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PROFIT-SHARING IN A COLLUSIVE INDUSTRY

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Abstract

We study a model in which collusive duopolists divide up the monopoly profit according to their relative bargaining power. We are particularly interested in how the negotiated profit shares depend on the sizes of the firms. If each can produce at the same constant unit cost up to its capacity, we show that the profit per unit of capacity of the small firm is higher than that of the large one. We also study how the ratio of the negotiated profits depends on the size of demand relative to industry capacity, and how this ratio changes with variations in demand.

PROFIT-SHARING IN A COLLUSIVE INDUSTRY*

by

Martin J. Osborne** and Carolyn Pitchik***

1. Introduction

Suppose that a number of heterogeneous oligopolists collude. How will they divide up the monopoly profit? This is the question which we address. In particular, we study how the profit shares which are agreed upon depend on the sizes of the firms. It is easy to find statements in the literature to the effect that agreements on the actions to be taken by collusive firms are the outcomes of "hard bargaining". For example, Bain [1948] criticizes Patinkin [1947] precisely because the latter suggests that output quotas are determined by efficiency considerations, rather than by the relative bargaining power of the firms. There has, however, been no attempt to use a formal bargaining model to analyze the issues. There is, in fact, no model which captures with complete success the subtle mixture of cooperation and competition involved in a bargaining process. However, in the case of two bargainers, there is one--due to Nash [1953]--which defines a plausible

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outcome. This solution has been axiomatized by Selten [1960], who establishes that it yields the only outcome satisfying a number of attractive conditions. It is also possible to interpret the solution as the outcome of a two-stage procedure in which players first announce threats, and then negotiate on the basis of these threats. Both players know this will happen, and hence simultaneously choose their threats so as to maximize their final payoffs given their opponent's threat. This solution can be extended to deal with the case of many individuals (see Selten [1964]), but the extra dimensions of the problem then make it less clear as to what is the correct formulation. For this reason we restrict attention to the case of two firms.

We wish to abstract from technological superiority, so we assume each firm can produce at constant unit cost up to its capacity. We find that, in the negotiated agreement, the profit per unit of capacity of the small firm is always at least as large as that of the large firm; if the industry capacity exceeds the monopoly output, then the inequality is strict (see Proposition A in Section 2). The fact that a large firm can make a more potent threat than a small one--it can more easily "flood the market"--means that its negotiated unit profit is higher than that of a small firm. However, to analyze the effect of such a threat, we have to take into account the possible retaliation by the small firm. Also, more importantly, we should consider the fact that both firms, irrespective of size, have an equal ability to disrupt an agreement. (This last fact has been used by political scientists in their analyses of the power of parties of various sizes.) It turns out that the balance of forces is in favor of the small firm in our model. We also find that the larger is industry capacity relative to demand, the higher is the unit profit of the small firm relative to that of the large one (see Proposition C).

Furthermore, a particular sort of decrease in demand which preserves the monopoly output also improves the position of the small firm. Thus our theory predicts that a small firm in a collusive duopoly will earn a higher unit profit than a large firm, and that its edge will be greater when demand is low relative to industry capacity.

Our model and its conclusions may be contrasted with others in the literature. In Stigler's classic paper ([1964]), and in Green and Porter's [1981] reinterpretation of it, as well as in Radner [1980] and others, an oligopoly is viewed as a noncooperative game. The main issue these papers address concerns the conditions under which oligopolists will collude. Given the fact that a firm which deviates from a collusive agreement can be "punished" by a retaliatory deviation, there are some circumstances at least under which collusion can be noncooperatively sustained. The problem with this approach is that there is in general a wide range of equilibrium agreements. Unless more or less ad hoc restrictions are placed on the threats or agreements which are allowed, specific predictions of the sort we want are difficult to obtain. The Nash variable-threat bargaining solution defines a unique outcome. Its disadvantage is that it is not grounded in a well-formulated noncooperative analysis, though it is supported by Selten's axiomatization. Our focus is on the outcome of negotiations among the firms, given that they collude; ideally one might hope for an analysis which gives a precise answer to both the questions[1/]: When will firms collude? What sort of agreement will they reach if they do so?

Our prediction concerning the higher unit profit of a small firm coincides with that of Stigler [1964]. However, the assumptions under which it is obtained are different. Stigler derives his result in a model in which there is incomplete information. The idea is that small firms can more easily cheat on an agreement without being detected. In our model there is complete information; the relative advantage of the small firm varies not with the extent of imperfect information, but with the capacity of the industry relative to demand. Our predictions are thus distinct from Stigler's.

The basic structure of the model of Radner [1977] is the same as ours. His model is different because it uses the core as a solution rather than the Nash bargaining solution. He shows that the outcome where unit profits are equal for all firms is in the core, though many other outcomes are also. Thus, as in the noncooperative models, no specific prediction is obtained.

We have not attempted to systematically collect evidence on the behavior of collusive firms. However, there is one striking example which accords well with our results. This is the Addyston Pipe cartel, in which the negotiated agreement (see Stevens [1913], pp. 205-209; Bittlingmayer [1982]) precisely favors the small firms in times of low demand, while providing a more equal division in times of high demand. Although the case of OPEC is somewhat more complex (it is even disputed that it should be regarded as behaving collusively) there is also some evidence that the small producers do better than the large ones (see for example Gately [1979], p. 311).

If firms can legally form a cartel, it is possible for them to achieve any division of the monopoly profit by making side-payments; such transfers are frequently made in actual cartels. If any agreement has to be tacit, however, it may be more reasonable to restrict the feasible outcomes to those which can be realized by independent actions by the firms. The distinction between the two cases is particularly significant (as, e.g. Bain [1948] notes) if the cost functions of the firms differ significantly. For then, in order to obtain the monopoly profit, the firms may have to produce widely varying outputs; some may have to close down altogether. In our analysis we assume side-payments are possible. However, given our assumption of identical unit costs we can in fact show (see Proposition B) that the outcome predicted by our model can be achieved without any such transfers. If the costs of the firms are different, then this may not be so; although we could apply the solution concept to such a situation even if transfers were prohibited, we have not attempted to do so.

