COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut 06520

COWLES FOUNDATION DISCUSSION PAPER NO. 663

THE EXACT DISTRIBUTION OF LIML: II

Peter C. B. Phillips

February 21, 1983

THE EXACT DISTRIBUTION OF LIML: II

by

P. C. B. Phillips*

Cowles Foundation
Yale University

0. ABSTRACT

This paper derives the exact probability density function of the limited information maximum likelihood (LIML) estimator of the coefficient vector of the endogenous variables in a structural equation containing $n+1$ endogenous variables and $L \geq 1$ degrees of overidentification. This generalizes the presently known results for the two endogenous variable case $(n+1 = 2)$ and the leading case analyses in the author's earlier paper (1982). Upon appropriate symbolic translation the results may be applied directly to the maximum likelihood estimator in the multivariate linear functional relationship.

---

*My thanks go to Glena Ames for her skill and effort in typing the manuscript of this paper.

1. INTRODUCTION

In recent years there has been a renewed interest in the limited information maximum likelihood (LIML) method of estimation in the simultaneous equations model. Some of this interest has been directed towards modifications of LIML which yield estimators with certain improved features. The improvements have been measured by the criterion of second order asymptotic efficiency and by the extent to which the modifications thin out the tails of the distribution, thereby reducing the probability of extreme outliers in finite sample LIML estimation. Studies of such modifications have been made by Fuller (1977), Kunitomo (1981) and Morimune (1981); and a review of this work may be found in Phillips (1983a). A second direction of interest has involved extensive numerical tabulations of the exact distributions of competing estimators in the case of a single equation with two endogenous variables. These tabulations have led to a reassessment of the relative merits of LIML and two stage least squares (2SLS) as competing estimators. In particular, the distribution of LIML is shown to have a superior central location and a more rapid approach to its asymptotic distribution than the distribution of 2SLS. The differences working in favor of LIML are most striking when the degree of equation overidentification is large and when there is a high correlation between the endogenous regressor and the structural equation error. The reader is referred to Anderson (1982) for a detailed account of this work.

The present paper is concerned with the distribution of the LIML estimator in the general single equation case. As such it is a sequel to an earlier paper by the author (1982) (hereafter referred to as

LIML: I) which dealt with a leading case of the general problem. The exact probability density function (p.d.f.) of the LIML estimator in an equation with two $(n+1 = 2)$ endogenous variables and an arbitrary degree of overidentification $(L \geq 1)$ was found by Mariano and Sawa (1972). In Basmann's (1974) notation their result characterizes the following subclass of distributions:

$$\bigcup_{L=1}^{\infty} M_{1,L}$$

where $M_{n,L}$ denotes the joint distribution on $\mathbb{R}^n$ of the LIML estimator of the coefficients of the n right-hand side endogenous variables in an equation with L degrees of overidentification. The results of the present paper characterize in the same notation the complete class of distributions

$$M = \bigcup_{n=1}^{\infty} \bigcup_{L=1}^{\infty} M_{n,L}$$

corresponding to a structural equation containing any number of endogenous variables and an arbitrary degree of overidentification.

2. THE MODEL AND NOTATION

As in LIML: I, we work with the structural equation

(1) $$y_1 = Y_2\beta + Z_1\gamma + u$$

where $y_1(T \times 1)$ and $Y_2(T \times n)$ are an observation vector and observation matrix, respectively, of $n+1$ included endogenous variables, Z_1

is a $T \times K_1$ matrix of included exogenous variables, and u is a random disturbance vector. The reduced form of (1) is written

$$(2) \qquad [y_1 \vdots Y_2] = [Z_1 \vdots Z_2]\begin{bmatrix} \pi_{11} & \Pi_{12} \\ \pi_{21} & \Pi_{22} \end{bmatrix} + [v_1 \vdots V_2] = Z\Pi + V ,$$

where Z_2 is a $T \times K_2$ matrix of exogenous variables excluded from (1). The rows of the reduced form disturbance matrix V are assumed to be independent, identically distributed, normal random vectors. We assume that the standardizing transformations (see Phillips (1983a) for full details) have been carried out, so that the covariance matrix of each row of V is the identity matrix and $T^{-1}Z'Z = I_K$ where $K = K_1 + K_2$. We also assume that $K_2 \geq n+1$ so that the degree of overidentification is $L = K_2 - n \geq 1$. When the equation is just identified $(K_2 = n)$ LIML reduces to indirect least squares and the exact distribution theory in Sargan (1976) and Phillips (1980) applies.

