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INFORMATION PATTERNS AND NASH EQUILIBRIA

IN EXTENSIVE GAMES

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by

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1. INTRODUCTION

In this paper we explore the relation between information patterns and Nash Equilibria in extensive games. By information we mean what players know about moves made by others, as well as by chance. For the most part we confine ourselves to pure strategies. But in Section 7 behavioral strategies are also examined. It turns out that they can be modeled as pure strategies of an appropriately enlarged game. Our results, applied to the enlarged game, can then be reinterpreted in terms of the behavioral strategies of the original game.

The extensive game model is of fundamental importance and captures the interplay between information and decision-making. Yet we find that its definition, as set forth by Kuhn in [5], is insufficient from certain points of view. It is unable to incorporate games with a continuum of players. Also it often makes for an unnaturally complex representation. For instance, a game in which n players move simultaneously can be described in the Kuhn framework. But first we would have to order the players artificially and then have them move in sequence with suitably

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enlarged information sets. If we try to carry this out when n is not finite but a continuum, the difficulty of the procedure becomes clear. Therefore we are led to develop a variant model which has the feature that several players can move simultaneously at any position in the game. Games of the type in [5] are, of course, included as a special case of our set-up.

In Section 2 we develop our model and illustrate it with an example. In the rest of the paper, we focus on the effect on Nash Equilibria (N.E.) that is caused solely by changes in the information pattern of an extensive game. In Section 3 we show that if information is refined, without increasing players' knowledge about chance moves,then the N.E.'s of the coarse game do not disappear. But the converse is not true: in general there is a rapid proliferation of new N.E.'s. In the next section, Section 4, we explore conditions under which this proliferation is arrested. The notion of "no informational influence" is introduced. It says that if a single player unilaterally changes his strategy, then the resultant new outcome tree does not pass through any other information set of the remaining players than the old one did. This is a purely set-theoretic condition and can hold not only in non-atomic, but also in finite, games--see the examples in Section 4. We prove that if it holds then a Nash outcome of the refined game is also that of its coarse form, i.e., is not a "new" N.E. brought about by the increased strategic (threat) possibilities. When we turn to non-atomic games, no informational influence holds in full force and we get: Nash outcomes are invariant of the information pattern (see Section 5). This leads to the "Anti-folk Theorem" in Section 6: the N.E.'s of a repeated game are precisely those which are N.E.'s in each stage.

2. EXTENSIVE GAMES IN SIMULTANEOUS MOVE FORM

2.1. Extensive Games: The Definition

An extensive game Γ in simultaneous move form is a seven-tuple:

$$\Gamma = (N \cup \{c\}, X, \pi, \{S^x\}_{x \in X}, \Phi, \{h_i\}_{i \in N}, \{I_i\}_{i \in N}) \ . \tag{2.1}$$

Let us explain our symbols. (Unless otherwise stated, all sets are assumed to be non-empty.)

(i) N is the set of all players, and c denotes chance $(c \notin N)$.

(ii) X is the set of all positions in the game, one of which, x_0 , is distinguished and represents the start of the game.

(iii) π maps X to $2^N \cup \{c\}$. If $\pi(x)$ is a non-empty subset of N then it denotes the set of players who move simultaneously at the position x . If $\pi(x) = \{c\}$ then chance moves at x . (Note that players and chance never move together.) Finally if $\pi(x) = \emptyset$ then x is an ending position of the game. It will be convenient to partition X into the three sets:

$$X_N = \{x \in X : \pi(x) \subset N, \pi(x) \neq \emptyset\}$$

$$X_c = \{x \in X : \pi(x) = \{c\}\}$$

$$X_E = \{x \in X : \pi(x) = \emptyset\}$$

(Note that X_c and X_E may be empty.)

(iv) For each $x \in X$, S^x is a set of functions from $\pi(x)$ to some set Y^x . We assume that $\pi(x) = \emptyset \Longleftrightarrow S^x = \emptyset$. Given $s^x \in S^x$, $t \in Y^x$ and $i \in \pi(x)$, denote by (s^x_{-i}, t) the function from $\pi(x)$ to Y^x which assigns t to i , and agrees with s^x elsewhere. Also let s^x_i stand for $s^x(i)$. Our assumption on S^x is:

(2.2) if $s^x, r^x \in S^x$, then $(s^x_{-i}, r^x_i) \in S^x$ for all $i \in \pi(x)$.

Define $S^x_i = \{s^x_i : s^x \in S^x\}$ for $i \in \pi(x)$. Note that, by (2.2),

(2.3) if $t \in S^x_i$ and $s^x \in S^x$, then $(s^x_{-i}, t) \in S^x$

and, by (2.3),

(2.4) if $\pi(x)$ is finite, $S^x = \prod_{i \in \pi(x)} S^x_i$ (where Π denotes Cartesian product).

S^x_i is <u>the set of moves available to player</u> i <u>at the position</u> x and S^x is the set of <u>move selections</u> at x by the players in $\pi(x)$ that are feasible in the game.

(v) Φ links positions to moves. Put $X^* = X \setminus \{x_0\}$. Let F be the collection of all finite sequences $(s^{x_0}, s^{x_1}, \ldots, s^{x_m})$ with $s^{x_k} \in S^{x_k}$ for $k = 0, 1, \ldots, m$. Then Φ is a one-to-one mapping,

$$\Phi : X^* \to F ,$$

such that:

(a) if $s^{x_0} \in S^{x_0}$, then $(s^{x_0}) \in \Phi(X^*)$;

(b) if $(s^{x_0}, s^{x_1}, \ldots, s^{x_{m-1}}, s^{x_m}) \in \Phi(X^*)$, then $(s^{x_0}, s^{x_1}, \ldots, s^{x_{m-1}}) = \Phi(x_m)$;

(c) if $\Phi(x) = (s^{x_0}, \ldots, s^{x_m})$ and $s^x \in S^x$, then $(s^{x_0}, \ldots, s^{x_m}, s^x) \in \Phi(X^*)$.

Since Φ is one-to-one, we will sometimes <u>identify</u> x <u>with</u>

$\Phi(x)$, and say that $x = (s^{x_0}, \ldots, s^{x_m})$ for $x \in X^*$. This should cause no confusion. Thus, if $\Phi(x) = (s^{x_0}, \ldots, s^{x_m})$, we will write (x, s^x) for $(s^{x_0}, \ldots, s^{x_m}, s^x)$, etc.

To describe the rest of the game we need to develop some more terminology. If $(x, s^x) = y$ for some $s^x \in S^x$ then we will say that y <u>immediately follows</u> x and write $x \underset{I}{\prec} y$. If there exist $x_1, \ldots, x_m$ in X such that $x \underset{I}{\prec} x_1 \underset{I}{\prec} \cdots \underset{I}{\prec} x_m \underset{I}{\prec} y$ then y <u>follows</u> x (or x <u>precedes</u> y), and we write $x \prec y$. (Then $\prec$ is a partial order on X with x_0 as its unique minimal element. Also note that for any $x \in X^*$ the set of all predecessors of x form a path, under $\prec$, from x_0 to x ; and $\Phi(x)$ lists them sequentially, along with the moves selected at each position that lead from it to its immediate follower.) Continuing with our definition

(vi) A--possibly finite--sequence $\{y_0, \ldots, y_k, \ldots\}$ is called a <u>play</u> if (a) $y_0 = x_0$

(b) $y_k \underset{I}{\prec} y_{k+1}$ for all k

(c) $y_\ell \in X_E$ if y_ℓ is the last element of the sequence.

(vii) A union of plays $\lambda = \bigcup_{\alpha \in A} p_\alpha$ is said to be an <u>outcome tree</u> (or, more simply, an <u>outcome</u>) if

(2.5) for $x \in \lambda \setminus X_E$,

$$\{s^x \in S^x : (x, s^x) \in \lambda\} = \begin{cases} S^x & \text{if } \pi(x) = \{c\} \\ \text{a singleton set otherwise.} \end{cases}$$

The set of all outcomes depends upon the five-tuple $\Gamma_- = (N, X, \pi, \{S^x\}_{x \in X}, \Phi)$ and will be denoted $\Lambda(\Gamma_-)$. Each h_i is simply a real-valued function on $\Lambda(\Gamma_-)$ and gives the

<u>payoff</u> to player i for any outcome of the game. (Normally chance moves have probabilities attached to them, with the result that λ gives rise to a probability distribution on plays in Γ. Then h_i is taken to be the expectation of a payoff defined <u>a priori</u> on plays. See Section 7.)

(viii) I_i is a partition of $X_i = \{x \in X : i \in \pi(x)\}$ and is called the <u>information partition of player</u> i. If x and y are two positions in the same set of player i's partition $(x, y \in u \in I_i)$, then this means that i cannot distinguish between x and y. It is natural to impose some constraints on the $\{I_i\}_{i\in N}$ in view of this interpretation. First

(2.6) $$\text{if } x, y \in u \in I_i, \text{ then } S_i^x = S_i^y .$$

If this were not so, then i could distinguish intrinsically between x and y. Given (2.6) we will, without confusion, talk of the set of moves S_i^u which is available to i at (each position in) his information set u. Next we assume

(2.7) No play passes through an information set more than once, i.e., for any play p and any information set u we must have $|p \cap u| \leq 1$, where $|\ |$ denotes cardinality. (For a discussion of (2.7) see Remark 1.)

This completes the definition of the game Γ.

Remarks

(1) The condition (2.7) follows from the assumption of perfect recall (see [5]). Take y in X, $\Phi(y) = (s^{x_0}, \ldots, s^{x_m})$. Then if $i \in \pi(x_\ell)$ for some $0 \leq \ell \leq m$, $s_i^{x_\ell} = t$, and $x_\ell \in v \in I_i$, we will say that "y follows from v via the move t of player i," and denote this by $(v,t) \underset{i}{\prec} y$. The perfect recall assumption may now be stated:

$$\text{(2.8)} \qquad \left\{ \begin{array}{c} x,\ y \in u \in I_i \\ (v,t) \underset{i}{\prec} x \end{array} \right\} \Rightarrow (v,t) \underset{i}{\prec} y \ .$$

It says that at any position each player can fully recall the entire history of his previous information and moves. Technically we need only the weaker condition (2.7), though we feel that it is more natural to postulate perfect recall. Indeed when we restrict ourselves in Section 7 to behavioral--rather than mixed--strategies, then (2.8) is implicitly assumed. For then, by Kuhn's theorem in [5], behavioral strategies suffice for the analysis of N.E.'s.

2.2. An Example

Consider:

$N = \{1,2,3,4\}$; $X = \{x_0, x_1, \ldots, x_{26}\}$;

$\pi(x_0) = \{1,2\}$, $\pi(x_1) = \{c\}$, $\pi(x_t) = \{3,4\}$ for $t = 2, \ldots, 6$,

and $\pi(x_t) = \emptyset$ for $t = 7, \ldots, 26$;

$S^{x_0} = \{(\alpha_1, \beta_1), (\alpha_1, \beta_2), (\alpha_2, \beta_1), (\alpha_2, \beta_2)\}$, i.e.,

$S_1^{x_0} = \{\alpha_1, \alpha_2\}$ and $S_2^{x_0} = \{\beta_1, \beta_2\}$;

$$S^{x_1} = \{c_1,\ c_2\}\ ,$$

$$S^{x_t} = \{(\gamma_1,\ \delta_1),\ (\gamma_1,\ \delta_2),\ (\gamma_2,\ \delta_1),\ (\gamma_2,\ \delta_2)\}\ ,\ \text{ i.e.,}$$

$$S_3^{x_t} = \{\gamma_1,\ \gamma_2\} \text{ and } S_4^{x_t} = \{\delta_1,\ \delta_2\} \text{ for } t = 2, \ldots, 6\ ;$$

$$S^{x_t} = \emptyset \text{ for } 7, \ldots, 26\ ;$$

$$\Phi(x_2) = (\alpha_1,\ \beta_1)\ ,\ \ \Phi(x_3) = ((\alpha_1,\ \beta_2),\ c_1)\ ,$$

$$\Phi(x_4) = ((\alpha_1,\ \beta_2),\ c_2)\ ,\ \ \Phi(x_5) = (\alpha_2,\ \beta_1)\ ,\ \text{ etc.;}$$

$$I_1 = I_2 = \{\{x_0\}\}\ ,\ \ I_3 = \{\{x_2,\ x_3\},\ \{x_4,\ x_5,\ x_6\}\} \text{ and}$$

$$I_4 = \{\{x_2,\ x_3,\ x_4,\ x_5\},\ \{x_6\}\}\ .$$

The tree of this game is:

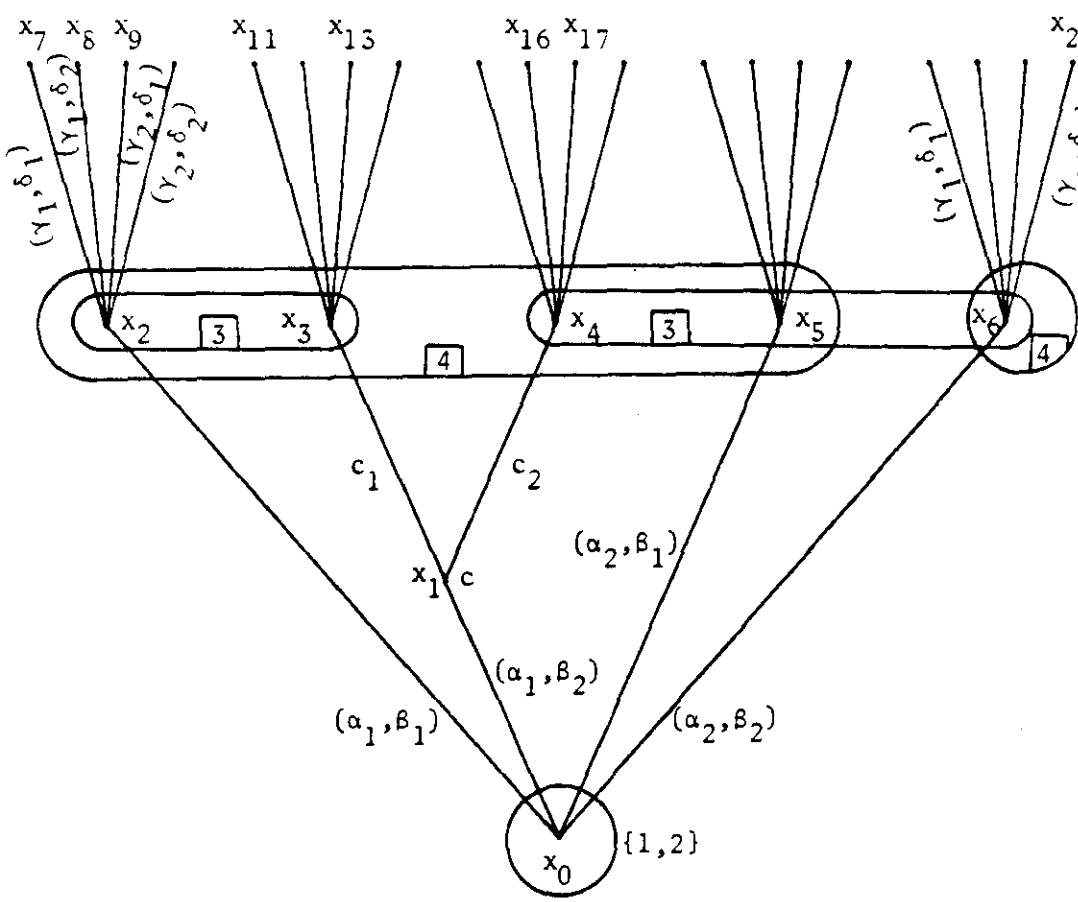

FIGURE 1

$\{\alpha_1, \alpha_2\}$, $\{\beta_1, \beta_2\}$, $\{\gamma_1, \gamma_2\}$, $\{\delta_1, \delta_2\}$ are the moves of players 1, 2, 3, 4.

$\{x_0, x_2, x_7\}$, $\{x_0, x_1, x_3, x_{11}, x_4, x_{17}\}$ are in $\Lambda(\Gamma_-)$.

2.3. Nash Equilibria

Fix a game $\Gamma = (N \cup \{c\}, X, \pi, \{S^x\}_{x \in X}, \Phi, \{h_i\}_{i \in N}, \{I_i\}_{i \in N})$ as in Section 2.1. The strategy-set of player i in Γ is made up of all possible choices of moves available to him in X_i under the proviso that he must make the same move at positions that are indistinguishable in his information set. It is the set $\Sigma_i(\Gamma)$ consisting of all maps σ_i from X_i to $\bigcup_{x \in X_i} S_i^x$ which satisfy

(i) $\sigma_i(x) \in S_i^x$

(ii) $\sigma_i(x) = \sigma_i(y)$ if $x, y \in u \in I_i$.

Thus we can also think of σ_i as a map $\sigma_i : I_i \to \bigcup_{x \in X_i} S_i^x$; and, without confusion, σ_i will be used in both senses. Given a strategy-choice $\sigma = \{\sigma_i\}_{i \in N}$, where each $\sigma_i \in \Sigma_i(\Gamma)$, abbreviate $\{\sigma_i(x)\}_{i \in \pi(x)}$ by $\sigma(x)$ for $x \in X_N$. Then σ is called feasible if

$$\sigma(x) \in S^x \quad \text{for all} \quad x \in X_N . \tag{2.9}$$

Let $\Sigma(\Gamma)$ be the set of all feasible strategy-choices in Γ . We assume throughout that $\Sigma(\Gamma)$ is not empty (see Remark 2). Define the strategy-to-outcome map $\xi : \Sigma(\Gamma) \to \Lambda(\Gamma_-)$ by:

$$\xi(\sigma) = \{x \in X : \text{if } \Phi(x) = (r^{x_0}, \ldots, r^{x_m}),\ x^\ell \in X_N, \text{ and } 0 \leq \ell \leq m, \text{ then } r^{x_\ell} = \sigma(x_\ell)\} . \tag{2.10}$$

(Note that chance always picks all of its moves in $\xi(\sigma)$.) From (2.5) it easily follows that $\xi(\sigma)$ is indeed an outcome in $\Lambda(\Gamma_{-})$. But not everything in $\Lambda(\Gamma_{-})$ need be achieved by strategies in $\Sigma(\Gamma)$. It will be useful to define $\Lambda(\Gamma) = \xi(\Sigma(\Gamma))$, the <u>set of outcomes that are feasible in</u> Γ . (In the example of Section 2.2, the outcome $\{x_0, x_1, x_3, x_{13}, x_4, x_{16}\}$ is in $\Lambda(\Gamma_{-})$ but not in $\Gamma(\Lambda)$.)

Next, given $\sigma \in \Sigma(\Gamma)$ and $\tau_i \in \Sigma_i(\Gamma)$, let $(\sigma|\tau_i)$ be the same as σ but with σ_i replaced by τ_i . By (2.3), $(\sigma|\tau_i)$ is also in $\Sigma(\Gamma)$, and therefore our next definition makes sense. The strategy-choice $\sigma \in \Sigma(\Gamma)$ is called a <u>Nash Equilibrium</u> (N.E.) of Γ if, for all $i \in N$:

(2.11) $$h_i(\xi(\sigma|\tau_i)) \leq h_i(\xi(\sigma)) \quad \text{for all} \quad \tau_i \in \Sigma_i(\Gamma) .$$

The outcome $\xi(\sigma)$ produced by an N.E. σ of Γ will be called a <u>Nash outcome</u> of Γ .

<u>Remarks</u>.

(2) If N is finite then, by (2.4), it follows that $\Sigma(\Gamma)$ is non-empty. But in our general set-up we have made no connection between the information sets and the feasibility condition (2.9), so it is not possible to deduce that $\Sigma(\Gamma)$ is non-empty. We find it more economical to assume non-emptiness here rather than to seek the extra conditions that will imply it.

3. PRESERVATION OF NASH OUTCOMES

We will focus on the effect on Nash outcomes that is caused solely by changes in the information pattern of the extensive game. For this purpose we take a pair of games Γ , Γ^* which are identical except for their information patterns $\{I_i\}_{i\in N}$, $\{I_i^*\}_{i\in N}$. Our sharpest result is in the case when N is finite, though many of its corollaries continue to hold in general. We therefore break up this section into two parts.

3.1. The Finite-Player Case

For simplicity, denote $\Sigma_i(\Gamma)$, $\Sigma(\Gamma)$, $\Sigma_i(\Gamma^*)$, $\Sigma(\Gamma^*)$ by Σ_i , Σ , Σ_i^* , Σ^* . For $\sigma \in \Sigma$, define:

$$R_i(\sigma) = \left\{x \in X_i \; : \; x \in \xi(\sigma|\tau_j) \text{ for some } \tau_j \in \Sigma_j \text{ and } j \in N\setminus\{i\}\right\}$$

i.e., $R_i(\sigma)$ is the set of positions that are reachable in Γ via unilateral deviations from σ by players in $N\setminus\{i\}$. Also define the sets $I_i(x)$, $I_i^*(x)$ to be the (unique) sets in I_i , I_i^* that contain x . (If $x \notin X_i$ then $I_i(x)$ is understood to be the empty set.)

Proposition 1. Assume

(i) N is finite

(ii) σ is an N.E. of Γ

(iii) For all i in N :

(a) $$\left.\begin{array}{l} x, y \in R_i(\sigma) \\ I_i(x) \neq I_i(y) \end{array}\right\} \Rightarrow I_i^*(x) \neq I_i^*(y)$$

(b) $$\left.\begin{array}{l} x,y \in \xi(\sigma|\tau_i) \cap X_i \\ \quad \text{for some } \tau_i \in \Sigma_i \\ I_i(x) = I_i(y) \end{array}\right\} \Rightarrow I_i^*(x) = I_i^*(y)$$

Then there is an N.E. σ^* of Γ^* such that $\xi^*(\sigma^*) = \xi(\sigma)$. (Here ξ^* , ξ are the strategy-to-outcome maps in Γ^* , Γ).

Proof. For any $i \in N$, put

$$A_i^* = \{I_i^*(x) \ : \ x \in R_i(\sigma)\}$$

$$B_i^* = I_i^* \setminus A_i^* .$$

Fix $v^* \in A_i^*$. Then, by (iii)(a),

$$x,\ y \in v^* \cap R_i(\sigma) \Rightarrow I_i(x) = I_i(y) . \tag{3.1}$$

Therefore we can define the map $\psi_i : A_i^* \to I_i$ by:

$$\psi_i(v^*) = I_i(x) \quad \text{for any} \quad x \in v^* \cap R_i(\sigma) .$$

Now construct $\sigma^* = \{\sigma_i^*\}_{i\in N}$ by:

$$\sigma_i^*(v^*) = \begin{cases} \sigma_i(\psi_i(v^*)) & \text{if} \quad v^* \in A_i^* \\ \text{arbitrary if} & v \in B_i^* . \end{cases} \tag{3.2}$$

By (i) and (2.4)

$$\sigma^* \in \Sigma^* . \tag{3.3}$$

Step 1. $\xi^*(\sigma^*) = \xi(\sigma)$.

Since $\xi^*(\sigma^*)$ and $\xi(\sigma)$ are outcome trees, neither $\xi^*(\sigma^*) \subsetneqq \xi(\sigma)$ nor $\xi(\sigma) \subsetneqq \xi^*(\sigma^*)$ is possible. Therefore, if $\xi^*(\sigma^*) \neq \xi(\sigma)$, there are plays p^* and p , with:

$$p^* \subset \xi^*(\sigma^*) \setminus \xi(\sigma)$$

$$p \subset \xi(\sigma) \setminus \xi^*(\sigma^*) .$$

Let x be the first position on p, starting from the root x_0, which is not on p^*, i.e., $x \equiv \Phi(x) = (s^{x_0}, \ldots, s^{x_m})$ has the property:

$$x \in p \setminus p^*$$

$$\{x_0, \ldots, x_m\} \subset p \cap p^* .$$

Clearly $\pi(x^m) \neq \emptyset$. If $\pi(x^m) = \{c\}$., then $x = (x^m, s^{x_m}) \in p \cap p^*$ by (2.5), a contradiction. If $\pi(x^m) \neq \{c\}$, then since $x^m \in R_i(\sigma)$ for all $i \in N$, we have, by (3.2),

$$\sigma_i^*(x^m) = \sigma_i(x^m) \quad \text{for all} \quad i \in \pi(x^m) ,$$

so $x \in p \cap p^*$, again a contradiction. This verifies Step 1. The Proposition will now follow from

<u>Step 2</u>. For any τ_j^* in Σ_j^* there exists a τ_j in Σ_j such that $\xi(\sigma|\tau_j) = \xi^*(\sigma^*|\tau_j^*)$.

Define $f_j : \xi^*(\sigma^*|\tau_j^*) \cap X_j \to \bigcup_{x \in \xi^*(\sigma^*|\tau_j^*) \cap X_j} S_j^x$ by

$$f_j(x) = \tau_j^*(x) . \tag{3.4}$$

We claim that for any $x, y \in \xi^*(\sigma^*|\tau_j^*)$

(3.5) If $x \in \xi^*(\sigma^*|\tau_j^*) \cap X_i$ and $i \in N \setminus \{j\}$, then $\sigma_i^*(x) = \sigma_i(x)$.

(3.6) If $x, y \in \xi^*(\sigma^*|\tau_j^*) \cap X_j$ and $I_j(x) = I_j(y)$, then $f_j(x) = f_j(y)$.

