COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut 06520

COWLES FOUNDATION DISCUSSION PAPER NO. 685



# DOMINANCE AND SHAREHOLDER UNANIMITY:

# A NEW APPROACH

Cynthia Van Hulle

December 3, 1983

DOMINANCE AND SHAREHOLDER UNANIMITY:

A NEW APPROACH*

by

Cynthia Van Hulle

Professor Hart's "On Shareholder Unanimity in Large Stock Market Economies" [8] and Professor Makowski's "Competitive Stock Markets" [9] are two outstanding papers both challenging the central role of Ekern-Wilson spanning in the literature on shareholder unanimity to corporate decisions. This research continues along the same route: it develops a new criterion, generally weaker than any previously proposed, to guarantee the validity of the net present value rule. In particular, it shows that when investors are sufficiently competitive such that market values are bid up "high enough" (see Section 3), maximization of the net present value of the company is preferred by all initial shareholders; simultaneously all final shareholders prove to be indifferent to the firm's actions. Contrary to all previous approaches this one does not require firms to be perfect competitors, neither in the spanning sense (i.e., firms do not affect the set of available return distributions by their decisions, nor the prices of these distributions),[1] nor in the sense of L. Makowski (i.e.,

*The research described in this paper is partially supported by ONR.

[1]In this paper the term "spanning" is generally used rather loosely, in the sense that, in most cases, it refers to the possibility of representing a distribution as a linear combination of available return distributions; strictly speaking this linear combination condition is weaker than the definition given above.

firms cannot affect the economy's reservation price of any return distribution) nor in the sense of O. Hart (i.e., firms are small relative to the market for their shares).

Consumption effects are not ruled out a priori; it only guarantees that whenever the consumption effect's impact on utility is contrary to that of the budget effect, the latter one dominates.

More specifically, whenever two projects with the same net present value are compared, this dominance criterion implicitly requires that at those two investment levels, the condition of one of the earlier mentioned unanimity approaches are met. This is not surprising: as a budget effect is absent, consumption effects necessarily have to be zero if they are to be dominated by the impact on wealth. However, when a change in net present value is involved, such strong notions of firm competition prove to be unnecessary. To establish a connection with the previous approaches, the Makowski criterion is formally shown to be a special case of dominance; simultaneously some conditions are developed under which Makowski's notion of company competition reduces to a spanning type constraint on corporate output. In order to clarify the links with Hart's model, the dominance approach is lifted from the essentially partial equilibrium level to a general equilibrium level.

It turns out that if Hart's assumptions are met and in addition investors are sufficiently competitive in the dominance sense, also in a finite economy, a firm cannot affect the market value of other companies by its decisions. Furthermore, as the size of the economy increases, this competitivity condition is more and more likely to be satisfied, until in the limit economy it holds automatically. Note that to be fully correct the preceding statement should replace the expression "Hart's assumptions" by "Hart's modified set of assumptions." For the model, as set up in [8], has associated with it a logical problem, sufficiently serious to possibly invalidate all of its conclusions.

Section 1 of this paper briefly recalls Makowski's results and examines the assumption of "perfect competition by firms" in some detail. After determining conditions under which this assumption reduces to the familiar Ekern-Wilson condition, Section 2 deals with the Hart model. After pointing out the earlier mentioned problem with this model and proposing a way to circumvent it, Section 3 develops the partial equilibrium version of the dominance criterion. Section 4 contains numerical examples illustrating the preceding findings. Finally, the finite economy counterpart of Hart's general equilibrium propositions is developed in Section 5.

Note that the present paper is concerned with a two period model in which only a single commodity basket is available. Generalization towards L. Makowski's multiperiod-multicommodity world of [9] is straightforward.

## 1. Makowski's Concept of Competition: A Generalization of Spanning

This section contains only a brief outline of Makowski's main findings in [9] concerning shareholder unanimity. It is solely intended to recall the results which are of direct interest to Sections 3, 4, and 5 and to show how the Makowski approach can be linked to spanning.

In [9] Professor Makowski claims that all initial shareholders of a perfectly competitive firm wish that company to choose a production-investment plan which maximizes net present value. Assuming everybody has correct expectations concerning the impact of a change in a company's investment plan and excluding short sales, a perfectly competitive firm is defined as one satisfying simultaneously the next conditions:

- none of its production decisions affects the prices of the consumption good at each date or the prices of traded company shares other than its own (i.e., condition (2a) in [9]), and
- for any of its investment plans, any individual with positive holdings in the firm can choose another portfolio, equally optimal, but which doesn't contain any shares of the company under consideration (i.e., Makowski's condition (2b)).

As well explained in [9], the second part, which is the heart of the definition, implies that, over the relevant region, firms have a horizontal demand curve for their shares, and therefore cannot affect the economy's reservation price for any of the income distributions they can offer. Makowski then shows that, assuming nonsatiation of wants, for any company satisfying (2a) and (2b), the net present value rule holds. This is so because under the preceeding conditions there is no room for consumption effects. Consequently initial shareholders are better off as the firm's net present value and hence their initial wealth increases; therefore also, since they are not affected by the firm's decision, all other investors are indifferent to the firm's actions. Basically Makowski's competition condition and subsequent derivations formalize the celebrated "clientele-argumentation": each firm attracts a "clientele" of investors for whom the return distribution of that firm's shares fits in with their plans; by changing its policy, a company may alter the clientele purchasing its stock but it can never affect the economy's reservation price for its shares.

By its very nature, any individual is quite strong in a unanimity problem, i.e., he can block any decision. However, it has not been fully recognized that in the current model, he has in addition an enormous strength over the exchange equilibrium itself, in the sense that, simply from observing the behavior of a single investor, often key information, especially information about distributions, can be inferred. Therefore not rarely, assumptions which seem to be merely "technical," have major "nontechnical" implications. Several examples of it will come up in this paper. In particular, this feature of the model makes it possible to connect in a simple way the unanimity approaches discussed here.

First however, in order to avoid any ambiguity and introduce notation, let us reconsider the standard state preference model underlying the current analysis.

The economy extends over two time periods, indexed respectively by 0 and 1. There are $I$ consumers, indexed by $i$ and $F$ firms, indexed by $f$ or $g$. At time 1 any one of a finite set of possible states of the world may occur, but at time 0 it is unknown into which state the economy will move.

At time 0 a finite supply of a single perishable commodity basket is available which, at that moment is usable for consumption and investment in firms; at time 1 companies produce an amount of this consumption basket which then can only be used for consumption. Each firm $f$ possesses a technology which determines the output $y_{1,f}$ at time 1, given the input at time 0, $y_{0,f}$, and the state of nature $e$. After occurrence of a particular state of nature, consumers are allocated output according to their holdings at time 1.

Investors are supposed to be nonsatiated utility maximizers and at time 0 they bring their initial resources of the consumption basket and initial holdings in firms to market.

Through a tatonnement process an exchange equilibrium, characterized by market clearing and consumers maximizing their utility subject to their budget constraints, is attained. In particular, each individual solves the next problem:

$$\max \quad U_i(x_0,\ x_1(1),\ \ldots,\ x_1(E))$$

$$\text{S.T.} \quad x_{i,0} + \sum_f s_{if} v_f \leqq \bar{x}_{i,0} + \sum_f \bar{s}_{if}(v_f - y_{0,f})$$

$$x_{i,1}(e) \leqq \sum_f s_{if} y_{1,f}(e)\ , \quad e = 1,\ \ldots,\ E\ ,$$

$$(x_{i,0},\ x_{i,1}) \in \mathbb{R}_+^{E+1}\ ; \quad s_{if} \in \mathbb{R}_+^F$$

with $U_i$ and $x_i$ denoting respectively utility and consumption, $\bar{x}_{i,0}$ and $\bar{s}_i$ initial endowment of the consumption basket and share holdings, $s_i$ holdings during period 0 and $v_f$ the total market value of the shares of company $f$.

Note that short selling of shares is excluded.

Finally let $X_{*i}(v - y_0)$ represent the set of optimal consumption plans of individual $i$ given the firms' announced plans and $S_{*i}(v - y_0)$ the corresponding optimal portfolios.

The next simple lemma shows a connection between Makowski's concept of firm competition and spanning.