In the next section we outline the model and state our main results. In Section 3 we provide the details of the analysis; in Section 4 we make concluding comments.

2. An Outline of the Model and Results

There are two firms, indexed $i = 1, 2$. Firm i has capacity k_i; we assume throughout that $k_1 \geqq k_2 > 0$ (i.e. firm 1 is larger). Each firm can produce the same good at the same, constant unit cost $u \geqq 0$ up to its capacity. Given the prices of all other goods, we assume that the aggregate demand for the output of the firms is a continuous, decreasing function of price, and that there exists a price such that demand is zero. This means that the maximal joint profit (subject to the restriction that output be at most $k_1 + k_2 = k$) is well-defined for each value of k; we denote it $\Pi^*(k)$.

The problem of the firms is to reach an agreement on how this profit should be shared. We allow side-payments (though the final outcome can always be attained without them), so that any division of this profit is possible. The strategic variable of the firms is the quantity of output[2/]

(as in Cournot). If the firms non-cooperatively select quantities q_i and q_j, let the profit of firm i be $h_i(q_i, q_j)$. Now suppose that firm i (= 1, 2) threatens to select q_i. Then according to the compromise rule in the Nash variable-threat (henceforth NVT) bargaining solution, the firms will agree to split the jointly maximal profit $\Pi^*(k)$ so that firm i receives

$$(2.1) \qquad h_i(q_i, q_j) + \tfrac{1}{2}[\Pi^*(k) - h_i(q_i, q_j) - h_j(q_j, q_i)]$$
$$= \tfrac{1}{2}[\Pi^*(k) + h_i(q_i, q_j) - h_j(q_j, q_i)]$$
$$= v_i(q_i, q_j), \text{ say.}$$

That is, the excess of $\Pi^*(k)$ over the sum of the profits if the threats are carried out is split equally between the firms. Intuitively, the payoffs when the threats are carried out reflect the bargaining power of the firms; since each firm can equally well disrupt an agreement and cause a reversion to the threat-point, the split of the excess is equal. (For a more elegant justification, see the axiomatization of Selten [1960].) Each firm knows the compromise rule, and so chooses its threat, given that of its opponent, to maximize $v_i(q_i, q_j)$. Thus, a pair of optimal threats is simply a Nash equilibrium of the game with payoff functions v_i. Since this game is constant-sum, the equilibrium payoffs are unique (even though there may be many pairs of optimal strategies). These equilibrium payoffs are the outcome of the bargaining--they are the negotiated payoffs, which we denote $v_i^*(k_1, k_2)$ (i = 1, 2). Note that the fact that the game with payoff functions v_i is constant-sum also means that the optimal threats guarantee the negotiated payoffs.

We shall now summarize our results; the assumptions we list are detailed in the next section. We begin by assuming (3.1), which is quite weak. We show that unless there is significant undercapacity in the industry, the negotiated profit per unit of capacity is larger for the small firm than for the large one, and it is always at least as large. Let the <u>unconstrained monopoly output</u> be that quantity produced by a monopolist with no capacity constraint.

<u>Proposition A</u>: <u>Assume (3.1). Then for each value of</u> (k_1, k_2) <u>we have</u> $v_1^*(k_1, k_2)/v_2^*(k_1, k_2) \leq k_1/k_2$. <u>If</u> $k_1 > k_2$, <u>and the total capacity</u> k <u>of the firms exceeds the unconstrained monopoly output, then the inequality is strict</u>.

We now make a stronger assumption, which allows us to obtain the optimal threats explicitly, and hence to establish some qualitative features of the solution. First, we show that the negotiated payoffs can be attained without any explicit transfers of payoff.

<u>Proposition B</u>: <u>Assume (3.1), (3.9), and (3.10). Then for each value of</u> (k_1, k_2) <u>there is a price</u> r <u>and an output</u> $q_i \leq k_i$ <u>for each firm</u> $i = 1, 2$ <u>such that</u> $v_i^*(k_1, k_2) = (r - u)q_i$.

Next, we show that the higher is industry capacity relative to demand, the better off is the small firm relative to the large one.

<u>Proposition C</u>: <u>Assume (3.1), (3.9), and (3.10). Then if the relative sizes of the firms are fixed, the larger the industry capacity relative to demand, the smaller is</u> $v_1^*(k_1, k_2)/v_2^*(k_1, k_2)$.

We can also analyze the effects of a change in the shape of the demand function. (For a definition of the "regions", see Diagram 2.)

Proposition D: Assume (3.1), (3.9), and (3.10). Suppose that demand falls at all prices below the monopoly price, which remains the same. Then the ratio $v_1^*(k_1, k_2)/v_2^*(k_1, k_2)$ falls for all values of (k_1, k_2) in regions II and III, and is unchanged in regions I and IV.

3. The Model

Let $D: \mathbb{R}_+ \to \mathbb{R}_+$ be the aggregate demand function for the good. Let p denote the excess of price over unit cost, and let $X = [-u, \infty)$; we shall frequently refer, somewhat loosely, to an element of X as a "price". Define $d: X \to \mathbb{R}_+$ by $d(p) = D(p + u)$; $d(p)$ is the demand for the good when its price exceeds the unit cost by p. We assume that[3/]

(3.1) d is continuous, there exists $p_o > 0$ such that $d(p) = 0$ if $p \geqq p_o$ and $d(p) > 0$ if $p < p_o$, and d is decreasing for $p \leqq p_o$.