We shall establish (3.5) and (3.6) by induction. If $x = (s^{x_0}, \ldots, s^{x_\ell})$, say that the length of x from x_0 is $\ell+1$. Put:

$$X^\ell = \{x \in X : \text{the length of } x \text{ from } x_0 \text{ is} \leq \ell\},$$

$$X_k^\ell = X_k \cap X^\ell \quad \text{for all} \quad k \in N.$$

Denote by $(3.5)^\ell$, $(3.6)^\ell$ the statements (3.5), (3.6) but with X_i, X_j replaced by X_i^ℓ, X_j^ℓ respectively. Observe that $(3.5)^0$ and $(3.6)^0$ are trivially true. So it suffices to show that

$$(*) \qquad (3.5)^\ell \text{ and } (3.6)^\ell \Rightarrow (3.5)^{\ell+1} \text{ and } (3.6)^{\ell+1}.$$

Let f_j^ℓ denote the restriction of f_j to $\xi^*(\sigma^*|\tau_j^*) \cap X_j^\ell$. By $(3.6)^\ell$ there is an extension of f_j^ℓ to a strategy $\overline{f}_j^\ell$ in Σ_j. Then, by $(3.5)^\ell$, (3.4) and the definitions of $\overline{f}_j^\ell$, ξ^* and ξ:

$$(3.7) \qquad \xi^*(\sigma^*|\tau_j^*) \cap X^{\ell+1} = \xi(\sigma|\overline{f}_j^\ell) \cap X^{\ell+1}.$$

By (3.7), it follows that

$$(3.8) \qquad \xi^*(\sigma^*|\tau_j^*) \cap X_i^{\ell+1} \subset R_i(\sigma) \quad \text{if} \quad i \in N \setminus \{j\}.$$

By (3.8) and (3.2), we get

$$\sigma_i^*(x) = \sigma_i(x) \quad \text{if} \quad x \in \xi^*(\sigma^*|\tau_j^*) \cap X_i^{\ell+1} \quad \text{and} \quad i \in N \setminus \{j\},$$

proving $(3.5)^{\ell+1}$. Next take $x, y \in \xi^*(\sigma^*|\tau_j^*) \cap X_j^{\ell+1}$ with $I_j(x) = I_j(y)$. By (3.7) and (iii)(b), $I^*(x) = I^*(y)$. Therefore, by (3.4), $f_j(x) = f_j(y)$, proving $(3.6)^{\ell+1}$. This establishes $(*)$, and thereby

(3.5) and (3.6). From (3.6) we see that f_j can be extended to a strategy τ_j in Σ_j . By (3.4) and (3.5), $\xi^*(\sigma^*|\tau_j^*) = \xi(\sigma|\tau_j)$. This verifies Step 2.

Q.E.D.

An Example

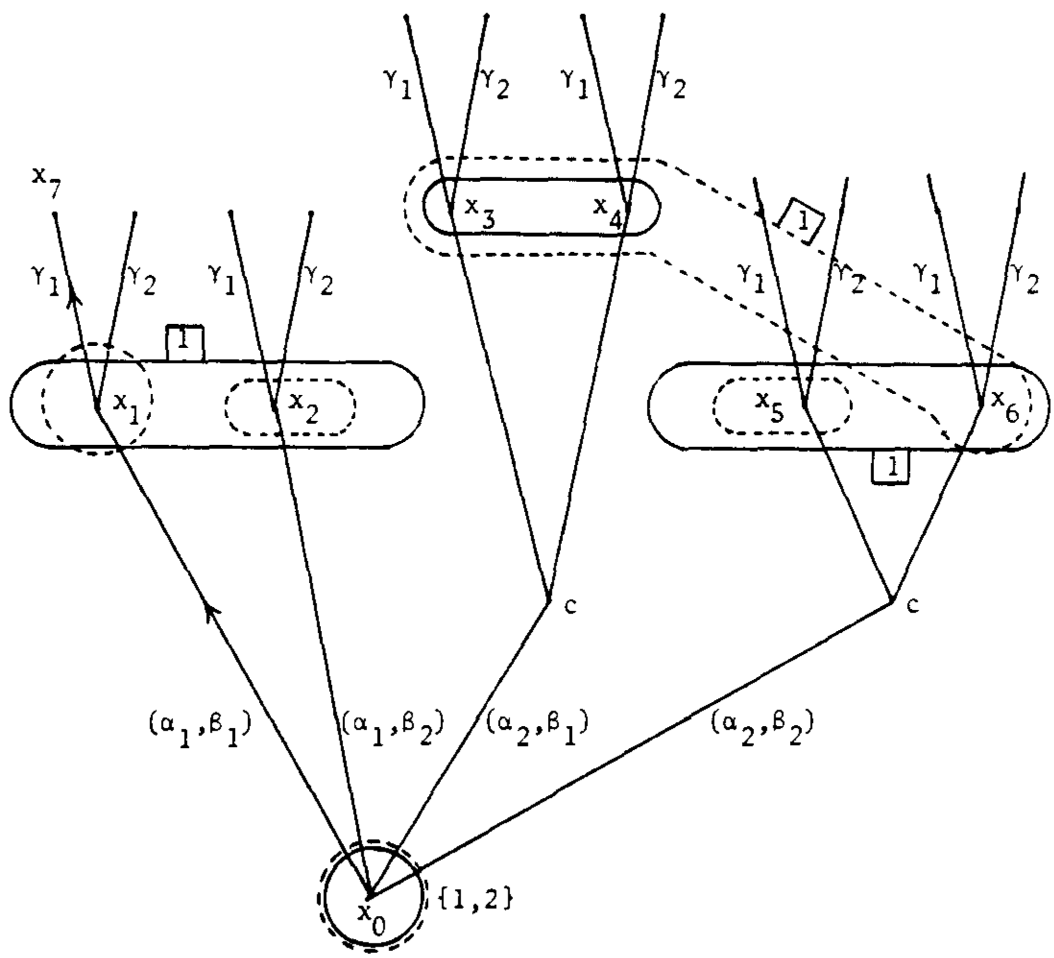

FIGURE 2

Here

$X_2 = \{x_0\}$

$X_1 = \{x_0, \ldots, x_6\}$

Γ , Γ^* have the solid, broken information patterns

α_1 , α_2 , γ_1 , γ_2 are moves of player 1

β_1 , β_2 are moves of player 2

$\xi(\sigma) = \{x_0, x_1, x_7\}$ = play marked with arrows.

$R_1(\sigma) = \{x_0, x_1, x_2\}$

$R_2(\sigma) = \{x_0\}$

$\xi(\sigma|\tau_2) \cap X_2 = \{x_0\}$ for k = 1, 2

$\xi(\sigma|\tau_1) \cap X_1 = \{x_0, x_1\}$ or $\{x_0, x_3, x_4\}$.

(σ is any strategy-choice that leads to the marked play $\xi(\sigma)$; and τ_1 , τ_2 range over all strategies of players 1, 2.)

It can now easily be checked that (iii)(a), (iii)(b) hold for Γ , Γ^* , σ . Thus if $\xi(\sigma)$ is Nash in Γ , it will also be Nash in Γ^* .

Remarks

(3) Observe that, by (2.7),

(3.9) No chance moves in the game $\Rightarrow$ (iii)(b) automatically holds.

Thus (iii)(b) says that there is no informational gain regarding chance moves in going from Γ to Γ^* at σ . However, this needs to be true for player i only under his own unilateral deviations.

(4) Say $\Gamma \rightarrowtail \Gamma^*$ if each I_i^* is a *refinement* of I_i, for all $i \in N$. Then

(3.10) $\quad \Gamma \rightarrowtail \Gamma^* \Rightarrow$ (iii)(a) automatically holds.

We can think of (iii)(a) as a weakening of $\Gamma \rightarrowtail \Gamma^*$. It requires that, in the region reached by *others'* unilateral deviations, there is *no informational loss* in going from Γ to Γ^* at σ.

(5) The scope of Proposition 1 will become clear later since *many of the propositions that follow will be its simple corollaries when* N *is finite*. Let us point out one such immediately. For any game Γ let $\eta(\Gamma)$ denote the *set of all its Nash outcomes*. They, by (3.9) and (3.10), we have

$$(3.11) \qquad \left\{ \begin{array}{l} N \text{ finite} \\ \Gamma \rightarrowtail \Gamma^* \\ \text{either (a) no chance moves} \\ \qquad \text{or (b) (iii)(b) holds at} \\ \qquad\qquad \text{each } \sigma \text{ in } \Sigma \end{array} \right\} \Rightarrow \eta(N) \subset \eta(\Gamma^*) .$$

(This, in the case of condition (a), is essentially the Proposition in [2].)

(6) The preceding remark leads one to investigate the possibility of Proposition 1 for the general N case. The difficulty arises in deducing (3.3) from (3.2). One would need to make more measurability-type assumptions on the structure of the game to overcome this difficulty. For instance consider:

$$(**) \qquad \left.\begin{array}{l} X' \subset X \\ \mathcal{B}_i = \{I_i(x) : x \in X' \cap X_i\} \ , \quad i \in N \\ y \in X \end{array}\right\} \Rightarrow \begin{array}{c} \{i \in \pi(y) : I_i(y) \in \mathcal{B}_i\} \\ \text{is in } \mathcal{C} \end{array}$$

Here $\mathcal{C}$ is an algebra of subsets of N which includes all singleton sets. Also require:

$$\pi(x) \in \mathcal{C} \text{ for all } x \in X_N ;$$

$$\left.\begin{array}{l} T \subset \pi(x) \\ T \in \mathcal{C} \\ s^x, r^x \in S^x \end{array}\right\} \Rightarrow \text{the combination } (s^x_T, r^x_{\pi(x)\setminus T}) \in S^x .$$

Finally enlarge $R_i(\sigma)$ in (iii)(a) to include positions reached by player i's own deviations. Then all these conditions together enable us to go from (3.2) to (3.3), and yield Proposition 1 for general N. Possibly (**) can be deduced from more elementary assumptions on the tree, though we have not explored this.

3.2. Nestedness of Nash Equilibria under Refinement

We now prove (3.11) without the assumption that N is finite. First note that if $\Gamma \dashv \Gamma^*$ there is a natural sense in which $\Sigma_i \subset \Sigma_i^*$: simply identify $\sigma_i \in \Sigma_i$ with $\sigma_i^* \in \Sigma_i^*$ where $\sigma_i^*(x) = \sigma_i(x)$ for all $x \in X_i$.

Proposition 2.1. Assume

(i) $\Gamma \dashv \Gamma^*$

(ii) σ is an N.E. of Γ

(iii) condition (iii)(b) of Proposition 1 holds at $\sigma \in \Sigma$.

Then σ is an N.E. of Γ^* .

Proof. Set $\sigma^* = \sigma$ and repeat, mutatis mutandis, the argument in Steps 1 and 2 of the proof of Proposition 1.

Q.E.D.

As an immediate corollary we get a global version of Proposition 2.1:

Proposition 2.2. Assume

(i) $\Gamma \twoheadrightarrow \Gamma^*$

(ii) condition (iii)(b) of Proposition 1 holds for every $\sigma \in \Sigma$.

Then $\eta(\Gamma) \subset \eta(\Gamma^*)$.

(Note: if there are no chance moves, then (iii), (ii) of Propositions 2.1, 2.2 automatically hold.)

Proposition 2.2 shows that, if we refine information and if there are no chance moves (or else (ii) holds), then the N.E.'s of the coarse game are not lost. But there is no dearth of examples to convince one that, more often than not, there is a rapid proliferation of new N.E.'s. Consider the three games Γ_1 , Γ_2 , Γ_3 with the information patterns given below. The payoffs are given in Figure 3.