We write the LIML estimator of β in (1) as β_{LIML} and define the matrices $W = X'(P_Z - P_{Z_1})X$, $S = X'(I - P_Z)X$ where $X = [y_1 \vdots Y_2]$ and $P_A = A(A'A)^{-1}A'$. β_{LIML} minimizes the ratio $\beta_\Delta' W \beta_\Delta / \beta_\Delta' S \beta_\Delta$, where $\beta_\Delta' = (1, -\beta')$, and satisfies the system

$$(3) \qquad (W - \lambda S)\beta_\Delta = 0$$

where λ is the smallest latent root of the matrix $S^{-1/2}WS^{-1/2}$. β_Δ in (3) also satisfies

$$(4) \qquad [S - f(W+S)]\beta_\Delta = 0$$

where $f = (1+\lambda)^{-1}$ is the largest latent root of $(W+S)^{-1/2}S(W+S)^{-1/2}$.

3. THE DISTRIBUTION OF LIML

The first steps of the derivation follow those of LIML: I. In particular, let the $m = n+1$ roots of the equation

$$\det[S - f(W+S)] = 0 \tag{5}$$

be ordered $f_1 > f_2 > \ldots > f_m > 0$ and assembled into the matrix $F = \text{diag}(f_1, f_2, \ldots, f_m)$. Further, let the corresponding vectors g_i satisfying $[S - f_i(W+S)]g_i = 0$ be normalized by $g_i'(W+S)g_i = 1$ and assembled into the matrix $G = [g_1, g_2, \ldots, g_m]$. We set $E = G^{-1}$ and define a transformation $(S,W) \to (E,F)$ by the equations

$$S = E'FE \; , \quad W = E'(I-F)E \; . \tag{6}$$

This transformation is made one to one by the imposition of a sign requirement on a particular column of $E = (e_{ij})$. We choose the final column of E (as in LIML: I) and set $e_{i,n+1} \geq 0$ for all i .

Our distribution theory begins with the joint p.d.f. of (W,S) . W and S are independent Wishart matrices. S is $W_m(T-K, I)$ (as in LIML: I) and W is noncentral Wishart $W_m(K_2, I, \overline{M})$. The noncentrality matrix $\overline{M}$ is given by

$$\begin{aligned} \overline{M} = MM' &= E(T^{-1/2}X'Z_2)E(T^{-1/2}Z_2'X) \\ &= T\Pi'\begin{bmatrix} 0 \\ \hline I_{K_2} \end{bmatrix}[0 \vdots I_{K_2}]\Pi \\ &= T\begin{bmatrix} \Pi_{21}' \\ \hline \Pi_{22}' \end{bmatrix}[\Pi_{21} \vdots \Pi_{22}] \\ &= T\begin{bmatrix} \beta' \\ \hline I_n \end{bmatrix}\Pi_{22}'\Pi_{22}[\beta, I] \; . \end{aligned} \tag{7}$$

The joint density of (W,S) is

$$(8) \qquad \mathrm{pdf}(W,S) = \frac{\mathrm{etr}\left(-\frac{1}{2}MM'\right)\mathrm{etr}\left\{-\frac{1}{2}(W+S)\right\}}{2^{m(T-K_1)/2}\Gamma_m\left(\frac{K_2}{2}\right)\Gamma_m\left(\frac{T-K}{2}\right)}$$

$$\cdot(\det W)^{(K_2-m-1)/2}(\det S)^{(T-K-m-1)/2}\,{}_0F_1\left(\frac{K_2}{2};\ \frac{1}{4}MM'W\right)\ .$$

Writing $\Pi_{22}'\Pi_{22}$ as $\bar{\Pi}_{22}'\bar{\Pi}_{22}$ where $\bar{\Pi}_{22}$ is an $n \times n$ matrix we find that

$$(9) \qquad \mathrm{etr}\left(-\frac{1}{2}MM'\right) = \mathrm{etr}\left\{-\frac{T}{2}(I+\beta\beta')\bar{\Pi}_{22}'\bar{\Pi}_{22}\right\}$$

and

$$(10) \qquad {}_0F_1\left(\frac{K_2}{2};\ \frac{1}{4}MM'W\right) = {}_0F_1\left(\frac{K_2}{2};\ \frac{T}{4}\bar{\Pi}_{22}[\beta \ \vdots\ I]W\left[\frac{\beta'}{I}\right]\bar{\Pi}_{22}'\right)\ .$$

