a^t}{1-a}.$$

The last inequality follows from:

$$P(\overset{\infty}{\cup}_{n=t} D_n) = P(\overset{\infty}{\cup}_{n=t} E_n) = P(\overset{\infty}{\cup}_{n=t} C_n) \leq 2A \frac{a^t}{1-a}.$$

Let $t' \geq t$ be the smallest even integer such that $6A \frac{a^{t'}}{1-a} \leq \rho'$, then

(8.3) is contradicted.

Step 2: (Definition of $(\bar{x}), (\bar{y})^{(\delta)}$.)

$$\text{Let } X_0 = \{x \in \mathcal{Z}_{\infty, L_c}^+(S, \mathcal{J}_0, P) \mid \|x\|_{\infty} \leq B\}$$

then I can apply the same argument as before to obtain the existence of

$$\bar{x}^{(\delta)} \in \mathcal{Z}_{\infty, L_c}^+(S, \mathcal{J}_0, P) \text{ such that}$$

$$d(x_t(K_0, \delta), \bar{x}^{(\delta)}) \rightarrow 0 .$$

Define $\bar{y}_0^{(\delta)} \in \mathcal{J}_{\infty, L}^-(S, \mathcal{J}_0, P)$

by $\bar{y}_0^{(\delta)} \equiv \bar{x}^{(\delta)} - \bar{K}^{(\delta)} - w .$

Step 3: (Feasibility and Interiority)

Since K_0 is closed in the topology of convergence in probability (Lemma (10)), it follows that $\bar{K}^{(\delta)} \in K_0$. Optimal programs satisfy $x_t = y_{t0} + K_t + w$, by convergence of the x_t 's and K_t 's and the invariance of w , it follows that $d(y_{t0}(K_0, \delta), \bar{y}_0^{(\delta)}) \rightarrow 0$. For all $t \geq 0$,

$$(y_{t0}(K_0, \delta), \sigma K_{t+1}(K_0, \delta)) \in Y .$$

The shift operator is measure preserving (i.e., $d(\sigma K_{t+1}(K_0, \delta), \sigma \bar{K}^{(\delta)}) \rightarrow 0$).

By Lemma (10), Y is closed in this topology, hence

$$(Y_0^{(\delta)}, \sigma \bar{K}^{(\delta)}) \in Y .$$

Step 4: (Optimality of $\{(\sigma^t \bar{x}), (\sigma^t \bar{y})\}_{t=0}^{\infty} \in \mathcal{F}(\bar{K}^{(\delta)})$.)

First, I recollect four facts derived from previous results that will be used in the proof of the optimality:

- (1) From corollary to Lemma (3), there exist $\bar{\epsilon} > 0$ and $E > 0$ such that if $\{(x_t)(y_t)\}_{t=0}^{\infty} \in \mathcal{F}(K)$ is an interior ρ -leisured program, and $\epsilon \in (0, \bar{\epsilon})$, then, if \tilde{K} is defined by $\tilde{K}^{(i)}(s) = \max \{K^{(i)}(s) - \epsilon, \gamma\}$, then it is possible to construct a program $\{(\tilde{x}_t)(\tilde{y}_t)\}_{t=0}^{\infty} \in \mathcal{F}(\tilde{K}_0)$ such that $E[\sum_{t=0}^{\infty} \delta^t (U(x_t(s), s) - U(\tilde{x}_t(s), s))] \leq \epsilon \cdot E$.

- (2) By continuity of $U(\cdot, s) : \mathbb{R}_t^{L_t} \rightarrow \mathbb{R}$ and a.s. convergence of $K_t(K_0, \delta)$ to $\bar{K}^{(\delta)}$, it follows that for all $\varepsilon > 0$ exists $\eta_\varepsilon > 0$ such that if

$$|K_t(s; K_0, \delta) - \bar{K}^{(\delta)}(s)| \leq \eta_\varepsilon \text{ then}$$

$$E\left[\sum_{t=0}^{\infty} \delta^t (U(x_t(\sigma^t s; K_0, \delta) - U(\bar{x}(\sigma^t s), s)) \mid \mathcal{J}_0\right] \leq \varepsilon .$$