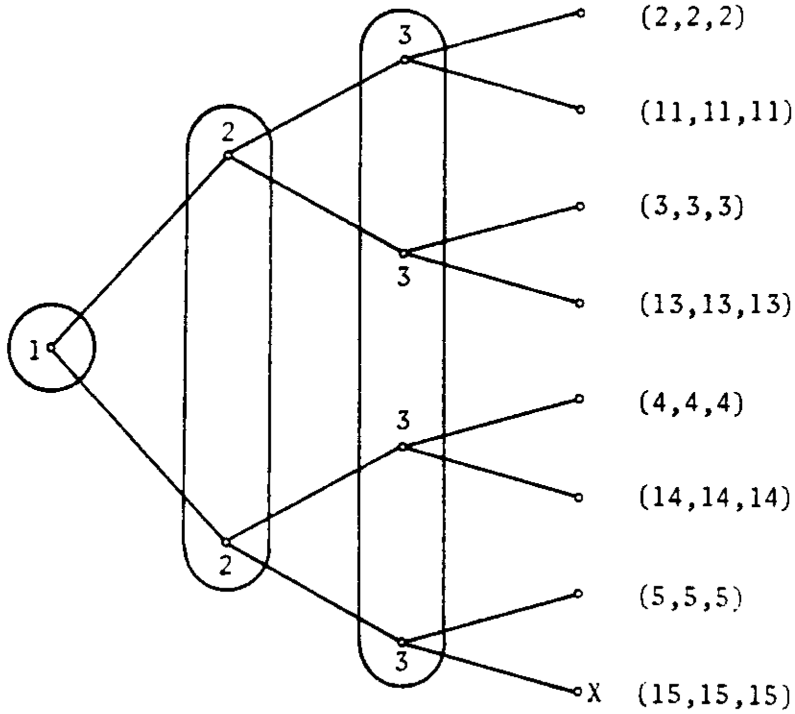

FIGURE 3. The Game Γ_1

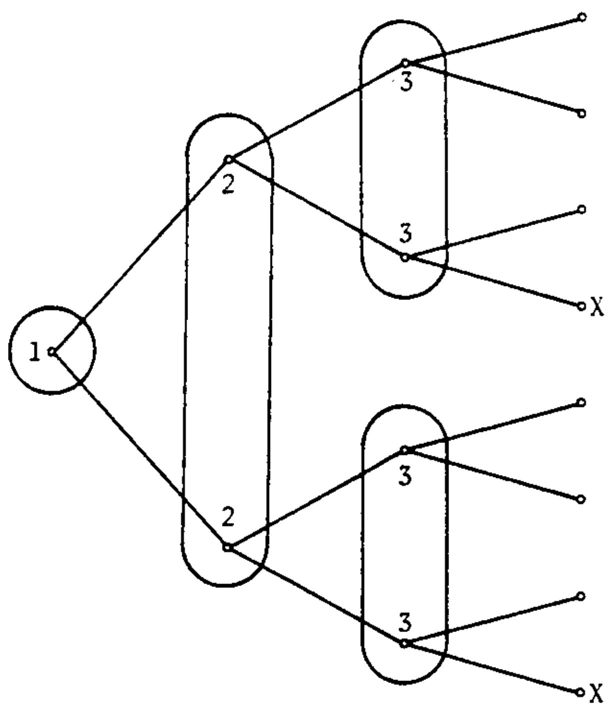

FIGURE 4. The Game Γ_2

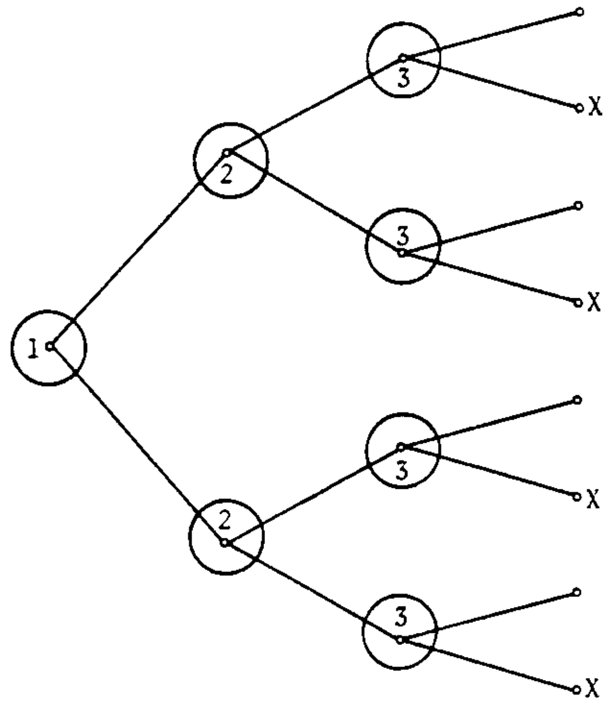

FIGURE 5. The Game Γ_3

The Nash plays in each case are marked by X . Those of Γ_ℓ are preserved in $\Gamma_{\ell+1}$ $(\ell = 1, 2)$ in accordance with Proposition 2.2.

4. NO INFORMATIONAL INFLUENCE

We are interested in investigating conditions under which this proliferation of Nash plays is arrested. The next proposition makes an advance in that direction, and constitutes a partial converse to Proposition 2.1. For $\tilde{\sigma} \in \Sigma^*$ define $D_j(\tilde{\sigma}) \subset I_j^*$ by:

$$(4.1) \qquad D_j(\tilde{\sigma}) = \{I_j^*(x) : x \in \xi^*(\tilde{\sigma}) \cap X_j\} ,$$

i.e., $D_j(\tilde{\sigma})$ is the collection of i's information sets through which the tree $\xi^*(\tilde{\sigma})$ passes. We say that i has <u>no informational influence on</u> j <u>at</u> σ^* <u>in</u> Γ^* if

$$(4.2) \qquad D_j(\sigma^*) \supset D_j(\sigma^*|\tau_i^*) \quad \text{for all } \tau_i^* \in \Sigma_i^* .$$

<u>Proposition 3.</u> Assume

(i) $\Gamma \dashv \Gamma^*$

(ii) σ^* is an N.E. of Γ^*

(iii) each player has no informational influence (on every other player) at σ^* in Γ^*

(iv) $\xi^*(\sigma^*) \in A(\Gamma)$, i.e., $\xi^*(\sigma^*)$ is feasible in Γ .

Then there is an N.E. σ of Γ such that $\xi(\sigma) = \xi^*(\sigma^*)$. (Indeed every $\sigma \in \Sigma(\Gamma)$, for which $\xi(\sigma) = \xi^*(\sigma^*)$, is an N.E. of Γ .)

<u>Proof.</u> By (iv) there is a σ in Σ such that $\xi(\sigma) = \xi^*(\sigma^*)$. Then it must be that for all $i \in N$:

(4.3) $$\sigma_i(x) = \sigma_i^*(x) \quad \text{if} \quad I_i^*(x) \in D_i(\sigma^*) \ .$$

Take any $\tau_j \in \Sigma_j$. Since $\Gamma \dashv \Gamma^*$ we can define $\tau_j^* \in \Sigma_j^*$ by

(4.4) $$\tau_j^*(x) = \tau_j(x) \quad \text{for} \quad x \in X_j \ .$$

The Proposition will follow if we can show that: $\xi^*(\sigma^*|\tau_j^*) = \xi(\sigma|\tau_j)$. If $\neq$ holds, then there is some $x \equiv \Phi(x) = (s^{x_0}, \ldots, s^{x_m})$ such that

(4.5) $$x \in \xi(\sigma|\tau_j) \quad \text{and} \quad x \notin \xi^*(\sigma^*|\tau_j^*)$$

and

(4.6) $$x_0, \ldots, x_m \in \xi(\sigma|\tau_j) \cap \xi^*(\sigma^*|\tau_j^*) \ .$$

Clearly, by (4.5) and (2.5), $\pi(x_m) \neq \{c\}$. By (4.6) and (iii), $I_i^*(x_m) \in D_i(\sigma^*)$ for all $i \in \pi(x_m) \setminus \{j\}$. Therefore, by (4.3),

(4.7) $$\sigma_i(x_m) = \sigma_i^*(x_m) \quad \text{if} \quad i \in \pi(x_m) \setminus \{j\} \ ;$$

and, by (4.4),

$$(4.8) \qquad \tau_j(x_m) = \tau_j^*(x_m) \quad \text{if} \quad j \in \pi(x_m) \ .$$

By (4.7) and (4.8),

$$(4.9) \qquad (\sigma^*|\tau_j^*)(x_m) = (\sigma|\tau_j)(x_m) \ .$$

By (4.6) and (4.9), the position $(x_m, (\sigma^*|\tau_j^*)(x_m)) = x$ and is in $\xi^*(\sigma^*|\tau_j^*) \cap \xi(\sigma|\tau_j)$, contradicting (4.5).

Q.E.D.

To clarify (4.2) consider the games in Figure 6.

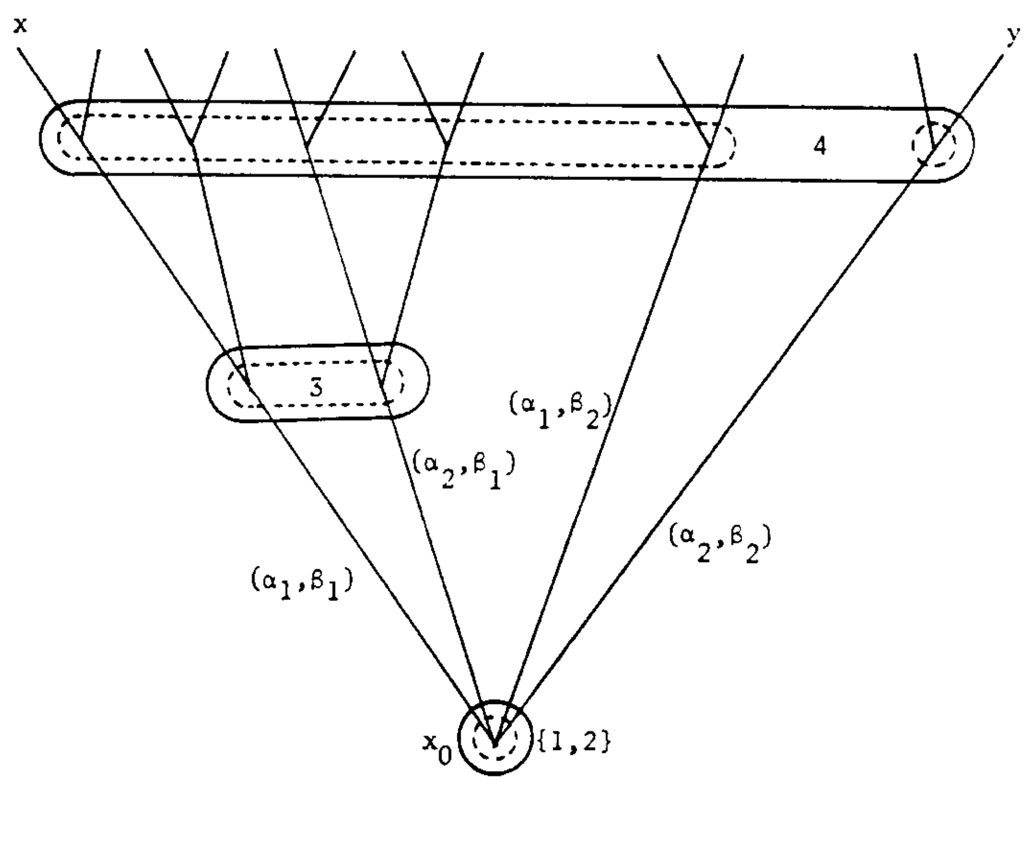

FIGURE 6

At x_0, $\{\alpha_1, \alpha_2\}$ and $\{\beta_1, \beta_2\}$ are the moves of players 1 and 2. The solid, broken lines give the information patterns of Γ, Γ^*. At any σ^* in Γ^* which gives the play x_0 to x as an outcome, no player has any informational influence. But if σ^* gives the outcome $\{x_0$ to $y\}$ then player 2 has informational influence on player 3.

The condition (iii) of Proposition 3 is undoubtedly severe, though it is a natural one in the context of a "large number of small players," not necessarily non-atomic. Suppose $N = \{1, \ldots, 1000\}$. Let $S = \{1, \ldots, 500\}$ and $T = \{501, \ldots, 1000\}$. The game Γ is as follows. First all players in S move simultaneously, and each $i \in S$ selects a real number r_i in the closed interval $[0,1]$. The players in S can observe $\sum_{i \in S} r_i$. But there is a grid on their scale which does not permit very fine measurements. They can tell only that $\sum_{i \in S} r_i$ lies in one of the intervals

$$[0,10), [10,20), \ldots, [490,500) .$$

After S has moved, then the players in T move simultaneously, and again each of them can select a real number in $[0,1]$. Suppose there is a Nash equilibrium in which $\sum_{i \in S} r_i = 145$. (One can easily concoct payoffs to make this so.) Then no player will have any informational influence at this N.E. The resulting N.E. play is marked in Figure 7. If any one player in S changes his strategy, this will change the play but no one in T can observe it because the new play continues to pass through $[140,150)$. If we call the below game Γ^* and let Γ be its coarsening in which players in T observe nothing (i.e. have the information set marked by dotted lines in Figure 7) then all the conditions of Proposition 3 are met.

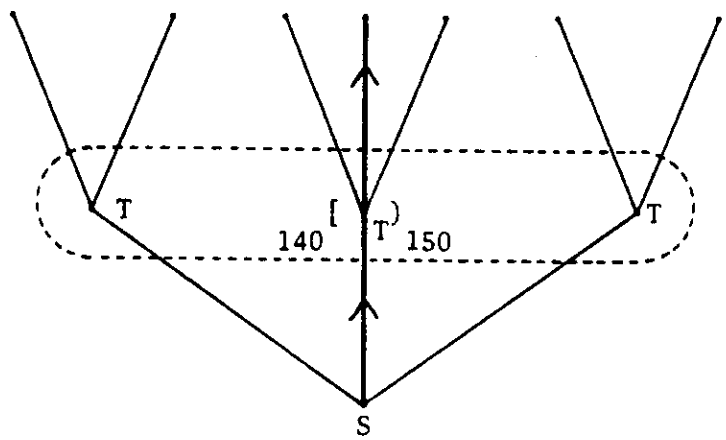

FIGURE 7. The Games Γ , Γ^*

Remarks

(7) If there are no chance moves, then Proposition 3 is a corollary of Proposition 1 when N is finite. In this case (iv) of Proposition 3 holds automatically. (The trouble, with chance moves, is that (iv) may not hold in general.)

(8) A stricter version of (4.2) is

$$(4.2)^* \qquad D_j(\sigma^*) = D_j(\sigma^* | \tau_i^*) \quad \text{for all} \quad \tau_i^* \in \Sigma_i^* .$$

Then we will say that i has strictly no informational influence on j . In the non-atomic case (Section 5, Lemma 2) it is in fact $(4.2)^*$ that obtains.