The jacobian of the transformation $(S,W) \to (E,F)$ is

$$(11) \qquad 2^m|\det E|^{m+2}\prod_{i<j}(f_i - f_j)$$

(as in equation (22) of LIML: I). We deduce from (8)-(11) that

$$(12) \qquad \mathrm{pdf}(E,F) = \frac{2^m\mathrm{etr}\left\{-\frac{T}{2}(I+\beta\beta')\bar{\Pi}_{22}'\bar{\Pi}_{22}\right\}}{2^{m(T-K_1)/2}\Gamma_m\left(\frac{K_2}{2}\right)\Gamma_m\left(\frac{T-K}{2}\right)}$$

$$\cdot\mathrm{etr}\left(-\frac{1}{2}E'E\right)[\det(E'E)]^{(T-K_1-m-1)/2}|\det E|^{m+2}$$

$$\cdot(\det F)^{(T-K-m-1)/2}[\det(I-F)]^{(K_2-m-1)/2}\prod_{i<j}(f_i - f_j)$$

$$\cdot{}_0F_1\left(\frac{K_2}{2};\ \frac{T}{4}\bar{\Pi}_{22}[\beta \ \vdots\ I]E'(I-F)E\left[\frac{\beta'}{I}\right]\bar{\Pi}_{22}'\right)\ .$$

We now introduce the same partition of E that is used in LIML: I (see equation (32)) and again employ the notation $\beta_{LIML} = r$:

$$E = \begin{array}{c} \\ 1 \\ n \end{array}\begin{array}{c} \begin{array}{cc} 1 & n \end{array} \\ \left[\begin{array}{c|c} e_{11} & e_{12}' \\ \hline E_{22}r & E_{22} \end{array}\right] \end{array}$$

We also partition F conformably as

$$F = \left[\begin{array}{c|c} f_1 & 0 \\ \hline 0 & F_2 \end{array}\right]$$

where $F_2 = \mathrm{diag}(f_2, f_3, \ldots, f_m)$. With this notation the argument of the ${}_0F_1$ function in (12) becomes

$$(13) \qquad (T/4)\bar{\Pi}_{22}[(1-f_1)(e_{11}\beta + e_{12})(e_{11}\beta' + e_{12}') + (I+\beta r')E_{22}'(I-F_2)E_{22}(I+r\beta')]\bar{\Pi}_{22}' .$$

Using the series representation of ${}_0F_1$ in zonal polynomials (Constantine (1963)) and the multinomial expansion of a zonal polynomial of a sum of matrices in terms of invariant polynomials of several matrix arguments (Davis (1980, 1981) and Chikuse (1980)) we deduce that

$$(14) \qquad {}_0F_1\left(\frac{K_2}{2}; \frac{T}{4}\bar{\Pi}_{22}[\beta \vdots I]E'(I-F)E\left[\frac{\beta'}{I}\right]\bar{\Pi}_{22}'\right)$$

$$= \sum_{f=0}^{\infty}\frac{1}{f!}\sum_{\varphi}\frac{1}{\left(\frac{K_2}{2}\right)_{\varphi}}\sum_{\substack{J[2] \\ (\varphi\in J[2])}}\frac{f!\theta_{\varphi}^{J[2]}}{j_1!j_2!}C_{\varphi}^{J[2]}\left(\frac{T}{4}(1-f_1)\bar{\Pi}_{22}(e_{11}\beta+e_{12})(e_{11}\beta'+e_{12}')\bar{\Pi}_{22}', \frac{T}{4}\bar{\Pi}_{22}(I+\beta r')E_{22}'(I-F_2)E_{22}(I+r\beta')\bar{\Pi}_{22}'\right) .$$

In this expression, φ represents an ordered partition of $f = j_1 + j_2$ into at most n (hereafter $\leq n$) parts and $J[2] = (J_1, J_2)$ where J_i

represents an ordered partition of the nonnegative integer j_i $(i=1, 2)$ into $\leq n$ parts. The notation $\varphi \in J[2]$ relates the two sets of partitions and is explained in Davis (1980, 1981). $C_{\varphi}^{J[2]}(X,Y)$ is a polynomial in the elements of the two matrices X and Y which is invariant under the simultaneous transformation $X \to H'XH$ and $Y \to H'YH$ for any orthogonal matrix H. These invariant polynomials in two matrix arguments are developed and tabulated to low orders by Davis (1980). The constants $\theta_{\varphi}^{J[2]}$ that appear in (14) are given by

$$(15) \qquad \theta_{\varphi}^{J[2]} = C_{\varphi}^{J[2]}(I,I)/C_{\varphi}(I)$$

(Davis (1980), equation (5.1)).