<u>Lemma 1</u>. Suppose that firm $f$ is a perfect competitor in the sense of Makowski. Consider two arbitrary input levels $y_{0,f}^{\ell_1}$ and $y_{0,f}^{\ell_2}$. Suppose that an individual $i$ with $s_{if}(y_{0,f}^{\ell_1}) > 0$ has $X_{i*}(v_f^{\ell_1} - y_{0,f}^{\ell_1})$ singleton; assume also some individual $j$ ( $j$ may but need not differ from $i$ ) with $s_{jf}(y_{0,f}^{\ell_2}) > 0$ has $X_{*j}(v_f^{\ell_2} - y_{0,f}^{\ell_2})$ singleton. Then $(y_{1,f}^{\ell_1} - y_{1,f}^{\ell_2})$ can be written as a linear combination of return distributions in the economy.

Proof. Because of the uniqueness of i's consumption pattern given $y_{0,f}^{1}$ and Makowski's assumption (2b), it must be true that

$$\sum_{\substack{g \\ g \neq f}} \alpha_g^{\ell_1} y_{1,g} = y_{1,f}^{\ell_1}, \quad \alpha_g^{\ell_1} \in \mathbb{R}_+, \quad \alpha_g^{\ell_1} \text{ not all zero.}$$

Similarly for plan 2:

$$\sum_{\substack{g \\ g \neq f}} \alpha_g^{\ell_2} y_{1,g} = y_{1,f}^{\ell_2}, \quad \alpha_g^{\ell_2} \in \mathbb{R}_+, \quad \alpha_g^{\ell_2} \text{ not all zero.}$$

Consequently the lemma's assertion holds. □

Hence it suffices for example that for each feasible plan, there is among the final shareholders each time some individual with a strictly concave utility function, for Makowski's notion of competition to imply spanning. Therefore, Makowski's condition (2b) collapses into the Ekern-Wilson linearity constraint unless everybody's preferences are sufficiently "ill behaved." Clearly the "technical" assumption concerning the specific mathematical properties of utility functions may have important "nontechnical" consequences for this model.

Nonetheless, one major contribution of [9] is that it shows that also in a finite economy spanning is not really a requirement for unanimity: all that matters is that given a common price perception, consumption effects can be sufficiently bounded.

Here it is important to realize that condition (2b) is a composite assumption: given investors' preferences it "constrains" the return distribution of a proposed change in investment; simultaneously it presumes the perceptions concerning pricing have the property that acceptance of the proposal doesn't generate consumption effects.

It is on the latter part of the condition the analysis of Sections 3, 4, and 5 focuses: it will turn out that whenever the choice of a plan involves a nonzero budget effect, the former part can be done away with.

## 2. Hart's Concept of Firm Competition and Its Relationship to the Makowski Model

Similar to the previous one, this section recalls the main findings of O. Hart's paper [8]. In addition, a logical problem which may invalidate the Hart findings is pointed out and a modification of the model tying it in more closely with Makowski's work and the results of the next sections, is suggested.

As mentioned in the introduction, Professor Hart derives shareholder unanimity without recourse to spanning by making each firm sufficiently small relative to the market for its shares.

This is done in either one of two ways:

a) both companies and individuals (within each type) are replicated;

b) solely the investors (within each type) are replicated while keeping the size of the firm below some finite level; this is achieved by allowing only a limited number of individuals to own initial holdings in the firm.

Similarity in tastes assumes that many, in the limit an infinite number of investors wish to buy a company's shares at the going prices. If in addition no short sales are permitted, equal treatment of all consumers necessarily implies that in equilibrium any one investor can only own a very small fraction in any firm, unless a great many perfect substitutes for the firm's shares are available. In any case, no single company's return distribution offers a consumption surplus to any investor, such that, if the market value of the other companies is not affected by its decisions, a firm can only influence its initial owners through its net present value. Therefore, as initial owner's initial holdings are bounded away from zero (this is achieved through the earlier mentioned assumption that in the pre-equilibrium situation ownership claims of any single firm are distributed over only a limited number of consumers), a higher net present value is always associated with an increase in the original shareholders' utility. Hence, if the value of the other comapnies stays unchanged, the net present value rule remain valid.

Furthermore, it is shown in [8] that when tastes are similar, no company can actually affect the prices of the other firms by its decisions. Intuitively one may expect similar people to react in a similar way. As everybody is treated equally and as companies are small, it is impossible that a firm's change in plans can significantly affect a large (in the limit an infinite) number of investors. In other words, under the stated conditions, the economy will absorb the change without really altering anybody's consumption pattern, hence implicit prices. Consequently market prices must remain unchanged.

To achieve similarity in tastes, O. Hart assumes that the wealth of all individuals (within the same class) is sufficiently "typical," i.e. similar to that of others. Thus, given typical wealth, in a large economy, the net present value rule holds and no firm can affect the economy's reservation price for return distributions over states. In addition, "typical" behavior implies that, whenever in equilibrium investors possess ownership claims bounded away from zero in any single firm, the company necessarily has many replicas such that next to (2a), Makowski's condition (2b) is automatically fulfilled also.

Consequently in a large economy firms would necessarily be perfect competitors in the Makowski sense.

However, there is one main difficulty with this assumption of "typical" wealth: whenever not all feasible investment plans of a firm are associated with the same net present value, "atypical" wealth cannot be ruled out. This is a serious problem because it may invalidate all of the results.

To see this, consider a large enough economy and suppose that in the current exchange equilibrium everybody's wealth is typical (there is no loss in generality to do this as the roles of the current and postchange equilibria can be interchanged). Hence, firms are small relative to the market for their shares and nobody derives a specific consumption surplus from the return distribution of a particular firm. Suppose that, say company f , decides to alter its plans, thereby affecting its net present value. Because initial holdings are bounded away from zero, necessarily the initial owners of the firm experience an alteration in their wealth which is bounded away from zero too. Consequently, as they were "typical" in the original situation, they will generally not be "typical" anymore after the change. It follows that the asymptotic properties of the large economy need not apply to them anymore and it is not excluded that, while readjusting their portfolio for the change in wealth, they bid up the market value of other firms: it can be shown formally (see [14]) that starting from an equilibrium in which everybody is "typical," a change in the input plan of a firm can only bring about upward pressures on the prices of other companies. This is so because, if the economy is large enough and even if some "outliers" decide to sell off some of their previous equilibrium holdings, there always remain enough investors keeping on to their shares to sustain the going prices. In addition, whenever such bidding up occurs, the initial owners in the firms experiencing an upward pressure on their market value may turn into "outliers" also and start off a second round in the process. Thus firms may affect the value of other companies by their decisions. Furthermore, if an initial shareholder in company f happens to own initial claims in a firm which value is bid up he need not necessarily experience a decrease in wealth when the net present value of firm f's new plan is below that of the original one. Also, as tastes and behavior are not necessarily "typical," it need not be true anymore that in the new equilibrium none of the consumers derives a specific consumption benefit from investment in the shares of a particular firm. Consequently, the net present value rule is generally not valid anymore either. A more mathematical description of the "outlier" problem is given in footnote 2.

---

[2] Assume w.l.o.g. that in the original equilibrium everybody is typical. If, as is supposed in [8], the two plans $y_{0,f}$ and $y_{0,f} + \Delta y_{0,f}$ have different net present value, there generally will be outliers in the exchange equilibrium corresponding to the firm's modified plan. Therefore, using Hart's notation, in the latter situation, it need not be true that every limit point of a sequence ${}^{r}z_{i\ell r}({}^{r}y_{0,f} + {}^{r}\Delta y_{0,f})$ lies in $\text{supp } n_{ir}(y_{0,f} + \Delta y_{0,f})$. Since $\sup n_{ir}(y_{0,f} + \Delta y_{0,f}) = \text{supp } n_{ir}(y_{0,f})$, it is possible that ${}^{r}z_{i\ell r}({}^{r}y_{0,f} + {}^{r}\Delta y_{0,f}) \notin \text{supp } n_{ir}(y_{0,f})$ for all sufficiently large $r$. Consequently there may exist $\pi_{i\ell r}(a, z(y_{0,f} + \Delta y_{0,f})) > \pi_{i\ell r}(a, z(y_{0,f})) + \xi$ for all sufficiently large $r$ and some $\xi > 0$.