5. NON-ATOMIC GAMES

5.1. The Definition

For simplicity we will assume, throughout Section 5, that there are no chance moves. (They will be incorporated in Remark 9.) We need to specialize the set-theoretic structure of Γ to treat non-atomic games. The player-set N is now equipped with a non-atomic measure. Precisely, we have a measure space $\{N,\mathcal{B},\mu\}$. $\mathcal{B}$ is an σ-field of subsets of N which includes the singleton sets $\{i\}$, $i \in N$; μ is a non-atomic probability measure on $\{N,\mathcal{B}\}$. Each Y^x (for $x \in X_N$) is also assumed to be a measurable space. We now add the following conditions on the constituents of Γ , over and above those in Section 2.1, (i)-(viii).

(ix) For any $x \in X_N$, $\pi(x)$ is a non-null[1] set in $\mathcal{B}$.

(x) For any $x \in X_N$, there is a measurable correspondence f^x from $\pi(x)$ to Y^x , and S^x consists of all measurable selections from f^x , i.e., of all functions $g : \pi(x) \to Y^x$ which satisfy:

(a) $g(i) \in f^x(i)$

(b) g is measurable.

(xi) For any $x, y \in X_N$, the set $\{i \in N : y \in I_i(x)\}$ is measurable.

These conditions are fairly innocuous. The sine qua non of the non-atomic assumption is in the next, and final, condition. It says that null sets of players and their moves cannot be observed by any of the others.

[1] $S \in \mathcal{B}$ is called null if $\mu(S) = 0$; non-null if it is not null.

(xii) If $x = (s^{x_0}, s^{x_1}, \ldots, s^{x_m})$, $y = (r^{y_0}, r^{y_1}, \ldots, r^{y_m})$, and $i \in N$ satisfy (where $y_0 \equiv x_0$):

(a) $x_\ell \in v \in I_i \Longleftrightarrow y_\ell \in v \in I_i$,

(b) if $x_\ell, y_\ell \in v \in I_i$, then $s_i^{x_\ell} = r_i^{y_\ell}$,

(c) $\mu(\{j \in \pi(x_\ell) \cap \pi(y_\ell) : s_j^{x_\ell} = r_j^{y_\ell}\}) = \mu(\pi(x_\ell)) = \mu(\pi(y_\ell))$

for $\ell = 0, 1, \ldots, m$, then

$$x \in v \in I_i \Longleftrightarrow y \in v \in I_i .$$

This completes our definition of a non-atomic game. Note that (x) easily implies

(5.1) $S_i^x = f^x(i)$

(5.2) If $\pi(x)$ is a disjoint union of measurable sets $\pi_1(x)$ and $\pi_2(x)$, and $g_1 : \pi_1(x) \to Y^x$, $g_2 : \pi_2(x) \to Y^x$ are measurable functions which satisfy $g_1(i) \in f^x(i)$ for $i \in \pi_1(x)$, $g_2(i) \in f^x(i)$ for $i \in \pi_2(x)$, then the function $g : \pi(x) \to Y^x$, obtained by putting together g_1 and g_2 , will belong to S^x .

It can be checked that (ix)-(xii) are consistent with the earlier assumptions in (i)-(viii), i.e., there are models of games that satisfy (i)-(xii). See the example in Section 6.

5.2. Invariance of Nash Plays on Information Patterns

We will establish that if (i)-(xii) hold for a game, then the Nash plays are invariant of the information pattern that the game is endowed with.

We prepare for this with

Lemma 1. Let Γ satisfy (i)-(xi). Then $\Lambda(\Gamma_-) = \Lambda(\Gamma)$.

(Note that, since there are no chance moves, outcome trees reduce simply to plays.)

Proof. Recall that a play is a sequence of immediate followers, starting with the root x_0 . Given our identification $x \equiv \Phi(x) = (s^{x_0}, \ldots, s^{x_m})$, let $p = (s^{x_0}, s^{x_1}, \ldots s^{x_m}, \ldots) \in \Lambda(\Gamma_-)$. Put $U_\ell = \bigcup_{i\in\pi(x_\ell)} I_i(x_\ell)$ and $U = \bigcup_\ell U_\ell$. For $x \in U_\ell$, let

$$\gamma^\ell(x) = \{i \in \pi(x) \cap \pi(x_\ell) : x \in I_i(x_\ell)\} .$$

By (2.7), if $\ell \neq \ell'$, then $\gamma^\ell(x) \cap \gamma^{\ell'}(x) = \emptyset$. By (xi), each $\gamma^\ell(x)$ is measurable. Therefore, by (xi), so is

$$\alpha(x) = \pi(x) - \bigcup_\ell \gamma^\ell(x) .$$

Let $\tilde{\sigma}$ be any element of $\Sigma(\Gamma)$ (which is non-empty by assumption) and now define σ on X_N by:

$$\sigma_i(x) = \begin{cases} s_i^{x_\ell} & \text{if } x \in U_\ell \text{ and } i \in \gamma^\ell(x) \text{ for some } \ell \geqq 0 , \\ \tilde{\sigma}_i(x) & \text{if } i \in \pi(x) \text{ but } i \notin \bigcup_\ell \gamma^\ell(x) . \end{cases}$$

Since $\{\gamma^\ell(x) : \ell = 0, 1, \ldots\}$ are disjoint, this σ is well-defined. It can be checked (inductively, starting at x_0) that $\xi(\sigma) = p$. It remains to verify that $\sigma \in \Sigma(\Gamma)$. It is clear that if $x, y \in u$ for some $u \in I_i$, then $\sigma_i(x) = \sigma_i(y)$. Therefore it is sufficient to show $\sigma(x) \in S^x$ for all $x \in X_N$. If $x \in X_N \setminus U$, then $\sigma(x) = \tilde{\sigma}(x)$

and $\sigma(x) \in S^x$ by assumption. If $x \in U_\ell$ for some $\ell \geqq 0$, then $\pi(x)$ is the disjoint union of $\{\gamma^\ell(x) : \ell = 0, 1, \ldots\}$ and $\alpha(x)$. By (2.6) and (5.1), $f^x(i) = f^{x_\ell}(i)$ for $i \in \gamma^\ell(x)$. Also, clearly $\bigcup_\ell \gamma^\ell(x) \subset \pi(x_\ell)$. But then by construction, the map $\sigma(x)$ on $\pi(x)$ (given by $\sigma_i(x)$ for $i \in \pi(x)$) coincides with s^{x_ℓ} on $\gamma^\ell(x)$ for all $\ell \geqq 0$. Hence $\sigma(x)$ is a measurable selection from f^x on $\bigcup_\ell \gamma^\ell(x)$. On the other hand, $\sigma(x)$ coincides with $\tilde{\sigma}(x)$ on $\alpha(x)$ and is, a fortiori, a measurable selection from f^x on $\alpha(x)$. Therefore by (5.2), $\sigma(x) \in S^x$.

Q.E.D.

Lemma 2. Suppose Γ satisfies (i)-(xii), and $\sigma \in \Sigma(\Gamma)$. Then each player has strictly no informational influence at σ in Γ.

Proof. Let $\sigma = \{\sigma(x)\}_{x \in X_N}$. Consider $\tau_j \in \Sigma_j(\Gamma)$. Put $\xi(\sigma) = (s^{x_0}, s^{x_1}, \ldots, s^{x_m}, \ldots)$ and $\xi(\sigma|\tau_j) = (r^{y_0}, r^{y_1}, \ldots, r^{y_m}, \ldots)$, where $y_0 \equiv x_0$. It will suffice to show that for any ℓ and any $i \in N \setminus \{j\}$, if $x = (s^{x_0}, \ldots, s^{x_\ell})$ and $y = (r^{y_0}, \ldots, r^{y_\ell})$ then (a), (b), (c) of (xii) are satisfied. Make the inductive hypothesis that we have shown this for $\ell = 0, 1, \ldots, k$ and consider the case[1] $\ell = k+1$. Now $x_{k+1} = (s^{x_0}, \ldots, s^{x_k})$ and $y_{k+1} = (r^{y_0}, \ldots, r^{y_k})$. Then, by (xii),

$$\text{(d)} \quad x_{k+1} \in v \in I_i \Longleftrightarrow y_{k+1} \in v \in I_i \quad \text{for } i \in N \setminus \{j\} .$$

Hence

---

[1]For $\ell = 0$ the hypothesis obviously holds.

(e) $\pi(x_{k+1}) \setminus \{j\} = \pi(y_{k+1}) \setminus \{j\} \quad (\equiv A_{k+1})$

(f) $s_i^{x_{k+1}} = r_i^{y_{k+1}}$ for $i \in A_{k+1}$.

From (e) and (f):

$$A_{k+1} = \{i \in \pi(x_{k+1}) \cap \pi(y_{k+1}) \; : \; s_i^{x_{k+1}} = r_i^{y_{k+1}}\}$$

hence, since $\mu(\{j\}) = 0$

(g) $\mu(A_{k+1}) = \mu(\pi(x_{k+1})) = \mu(\pi(y_{k+1}))$.

This verifies the hypothesis for $\ell = k+1$.

Q.E.D.

Fix a <u>six</u>-tuple $L = \{N, X, \pi, \{S^x\}_{x \in X}, \Phi, \{h_i\}_{i \in N}\}$ for which all the assumptions in (i)-(viii), <u>as well as</u> (ix), (x) hold. Denote by $\nabla(L)$ the set of all games obtained by adding information patterns to L subject to (2.6) and (2.7), <u>as well as</u> (xi) and (xii). For any $\Gamma \in \nabla(L)$, recall that $\eta(\Gamma)$ is the set of all its Nash plays.

<u>Proposition 4.1</u>. $\eta(\Gamma) = \eta(\tilde{\Gamma})$ for any Γ , $\tilde{\Gamma}$ in $\nabla(L)$.

<u>Proof</u>. Denote by $\{I_i\}_{i \in N}$, $\{\tilde{I}_i\}_{i \in N}$ the information patterns in Γ , $\tilde{\Gamma}$. For each $i \in N$, let I_i^* be the common refinement of I_i and $\tilde{I}_i$, i.e.,

$$I_i^* = \{v^* \subset X_i \; : \; v^* \neq \emptyset, \; v^* = v \cap \tilde{v} \text{ for some } v \in I_i \text{ and } v \in \tilde{I}_i\} \; .$$

Consider the game Γ^* obtained by adding I_i^* to L . We will show that $\Gamma^* \in \Gamma(L)$. Clearly I_i^* is a partition of X_i . For any $x, y \in X_N$: $\{i \in N : y \in I_i^*(x)\} = \{i \in N : y \cap I_i(x) \cap \tilde{I}_i(x)\}$ $= \{i \in N : y \in I_i(x)\} \cap \{i \in N : y \in \tilde{I}_i(x)\}$. Since each of the last two sets is measurable, so is the first, and thus I_i^* satisfies (xi). We omit the straightforward check that I_i^* satisfies (2.7). Finally take $x = (s^{x_0}, s^{x_1}, \ldots, s^{x_m})$, $y = (r^{y_0}, r^{y_1}, \ldots, r^{y_m})$, and $i \in N$ (where $x_0 \equiv y_0$) such that

(a*) $x_\ell \in v^* \in I_i^* \Longleftrightarrow y_\ell \in v^* \in I_i^*$;

(b*) If $x_\ell, y_\ell \in v^* \in I_i^*$, then $s_i^{x_\ell} = r_i^{y_\ell}$;

(c*) Condition (c) of (xii) holds.

In (a*) let $v^* = v \cap \tilde{v}$ for $v \in I_i$, $\tilde{v} \in \tilde{I}_i$. Then $x_\ell \in v^* \Rightarrow x_\ell \in v$, and $y_\ell \in v^* \Rightarrow y_\ell \in v$. From this it follows that (a*) implies:

$$x_\ell \in v \in I_i \Longleftrightarrow y_\ell \in v \in I_i ,$$

i.e., (a) of (xii) holds for I_i. In the same manner (a) of (xii) holds for $\tilde{I}_i$, and (b) of (xii) holds for both I_i, $\tilde{I}_i$. (c) is independent of the information pattern and depends only on x and y. To sum up, (a), (b), (c) of (xii) are satisfied for x, y, and i in both Γ, $\tilde{\Gamma}$. Then by (xii),

(d*) $x \in v \in I_i \Longleftrightarrow y \in v \in I_i$;

(e*) $x \in \tilde{v} \in \tilde{I}_i \Longleftrightarrow y \in \tilde{v} \in \tilde{I}_i$.

Let $x \in w^* \in I_i^*$, $w^* = w \cap \tilde{w}$ for $w \in I_i$, $\tilde{w} \in \tilde{I}_i$. Then $x \in w$ and from (d*), $y \in w$. Similarly, $y \in \tilde{w}$. Hence $y \in w^*$. In the same way, $y \in w^* \Rightarrow x \in w^*$. This proves that condition (xii) is also satisfied by Γ^*. Consequently $\Gamma^* \in \nabla(L)$.