- (3) Let $A_\varepsilon^{(n)} = \{s : |K_n(s; K_0, \delta) - \bar{K}^{(\delta)}(s)| \leq \varepsilon\}$. Then the following equicontinuity property is satisfied: For any $\varepsilon > 0$, there exists $\bar{\eta}_\varepsilon > 0$ such that if $P((A_\varepsilon^{(n)})^c) \leq \bar{\eta}_\varepsilon$ then

$$\int_{(A_\varepsilon^{(n)})^c} \sum_{t=0}^{\infty} \delta^t (U(x_{n+t}(\sigma^t s; K_n(K_0, \delta), \delta), s) - U(\bar{x}(\sigma^t s), s)) P(ds) \leq \varepsilon$$

- (4) By Lemma (2) any ρ -leisured γ -interior program can be reached in a finite number of periods from any interior capital stock. In particular, exists $\bar{c} > 0$ such that if $\{(x_t)(y_t)\}_{t=0}^{\infty}$ is a ρ -leisured γ -interior program, then

$$E\left[\sum_{t=0}^{\infty} \delta^t (U(x_t(s), s) - U(\sigma^t x_t(s; K_0, \delta), s)) \mid \mathcal{J}_0\right] \leq \bar{c} .$$

Now I prove the optimality of $\{(\sigma^t \bar{x})(\sigma^t \bar{y})\}_{t=0}^{\infty} \in \mathcal{F}(K^{(\delta)})$ by contradiction. Assume that there exist a program

$$\{(x_t)(y_t)\}_{t=0}^{\infty} \in \mathcal{F}(\bar{K}^{(\delta)}) \text{ and an } \varepsilon > 0$$

such that

$$(8.5) \quad E \left[\sum_{t=0}^{\infty} \delta^t (U(s_t(s), s) - U(\bar{x}(\sigma^t s), s)) \right] \geq \varepsilon > 0 .$$

Remark:

There is no loss in generality in assuming that the program $\{(x_t)(y_t)\}_{t=0}^{\infty}$ is ρ -leisured and γ -interior. If such a program exists, then (8.5) will be true for an optimal ρ -leisured program $\{(x_t)(y_t)\}$, if such a program is not γ -interior it can be modified using the interiority of $\{(\bar{x}), (\bar{y})\}$ in order to define another Pareto-superior γ -interior program. Now I show that by a suitable decomposition of (8.5) the optimality of some program $\{x_t(K_0, \delta), y_t(K_0, \delta)\}_{t=0}^{\infty}$ is contradicted. First, I choose the ε 's, etc.

Let $\hat{\varepsilon} \leq \varepsilon/8$. By (2) $\eta_{\hat{\varepsilon}}$ is defined.

Let $\eta \leq \min \{\eta_{\hat{\varepsilon}}, \varepsilon/8 \cdot E\}$, where E is as in (1).

By (3), $\eta_{\hat{\varepsilon}}$ is also defined. Choose n large enough such that $P((A_{\eta}^{(n)})^c) \leq \min\{\bar{\eta}_{\hat{\varepsilon}}, \hat{\varepsilon}/\bar{c}\}$, where $A_{\eta}^{(n)}$ is defined in (3) and \bar{c} in (4). Since n and η had been fixed, I simplify notation by writing $A = A_{\eta}^{(n)}$. Define \bar{K}_{η} by $\bar{K}_{\eta}^i = \max\{\bar{K}^i - \eta, \gamma\}$ $i \in L_p$, (i.e., the stationary capital stock is decreased). Define the program $\{(\tilde{x}_t)(\tilde{y}_t)\}_{t=0}^{\infty}$ as follows:

If $s \in A^c$, then $\{(\tilde{x}_t(s)), (\tilde{y}_t(s))\}_{t=0}^{\infty} \equiv \{(x_{t+n}(\sigma^t s; K_n(K_0, \delta), \delta)),$

$$(y_{t+n,0}(\sigma^t s; K_n(K_0, \delta), \delta), y_{t+n,1}$$

$$(\sigma^{(t+1)} s; K_n(K_0, \delta), \delta))\}_{t=0}^{\infty}$$

and $\tilde{K}_0(s) \equiv \tilde{Y}_{-1,1}(s) = K_n(s; K_0, \delta)$.