Since $\gamma(a, y_{0,f} + \Delta y_{0,f}) = \max_{i,\ell} \pi_{i\ell}(a, z(y_{0,f} + \Delta y_{0,f}))$, it is possible that

$$\liminf_{r\to\infty} {}^{r}\gamma(a, {}^{r}y_{0,f} + {}^{r}\Delta y_{0,f}) \to \gamma(a, y_{0,f} + \Delta y_{0,f}) > \gamma(a, y_{0,f}) .$$

Therefore Hart's equations (17), (19) and the immediately following one need not hold. Since not necessarily a $z \in \text{supp } n_{ir}(y_{0,f} + \Delta y_{0,f})$ exists

Several ways to circumvent the consequences of atypical wealth may be thought of. One which seems particularly appealing, since it is in line both with Makowski's work and the analysis in Section 5 of this paper, is to drop Hart's assumption about similarity of wealth altogether and replace it by the next modified version of Makowski's clever assumption 3 (page 316 of [9]): "suppose that for any plan in firm f's opportunity set one can always find a typical consumer wishing to hold at least an arbitrary small fraction of firm f's shares." Then as this investor behaves typically, consistency with this investor's optimal behavior requires that the market price of firm f behaves typically also, i.e., stays the same. Consequently, the economy's reservation price for all of the distributions remains unchanged. Since due to this nonisolation assumption any period one income distribution necessarily gets distributed over a large, in the limit an infinite number of investors and hence no particular consumption benefits can be associated with holdings in a specific firm, the net present value rule holds.[3,4]

---

such that $\lim_{r\to\infty} {}^r\gamma(a) = \pi_i(a,z)$, the second equation on page 1082 of [8] may make no sense and invalidate subsequent assertions.

[3]More formally, drop assumption 8 in [8] and replace it by the following one: for all feasible $y_{0,f}$, there exists for all $r \geq R$ with $R$ some large positive number, some $i\ell_r \in \{i\ell | \pi_{i\ell r}({}^r y_{1,f}, {}^r z_i) = {}^r p(a)\}$ and simultaneously ${}^r z_{i\ell r} \in \text{supp}\ {}^r n_r({}^r y_{0,f})$.

Then for any distribution " a " there exists $z \in \text{sup}\ n_r$ and Hart's equation (17) and subsequent ones are valid.

[4]Anticipating the findings in Section 3, note that, if the economy is large enough, one can always find a typical consumer wishing to hold at least an arbitrary small fraction of firm f's shares when, for example, utility functions can be written as $U_i(c_{0i}, c_{1i}(1), \ldots, c_{1i}(E)) = \sum_{e=1}^{E} f_i(e) u_{ie}(c_{0i}, c_{1i}(e))$ with $f_i(e)$ individual i's subjective probability of state e occurring and $u_{ie}(\cdot)$ a state dependent, pseudo-concave and homogeneous (of degree 1)

It is quite remarkable that, even when a large economy is involved, the power of a single individual proves to be very significant: one typical consumer suffices to fix the economy's reservation price for a firm's shares.

Finally, note for further reference that, by the same argument as the one used to prove that only upbidding may occur whenever originally everybody is typical, it can be shown that the same statement still holds when in the pre-change exchange equilibrium the nonisolation assumption is satisfied.

## 3. Shareholder Agreement: The Dominance Approach

The structure of this section is as follows: to unburden the discussion related to the dominance approach (Theorem 1), first one additional tool will be developed with the aid of a preliminary proposition; then the analysis is carried one step further into Theorem 1, the main result of this section.

Consider the next intuitive argument underlying Proposition 1. Suppose that in a finite economy, a firm proposes a project which return distribution may not satisfy a spanning constraint. Suppose also that all investors know the current exchange equilibrium well and understand that, as each of them is small and there are many of them competing for the same income distributions, prices will be bid up at least to the level where, unless their initial wealth changes, it doesn't pay any one of them to adjust their equilibrium consumption pattern in response to the adoption

---

von Neumann-Morgenstern utility function. In that case none of the initial shareholders of the firm changing its plan will engage in upbidding (this follows from Lemma 2 of Section 3).

of the project.[5] Hence, assuming in addition it is sufficiently small not to affect significantly the wealth of many investors and the economy's input and output, one would not expect massive readjustments in optimal portfolios in response to the introduction of the new investment opportunity.

In such a case, in spite of the fact that the proposal may generate a new type of future income distribution, it does not seem likely it would bring about perceptible changes in the market price of firms other than possibly the company proposing it.

Proposition 1. Reconsider the model described in Section 1 and add the following assumptions:

- for any $(y_{0,f}, y_{1,f})$ in firm f's opportunity set, there exists an exchange equilibrium such that for each firm $g \neq f$, $v_g(y_{0,f}, y_{1,f}) = \bar{v}_g$ (every individual correctly perceives no impact of firm f's decisions on the market prices of the other firms, i.e. Makowski's assumption (2a));
- for the current $(y_{0,f}, y_{1,f})$ in the production set, for each individual $i$ such that $s_{if} > 0$, there exists another $(x^i_*, s^i_*) \in (X_{*i}, S_{*i})$ satisfying $s_{*if} = 0$ and investors correctly conjecture that the future income distribution of any firm f's projects will be priced to reflect at least direct consumption benefits (see below); this assumption is a weakened version of Makowski's assumption $(2\hat{b})$ and the first part of it will be referred to as (2b);

[5] Cf. the competitive price perception hypothesis of Grossman and Hart [6].

– $U_i$ is pseudo-concave[6] and strictly increasing in its arguments for all $i \in I$ (assumption 3);

– $(x_{0,i}, x_{1,i}) \in R_{++}^{E+1}$ for all $i \in I$, i.e. each individual consumes something in each period and event $e$ (i.e. Makowski's assumption 2 in [9]) (assumption 4).

Then all initial shareholders prefer acceptance of a project by company $f$ if and only if it increases the net present value $(v_f - y_{0,f})$ ; all other investors are indifferent as to the firm's actions. Furthermore, the pricing satisfies the next weak inequality:

$$(1) \qquad v_f(y_{0,f} + \Delta y_{0,f}) \geqq \max_i \sum_e \pi_i(y_{0,f}, e) y_{1,f}(y_{0,f} + \Delta y_{0,f})$$

with $\pi_i(y_{0,f}, e)$ investor i's implicit prices in the current, i.e. prechange, equilibrium. In addition, if one assumes also that in case the project is accepted, for some individual $i$ such that $\bar{s}_{if} = 0$, it is true that $s_{if}(y_{0,f} + y_{0,f}) \quad 0$ (i.e. Makowski's assumption 3 on non-isolation of initial shareholders), the preceding weak inequality holds as an equality.

Recall that if the preceding assumption on utility functions is met and $(x_{0,i}, x_{1,i}) \in R_{++}^{E+1}$ $(\forall i \in I)$, at equilibrium the following relationships are satisfied for all $i$ and all $f$ :

$$(2) \qquad \sum_e \pi_i(e) y_{1,f}(e) + \eta_{i,f} = v_f$$

---

[6] A numerical function $\theta$, defined on a set $\Gamma$ in $\mathbf{R}^n$ is pseudo-concave at $\bar{x} \in \Gamma$ if it is differentiable at $\bar{x}$ and if, for any $x \in \Gamma$ such that $\nabla\theta(\bar{x}) \cdot (x - \bar{x}) \leqq 0$, it follows that $\theta(x) \leqq \theta(\bar{x})$ ( $\nabla\theta(\bar{x})$ denotes the gradient of $\theta$ evaluated at the point $\bar{x}$ ). Pseudo-concavity implies strict quasi concavity but, given differentiability of the function, is weaker than concavity (O. Mangasarian [11], pp. 141, 147).

with $\eta_{i,f}$ a nonnegative Lagrange multiplier connected with the nonnegativity constraints on $s_{if}$ and $\pi_i(e) \in \mathbb{R}_{++}$, individual i's implicit price for a unit of consumption if and only if state e occurs, i.e.

$$\text{(3)} \qquad \pi_i(e) = \frac{\partial U_i/\partial x_{1,i}(e)}{\partial U_i/\partial x_{0,i}} \quad .$$

Recall also that, as indicated in [9] (p. 316), the $\pi_i$-vectors of a particular individual are the same for all $(x_{0,i}, x_{1,i}) \in X_{*i}$ .