By construction, $\Gamma \dashv \Gamma^*$ and $\tilde{\Gamma} \dashv \Gamma^*$. By Proposition 2.2, $\eta(\Gamma) \subset \eta(\Gamma^*)$ and $\eta(\tilde{\Gamma}) \subset \eta(\Gamma^*)$. Let σ^* be any N.E. of Γ^*. In the wake of Lemmas 1 and 2, we can apply Proposition 3. This tells us that there are N.E.'s σ in Γ and $\tilde{\sigma}$ in $\tilde{\Gamma}$, such that $\xi(\sigma) = \xi^*(\sigma^*) = \tilde{\xi}(\tilde{\sigma})$. Since σ^* was arbitrary, $\eta(\Gamma^*) \subset \eta(\tilde{\Gamma})$ and $\eta(\Gamma^*) \subset \eta(\tilde{\Gamma})$, therefore $\eta(\Gamma) = \eta(\Gamma^*) = \eta(\tilde{\Gamma})$.

Q.E.D.

5.3. A Variation on the Theme

The condition (xii) is fairly stringent. Each player has no informational influence on others, not even on a null set. On the other hand absolutely no assumption was made on the payoff functions in proving Proposition 4.1. We now relax (xii) to (xii)* but at the expense of having to add conditions (xiii) and (xiv) below. Then Proposition 4.1 can still be retrieved, as Proposition 4.2.

Condition (xiii) says, roughly, that if two positions differ only on account of null sets not containing a particular player i, then i cannot tell them apart.

(xiii) If $\Phi(x) = (s^{x_0}, s^{x_1}, \dots, s^{x_m})$ and $\Phi(y) = (r^{y_0}, r^{y_1}, \dots, r^{y_m})$ satisfy, for $i \in \pi(x)$ (with $y_0 \equiv x_0$),

(f) $\mu(\{j \in \pi(x_\ell) \cap \pi(y_\ell) : s_j^{x_\ell} = r_j^{y_\ell}\}) = \mu(\pi(x_\ell)) = \mu(\pi(y_\ell))$ for $\ell = 0, 1, \dots, m$;

(g) for all $\ell = 0, 1, \dots, m$, $i \in \pi(x_\ell) \Leftrightarrow i \in \pi(y_\ell)$;

(h) for all $\ell = 0, 1, \dots, m$, $i \in \pi(x_\ell) \Rightarrow s_i^{x_\ell} = r_i^{y_\ell}$;

then $i \in \pi(y)$ and $S_i^x = S_i^y$.

The next condition (xiv) is on payoffs. It says that they depend on plays "modulo" null sets.

(xiv) If two plays $p = (s^{x_0}, s^{x_1}, \ldots)$ and $p' = (r^{y_0}, r^{y_1}, \ldots)$ satisfy

(i) $\mu(\{j \in \pi(x_\ell) \cap \pi(y_\ell) : s_j^{x_\ell} = r_j^{y_\ell}\}) = \mu(\pi(x_\ell)) = \mu(\pi(y_\ell))$ for all $\ell \geqq 0$;

(j) $i \in \pi(x_\ell) \Longleftrightarrow i \in \pi(y_\ell)$, for all $\ell \geqq 0$;

(k) $i \in \pi(x_\ell) \Rightarrow s_i^{x_\ell} = r_i^{y_\ell}$, for all $\ell \geqq 0$;

then $h_i(p) = h_i(p')$.

In the light of (xiii) and (xiv) we weaken (xii) to:

(xii)* <u>No positive informational influence</u>. Each player i has strictly no informational influence on almost all other players (i.e. all except a null set).

Let L^* be a six-tuple as before, but assume this time that the assumptions (i)-(viii), (ix)-(xi), <u>as well as</u> (xiii), (xiv) hold. Define $\nabla^*(L^*)$ exactly as $\nabla(L)$ but with (xii) replaced by the weaker (xii)*.

<u>Proposition 4.2</u>. $\eta(\Gamma) = \eta(\tilde{\Gamma})$ for any Γ , $\tilde{\Gamma}$ in $\nabla^*(L^*)$.

<u>Proof</u>. It is sufficient to show that for any N.E. σ of Γ , there is a N.E. $\tilde{\sigma}$ of $\tilde{\Gamma}$ such that $\xi(\sigma) = \tilde{\xi}(\tilde{\sigma})$.

Let $\xi(\sigma) = (s^{x_0}, s^{x_1}, \ldots)$. Select a $\tilde{\sigma}$ in $\Sigma(\tilde{\Gamma})$ such that $\tilde{\xi}(\tilde{\sigma}) = \xi(\sigma)$. This is possible by Lemma 1.

Suppose $\tilde{\sigma}$ is not a N.E. of $\tilde{\Gamma}$. Then there is an $\tilde{\tau}_i \in \Sigma_i(\tilde{\Gamma})$ for some $i \in N$ such that $h_i(\tilde{\xi}(\tilde{\sigma}|\tilde{\tau}_i)) > h(\tilde{\xi}(\tilde{\sigma}))$. Let $\tilde{\xi}(\tilde{\sigma}|\tilde{\tau}_i) = (r^{y_0}, r^{y_1}, \ldots)$ $(y_0 \equiv x_0)$. Then condition (xii)* implies

$$\mu(\{j \in \pi(x_\ell) \cap \pi(y_\ell) : s_j^{x_\ell} = r_j^{y_\ell}\}) = \mu(\pi(x_\ell)) = \mu(\pi(y_\ell))$$
$$\text{for all} \quad \ell \geqq 0 \ .$$

Choose an $\tau_i \in \Sigma_i(\Gamma)$ such that if $\Phi(x) = (t^{w_0}, t^{w_1}, \ldots, t^{w_m})$ satisfies

(f*) $\mu(\{j \in \pi(y_\ell) \cap \pi(w_\ell) : r_j^{y_\ell} = t_j^{w_\ell}\}) = \mu(\pi(y_\ell)) = \mu(\pi(w_\ell))$ for all $\ell = 0, 1, \ldots, m$;

(g*) for $\ell = 0, 1, \ldots, m$, $i \in \pi(y_\ell) \Longleftrightarrow i \in \pi(w_\ell)$;

(h*) for $\ell = 0, 1, \ldots, m$, if $i \in \pi(w_\ell)$ then $r_i^{y_\ell} = t_i^{w_\ell}$;

then $\tau_i(x) = \tilde{\tau}_i(y_{m+1})$. Assumption (xiii) ensures that this choice of τ_i is possible. Let $\xi(\sigma|\tau_i) = (q^{a_0}, q^{a_1}, \ldots)$, with $a_0 \equiv x_0$. From the construction, it is clear that

(i*) for all $\ell \geqq 0$, $\mu(\{\ell \in \pi(a_\ell) \cap \pi(y_\ell) : q_j^{a_\ell} = r_j^{y_\ell}\})$ $= \mu(\pi(a_\ell)) = \mu(\pi(y_\ell))$;

(j*) for all $\ell \geqq 0$, $i \in \pi(a_\ell) \Longleftrightarrow i \in \pi(y_\ell)$;

(k*) for all $\ell \geqq 0$, if $i \in \pi(a_\ell)$, then $q_i^{a_\ell} = r_i^{y_\ell}$.

Therefore, by (xiv), we have $h_i(\xi(\sigma|\tau_i)) = h_i(\tilde{\xi}(\tilde{\sigma}|\tilde{\tau}_i))$. That is, $h_i(\xi(\sigma|\tau_i)) = h_i(\tilde{\xi}(\tilde{\sigma}|\tilde{\tau}_i)) > h_i(\tilde{\xi}(\tilde{\sigma})) = h_i(\xi(\sigma))$. This is a contradiction.

Q.E.D.

If (xii), (xii)* are violated then Propositions 4.1, 4.2 break down. Non-trivial counterexamples can easily be obtained by modifying the "dilemma game with rumour" in [3].

The careful reader must have noticed that we have defined a Nash

Equilibrium by requiring that all[1]--as opposed to "almost all"--players must be optimal in accordance with (2.11). This is because, in our opinion, the very basis of an N.E. is individual optimization, and ignoring even a single player would go against the grain of this notion.

Remarks

(9) If chance moves are always countable, then an analogue of Proposition 4.1 (or 4.2) is possible. Take two non-atomic games Γ, Γ^* differing only in information. Suppose (iii)(b) of Proposition 1 holds at all strategies in both directions, i.e., in going from Γ to Γ^* and Γ^* to Γ. Then we can show that $\eta(\Gamma) = \eta(\Gamma^*)$. (Naturally, condition (xiv) has to be strengthened to apply to outcome trees, rather than just plays.) If chance moves are uncountable then we would need additional measurability assumptions in the spirit of Remark 6.

(10) An asymptotic version of the non-atomic result has been examined in part II of [1].

6. THE ANTI-FOLK THEOREM[2]

Let Γ be a non-atomic game in strategic form, i.e., $\pi(x_0) = N$ and every $s^{x_0} \in S^{x_0}$ constitutes an ending position. Further assume that the condition (xiv) holds. In this context that simply says: if $\mu(\{j \in N : s_j^{x_0} \neq r_j^{x_0}\}) = 0$ and $s_i^{x_0} = r_i^{x_0}$ then $h_i(s^{x_0}) = h_i(r^{x_0})$, i.e., the payoff to any player depends on his strategy and the measurable function of others strategies modulo null sets.

Consider an infinite repetition Γ^∞ of Γ, in which each player

[1]That is why the "almost all" variations of assumptions (xii) and (xii)*, (xiii), (iv) would not suffice for our results.

[2]For a further discussion of this topic see [4].

can observe at each stage the entire past history of (a) his own moves and payoffs, (b) the measurable functions of others' moves, modulo null sets. The payoffs to plays in Γ^{∞} are assigned by some rule (e.g., lim inf , discounted sum)...it doesn't much matter. Then Γ^{∞} satisfies (xii), (xii)* (and, also, of course (i)-(xi), (xiii), (xiv)). Consider the game Γ_C^{∞} obtained by coarsening Γ^{∞} as shown in Figure 8 i.e., each player observes <u>nothing</u> at the end of any stage in Γ_C^{∞} . Clearly both Propositions 4.1 and 4.2 apply.

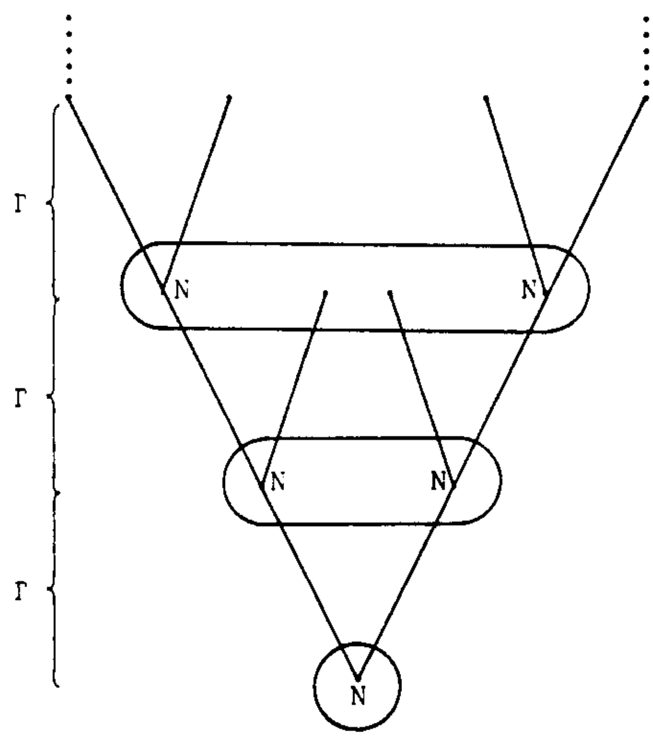

FIGURE 8

This says that the Nash plays of Γ_C^{∞} are identical with the Nash plays of Γ^{∞} . If we denote the strategy set of i in Γ by Σ_i , then clearly his strategy set in Γ_C^{∞} is $(\Sigma_i)^{\infty}$, i.e. a strategy for him is to simply pick an infinite sequence each of whose elements is in Σ_i . It is a short step from this to verify that the Nash plays of

Γ_C^∞ (hence of Γ^∞) are typically "small." Indeed if we assign the payoff to a play of Γ^∞ by the discounted sum[1] of payoffs in each stage, then it is obvious that

$$(\sigma^1, \sigma^2, \ldots, \sigma^n, \ldots) \text{ is an N.E. of } \Gamma^\infty \Longleftrightarrow \text{each } \sigma^\ell \text{ is an N.E. of } \Gamma \text{ for } \ell = 1, 2, \ldots .$$

This is in sharp contrast with the "folk theorem" ([4], [6]). These players have enormous informational influence, and a stupendous proliferation of Nash plays is obtained in Γ^∞.

7. BEHAVIORAL STRATEGIES

Our description of extensive games in Section 2 permits us to model behavioral strategies in Γ as pure strategies of an associated game $\hat{\Gamma}$, in the case when N is finite. The preceding results then apply to $\hat{\Gamma}$ and can be reinterpreted within Γ. For ease of exposition we shall make the restrictions:

(7.1) X is a finite set

(7.2) There are no chance moves.

Note that (7.1) implies that not only N, but also players' moves and the length of the game are all finite. However only the restriction that N is finite, is substantial, all the others are made for notational convenience.

[1] Assuming this will always exist, e.g. by requiring that the payoffs are uniformly bounded in Γ.