It is more convenient here to work with the inverse demand function $P: [0, \infty) \to X$ defined by $P(q) = d^{-1}(q)$ if $0 < q \leqq d(-u)$, $P(0) = p_o$, and $P(q) = -u$ otherwise. (This incorporates the assumption that if a quantity in excess of $d(-u)$ is offerred for sale then it is all sold at the price $-u$.) By virtue of (3.1), P is well-defined, and decreasing on $(0, d(-u))$. Let $\Pi(q) = qP(q)$ (the profit associated with q), and let $\Pi^*(k)$ be the maximal joint profit of the firms--i.e.

$$\Pi^*(k) = \max_q \{\Pi(q): \ 0 \leqq q \leqq k\}$$

(recall that $k = k_1 + k_2$). We normalize the units in which quantity is measured so the unconstrained maximizer[4/] of Π (the "unconstrained monopoly output", which exists by (3.1)) is 1; we choose the units for price so that the unconstrained maximal profit is also 1 (so that $d(1) = 1$).

Now, if firm i chooses the output q_i and firm j chooses q_j, the market sets the price $P(q_i+q_j)$, so that the payoff to firm i is $h_i(q_i,q_j) = q_iP(q_i+q_j)$. Let $S(x) = [0, x]$, the set of possible outputs of a firm with capacity x, and let $H(k_1, k_2)$ be the game in which the strategy set of $i = 1, 2$ is $S(k_i)$ and its payoff function is h_i. For each value of (k_1, k_2) we are interested in the NVT bargaining solution of $H(k_1, k_2)$. If q_i and q_j are chosen as threats in this game, this solution involves a compromise payoff to firm i of

(3.2) $$v_i(q_i,q_j;k) = \tfrac{1}{2}[\Pi^*(k) + (q_i - q_j)P(q_i + q_j)]$$

(see (2.1)). Let $V(k_1, k_2)$ be the game in which the strategy set of firm $i = 1, 2$ is $S(k_i)$, and its payoff function is v_i. Then the NVT bargaining solution of $H(k_1, k_2)$ is a Nash equilibrium of $V(k_1, k_2)$, the equilibrium payoffs being the "negotiated payoffs" of our model. Denote the negotiated payoff of firm i by $v_i^*(k_1, k_2)$.

For each pair (k_1, k_2), each v_i is continuous in (q_i, q_j), so that if we allow the firms to use mixed strategies (i.e. probability distributions on $S(k_i)$), then the game $V(k_1, k_2)$ has an equilibrium. (Below we shall give conditions on the inverse demand function P under which there is an equilibrium in pure strategies, but we do not need these restrictions to establish our result on the relative sizes of the equilibrium payoffs.) Thus we have:

Remark 3.3: Under (3.1), the game $V(k_1, k_2)$ has an equilibrium for each value of (k_1, k_2).

We now claim that if a firm increases in size (the size of the other firm fixed), its negotiated payoff does not decrease. First note that the negotiated payoff $v_i^*(k_1, k_2)$ is equal to $\frac{1}{2}\Pi^*(k)$ plus the equilibrium payoff to i in the zero-sum game with payoffs $\frac{1}{2}(q_i-q_j)P(q_i+q_j)$. If k_i increases, k_j fixed, then $S(k_i)$ expands, while $S(k_j)$ is constant, so that the equilibrium payoff to i in the zero-sum game does not decrease. Since $\Pi^*(k)$ is either constant, or increases, this means that $v_i^*(k_1, k_2)$ is non-decreasing. Since both firms are identical if $k_1 = k_2$, we must also then have $v_i^*(k_1, k_2) = \frac{1}{2}\Pi^*(k)$. Thus we have:

Remark 3.4: For $i = 1, 2$, $v_i^*(k_1, k_2)$ is non-decreasing in k_i, with $v_i^*(k_1, k_2) = \frac{1}{2}\Pi^*(k)$ if $k_1 = k_2$, and hence $v_1^*(k_1, k_2) \geqq v_2^*(k_1, k_2)$.

We now study the negotiated profit of firm 1 (the large firm). We show that it is at most $(k_1/k)\Pi^*(k)$ by arguing that even if firm 2 always threatens $q_2 = k_2$, rather than using its best response to the threat of firm 1, it is at most this amount.

Lemma 3.5: Under (3.1), for each value of (k_1, k_2) we have $v_1^*(k_1, k_2) \leqq v_1(m,k_2;k) \equiv \frac{1}{2}[\Pi^*(k) + (m-k_2)P(m+k_2)]$ for some $k_2 \leqq m \leqq k_1$.

Proof: For every value of (k_1, k_2) we have

$$v_1^*(k_1, k_2) = \min_{F_2} \max_{q_1} \{v_1(q_1,F_2;k): \ F_2 \in \mathcal{S}(k_2) \ \text{ and } \ q_1 \in S_q(k_1)\},$$

where $\mathcal{S}(k_2)$ is the set of mixed strategies of firm 2 (i.e. the probability distributions on $S(k_2)$), and we have extended v_1 to this set in the natural way. Hence certainly

$$v_1^*(k_1, k_2) \leqq \max_{q_1 \leqq k_1} v_1(q_1,k_2;k) = \max_{q_1 \leqq k_1} \tfrac{1}{2}[\Pi^*(k) + (q_1 - k_2)P(q_1 + k_2)]$$

$$= \tfrac{1}{2}[\Pi^*(k) + (m - k_2)P(m + k_2)], \text{ say.}$$

If $q_1 < k_2$ then the second term in $v_1(q_1,k_2;k)$ is negative, so it is clear that the maximizer m satisfies $m \geqq k_2$. This completes the proof.

The bound on $v_1(x,y;x+y)$ provided in the following will be used to bound $v_1(m,k_2;k)$, so that, using Lemma 3.5, we can establish the bound on $v_1^*(k_1, k_2)$ which we desire.

Lemma 3.6: Under (3.1), for any x, y with $x \geqq y > 0$ we have $v_1(x,y;x+y) = \frac{1}{2}[\Pi^*(x+y) + (x-y)P(x+y)] \leqq (x/(x+y))\Pi^*(x+y)$, with strict inequality if $(x+y)P(x+y) < \Pi^*(x+y)$ and $x > y$.