Noting that $\det E = (\det E_{22})(e_{11} - e_{12}'r)$ we transform $E \to (e_{11}, e_{12}, r, E_{22})$ and find the density:

$$(16) \quad \mathrm{pdf}(e_{11},e_{12},r,E_{22},F) = \frac{\mathrm{etr}\left\{-\frac{T}{2}(I+\beta\beta')\overline{\Pi}_{22}'\overline{\Pi}_{22}\right\} e^{-\frac{1}{2}(e_{11}^2+e_{12}'e_{12})}}{2^{m(T-K_1)/2-m}\,\Gamma_m\left(\frac{K_2}{2}\right)\Gamma_m\left(\frac{T-K}{2}\right)}$$

$$\cdot \mathrm{etr}\left\{-\frac{1}{2}(I+rr')E_{22}'E_{22}\right\}[\det(E_{22}'E_{22})]^{(T-K_1-m-1)/2+1}|e_{11}-e_{12}'r|^{T-K_1-m}$$

$$\cdot (\det F)^{(T-K-m-1)/2}[\det(I-F)]^{(K_2-m-1)/2}\prod_{i<j}(f_i-f_j)$$

$$\cdot \sum_{f=0}^{\infty}\sum_{\varphi}\frac{1}{\left(\frac{K_2}{2}\right)_{\varphi}}\sum_{\substack{J[2] \\ (\varphi\in J[2])}}\frac{\theta_{\varphi}^{J[2]}}{j_1!j_2!}(1-f_1)^{j_1}$$

$$\cdot C_{\varphi}^{J[2]}\left(\frac{T}{4}\overline{\Pi}_{22}(e_{11}\beta+e_{12})(e_{11}\beta'+e_{12}')\overline{\Pi}_{22}',\ \frac{T}{4}\overline{\Pi}_{22}(I+\beta r')E_{22}'(I-F_2)E_{22}(I+r\beta')\overline{\Pi}_{22}'\right).$$

We transform $E_{22} \to H'E_{22} = D_{22}$ where H is an orthogonal matrix and then integrate over the orthogonal group, $O(n)$, normalized so that the measure over the whole group is unity. In performing this step we utilize the following integral, where $(\underline{dH})$ denotes the normalized invariant measure on $O(n)$:

$$(17)\quad \int_{O(n)} C^{\kappa,\lambda}(A'H'XHA,B)(\underline{dH}) = C^{\kappa,\lambda}(A'A,B)C_\kappa(X)/C_\kappa(I)$$

(Davis (1980), equation (5.13)). We find

$$(18)\quad \mathrm{pdf}(e_{11},e_{12},r,D_{22},F)$$

$$= \frac{\mathrm{etr}\left\{-\frac{T}{2}(I+\beta\beta')\overline{\Pi}_{22}'\overline{\Pi}_{22}\right\} e^{-\frac{1}{2}(e_{11}^2+e_{12}'e_{12})}}{2^{m(T-K_1)/2-m}\,\Gamma_m\left(\frac{K_2}{2}\right)\Gamma_m\left(\frac{T-K}{2}\right)}$$

$$\cdot\mathrm{etr}\left\{-\frac{1}{2}(I+rr')D_{22}'D_{22}\right\}[\det(D_{22}'D_{22})]^{(T-K_1-m+1)/2}|e_{11}-e_{12}'r|^{T-K_1-m}$$

$$\cdot(\det F)^{(T-K-m-1)/2}[\det(I-F)]^{(K_2-m-1)/2}\prod_{i<j}(f_i-f_j)$$

$$\cdot\sum_{f=0}^{\infty}\sum_{\varphi}\frac{1}{\left(\frac{K_2}{2}\right)_\varphi}\sum_{\substack{J[2] \\ (\varphi\in J[2])}}\frac{\theta_\varphi^{J[2]}}{j_1!j_2!}(1-f_1)^{j_1}$$

$$\cdot C^{J[2]}\left(\frac{T}{4}\overline{\Pi}_{22}(e_{11}\beta+e_{12})(e_{11}\beta'+e_{12}')\overline{\Pi}_{22}',\frac{T}{4}\overline{\Pi}_{22}(I+\beta r')D_{22}'D_{22}(I+r\beta')\overline{\Pi}'\right)C_{J_2}(I-F_2)/C_{J_2}(I)$$