The idea behind going from Γ to $\hat{\Gamma}$ is roughly as follows. Consider the game Γ where $\{\alpha_1, \alpha_2\}$, $\{\beta_1, \beta_2\}$ are the moves of

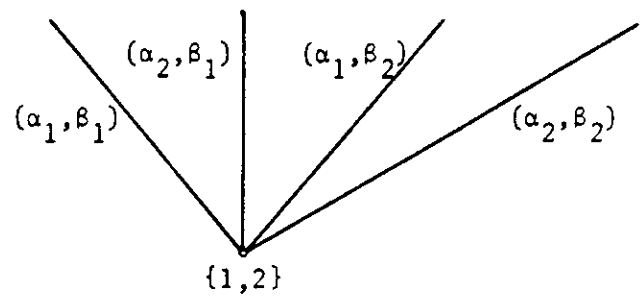

FIGURE 9

1, 2. The behavioral strategies of 1, 2 are the sets

$$B_1 = \{b_1 = (b_1^1, b_1^2) \in R_+^2 : b_1^1 + b_1^2 = 1\}$$

$$B_2 = \{b_2 = (b_2^1, b_2^2) \in R_+^2 : b_2^1 + b_2^2 = 1\} .$$

Construct the game $\hat{\Gamma}$ as follows:

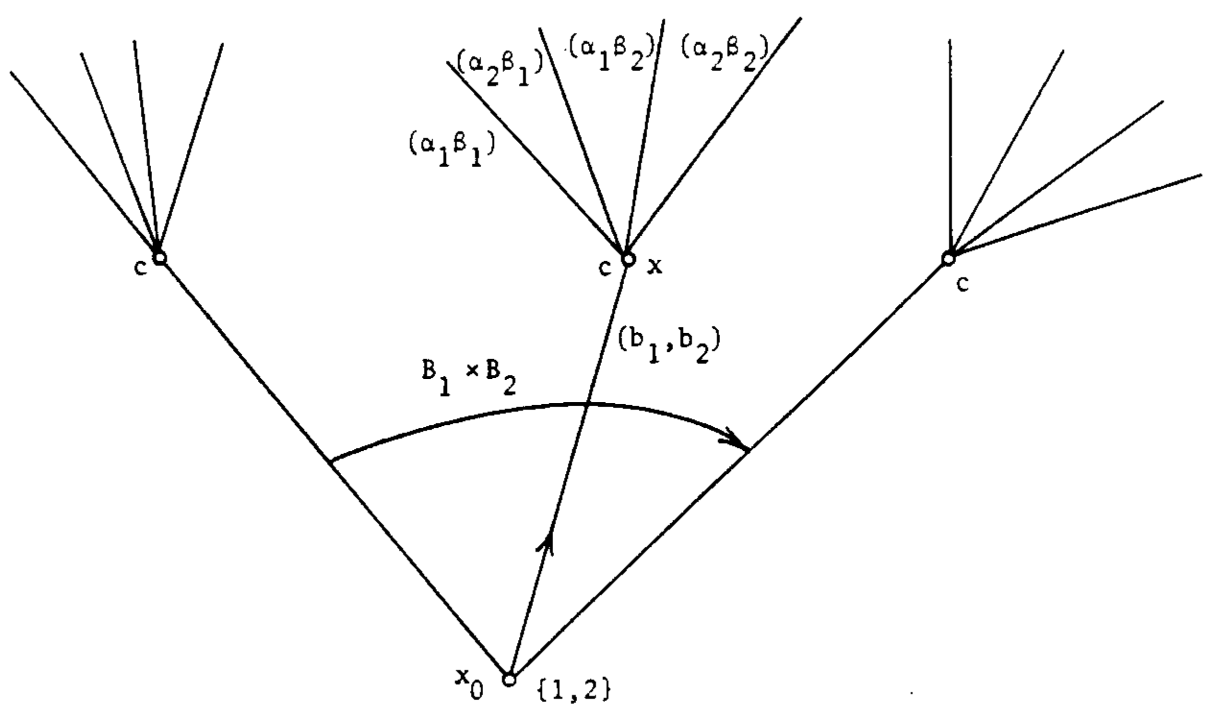

FIGURE 10

At x_0 players 1, 2 choose b_1, b_2 from B_1, B_2. At the resultant position x, c picks (α_1, β_1), (α_2, β_1), (α_1, β_2), (α_2, β_2) with probabilities $b_1^1 \times b_2^1$, $b_1^2 \times b_2^1$, $b_1^1 \times b_2^2$, $b_1^2 \times b_2^2$. The payoff to the outcome arising from (b_1, b_2) in $\hat{\Gamma}$ is $\hat{h}_i(b_1, b_2) = \sum_{k=1}^{2} \sum_{\ell=1}^{2} b_1^k b_2^\ell h_i(\alpha_k, \beta_\ell)$ where h_i is the payoff function of i in Γ. Then the pure strategies of $\hat{\Gamma}$ correspond exactly to the behavioral strategies of Γ. We now extend this picture to the general case (assuming (7.1), (7.2)).

A behavioral strategy b_i of player i is a function on X_i which assigns to each $x \in X_i$ a probability distribution $b_i(x)$ on S_i^x, i.e.,

$$\sum_{t \in S_i^x} b_{it}(x) = 1 \quad \text{and} \quad b_{it}(x) \geq 0 \quad \text{for all} \quad t \in S_i^x .$$

This must also satisfy $b_i(x) = b_i(y)$ if $x, y \in u \in I_i$. Denote by B_i the set of all behavioral strategies of player i. Put $B = \prod_{i \in N} B_i$. Any $b \in B$ induces a map $\mathbf{P}_b : X \to \mathbf{R}$ where $\mathbf{P}_b(x)$ is the product of the probabilities on all the arcs, going from x_0 to x, assigned according to b. If we restrict $\mathbf{P}_b$ to X_E then we get a probability distribution on X_E. The payoff to i in Γ^B is the expectation:

$$h_i^B = \sum_{x \in X_E} \mathbf{P}_b(x) h_i(x) .$$

(Since Γ has no chance moves and is of finite length, outcomes can be identified with points in X_E and we may view h_i as defined on X_E.)

We now proceed to construct $\hat{\Gamma}$ which will represent Γ^B in the format presented in Section 2. A cap will consistently be used to distinguish constituents of $\hat{\Gamma}$ from Γ .

(a) The player-sets are identical: $\hat{N} = N = \{1, \ldots, n\}$.

(b) There is an onto map $\delta : \hat{X} \setminus \hat{X}_c \to X$ which preserves positions in the sense: $\delta(\hat{X}_i) = X_i$ for $i \in N$ and $\delta(\hat{X}_E) = X_E$.

(c) Followers are preserved under δ , i.e.,

$$\hat{x} \succ \hat{y} \Rightarrow \delta(\hat{x}) \succ \delta(\hat{y}) \quad \text{for all } \hat{x}, \hat{y} \in \hat{X}_N \cup \hat{X}_E .$$

(d) $\hat{\pi}(\hat{x}) = \pi(\delta(x))$ for all $\hat{x} \in \hat{X}_N$.

(e) Chance moves come immediately after players' moves, and only then, i.e.,

(i) $\hat{x} \in \hat{X}_N$, $\hat{y} \underset{I}{\succ} \hat{x} \Rightarrow \hat{\pi}(\hat{y}) = \{c\}$

(ii) $\hat{\pi}(\hat{y}) = \{c\}$ there is an $\hat{x} \in \hat{X}_N$ such that $\hat{y} \underset{I}{\succ} \hat{x}$.

(f) For $\hat{x} \in \hat{X}_i$ the moves in $\hat{\Gamma}$ at $\hat{x}$ are precisely probability distributions on the pure strategies $S_i^{\delta(\hat{x})}$ available to i in Γ , i.e.,

$$\hat{S}_i^{\hat{x}} = \left\{ \hat{s}_i^{\hat{x}} : \sum_{t \in S_i^{\delta(\hat{x})}} \hat{s}_i^{\hat{x}}(t) = 1,\ \hat{s}_i^{\hat{x}}(t) \geq 0 \text{ for all } t \right\} .$$

(g) Chance moves in $\hat{\Gamma}$ mimic the moves picked with positive probability in Γ by the immediately preceding players. In other words, suppose $\hat{y} = (\hat{x}, \hat{s}^{\hat{x}})$ and $\hat{\pi}(\hat{y}) = \{c\}$. By (e)(ii) we have $\hat{x} \in \hat{X}_N$. We require

$$\hat{S}^{\hat{y}} = \prod_{i \in \pi(\hat{x})} \left\{ t \in S_i^{\delta(\hat{x})} : \hat{s}_i^{\hat{x}}(t) > 0 \right\} .$$

$$\left.\begin{array}{ll}\text{(h)} & \hat{x} \in \hat{X}_N \\ & \hat{x} \underset{I}{\prec} \hat{y} \\ & \hat{z} = (\hat{y}, \hat{s}) \ , \quad \hat{s} \in \hat{S}^{\hat{y}}\end{array}\right\} \Rightarrow \delta(\hat{z}) = (\delta(\hat{x}), \hat{s}) \ .$$

(Note: $\hat{S}^{\hat{y}} \subset S^{\delta(\hat{x})}$ by (g).)

(i) Since $\delta : \hat{X}_i \to X_i$ is onto, and I_i partitions X_i, δ^{-1} yields the partition $\hat{I}_i$ of $\hat{X}_i$, i.e.,

$$\hat{I}_i = \{\delta^{-1}(v) : v \in I_i\} \ .$$

It can be checked that, starting from the root $\hat{x}_0 \equiv x_0$, the properties (a)-(i) give a (unique) recursive construction of the tree of $\hat{\Gamma}$ in terms of Γ. To complete the definition of $\hat{\Gamma}$ it now remains to specify the payoff functions $\hat{h}_i$, $i \in \hat{N}$.

There is a one-to-one onto map from behavioral strategies of player i in Γ^B to his pure strategies in $\hat{\Gamma}$. This map $\psi_i : B_i \to \Sigma_i(\hat{\Gamma})$ is given by:

$$\psi_i(b_i) = \hat{\sigma}_i$$

where

$$\hat{\sigma}_i(\hat{x}) = b_i(\delta(\hat{x})) \quad \text{for all} \quad \hat{x} \in \hat{X}_i \ .$$

Put $\psi = (\psi_1, \ldots, \psi_n)$ i.e. ψ maps B to $\Sigma(\hat{\Gamma})$ with $\psi(b) \equiv \psi(b_1, \ldots, b_n) = (\psi_1(b_1), \ldots, \psi_n(b_n))$. Take any $\lambda \in \Lambda(\hat{\Gamma})$. There there is some $\hat{\sigma} \in \Sigma(\hat{\Gamma})$ such that $\hat{\xi}(\hat{\sigma}) = \hat{\lambda}$. Define[1]

$$\hat{h}_i(\hat{\lambda}) = h_i^B(\psi^{-1}(\hat{\sigma})) \ . \tag{7.3}$$

---

[1]We define $\hat{h}_i$ only on $\Sigma(\hat{\Gamma})$ rather than on $\Sigma(\hat{\Gamma}_-)$. This is sufficient for the current purpose.

It can be easily checked that if $\hat{\xi}(\hat{\sigma}) = \hat{\xi}(\hat{\sigma}')$ for $\hat{\sigma}$, $\hat{\sigma}'$ in $\Sigma(\hat{\Gamma})$, then $P_{\psi^{-1}(\hat{\sigma})} = P_{\psi^{-1}(\hat{\sigma}')}$. Therefore $\hat{h}_i(\hat{\lambda})$ is invariant of the choice of $\hat{\sigma}$, and (7.3) serves as a definition of $\hat{h}_i$. Note that payoffs are faithfully preserved under ψ:

$$h_i^B(b) \equiv h_i^B(b_1,\ldots,b_n) = \hat{h}_i(\hat{\xi}(\psi_1(b_1),\ldots,\psi_n(b_n))) \equiv \hat{h}_i(\hat{\xi}(\psi(b))) .$$

Hence

(7.4) $\quad$ b is an N.E. of $\Gamma^B \leftrightarrow \psi(b)$ is an N.E. of $\hat{\Gamma}$.

Thus, to analyze behavioral strategy N.E.'s of Γ, it suffices to consider pure-strategy N.E.'s of $\hat{\Gamma}$.

From the initial pair of games Γ, Γ^* we derive $\hat{\Gamma}$, $\hat{\Gamma}^*$. Propositions 1, 2, 3 can be applied to $\hat{\Gamma}$, $\hat{\Gamma}^*$. Using the isomorphism ψ, they can then be transferred to Γ^B, Γ^{*B}. We shall work this out in detail for some cases. First, for any behavioral strategy-choice b denote by $\xi^B(b)$ the support of P_b, i.e., $\xi^B(b) = \{x \in X : P_b(x) > 0\}$ is the set of positions reached with positive probability under b. Consider Proposition 3, for instance. To interpret it with behavioral strategies, take $\Gamma \dashv \Gamma^*$. It follows immediately that

(i)' $\hat{\Gamma} \dashv \hat{\Gamma}^*$.

Further suppose

(ii)' $\hat{\sigma}^*$ is an N.E. of $\hat{\Gamma}^*$.