Proof: The inequality follows directly, noting that by definition $\Pi^*(x+y) \geqq (x+y)P(x+y)$.

Proposition 3.7: Assume (3.1) is satisfied. Then for each value of (k_1, k_2) we have $v_1^*(k_1, k_2) \leqq (k_1/k)\Pi^*(k)$, so that $v_1^*(k_1, k_2)/v_2^*(k_1, k_2) \leqq k_1/k_2$; if $\Pi^*(k) > kP(k)$, and hence in particular if $k > 1$, the inequalities are strict whenever $k_1 > k_2$.

Proof: Setting $x = m$ (see Lemma 3.5) and $y = k_2$ in Lemma 3.6 we have

$$(3.8)\quad v_1(m,k_2;m+k_2) = \tfrac{1}{2}[\Pi^*(m+k_2) + (m-k_2)P(m+k_2)] \leqq (m/(m+k_2))\Pi^*(m+k_2),$$

with strict inequality if $m = k_1 > k_2$ and $kP(k) < \Pi^*(k)$. But $m \leqq k_1$, so $\Pi^*(m+k_2) \leqq \Pi^*(k)$. Hence the inequality in (3.8) implies that

$$\tfrac{1}{2}[\Pi^*(k) + (m-k_2)P(m+k_2)] \leqq (m/(m+k_2))\Pi^*(k).$$

Using Lemma 3.5 we have $v_1^*(k_1, k_2) \leqq (m/(m+k_2))\Pi^*(k) \leqq (k_1/k)\Pi^*(k)$, with the first inequality strict if $m = k_1 > k_2$ and $kP(k) < \Pi^*(k)$, and the second one strict if $m < k_1$. This completes the proof.

Proposition A (see Section 2) is simply a restatement of Proposition 3.7. This result tells us that the negotiated unit profit of the small firm is always at least equal to that of the large firm, and that if the firms together can produce at least the unconstrained monopoly output, then the small firm is strictly better off. Without making more assumptions about the inverse demand function we cannot say much more about the negotiated profits; however, we do have the following.

Lemma 3.8: Under (3.1), if $k_2 \geqq \frac{1}{2}d(0)$ (and hence $k_1 \geqq \frac{1}{2}d(0)$), then $(\frac{1}{2}d(0), \frac{1}{2}d(0))$ is a pair of optimal threats, and the negotiated profits are $v_i^*(k_1, k_2) = \frac{1}{2}\Pi^*(k)$, $i = 1, 2$.

Proof: It is easy to check, using (3.2), that for this range of (k_1, k_2), $(\frac{1}{2}d(0), \frac{1}{2}d(0))$ is an equilibrium of $V(k_1, k_2)$, with payoff $\frac{1}{2}\Pi^*(k)$ to each firm.

Thus any capacity in excess of half the quantity demanded at zero price has no effect on the negotiated profits, which are always subsequently split equally.

We now make an additional assumption on the inverse demand function P which ensures that each payoff function v_i is quasi-concave in q_i, so that there exists an equilibrium in pure strategies.

(3.9) Π is concave.

(Recall that $\Pi(q) = qP(q)$.) For convenience we also assume that

(3.10) P is smooth on $(0, d(-u))$.

Proposition 3.11: Under assumptions (3.1), (3.9), and (3.10), for each value of (k_1, k_2) each payoff function v_i is quasi-concave in q_i, so that the game $V(k_1, k_2)$ has a pure strategy equilibrium.

Proof: We have $v_i(q_i,q_j;k) = \frac{1}{2}[\Pi^*(k) + (q_i-q_j)P(q_i+q_j)]$, so that

$$\partial v_i(q_i,q_j;k)/\partial q_i = \tfrac{1}{2}[P(q_i+q_j) + (q_i-q_j)P'(q_i+q_j)].$$

Hence if $q_i \leqq q_j$, v_i is nondecreasing in q_i. Also

$$\begin{aligned} \text{(3.12)} \qquad \partial^2 v_i(q_i,q_j;k)/\partial q_i^2 &= \tfrac{1}{2}[2P'(q_i+q_j) + (q_i-q_j)P''(q_i+q_j)] \\ &= \tfrac{1}{2}[2P'(q_i+q_j) + (q_i+q_j)P''(q_i+q_j) - 2q_jP''(q_i+q_j)]. \end{aligned}$$

If $q_i > q_j$, and $P''(q_i+q_j) \leqq 0$ then by the first line of (3.12) this second derivative is nonpositive, while if $P''(q_i+q_j) \geqq 0$ it is nonpositive by the second line (using the concavity of Π, which implies that $2P'(q) + qP''(q) \leqq 0$ for all q). Thus v_i is quasi-concave in q_i, completing the proof.

In the remainder of this section, we maintain assumptions (3.9) and (3.10). We then have

$$\text{(3.13)} \qquad \Pi^*(k) = \begin{cases} kP(k) & \text{if } 0 \leqq k \leqq 1 \\ 1 & \text{if } k > 1 \end{cases}$$

(recall that the unconstrained monopoly output is normalized to 1). Furthermore, the quasi-concavity of the payoff functions implies that for each value of q_j, firm i has a pure best response, say $B_i(q_j)$, which has the form $B_i(q_j) = \min(k_i, A(q_j))$ (where $A(q_j)$ is the "unconstrained best response"--i.e. global maximizer of $v_i(\cdot,q_j;k)$). From (3.2) it is easy to argue that $A(0) = 1$, and $\min(q_j, 1-q_j, \frac{1}{2}d(0)-q_j) \leqq A(q_j) \leqq$