Proof. Two steps are involved.

Step 1: Consider the next lemma

Lemma 2. Suppose assumptions (2a), ($\hat{2}$b), (3) and (4) are satisfied. Suppose also that investor i having currently $s_{if} > 0$ believes that the project will be priced such that

$$\text{(4)} \qquad \sum_e \pi_i(e)\Delta y_{1,f}(e) \leqq \Delta v_f$$

with $\Delta$ the difference operator and the $\pi_i(e)$ i's prechange implicit prices.

Then abstracting from a possible change in preferences induced by a change in initial wealth, all of the prechange equilibrium patterns in $X_{*i}(y_{0,f})$ which remain feasible, remain optimal also.

Proof of the Lemma. To do away with a possible wealth effect, imagine for a moment that either $\bar{s}_{if} = 0$ or $\bar{s}_{if}(\Delta v_f - \Delta y_{0,f})$ is compensated for by a change in $\bar{x}_{0,1}$ . Indicate the so obtained (post change-no budget effect) consumption opportunity set by ${}_{nb}X_i(y_{0,f} + \Delta y_{0,f})$ .

Case 1: Suppose that there is some $x_i \in X_{*i}(y_{0,f})$ such that simultaneously $x_i \in {}_{nb}X_i(y_{0,f} + y_{0,f})$ and $x_i$ can be realized by adopting a financial plan with $s_{if} = 0$ .

Consider an arbitrary ${}_{nb}x_i \in {}_{nb}X_i(y_{0,f} + \Delta y_{0,f})$ with a corresponding portfolio ${}_{nb}s_i$ .

Denoting the gradient by $\nabla$ and using equations (2) and (3) it is easy to show that

$$(5)\qquad \frac{\partial U_i(x_{*i}(s_i))}{\frac{\partial U_i(x_{*i})}{\partial x_{0,i}}} \cdot ({}_{nb}x_i - x_{*i})$$

$$= -\sum_{\substack{g\\ g\neq f}} \eta_{ig} \cdot ({}_{nb}s_{ig} - s_{ig}) + (\sum_e \pi_i(e)\Delta y_{1,f}(e) - \Delta v_f) \cdot ({}_{nb}s_{if} + s_{if}) \ .$$

Because of the optimality of $s_i$ in the preproject situation, one may have $({}_{nb}s_{ig} - s_{ig}) < 0$ only if $\eta_{if} = 0$ . Consequently every one of the terms $-\eta_{ig}\ ({}_{nb}s_{ig} - s_{ig})$ is nonpositive. Since by assumption $s_{if} = 0$ , it follows from condition (4) that also the last term on the RHS is nonpositive. Hence, as $\frac{\partial U_i(x_{*i})}{\partial x_{0,i}} > 0$ , necessarily $\nabla U_i(x_{*i}(s_i)) \cdot ({}_{nb}x_i - x_{*i}) \in \mathbb{R}_-$ . By pseudo concavity $U_i({}_{nb}x_i) \leqq U_i(x_{*i}(s_i))$ . As this inequality holds for all ${}_{nb}x_i \in {}_{nb}X_i(y_{0,f} + \Delta y_{0,f})$ , $x_{*i}$ remains optimal.

Case 2: Suppose $x_{*i} \in X_{*i}(y_{0,f})$ and $x_{*i} \in {}_{nb}X_i(y_{0,f} + \Delta y_{0,f})$ would have a corresponding financial plan with $s_{if} > 0$ .

As in that case there exists $s_i \in S_*(y_{0,f})$ and ${}_{nb}s_i \in {}_{nb}S(y_{0,f} + \Delta y_{0,f}$ such that $x_{*1i} = \sum_g s_{ig}y_{1g} = \sum_{\substack{g\\ g\neq f}} {}_{nb}s_{ig}y_{1g} + {}_{nb}s_{if}(y_{1,f} + \Delta y_{1,f})$ , there necessarily exist $\alpha$ (not all zero) such that $\Delta y_{1,f} = \sum_g \alpha_g y_{1g}$ .

Consequently, using (4) and (3):

(6) $$\sum_g \alpha_g v_g = \sum_g \sum_e \pi_i(e)\alpha_g y_{1g} \leqq v_f \ .$$

Therefore, in comparison with the preproject situation, after the change and with the portfolio ${}_{nb}s_i$, $x_{*1i}$ would require more period 0 outlays if the inequality (4) and hence (6) would be strict. This in turn is contradictory to the fact that, as the wealth remains unchanged and consumers are nonsatiated, $x_{*i} \in X_{*i}(y_{0,f})$. Hence (4) and (6) must hold as strict equalities. Consequently the last term on the RHS of (5) is 0 and again $x_{*i}$ remains optimal w.r.t. ${}_{nb}X_i(y_{0,f} + \Delta y_{0,f})$. □

Step 2: Proposition 1 follows from applying a similar reasoning as in the proofs of Makowski's theorems 1 and 2 and using the fact that because of Lemma 2, any $x_{*i} \in {}_{nb}X_i(y_{0,f} + \Delta y_{0,f})$ remains optimal within this set. □

Reconsider Lemma 2. This lemma, which will prove to be the main workhorse in the subsequent analysis, shows that $\Delta v_f = \sum_e \pi_i(y_{0,f}, e)\Delta y_{1,f}(e)$ is the minimal time 0 price for the project's distribution such that--apart from a wealth effect and/or the proposal destroying the feasibility of the investor's previously chosen income pattern (i.e. $X_{*i} \cap {}_{nb}X_i$ is empty)--individual i does not wish to move away from his preproject consumption pattern.

Any benefit the investor may receive if this pricing condition is not satisfied is henceforth referred to as a "direct consumption effect." A change in utility caused by $X_{*i} \cap {}_{nb}X_i$ being empty is indicated by an "indirect consumption effect."

The proposition thus implies that, if no direct consumption benefits can be gained and a move away from firm f's current investment level does not entail an indirect consumption effect, f's production opportunities can "objectively" (i.e. everybody agrees) be subdivided on the basis of net present value into two nonoverlapping sets "plans at least as good as the present one" and "plans worse than the current one." Therefore when pitted against the current proposal, a net present value maximizing plan would always be the winner. However when such a maximizer is compared to another proposal from the set "plans at least as good as the present one" it seems obvious that, since condition ($\hat{2}$b) and the "no direct consumption benefit" assumption need not be satisfied at those input levels, there is no reason to believe that the maximizing proposal would be unanimously preferred. This then would be the price to be paid for weakening Makowski's condition (2b).

With all this in mind the next theorem may appear somewhat counterintuitive.

<u>Theorem 1</u>. Consider a firm f with L feasible input levels in its investment opportunity set, i.e. $Y_f = \{(y^1_{0,f}, y^1_{1,f}), \ldots, (y^L_{0,f}, y^L_{1,f})\}$ . Suppose that investors know that $(v^1_f - y^1_{0,f}) \geqq (v^2_f - y^2_{0,f}) \geqq \cdots \geqq (v^L_f - y^L_{0,f})$ . Assume this pricing vector over projects satisfies the condition that no individual receives a direct consumption benefit ever when the firm moves consecutively from project 1 to project 2, from project 2 to project 3, ..., from project L-1 to project L . More specifically it is assumed that for any $i \in I$ there exist vectors of implicit prices for which the following holds:

$$(7)\quad v_f^2 - v_f^1 = \sum_e \pi_i(y_{0,f}^1,e)\cdot(y_{1,f}^2-y_{1,f}^1);\ (v_f^3-v_f^2) \geqq \sum_e \pi_i(y_{0,f}^2,e)\cdot(y_{1,f}^3-y_{1,f}^2);$$

$$\ldots;\ v_f^L - v_f^{L-1} = \sum_e \pi_i(y_{0,f}^{L-1},e)\cdot(y_{1,f}^L-y_{1,f}^{L-1})$$

(i.e., investors perceive pricing of each plan relative to the just preceeding one to be high enough such that in the direction of decreasing net present value direct consumption benefits are excluded).