(iii)' No player has informational influence at $\hat{\sigma}^*$ in $\hat{\Gamma}^*$.

(iv)' $\hat{\xi}^*(\hat{\sigma}^*) \in \Lambda(\hat{\Gamma})$.

Then, by Proposition 3, any $\hat{\sigma} \in \Sigma(\hat{\Gamma})$ with $\hat{\xi}(\hat{\sigma}) = \hat{\xi}^*(\hat{\sigma}^*)$ is an N.E. of $\hat{\Gamma}$.

Put $b = \psi^{-1}(\hat{\sigma})$, $b^* = \psi^{-1}(\hat{\sigma}^*)$. From our construction of $\hat{\Gamma}^*$

one can easily verify that (iii)' is tantamount to:

(7.5) For any $j \in N$:

$$\{I_j^*(x) : x \in \xi^{*B}(b^*) \cap X_j\} \supset \{I_j^*(x) : x \in \xi^{*B}(b^*|c_i^*) \cap X_j\}$$

$$\text{for all } c_i^* \in B_i^* \text{ and } i \in N\setminus\{j\} .$$

To interpret (iv') first note that:

(7.6) $$\hat{\xi}(\hat{\tau}) = \hat{\xi}^*(\hat{\tau}^*) \Longleftrightarrow P_{\psi^{-1}(\hat{\tau})} = P_{\psi^{-1}(\hat{\tau}^*)} .$$

Therefore (iv)' says:

(7.7) $$P_{b^*} = P_d \quad \text{for some } d \in B .$$

Now, using (7.4) and (7.6), we have

$$\left\{\begin{array}{l} \Gamma \dashrightarrow \Gamma^* \\ b^* \text{ is an N.E. of } \Gamma^{*B} \\ (7.5) \text{ and } (7.7) \text{ hold} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} d, \text{ given by (7.7), is} \\ \text{an N.E. of } \Gamma^B \end{array}\right\}$$

which is Proposition 3 in terms of behavioral strategies. As an example, reconsider the game of Figure 6, but with behavioral strategies for players 3 and 4 in Γ^* (the refined game), as shown in Figure 11. (Arrows indicate pure moves.)

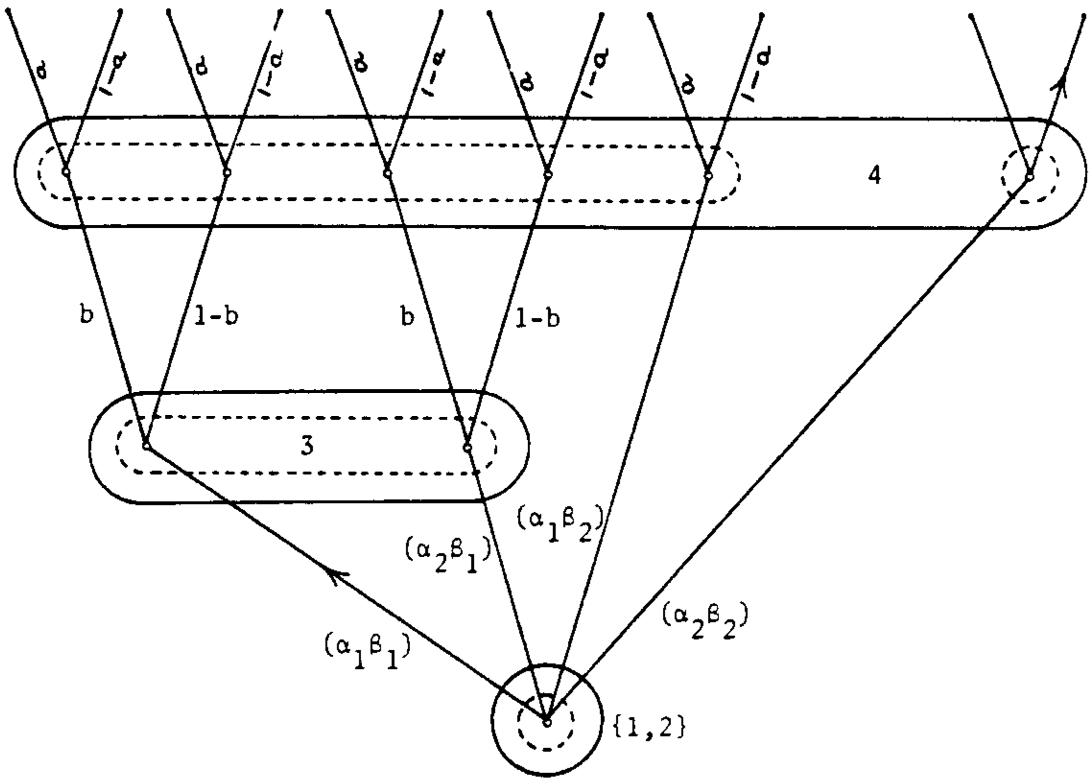

FIGURE 11

The conditions (7.5) and (7.7) are met at these strategies. Therefore if the outcome is Nash in the refined game it will also be Nash in the coarsening.

Next consider Proposition 2.1 for $\hat{\Gamma}$, $\hat{\Gamma}^*$. Condition (iii) of Proposition 2.1 says (in the context of Γ^B) that for all $i \in N$:

$$(7.8) \quad \left.\begin{array}{l} x,\, y \in \xi^B(b|c_i) \cap X_i \quad \text{for some} \quad c_i \in B_i \\ I_i(x) = I_i(y) \end{array}\right\} \Rightarrow I_i^*(x) = I_i^*(y) \;.$$

So we have, translating Proposition 2.1 from $\hat{\Gamma}$, $\hat{\Gamma}^*$ to Γ^B , Γ^{*B} :

$$\left\{\begin{array}{l} \Gamma \dashrightarrow \Gamma^* \\ b \text{ is an N.E. of } \Gamma^B \\ (7.8) \text{ holds} \end{array}\right\} \Rightarrow \{b \text{ is an N.E. of } \Gamma^{*B}\} \;.$$

Thus, in the coarse game in Figure 12, (7.8) holds at the strategies indicated. We conclude that if they are Nash in Γ, then they remain Nash in Γ^*.

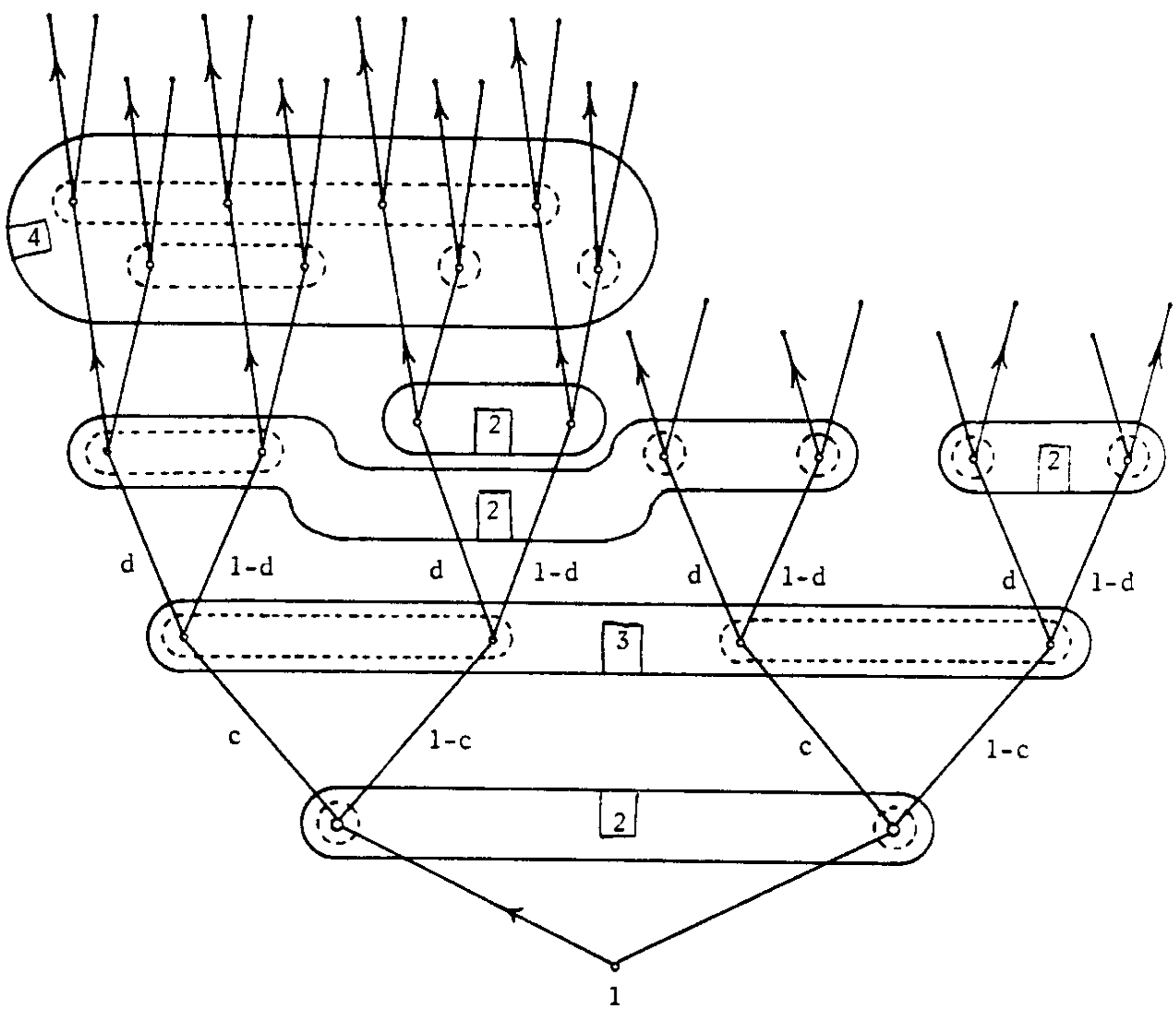

FIGURE 12

Similarly Proposition 1, applied to $\hat{\Gamma}$ and $\hat{\Gamma}^*$, can be interpreted in Γ^B and Γ^{*B}. We leave this to the reader.

LIST OF NOTATIONS

For the reader's convenience we append a list of notations which are used frequently.

Γ = extensive game = $(N \cup \{c\},\ X,\ \pi,\ \{S^x\}_{x \in X},\ \Phi,\ \{h_i\}_{i \in N},\ \{I_i\}_{i \in N})$

N = player-set

c = chance

X = set of all positions

x_0 = root = start of game $(x_0 \in X)$

$\pi(x)$ = set of players who move simultaneously at x, or $\{c\}$, or empty

X_i = set of player i's positions = $\{x \in X : i \in \pi(x)\}$

X_N = players' positions = $\bigcup_{i \in N} X_i$

X_c = positions for chance moves = $\{x \in X : \pi(x) = \{c\}\}$

X_E = ending positions = $\{x \in X : \pi(x) = \emptyset\}$

S^x = set of move-selections at x by $\pi(x)$ (Note: $S^x = \emptyset \Longleftrightarrow \pi(x) = \emptyset$.)

S_i^x = set of moves of i at x (for $i \in \pi(x)$)

$\Phi(x) = (s^{x_0}, \ldots, s^{x_m})$ = path from x_0 to x and moves picked along it

$x \underset{I}{\prec} y$ = y immediately follows x (i.e., $y = (x, s^x)$ for some $s^x \in S^x$)

$x \prec y$ = y follows x (x precedes y)

$p = (x_0, \ldots, x_k, \ldots)$ = play (i.e., $x_0 \underset{I}{\prec} \ldots\ldots \underset{I}{\prec} x_k \underset{I}{\prec} \ldots\ldots$)

λ = outcome tree (union of plays on which chance picks all its moves)

Γ_- = same as Γ without $\{h_i\}_{i \in N}$ and $\{I_i\}_{i \in N}$

$\Lambda(\Gamma_-)$ = set of outcome trees in Γ_-

$h_i : \Lambda(\Gamma_-) \to R$ = payoff function of player i

I_i = player i's information partition on X_i

$\Sigma_i(\Gamma)$ = strategy-set of player i in Γ

$\Sigma(\Gamma)$ = strategy-selections feasible in Γ

σ = element of $\Sigma(\Gamma)$

σ_i = player i's strategy in σ

$\xi : \Sigma(\Gamma) \rightarrow \Lambda(\Gamma)$ = strategy-to-outcome map

$\Lambda(\Gamma) = \xi(\Sigma(\Gamma))$ = set of outcomes feasible in Γ

$I_i(x)$ = information set of player i that contains x ($I_i(x)$ is empty if $x \notin X_i$)

$\eta(\Gamma)$ = set of Nash outcomes in Γ

$\Gamma \rightarrow \Gamma^*$ = Γ^* is an information-refinement of Γ

Γ^B = the game with behavioral-strategies on Γ

$\hat{\Gamma}$ = enlargement of Γ so that Γ^B corresponds to $\hat{\Gamma}$ (Note: in $\hat{\Gamma}$ we consider only pure strategies)

B_i = set of behavioral strategies of i in Γ

B = product of B_i over $i \in N$

b = element of B

b_i = player i's behavioral strategy in b

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