max $(q_j, \frac{1}{2}d(0)-q_j)$ for all q_j. (An example is shown in Diagram 1; $\frac{1}{2}d(0) > 1/2$, though it may be larger or smaller than 1, depending on the shape of P.) From this it follows that the optimal threats (Nash equilibrium strategies in the game $V(k_1, k_2)$) are $(\min(k_1, A(k_2)), k_2)$ if $k_2 \leqq \frac{1}{2}d(0)$ (recall that $k_1 \geqq k_2$) and $(\frac{1}{2}d(0), \frac{1}{2}d(0))$ if $k_2 \geqq \frac{1}{2}d(0)$ (the latter is true even without assumptions (3.9) and (3.10): see Lemma 3.8). Hence, using (3.13), the negotiated payoff of firm i is

$$(3.14)\quad v_i^*(k_1, k_2) = \begin{cases} \frac{1}{2}[kP(k) + (k_i - k_j)P(k)] = k_iP(k) = (k_i/k)\Pi^*(k) & \text{in region I} \\ \frac{1}{2}[1 + (k_i - k_j)P(k)] & \text{in region II} \\ \frac{1}{2}[1 + (-1)^{i+1}(A(k_2) - k_2)P(A(k_2) + k_2)] & \text{in region III} \\ 1/2 & \text{in region IV,} \end{cases}$$