The next step is to integrate F out of (18). The required integral is:

$$(19)\quad \int_{0<F<I}(1-f_1)^{j_1}(\det F)^{(T-K-m-1)/2}[\det(I-F)]^{(K_2-m-1)/2}C_{J_2}(I-F_2)\prod_{i<j}(f_i-f_j)dF\ .$$

First we transform $F_2 \to \overline{F}_2$ by defining $F_2 = f_1\overline{F}_2$. The jacobian is f_1^n and (19) becomes

$$(20)\quad \int_0^1 f_1^{m(T-K)/2-1}(1-f_1)^{(K_2-m-1)/2+j_1} \int_{0<\overline{F}_2<I}$$

$$\cdot(\det \overline{F}_2)^{(T-K-m-1)/2}\det(I-\overline{F}_2)[\det(I-f_1\overline{F}_2)]^{(K_2-m-1)/2}$$

$$\cdot C_{J_2}(I-f_1F_2) \prod_{2\leq i<j}(\overline{f}_i-\overline{f}_j)d\overline{F}_2 df_1$$

where $\overline{F}_2 = \text{diag}(\overline{f}_2, \ldots, \overline{f}_m)$ and $\overline{f}_2 > \overline{f}_3 > \ldots > \overline{f}_m$. We write out the expansions:

$$(21)\quad [\det(I-f_1\overline{F}_2)]^{(K_2-m-1)/2} = \sum_{j_3}\sum_{J_3}\frac{\left(\frac{-K_2+m+1}{2}\right)_{J_3}}{j_3!}C_{J_3}(f_1\overline{F}_2)$$

where the summation over j_3 is finite if $K_2 - m - 1$ is even and infinite otherwise, and where J_3 is an ordered partition of j_3 with $\leq n$ parts; and

$$(22)\quad C_{J_2}(I-f_1\overline{F}_2) = C_{J_2}(I)\sum_{j_4=0}^{j_2}\sum_{J_4}\binom{J_2}{J_4}C_{J_4}(-f_1\overline{F}_2)/C_{J_4}(I)$$

where J_4 is a partition of j_4 with $\leq n$ parts and $\binom{J_2}{J_4}$ denotes the generalized binomial coefficient introduced by Constantine (1966). The product formula

$$(23)\quad C_{J_3}(f_1\overline{F}_2)C_{J_4}(-f_1\overline{F}_2) = (-1)^{j_4}f_1^{j_3+j_4}\sum_{(J_5\in J_3\cdot J_4)}\left(\theta_{J_5}^{J_3,J_4}\right)^2 C_{J_5}(\overline{F}_2)$$

(Davis (1980), equation (5.10)) enables us to write (20) in the form

$$(24)\quad C_{J_2}(I)\sum_{j_3}\sum_{J_3}\frac{\left(\frac{-K_2+m+1}{2}\right)_{J_3}}{j_3!}\sum_{j_4=0}^{j_2}\sum_{J_4}\binom{J_2}{J_4}(-1)^{j_4}\sum_{(J_5\in J_3\cdot J_4)}\left(\theta_{J_5}^{J_3,J_4}\right)^2$$

$$\cdot\int_0^1 f_1^{m(T-K)/2+j_3+j_4-1}(1-f_1)^{(K_2-m-1)/2+j_1}df_1$$

$$\cdot\int_{0<\overline{F}_2<I}(\det\overline{F}_2)^{(T-K-m-1)/2}[\det(I-\overline{F}_2)]C_{J_5}(\overline{F}_2)\prod_{i<j}(\overline{f}_i-\overline{f}_j)d\overline{F}_2/C_{J_4}(I)$$

To evaluate the integral over $\overline{F}_2$ we note that

$$(25)\quad \int_0^I(\det R)^{a-(n+1)/2}[\det(I-R)]^{b-(n+1)/2}C_J(R)dR = \frac{\Gamma_n(a,J)\Gamma_n(b)}{\Gamma_n(a+b,J)}C_J(I)$$

where the integration is over all positive definite R for which $0 < R < I$ (Constantine (1963), Theorem 3). In (25)

$$(26)\quad \Gamma_n(a,J) = \pi^{\frac{1}{4}n(n-1)}\prod_{i=1}^{n}\left(a+j_i-\frac{1}{2}(i-1)\right)$$

for the partition $J = (j_1, j_2, \ldots, j_n)$. Following the approach used in LIML: I (equations (25)-(28)), we deduce from (25) a corresponding integral in terms of the latent roots $(r_1 > \ldots > r_n)$ of R :

$$(27)\quad \int_0^I(\prod_i r_i)^{a-(n+1)/2}(\prod_i(1-r_i))^{b-(n+1)/2}C_J(R)\prod_{i<j}(r_i-r_j)\prod_i dr_i$$

$$= \frac{\Gamma_n(a,J)\Gamma_n(b)C_J(I)\Gamma_n\left(\frac{n}{2}\right)}{\Gamma_n(a+b,J)\pi^{n^2/2}}\ .$$