Suppose in addition that assumptions (2a), (3) and (4) are satisfied. Then all initial shareholders unanimously prefer the firm to maximize its net present value. This preference is independent of the order in which projects are proposed and nobody has an incentive to misrepresent his preferences. In addition, all noninitial shareholders are indifferent as to the firm's policy.

Thus all the theorem requires is that prices are "high enough." There are no explicit assumptions on distributions or portfolios, nor are consumption effects ruled out. In particular, note that

- contrary to the preceding proposition, assumption ($\hat{2}$b) has been deleted;
- when going down the sequence from project 1 to project 2, ..., from project L-1 to project L, indirect consumption effects are not excluded;
- when moving upward along the sequence, next to indirect consumption effects, direct consumption benefits may appear also;
- the type of unanimity is generally stronger than the one guaranteed by Proposition 1 in the sense that presently everybody's preference ordering over feasible plans corresponds to net present value.

Proof. Step 1: Suppose that the current input level corresponds to $y^{\ell}_{0,f}$ $(1 \leqq \ell \leqq L)$ ; if there is no other input level with the same or lower net present value, proceed to Step 2. Otherwise, denote the change in utility for some arbitrary investor when the firm moves from $y^{\ell}_{0,f}$ to $y^{\ell+1}_{0,f}$ by $\Delta U_i(y^{\ell}_{0,f}, y^{\ell+1}_{0,f})$ .

Compare this with the hypothetical situation in which condition $(2\hat{b})$ is satisfied at $y^{\ell}_{0,f}$ and denote the effect on utility in that (hypothetical) case by $\Delta\hat{U}_i(y^{\ell}_{0,f}, y^{\ell+1}_{0,f})$ . Clearly $\Delta\hat{U}_i(y^{\ell}_{0,f}, y^{\ell+1}_{0,f}) \geqq \Delta U_i(y^{\ell}_{0,f}, y^{\ell+1}_{0,f})$ . As Proposition 1 applies to this hypothetical situation, it follows that $0 \geqq \Delta\hat{U}_i(y^{\ell}_{0,f}, y^{\ell+1}_{0,f}) \geqq \Delta U_i(y^{\ell}_{0,f}, y^{\ell+1}_{0,f})$ . A similar argument consecutively applied to a change from $y^{\ell+1}_{0,f}$ to $y^{\ell+2}_{0,f}, \ldots, y^{L-1}_{0,f}$ to $y^{L}_{0,f}$ shows that $U_i(y^{\ell}_{0,f}) \geqq U_i(y^{\ell+1}_{0,f}) \geqq \ldots \geqq U_i(y^{L}_{0,f})$ .

Step 2: Consider a net present value maximizing input level $y^{\ell m}_{0,f}$ $(1 \leqq \ell m \leqq \ell \leqq L)$ . If $\ell m = \ell$ , go to Step 3. Otherwise consider the sequence $y^{\ell m}_{0,f}, \ldots, y^{\ell}_{0,f}$ and, using Step 1's argumentation, find that $U_i(y^{\ell m}_{0,f}) \geqq \ldots \geqq U_i(y^{\ell}_{0,f})$ .

Step 3: Suppose there are several net present value maximizing inputs (if not proceed to Step 4). Again the same argumentation shows that $U_i(y^{1}_{0,f}) \geqq \ldots \geqq U_i(y^{\ell m}_{0,f})$ with $\{y^{1}_{0,f}, \ldots, y^{Lm}_{0,f}\}$ the set of maximizers.

Switching the roles of $y^{1}_{0,f}$ and $y^{s}_{0,f}$ in the preceding reasoning shows that necessarily $U_i(y^{1}_{0,f}) = U_i(y^{2}_{0,f})$ . Similarly $U_i(y^{2}_{0,f}) = \ldots = U_i(y^{Lm}_{0,f})$ .

Step 4: In the same way as in Step 3 it can be shown that the individual i is always indifferent between investment levels yielding the same net present value.

Step 5: Steps 1 to 4 have shown that individual i's preferences over feasible production plans are completely ordered according to net present value. Since the preceding reasoning applies to any investor, all individuals have the same preference ordering over projects. This preference ordering does not depend on the current plan. And since for every investor voting nonsincerely can only result in company decisions moving away from the direction of his preferences, nobody has an incentive to misrepresent his preferences. □

The preceding theorem may appear a bit extravagant, especially since it seems to require weaker conditions than Proposition 1, while yielding stronger results. Since also in model building one would not expect "free-lunch" opportunities, some additional investigation into what is really going on seems appropriate. In particular what is the implicit structure imposed by the pricing assumption such that so strong a conclusion can be arrived at?

To facilitate the analysis of this problem, distinguish between the following special cases:

Case 1: Suppose w.l.o.g. that firm f's current plan is $\ell$ and that it considers switching to the next one down the sequence, i.e. $(\ell+1)$. Assume that $v_f^{\ell} - y_{0,f}^{\ell} = v_f^{\ell+1} - y_{0,f}^{\ell+1}$.

Consider the following classes of investors (without distinguishing between initial and noninitial shareholders):

(1.a) $s_{if}(y_{0,f}^{\ell}) > 0$ and $s_{if}(y_{0,f}^{\ell+1}) > 0$ with $s_{if}(y_{0,f}^{\ell})$ respectively $s_{if}(y_{0,f}^{\ell+1})$ i's optimal holdings in company f given plan $\ell$ respectively $(\ell+1)$ ;

(1.b) $s_{if}(y_{0,f}^{\ell}) > 0$ and $s_{if}(y_{0,f}^{\ell+1}) = 0$ where again these holdings are optimal;

similarly (1.c) $s_{if}(y^{\ell}_{0,f}) = 0$ and $s_{if}(y^{\ell+1}_{0,f}) > 0$ ;

and (1.d) $s_{if}(y^{\ell}_{0,f}) = 0 = s_{if}(y^{\ell+1}_{0,f})$ .

Case 2: Consider again the same situation but suppose that

$$v^{\ell}_f - y^{\ell}_{0,f} > v^{\ell+1}_f - y^{\ell+1}_{0,f}$$

(2.a), (2.b), (2.c), (2.d) refers to the same classification as the one above.

Look first at the class (1.a). The theorem implies that for any one of its individuals

(8) $$U_i(y^{\ell}_{0,f}) = U_i(y^{\ell+1}_{0,f}) \quad \text{and}$$

(9) $$\sum_e \pi_i(y^{\ell}_{0,f}, e)\cdot(y^{\ell+1}_{0,f}(e) - y^{\ell}_{0,f}(e)) \leqq y^{\ell+1}_{0,f} - y^{\ell}_{0,f} = v^{\ell+1}_f - v^{\ell}_f .$$

In addition, optimality of their portfolios requires:

(10) $$\sum_e \pi_i(y^{\ell}_{0,f}, e)\cdot y^{\ell}_{1,f}(e) = v^{\ell}_f$$

(11) $$\sum_e \pi_i(y^{\ell+1}_{0,f}, e)\cdot y^{\ell+1}_{1,f}(e) = v^{\ell+1}_f .$$

However, because of equation (3), Lemma 2 implies:

$\sum_e \pi_i(y^{\ell}_{0,f}, e)\cdot(y^{\ell+1}_{1,f} - y_{1,f}) \geqq y^{\ell+1}_{0,f} - y^{\ell}_{0,f}$ , such that (4) must hold as an equality. Switching the roles of $\ell$ and $\ell+1$ , it is easy to see the following holds true also:

(12) $$\sum_e \pi_i(y^{\ell+1}_{0,f}, e)\cdot(y^{\ell}_{1,f} - y^{\ell+1}_{1,f}) = y^{\ell}_{0,f} - y^{\ell+1}_{0,f} .$$

Hence, the theorem implicitly requires that for the individuals of this class, both downward and upward direct consumption benefits are absent. In addition, because of (8), Lemma 2, equations (9) and (12), necessarily $\hat{U}_i(y^{\ell}_{0,f}) = \hat{U}_i(y^{\ell}_{0,f}, y^{\ell+1}_{0,f}) = U_i(y^{\ell+1}_{0,f}) = \hat{U}_i(y^{\ell+1}_{0,f}, y^{\ell}_{0,f})$ . Consequently, Lemma 2 implies that for arbitrary $y^{\ell}_{1,f}$ and $y^{\ell+1}_{1,f}$ , one may not expect, neither $s_{if}(y^{\ell}_{0,f})$ nor $s_{if}(y^{\ell+1}_{0,f})$ to be bounded away from zero.