where the various regions are as indicated in Diagram 2. If we fix k_2, and increase k_1, then the negotiated payoff of firm 1 as a fraction of the maximal joint profit (i.e. $v_1^*(k_1, k_2)/\Pi^*(k)$) varies as shown in Diagram 3. (The diagram is drawn for the case $k_2 < 1/2$.) In particular, it is a concave function of k_1: additions to capacity increase firm 1's share of the total payoff at a decreasing rate. We can also use (3.14) to examine the effect of increasing industry capacity relative to demand, keeping the relative sizes of the firms fixed. We obtain the following, which gives us Proposition C, and is illustrated in Diagram 4.

<u>Proposition 3.15</u>: <u>Let</u> $k_1 = ck$, $k_2 = (1-c)k$, <u>where</u> $c > 1/2$ <u>is a constant, so that</u> $k_1/k_2 = c/(1-c)$ <u>is constant. Then the ratio of negotiated profits</u> $v_1^*(ck, (1-c)k)/v_2^*(ck, (1-c)k)$ <u>is constant (equal to</u> $c/(1-c)$) <u>if</u> $k \leqq 1$, <u>decreasing in</u> k <u>if</u> $1 \leqq k \leqq d(0)/2(1-c)$, <u>and constant (equal to</u> 1) <u>thereafter</u>.

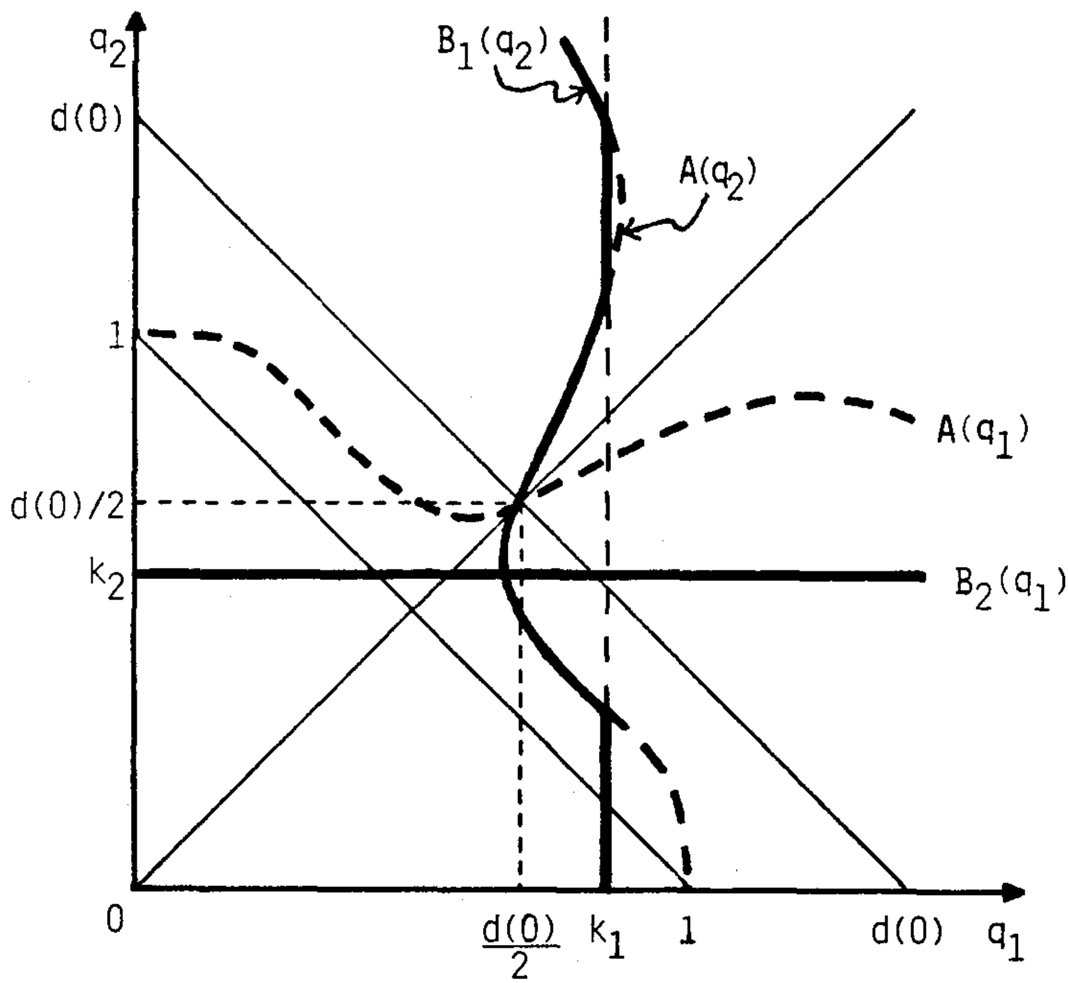
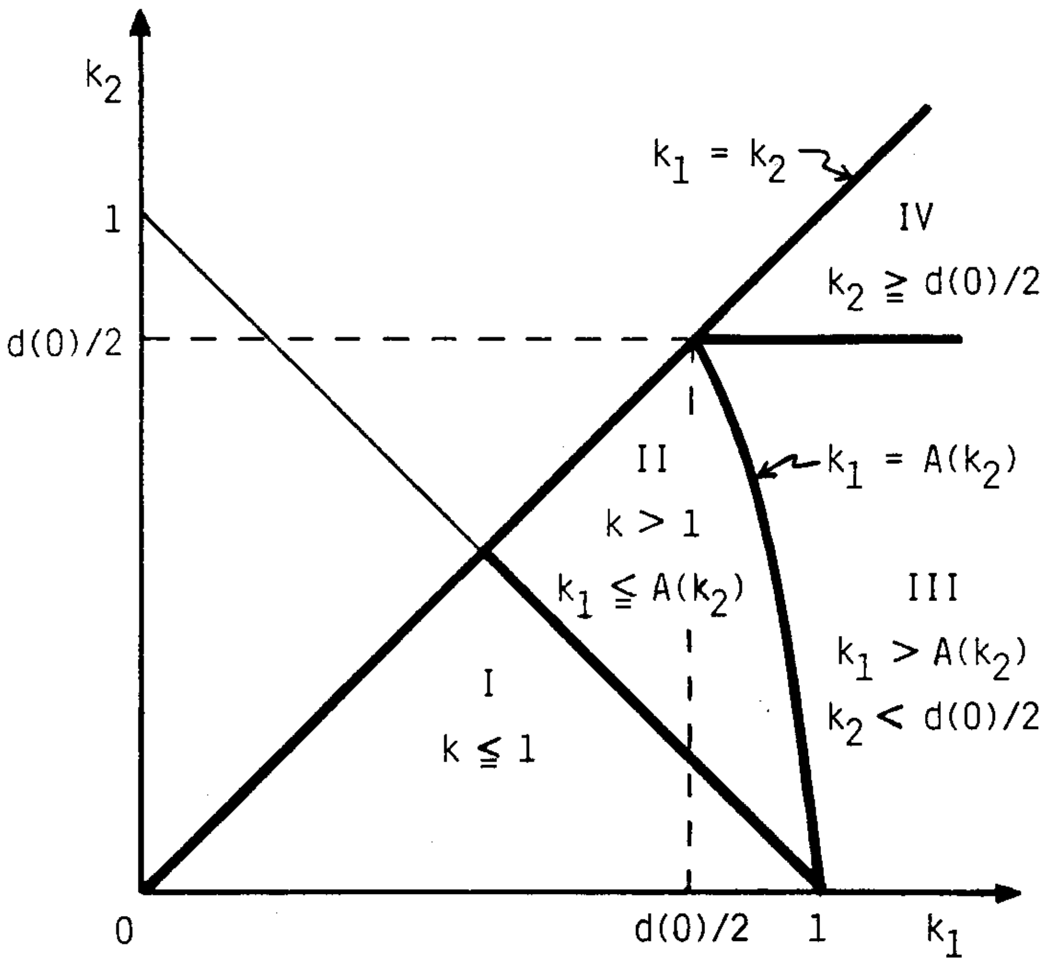