If $s \in A$, then $\tilde{K}_0(s) \equiv \tilde{Y}_{-1,1}(s) = \bar{K}_\eta(s)$ and the path $\{(\tilde{x}_t(s)), (\tilde{y}_t(s))\}_{t=0}^\infty$ satisfies (1) (i.e., reaches $\{(x_t(s), y_t(s))\}$). For $s \in A$, $|K_n(s; K_0, \delta) - \bar{K}^{(\delta)}(s)| \leq \eta$. Optimality of $\{(\sigma^t x_t(K_0, \delta)), (\sigma^t y_{t0}(K_0, \delta)), \sigma^{(t+1)} y_{t1}(K_0, \delta)\}$ requires

$$(8.6) \quad S \equiv E \left[\sum_{t=0}^{\infty} \delta^t (U(x_{t+n}(\sigma^t s; K_n(K_0, \delta), \delta), s) - U(\tilde{x}_t(s), s)) \right] \geq 0$$

Now I decompose (8.5) and (8.6):

$$(8.7) \quad \begin{aligned} \epsilon &\leq E \left[\sum_{t=0}^{\infty} \delta^t (U(x_t(s), s) - U(\bar{x}(\sigma^t s), s)) \right] = \\ &= \int_A \sum_{t=0}^{\infty} \delta^t (U(x_t(s), s) - U(\bar{x}(\sigma^t s), s)) P(ds) \\ &+ \int_{A^c} \sum_{t=0}^{\infty} \delta^t (U(x_t(s), s) - U(\bar{x}(\sigma^t s), s)) P(ds) \equiv S_1 + S_2 \end{aligned}$$

$$(8.8) \quad \begin{aligned} S_1 &= \int_A \sum_{t=0}^{\infty} \delta^t (U(x_t(s), s) - U(\tilde{x}_t(s), s)) P(ds) \\ &+ \int_A \sum_{t=0}^{\infty} \delta^t (U(\tilde{x}_t(s), s) - U(x_{t+n}(\sigma^t s; K_n(K_0, \delta), \delta), s)) P(ds) \\ &+ \int_A \sum_{t=0}^{\infty} \delta^t (U(x_{t+n}(\sigma^t s; K_n(K_0, \delta), \delta), s) - U(\bar{x}(\sigma^t s), s)) P(ds) \\ &\equiv S_3 + S_4 + S_5 \end{aligned}$$

$$\begin{aligned}
(8.9) \quad S_2 &= \int_{A^c} \sum_{t=0}^{\infty} \delta^t (U(x_t(s), s) - U(x_{t+n}(\sigma^t s, K_n(K_0, \delta), \delta), s))) P(ds) \\
&+ \int_{A^c} \sum_{t=0}^{\infty} \delta^t (U(\alpha_{t+n}(\sigma^t s, K_n(K_0, \delta), \delta), s) - U(\bar{x}(\sigma^t s), s)) P(ds) \\
&\equiv S_6 + S_7
\end{aligned}$$

Finally, from (8.6),

$$\begin{aligned}
(8.10) \quad -S &= S_4 + \int_{A^c} \sum_{t=0}^{\infty} \delta^t (U(\tilde{x}_t(s), s) - U(x_{t+n}(\sigma^t s, K_n(K_0, \delta), \delta), s))) P(ds) \\
&= S_4 + S_8
\end{aligned}$$

It follows that

$$(8.11) \quad S \leq -\varepsilon + S_3 + S_5 + S_6 + S_7 - S_8$$

Given our choice of A and the definition of $\{(\tilde{x}_t)(\tilde{y}_t)\}$, we have:

$$S_8 = 0$$

by construction.

$$S_7 \leq \hat{\varepsilon} \leq \varepsilon/8 .$$

This follows from (3) (equicontinuity) and the fact that $P(A^c) \leq \bar{\eta}_{\hat{\varepsilon}}$.

$$S_6 \leq P(A^c) \cdot \bar{\varepsilon} \leq \hat{\varepsilon} \leq \varepsilon/8 .$$

This follows from (4) and the choice of A .

$$S_5 \leq \hat{\epsilon} \leq \epsilon/8 .$$

From (2) and the fact that $\eta \leq \eta_{\hat{\epsilon}}$.

$$S_3 \leq \eta \cdot E \leq (\epsilon/8 \cdot E)^E = \epsilon/8 .$$

Using (1) . Substituting all these inequalities in (8.11), we finally arrive to

$$S \leq -\epsilon/2 ,$$

which contradicts (8.6).

Q. E. D.

9. Proof of Theorem 3:

It is a direct consequence of combining Theorem (1) and Theorem

(2). There is nothing to be proved!

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