To guarantee consistency of the model with, for example $s_{if}(y^{\ell}_{0,f})$ being bounded away from zero for investor $i$ , one has to assume that either the individual can "undo" on his own account the indirect consumption effects of $(y^{\ell}_{1,f} - y^{\ell+1}_{1,f})$ on his portfolio without having to reduce $s_{if}(y^{\ell}_{0,f})$ to zero or, he is indifferent between having $s_{if}(y^{\ell}_{0,f})$ arbitrary small and having it bounded away from zero (i.e. assumption $(\hat{2}b)$ is satisfied at the investment level of $y^{\ell}_{0,f}$ ). Obviously the same argumentation applies to $s_{if}(y^{\ell+1}_{0,f})$ .

Reasoning along similar lines, it is not difficult to see that for an individual of class (1.b) to have $s_{if}(y^{\ell}_{0,f})$ bounded away from zero, consistency requires that he perceives a perfect substitute portfolio at firm f's investment level $y^{\ell}_{0,f}$ . Likewise, for investors in (1.c), "larger" individual holdings in the firm presumes they perceive condition $(2\hat{b})$ to be met at the level $y^{\ell+1}_{0,f}$ .

Nothing interesting can be inferred concerning investors in (1.d).

Hence for the choice between projects with the same net present value to be a matter of indifference and simultaneously have consistency with holdings being bounded away from zero, Theorem 1 implicitly requires that either the Makowski condition concerning substitute portfolios is satisfied at these projects respective input levels for individuals with "larger" holdings or that some personal "undoing" of the change is possible. Since no budget effect is involved, these conclusions are independent of the fact whether or not the investor is an initial shareholder.

Consider now the case where the choice of the plan involves a change in net present value, i.e. $v_f^\ell - y_{0,f}^\ell > v_f^{\ell+1} - y_{0,f}^{\ell+1}$. Clearly, as the non initial shareholders' wealth is not affected by the adopted plan, the preceding analysis still applies to them. However, for initial shareholders quite a change occurs. In particular, as there is a budget effect and the theorem predicts that $U_i(y_{0,f}^\ell)$ may be strictly greater than $U_i(y_{0,f}^{\ell+1})$ the analysis and inferences of (1.a) generally don't hold anymore. Therefore, for initial shareholders to agree on the relative desirability of different investment levels and this ranking to correspond to the ranking according to net present value, consistency with shareholdings being bounded away from zero does not necessarily require the Makowski-type or "undoing"-type condition of Case 1. The reason is twofold. First, if there is a difference in net present value, Theorem 1 only requires that the no-direct-consumption-benefit condition is met in the downward direction (i.e., in the direction of the project with lower net present value). Therefore direct consumption benefits in the upward direction are not excluded and Lemma 2 does not imply arbitrary small holdings anymore. In addition, as it is not required that utility remains unchanged, indirect consumption effects are not implicitly ruled out, adding to compatibility of holdings being bounded away from zero with the theorem, without the earlier mentioned implicit conditions. Second, changes in wealth may bring about changes in tastes: even if at one wealth level investing more than only a small fraction of resources in a particular type of distribution is nonoptimal, this need not be so anymore at another level of wealth (i.e. remember that the conclusions of Lemma 2 only applied if the budget effect was abstracted from).

Similarly, for the individuals in (2.b) and (2.c), none of the earlier conclusions hold; again nothing much can be said about investors in (2.d). Thus as soon as a budget effect is involved, Theorem 1 does not exclude consumption benefits: it only guarantees that the former dominates the latter.

Remark 1. It is not difficult to see that Lemma 2 and hence Theorem 1 would also hold when short selling would be possible. The only difference is that the no-direct-consumption-benefits constraint would require condition (5) to hold as a strict equality. One would also find that Theorem 1's implicit constraints are more stringent, i.e. in addition to the condition the market price is "high enough," the price may not be "too high" either.

Remark 2. Whenever the change in investment does not entail a budget effect, Proposition 1 also implicitly requires some "undoing" possibilities or perfect substitute portfolios in the Makowski sense at the new input level for holdings associated with this plan to be bounded away from zero.

Remark 3. As soon as a budget effect is present, neither the proposition nor the theorem imply that companies face a horizontal demand curve for their shares.

It has been mentioned on several occasions that, except for the stronger assumptions on preferences, Makowski's competition condition is more stringent than dominance.

Although the preceding analysis strongly indicates this statement is true, a formal proof has not yet been provided. The following very convenient corollary will turn this task into a most simple one.

Corollary 1. Recall the sequence of $L$ feasible investment plans from Theorem 1 and consider in particular the two consecutive plans $\ell$ (the current level) and $\ell+1$. Suppose that for investor $i$ there exists an optimal portfolio associated with the plan $\ell+1$ with the property that none of its components offers him a direct consumption benefit if the firm would switch to the plan $\ell+1$. Assume this condition is met for all investors and every two consecutive proposals in the sequence. If in addition assumptions (2a), (3) and (4) are satisfied, the net present value rule holds.

Proof. Similar to that of Theorem 1. □

Instead of focusing on the pricing of the company undertaking the project, the corollary transfers the no direct consumption benefit condition to each component of an optimal portfolio which may or may not contain any $f$ shares. By doing so, it slightly weakens the no-direct-consumption-benefit-condition-in-the-downward-direction (7) for investors in the classes (1.c), (2.c), and (1.d), (2.d). To see this, assume that an initial shareholder $i$ with $\eta_{if}$ strictly positive for the plan $\ell$ (see equation (2))--and hence sells off all of his holdings in $f$ if that plan is accepted--may find that, when the firm switches to $(\ell+1)$, condition (7) is violated. As long as the surplus in consumption value does not fully offset $\eta_{if}$ however, he cannot derive a benefit from repurchasing shares in $f$. Thus, although he perceives a direct consumption benefit, this investor is not able to take advantage of it.

Because of Makowski's condition (2a), the perfect substitute portfolios associated with each of firm $f$'s plans (condition (2b)), necessarily satisfy the Corollary's requirement. Hence the market value rule holds in the Makowski model.

## 4. Some Numerical Examples

Example 1. Consider an economy with F firms (F > 204) and I individuals (I > 2000) . Suppose the set of time one states of the world consists of only two elements. Only 3 types of firms exist; within each class a typical firm produces respectively

$$y_{1,1} = \begin{bmatrix} 100 \\ 1100 \end{bmatrix} ; \quad y_{1,2} = \begin{bmatrix} 1100 \\ 100 \end{bmatrix} ; \quad y_{1,3} = \begin{bmatrix} 200000 \\ 200000 \end{bmatrix}$$

and requires inputs of respectively

$$y_{0,1} = 400 ; \quad y_{0,2} = 500 ; \quad y_{0,3} = 100000 .$$

There is only one type 3 firm in operation. No separate lending and borrowing market is available and inputs are fully equity financed; short selling is excluded.

There are at least two classes of individuals. The two types this example will be concerned with have the respective utility functions:

$$U_1 = 20c_1 + 8c_2(1) - 1/20(c_2(1))^2 + 24c_2(2) - 1/40(c_2(2))^2$$

$$U_2 = 20c_1 + 24c_2(1) - 1/40(c_2(2))^2 + 8c_2(2) - 1/20(c_2(2))^2 .$$

Every one of the first thousand individuals in each of these two classes owns an initial endowment of the consumption basket of 150 units. In addition they each own initially a 1/2000 proportion of the shares in every of the first hundred companies of type 1 and type 2; also, before trading, each of them owns a 1/4000 portion of the type 3 company.

Market price for firms of type 1, 2 and the type 3 company is respectively $v_1 = 890 = v_2$ ; $v_3 = 180000$ . Note that market pricing is not additive but subadditive (i.e. $166.67 \times [y_{1,1} + y_{1,2}] = \begin{bmatrix} 200000 \\ 200000 \end{bmatrix} = y_{1,3}$ but $166.67 \times [v_1 + v_2] = 296672.6 > v_3 = 180000$ ). As is well known, this phenomenon may occur whenever short sales are excluded.