Diagram 1



k_2
1
d(0)/2
0
d(0)/2
1
k_1
$k_1 = k_2$
IV
$k_2 \geqq d(0)/2$
$k_1 = A(k_2)$
II
$k > 1$
$k_1 \leqq A(k_2)$
III
$k_1 > A(k_2)$
$k_2 < d(0)/2$
I
$k \leqq 1$

Diagram 2

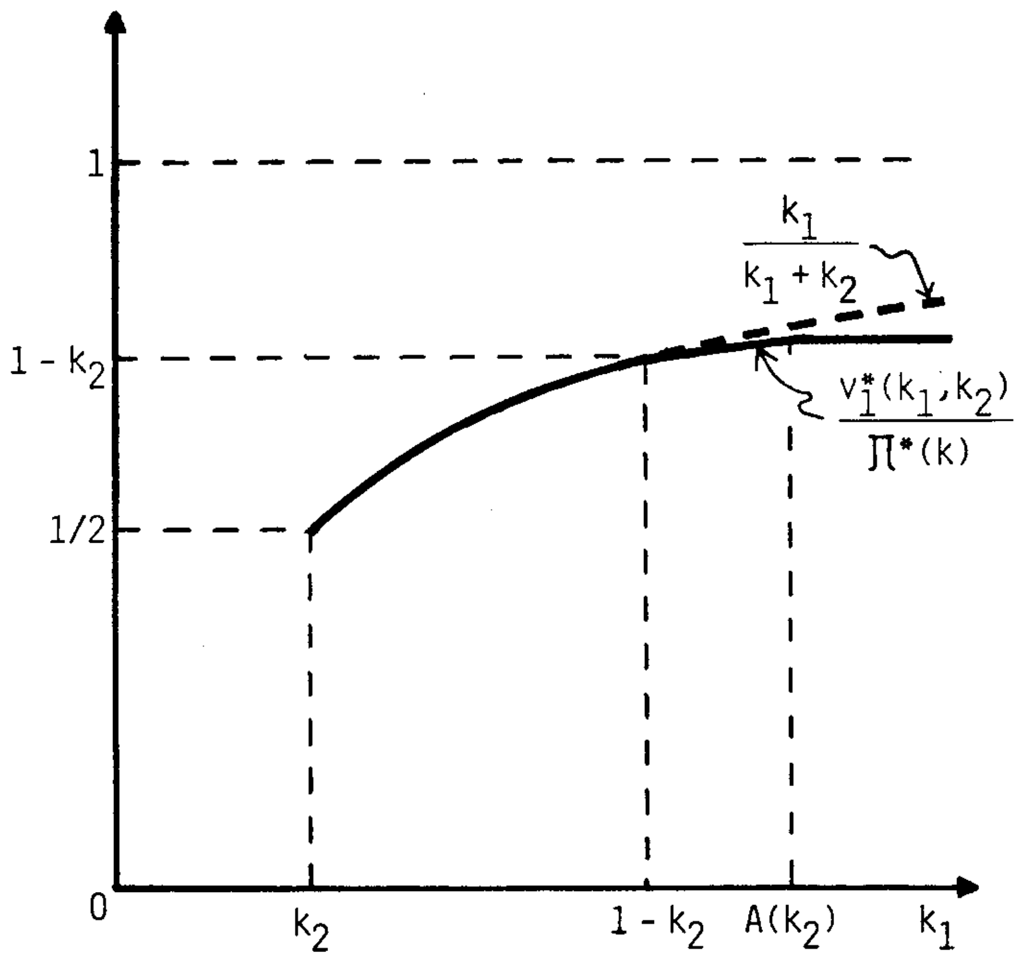


Diagram 3

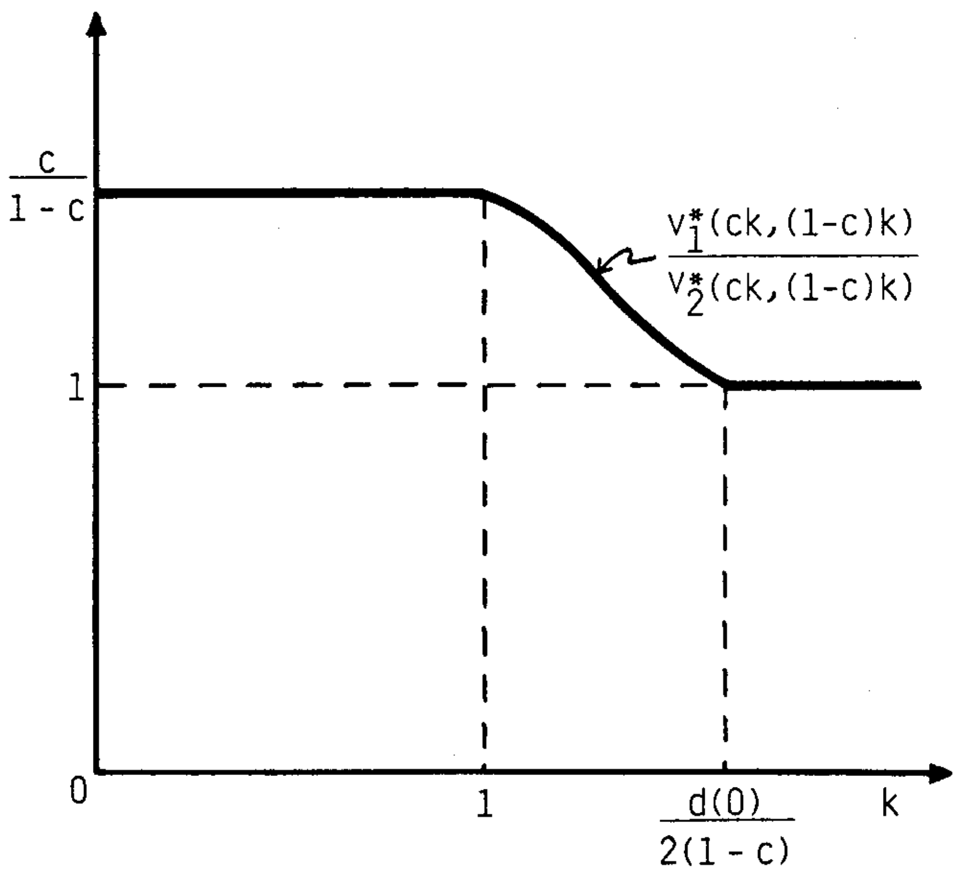


Diagram 4

Finally, we consider the effect of changing the demand function while keeping the capacities of the firms fixed relative to the unconstrained monopoly output. If $k > 1$, then the total output under the optimal threats exceeds the output which maximizes joint profits, so that if the inverse demand curve shifts downwards to the right of $q = 1$, the payoffs at the threat point are reduced for both firms, and the difference between them must also decrease. Because of this last effect, the ratio of the negotiated profits turns in favor of the small firm. Formally, we have the following, and hence Proposition D.

Proposition 3.16: *Let* P_o *and* P_1 *be inverse demand functions which satisfy (3.1), (3.9), and (3.10). Assume that* $P_o'(1) = P_1'(1) = -1$, *so that joint profits are maximized at* $q = 1$ *and the maximum is* 1 *in each case; also assume that* $P_1(q) = P_o(q)$ *if* $q \leqq 1$ *or* $q \geqq P_o^{-1}(-u)$, *and* $P_1(q) < P_o(q)$ *if* $1 < q < P_o^{-1}(-u)$. *Then for any fixed* (k_1, k_2) *in regions II or III before the change, the ratio* $v_1^*(k_1, k_2)/v_2^*(k_1, k_2)$ *is lower for the inverse demand* P_1 *than for* P_o; *for other values of* (k_1, k_2) *the ratio is unchanged.*

Proof: First consider those cases where the capacities are such that after the change in P, and hence in the shapes of the regions, (k_1, k_2) lies in the same region as initially. If $k > 1$ then the assumption on demands implies that $P_1(k) < P_o(k)$, so that if (k_1, k_2) is in region II, the negotiated profit of firm 1 falls, while that of firm 2 rises (since $k_1 - k_2 > 0$). For (k_1, k_2) in region III, note that q_1 is chosen by firm 1 to be $A(k_2)$ in order to maximize $(q_1 - k_2)P(q_1 + k_2)$, and $A(k_2) + k_2 > 1$ (see the properties of A). Hence the change in P causes a decrease in this maximum and hence a decrease in the negotiated profit of firm 1 and an increase in that of firm 2. If (k_1, k_2) is in regions I or IV, there is no change in either firm's negotiated profit. Finally, if the change in P causes (k_1, k_2) to fall in a different region, it is clear that the change in the ratio of the payoffs is in the same direction. This completes the proof.

Finally, we can show that, under (3.9), the negotiated payoffs can be attained without any transfers of payoff (see Proposition B). In other words, each firm has enough capacity to produce what is required, at the monopoly price, to generate its negotiated profit; or $v_i^*(k_1, k_2) \leqq k_i p_m(k)$, where $p_m(k)$ is the monopoly price, for $i = 1, 2$.

Proposition 3.17: Under assumptions (3.1), (3.9), and (3.10), we have $v_i^*(k_1, k_2) \leqq k_i p_m(k)$ $(i = 1, 2)$, for all pairs (k_1, k_2).

Proof: First note that if $k \leqq 1$ then $v_i^*(k_1, k_2) = k_i P(k)$ (see (3.14)) and $p_m(k) = P(k)$, so that the result is certainly true. Also note that since $v_1^*(k_1, k_2)/v_2^*(k_1, k_2) \leqq k_1/k_2$ (see Proposition A), it is enough to show that $v_2^*(k_1, k_2) \leqq k_2 p_m(k)$. Now, if $k > 1$ and $k_2 \geqq 1/2$, then the result is certainly true since $p_m(k) = 1$ and $v_2^*(k_1, k_2) \leqq 1/2$ (the latter from the facts that $v_1^*(k_1, k_2) \geqq v_2^*(k_1, k_2)$ (see Remark 3.4) and $\Pi^*(k) = 1$). If $k > 1$ and $k_2 < 1/2$ then we have $v_2^*(k_1, k_2) \leqq v_2^*(1-k_2, k_2)$, since $v_2^*(k_1, k_2)$ is nonincreasing in k_1 (from Remark 3.4). But by the argument above for $k \leqq 1$, we have $v_2^*(1-k_2, k_2) = k_2$, so $v_2^*(k_1, k_2) \leqq k_2$, completing the proof.

4. Discussion

We have analyzed a model of duopoly in which the firms' behavior is basically cooperative, but the outcome depends on the fact that each firm can threaten to act "disruptively". Here we make two brief concluding comments.

The game-theoretic solution concept we have used (the NVT bargaining solution) has a number of attractive features. It involves the Nash equilibrium of a strictly competitive game, so that each firm guarantees its equilibrium payoff by using an equilibrium strategy. Also, Selten [1960] shows that it satisfies properties of strategic monotonicity (more strategies cannot hurt a player), and payoff monotonicity (if a player's payoffs to all strategy pairs increase, while the maximal sum of the payoffs to both players stays the same, then that player's negotiated payoff increases). In fact, Selten characterizes the solution by a number of such axioms. One feature of the solution which may at first seem unreasonable concerns the coefficients 1/2 in (2.1). Since there is a natural measure of "size" of a player in our specific context, it might be thought that the 1/2 coefficients are inappropriate, especially when the solution is interpreted as a two-stage bargaining process. However, Selten's axiomatization shows that this may be an erroneous conclusion. The 1/2 coefficients come from a symmetry axiom (in conjunction, of course, with the other axioms), which is very natural to impose. It says that players can get different payoffs only because of differences in their strategy sets or payoff functions. Thus in our case firm 1 receives a higher negotiated payoff than firm 2 (if $k_1 > k_2$) because it has different strategies available to it, and not simply because it is larger: largeness only conveys bargaining power because it is associated with different strategic options.

Now consider the effect of allowing increasing, rather than constant returns to scale (up to capacity). In a simple case, the solution given by our model can be applied. Suppose the cost function is of the form $C(x) = s + ux$ (if $x \leqq k_i$), where $s, u > 0$, so that there is a fixed cost s. If we let p be the excess of price over u, then the payoffs in $H(k_1, k_2)$ are exactly s less than they are in our model. The change in the maximal joint payoff depends on the interpretation given to the fixed cost s. If a firm can close down and avoid paying s in the time period relevant for our model, then it might be best for this to happen, and for all the production to take place in the larger firm, or for both firms to close down. In any case, the maximal joint payoff is lower than under our original assumptions (by at most $2s$). Thus for any given pair (k_1, k_2) the payoffs in the bargaining game $V(k_1, k_2)$ fall by the same amount (at most s) for every pair (q_1, q_2). This means that the optimal strategies in each case are identical to those under our earlier assumptions. The only change is in the negotiated payoffs, which both decrease by the same amount, so that the ratio of firm 1's payoff to that of firm 2 increases. Hence if the ratio is sufficiently close to k_1/k_2, it may exceed this under the new assumption. In fact, in our model, the ratio is precisely k_1/k_2 if $k \leqq 1$, and decreases as k_1 increases, k_2 fixed, if $k > 1$. Thus in the new situation the large firm earns a higher unit profit than the small one if the total capacity in the industry is small (relative to the optimal output of a monopolist without a capacity constraint), but a lower unit profit if the industry capacity is sufficiently large.

Footnotes

1/ In a static setting, D. K. Osborne [1976] claims that the outputs defined by joint profit maximization, together with some particular threats which are to be used in the event of deviations, give firms the proper incentives to collude, and define a reasonable outcome in this case (in his terminology, they provide a solution to the "deterrence" and "sharing" problems). However, as he points out, the threats which support his outcome are not completely rational ("perfect") (see p. 839). Moreover, as we have pointed out above (and as was noted by Bain [1948]), the outputs entailing joint profit maximization do not plausibly reflect the relative bargaining positions of the firms.

2/ We have also analyzed the case where price is the strategic variable (as in Bertrand). Qualitative results analagous to those here can be obtained; we shall report them in a subsequent paper.

3/ We could relax the continuity of d, and subsequently appeal to the results of Dasgupta and Maskin [1982] rather than the classical result.

4/ If there is more than one, take the largest.

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