Some computation shows that the optimal consumption pattern for every member in the two groups of 1000 individuals is respectively $c_1 = \begin{bmatrix} 80 \\ 60 \\ 160 \end{bmatrix}$ and $c_2 = \begin{bmatrix} 80 \\ 160 \\ 60 \end{bmatrix}$ yielding a level of utility of 5100 and an implicit price vector $\pi_1 = \begin{bmatrix} 0.1 \\ 0.8 \end{bmatrix}$ respectively $\pi_2 = \begin{bmatrix} 0.8 \\ 0.1 \end{bmatrix}$. At equilibrium individuals of class 1 only own shares in firms of type 1 and 3 whereas those in class 2 only hold ownership claims in companies of type 2 and 3.

Note that the single firm of type 3 cannot be considered as a small one: some computation shows that each member of the group under consideration will receive, at time 1, 50 units of the consumption good from his equilibrium holdings in that company; a glance at the respective optimal consumption patterns shows that this is a large portion of the individual's time 1 income, whatever the state that obtains. More importantly however, if this firm would not exist, <u>ceteris paribus</u>, the maximal obtainable utility level for both groups would only be about 4990, even if, through compensation, initial wealth would be kept unchanged. Thus the company of type 3 definitely offers investors a nonzero consumption benefit.

<u>Case 1.1</u>: Suppose that the third type firm considers a project which would affect its output and input in the following way: $\Delta y_{1,3} = \begin{bmatrix} 800 \\ 8800 \end{bmatrix}$ ; $\Delta y_{0,3} = 7120$ . Assume that the project is perceived to change $v_3$ by the amount $\Delta v_3 = 7120$ implying that the firm's net present value is not affected. Suppose also the value of the other firms remains unchanged.

It is not difficult to check that the members of the first class of thousand individuals are able to "undo" the project, both distribution and price wise and that, since $\Delta(v_3 - y_{0,3}) = \pi_1(1)\Delta y_{1,3}(1) + \pi_1(2)\Delta y_{1,3}(2) - \Delta y_{0,3} = 0.1 \times 800 + 0.8 \times 8800 - 7120 = 0$, they would be indifferent. Recomputation of the optimal solution indeed shows that their consumption and hence utility remains unaffected.

For the second class of thousand individuals however, although pricing is "high enough" (i.e. the minimal no direct consumption benefit value change for these investors is $\pi_2(1)\Delta y_{1,3}(1) + \pi_2(2)\Delta y_{1,3}(2) = 0.8 \times 800 + 0.1 \times 8800 = 1520 < 7120 = \Delta v_3$), because of the subadditivity the implicit conditions of Theorem 1 associated with the case of no change in net present value, are not met: one can check that neither the Makowski condition, nor the "undoing" condition nor Hart's smallness constraint hold.

However the preceding analysis still allows for the most interesting inferences:

- since pricing is "high enough" and no budget effect occurs, the same argumentation as in the proof of Theorem 1 would imply that the project cannot make the second class individuals better off; hence as the first class is indifferent, the two thousand investors considered, still all agree;
- if in the post change situation utility is strictly below the initial level and as a budget effect is absent, Lemma 2 would imply that moving back to the original input-output level would entail a direct consumption benefit for the investors in the second class.

Recomputing the postchange optimum for these investors, shows that their optimal consumption pattern transforms into: $c' = \begin{bmatrix} 78.6 \\ 161.1 \\ 54.1 \end{bmatrix}$ and their vector of implicit prices into: $\pi' = \begin{bmatrix} 0.79725 \\ 0.1295 \end{bmatrix}$ .

Their postchange utility levels turn out to be 5076, thus strictly below the one in the original situation. As predicted $\Delta(v_3 - y_{0,3}) = 0 < -[0.79725 \times 800 + 0.1295 \times 8800 - 7120] = 5342.6$ .

Case 1.2: Suppose that the firm of type 3 would instead of the preceding one, propose the following project:

$$\Delta y_{1,3} = \begin{bmatrix} -25600 \\ 6400 \end{bmatrix} ; \quad \Delta y_{0,3} = 12560 \quad \text{and} \quad \Delta v_3 = 2560 .$$

Consequently, the net present value would decrease with 10000 units if the proposal is implemented.

It is easy to check that for the two groups which have been under consideration, the no direct consumption benefit condition is met. Again this case is not covered by the Makowski and Hart models. Theorem 1 however predicts unanimous rejection by the two groups.

Recalculating the respective optimal solutions shows that each investor's utility indeed goes down.

In particular, the first class of consumers changes its optimal consumption pattern to $c_1' = \begin{bmatrix} 75 \\ 60 \\ 160 \end{bmatrix}$ , yielding a utility level of $U_1' = 5000 < U_1 = 5100$ ; the second class changes it to $c_2' = \begin{bmatrix} 71.4 \\ 164.3 \\ 36.5 \end{bmatrix}$ with corresponding utility level $U_2' = 4921.7 < U_2 = 5100$ .

Their respective new implicit prices are $\pi_1' = \begin{bmatrix} 0.1 \\ 0.8 \end{bmatrix}$ and $\pi_2' = \begin{bmatrix} 0.78925 \\ 0.21750 \end{bmatrix}$ . It is easy to check that a move back to the original situation, would involve next to a budget effect also a consumption benefit for the second group; for the first one only a change in wealth seems to be involved.

Example 2. The preceding example only dealt with projects which distribution satisfied a "spanning"-type condition. To show that this is immaterial for the results, reconsider Example 1 except that presently the set of possible states of nature contains three elements. The output vector of the three types of firms changes into respectively

$$y_{1,1} = \begin{bmatrix} 100 \\ 1100 \\ 600 \end{bmatrix} ; \quad y_{1,2} = \begin{bmatrix} 1100 \\ 100 \\ 600 \end{bmatrix} ; \quad y_{1,3} = \begin{bmatrix} 200000 \\ 200000 \\ 200000 \end{bmatrix}$$

requiring the same respective inputs as before.

The corresponding market values of the firms are equal to: $v_1 = 715 = v_2$ and $v_3 = 180000$ . It is easy to check that market pricing is again subadditive. Presently the two groups of investors looked at possess respectively the utility functions:

$$U_1 = 40c_1 + 8c_2(1) - \frac{1}{20}(c_2(1))^2 + 24c_2(2) - \frac{1}{40}(c_2(2))^2 + 20c_2(3) - \frac{1}{110}(c_2(3))^2$$

$$U_2 = 40c_1 + 24c_2(1) - \frac{1}{40}(c_2(1))^2 + 8c_2(2) - \frac{1}{20}(c_2(2))^2 + 20c_2(3) - \frac{1}{110}(c_2(3))^2 .$$

Initial endowments are as before.

Some computational work shows that the optimal consumption pattern for both groups is respectively $c_1 = \begin{bmatrix} 80 \\ 60 \\ 160 \\ 110 \end{bmatrix}$ and $c_2 = \begin{bmatrix} 80 \\ 160 \\ 60 \\ 110 \end{bmatrix}$ with corresponding implicit price vectors: $\pi_1 = \begin{bmatrix} 0.05 \\ 0.4 \\ 0.45 \end{bmatrix}$ and $\pi_2 = \begin{bmatrix} 0.4 \\ 0.05 \\ 0.45 \end{bmatrix}$ . The individual utility levels amount to $U_1 = U_2 = 8790$ .

Again assume the type 3 firm proposes a project: $y_{1,3} = \begin{bmatrix} -30000 \\ +24000 \\ +30000 \end{bmatrix}$ , $\Delta y_{0,3} = 43600$ ; $\Delta v_3 = 21600$ . Hence, acceptance of the proposal would result in the firm's net present value to decrease with 22000 units. This case is not covered by the Makowski and Hart models. As it turns out that pricing satisfies the no direct consumption benefit condition, Theorem 1 predicts unanimous rejection of the proposal. After some laborous computations, it turns out that the new optimal consumption patterns for individuals in the respective groups are equal to $c_1' = \begin{bmatrix} 71.32 \\ 58.9 \\ 158.5 \\ 118.5 \end{bmatrix}$, $c_2' = \begin{bmatrix} 77.1 \\ 168.6 \\ 25.1 \\ 98.4 \end{bmatrix}$ yielding utility levels $U_1' = 8568.83$ and $U_2' = 8469.1$. Hence both classes are worse off.

Their new vectors of implicit prices are as follows:

$$\pi_1' = \begin{bmatrix} 0.05275 \\ 0.401875 \\ 0.4461363 \end{bmatrix} \text{ and } \pi_2' = \begin{bmatrix} 0.38925 \\ 0.1460381 \\ 0.4552727 \end{bmatrix}.$$

Although the project increases the space "spanned" by the firms' respective output vectors, it can be checked in the same way as before that returning to the preproject and hence smaller "spanned" space yields, next to a positive budget effect for all investors, also a consumption benefit.

## 5. Finite Economies and Actual Price Independence

The dominance approach, as developed in Section 3, presupposes that a firm, by its decisions, cannot affect market values other than its own. It would be interesting to investigate when this assumption will actually be satisfied: if the implicit properties needed for the system to meet such a condition are unknown, one doesn't really understand the results. This section can only be regarded as a first step into the direction of "better understanding the model," because it replaces one assumption implicitly presuming much structure by other ones having the same feature.

In the process however, some additional insight is gained, especially in the relationship between a finite and large economy. In particular, the next Theorem 2 which can also be regarded as a generalization of Makowski's Theorem 2 in [9], is the finite model counterpart of Hart's Theorems 1 and 2 in [8].

Theorem 2. Recall again Theorem 1's sequence of firm $f$'s feasible investment plans and consider in particular the two consecutive projects $\ell$ (the current one) and $\ell+1$.

Assume that pricing in response to firm $f$ switching to its policy $\ell+1$ is such that no asset offers a direct consumption benefit in comparison with the original situation. Suppose also that among the final shareholders of each firm there is an individual $i$, having nonempty $X_{i*}(y_{0,f}^{\ell+1}) \cap X_{i*}(y_{0,f}^{\ell})$. Then if assumptions (3) and (4) are met in addition, the economy displays actual price independence w.r.t. company $f$'s move from plan $\ell$ to $\ell+1$ and moreover, as the net present value associated with $\ell+1$ is below (not necessarily strictly) the one associated with $\ell$, all initial shareholders of firm $f$ prefer $\ell$, while all other investors are indifferent. These conclusions extend to any feasible change in $f$'s plan whenever for every two consecutive plans the preceding conditions are met.

Proof. The no direct consumption benefit condition implies that

$$(13) \qquad v_g(y_{0,f}^{\ell+1}) - v_g(y_{0,f}^{\ell}) \geqq \sum_e \pi_i(y_{0,f}^{\ell},e)\cdot(y_{1g}(y_{0,f}^{\ell+1}, e) - y_{1g}(y_{0,f}^{\ell}, e)) .$$

Since for all $g \neq f$, $y_{1g}(y_{0,f}^{\ell+1}) = y_{1g}(y_{0,f}^{\ell})$, (13) implies

(14) $$v_g(y_{0,f}^{\ell+1}) > v_g(y_{0,f}^{\ell}) \quad \text{for all} \quad g \neq f .$$

Consider a final shareholder in some company $g^* \neq f$ with

$$X_{*i}(y_{0,f}^{\ell+1}) \cap X_{*i}(y_{0,f}^{\ell}) \neq \phi .$$

This latter condition implies that for this individual $\pi_i(y_{0,f}^{\ell+1}) = \pi_i(y_{0,f}^{\ell})$ (see Makowski [9], p. 316). Hence, using equation (2), it follows that $v_{g*}(y_{0,f}^{\ell+1}) = \sum_e \pi_i(y_{0,f}^{\ell+1}, e) \cdot y_{1g*}(e) = \sum_e \pi_i(y_{0,f}^{\ell}, e) \cdot y_{1g*}(e) \leqq y_{g*}(y_{0,f}^{\ell})$ . Then, because of (14), obviously $v_{g*}(y_{0,f}^{\ell+1}) = v_{g*}(y_{0,f}^{\ell})$ . Since this is true for all $g^* \neq f$ , Theorem 1 is applicable and the statements concerning preferences over projects $\ell$ and $\ell+1$ hold.[7] □

Hence whenever no one reaps direct consumption benefits in response to a company changing its plans, some individuals are not affected directly or indirectly and trading is sufficiently broad, the finite economy displays actual price independence and in addition the market value rule holds in general.

The correspondences between the assumptions of this theorem and those of the large economy model are straightforward.

First, this model's final shareholder having a prechange optimal consumption pattern coinciding with a postchange optimal one, clearly is the finite economy's counterpart of the "typical" final shareholder. The only difference between the two versions is that, contrary to the large economy case, here it is necessary to assume a non affected investor exists. For, in [8] the presence of typical consumers directly follows from the

---

[7] Extension to several firms changing their input plans simultaneously is straightforward.

boundedness of endowments and plans (this is implicitly presumed in this research also) and a uniqueness assumption concerning exchange equilibria. The intuitive explanation is again based on the argument invoked over and over again in Section 2: as a single firm is small relative to the market, it seems unlikely that it would be able to significantly affect a large, in the limit, an infinite number of investors.

However, in the finite economy model, the proportion of investors whose initial endowment of time 0 good and time 1 claims remains untouched by firm f's decision, is generally not large enough to be able to infer from the uniqueness of equilibrium that necessarily most people's equilibrium consumption pattern remains unchanged also.[8] In other words, the uniqueness assumption does not provide any mileage in the small economy model. Nevertheless in both settings the role of the typical consumer among the final shareholders is the same: guarantee that "not being affected" spreads sufficiently through the new exchange equilibrium.

Second, in contrast to the Hart model, the finite economy case needs to assume the absence of direct consumption benefits. Intuitively, as it is small relative to the market, it seems very unlikely a company's price would fall since such a decrease is necessarily associated with the firm offering a nonzero direct consumption benefit to a large, in the limit, an infinite number of consumers. Hence as the economy grows, one would expect the no direct consumption benefit condition to become less and less stringent, until in the limit model, it holds automatically. More technically, the cause for the difference is the fact that, in the finite setting, the nonisolation condition at the current (prechange) equilibrium

[8] I.e. Hart's step 2, [8], p. 1080, in his proof of his first theorem does not hold in a small economy.

is useless. As the economy expands however, this assumption starts doing an increasing amount of work by guaranteeing that any type of traded claim of future income is distributed over more and more, in the limit an infinite number of investors. Once in this stage, previous analysis has shown that prices can only move upward (see Section 2).

# BIBLIOGRAPHY

[1] D. P. Baron, "On the Relationships between Complete and Incomplete Financial Market Models," International Economic Review (February 1979): 105-118.

[2] H. DeAngelo, "Competition and Unanimity," American Economic Review (March 1981): 18-27.

[3] S. Ekern and R. Wilson, "On the Theory of the Firm in Incomplete Markets," Bell Journal of Economics (Spring 1974): 171-180.

[4] E. Fama and A. Laffer, "The Number of Firms and Competition," American Economic Review (September 1972): 670-674.

[5] E. Fama and M. Miller, The Theory of Finance (Dryden Press, 1972).

[6] S. Grossman and O. Hart, "A Theory of Competitive Equilibrium in Stock Market Economies," Econometrica (March 1979).

[7] O. Hart, "Monopolistic Competition in a Large Economy with Differentiated Commodities," Review of Economic Studies (1979): 1-30.

[8] O. Hart, "On Shareholder Unanimity in Large Stock Market Economies," Econometrica (September 1979): 1057-1084.

[9] L. Makowski, "Competitive Stock Markets," Review of Economic Studies (1983): 305-330.

[10] L. Makowski, "Competition and Unanimity Revisited," American Economic Review (June 1983): 329-339.

[11] O. Mangasarian, Non Linear Programming (McGraw-Hill, 1969).

[12] K. Parthasarathy, Probability Measures on Metric Spaces (New York: Academic Press, 1967).

[13] R. Radner, "A Note on Unanimity of Stockholders' Preferences among Alternative Production Plans: A Reformulation of the Ekern-Wilson Model," Bell Journal of Economics, 5 (Spring 1974): 181-184.

[14] C. Van Hulle, "Shareholders' Unanimity and Optimal Corporate Decision Making in Imperfect Capital Markets," Unpublished Ph.D. dissertation, D.T.E.W., Leuven University, 1983.