This integral may now be used to reduce (24), leading in fact to

(28) $$C_{J_2}(I)\sum_{j_3}\sum_{J_3}\frac{\left(\frac{-K_2+m+1}{2}\right)_{J_3}}{j_3!}\sum_{j_4=0}^{j_2}\sum_{J_4}\binom{J_2}{J_4}(-1)^{j_4}\sum_{(J_5\in J_3\cdot J_4)}\left(\theta_{J_5}^{J_3,J_4}\right)^2$$

$$\cdot B\left(m\frac{T-K}{2}+j_3+j_4,\ \frac{K_2}{2}-\frac{m+1}{2}+j_1+1\right)$$

$$\cdot\frac{\Gamma_n\left(\frac{T-K}{2}-\frac{1}{2},\ J_5\right)\Gamma_n\left(\frac{n+1}{2}+1\right)\Gamma_n\left(\frac{n}{2}\right)C_{J_5}(I)}{\Gamma_n\left(\frac{T-K+n+2}{2},\ J_5\right)\pi^{n^2/2}C_{J_4}(I)}$$

$$= C_{J_2}(I)\omega_n(j_1,\ J_2)\ ,\quad \text{say.}$$

From this expression and (18) we deduce

(29) $$\mathrm{pdf}(e_{11},e_{12},r,D_{22})$$

$$=\frac{\mathrm{etr}\left\{-\frac{T}{2}(I+\beta\beta')\overline{\Pi}_{22}'\overline{\Pi}_{22}\right\}e^{-(e_{11}^2+e_{12}'e_{12})/2}}{2^{m(T-K_1)/2-m}\Gamma_m\left(\frac{K_2}{2}\right)\Gamma_m\left(\frac{T-K}{2}\right)}$$

$$\cdot\mathrm{etr}\left\{-\frac{1}{2}(I+rr')D_{22}'D_{22}\right\}[\det(D_{22}'D_{22})]^{(T-K_1-m+1)/2}\left|e_{11}-e_{12}'r\right|^{T-K_1-m}$$

$$\cdot\sum_{f=0}^{\infty}\sum_{\varphi}\frac{1}{\left(\frac{K_2}{2}\right)_{\varphi}}\sum_{\substack{J[2]\\(\varphi\in J[2])}}\frac{\theta_{\varphi}^{J[2]}}{j_1!j_2!}\omega_n(j_1,J_2)$$

$$\cdot C_{\varphi}^{J[2]}\left(\frac{T}{4}\overline{\Pi}_{22}(e_{11}\beta+e_{12})(e_{11}\beta'+e_{12}')\overline{\Pi}_{22}',\frac{T}{4}\overline{\Pi}_{22}(I+\beta r')D_{22}'D_{22}(I+r\beta')\overline{\Pi}_{22}'\right)$$

We transform $D_{22}\to(H,D)$, where H is orthogonal and $D=(D_{22}'D_{22})^{1/2}$ according to the unique decomposition $D_{22}=HD$. The measure changes according to (see equation (34) of LIML: I)

(30) $$dD_{22}=2^{-n}(\det D)^{-1/2}dD(dH)$$

where (dH) is the invariant measure on O(n) . Hence, by integrating over H and using (James (1954))

$$\text{(31)} \qquad \mathrm{vol}[O(n)] = \int_{O(n)} (dH) = \frac{2^n \pi^{n^2/2}}{\Gamma_n\left(\frac{n}{2}\right)}$$

we obtain

(32) $\mathrm{pdf}(e_{11}, e_{12}, r, D)$

$$= \frac{2^n \pi^{n^2/2} \mathrm{etr}\left\{-\frac{T}{2}(I+\beta\beta')\overline{\Pi}_{22}'\overline{\Pi}_{22}\right\} e^{-(e_{11}^2 + e_{12}'e_{12})/2}}{2^{m(T-K_1)/2-m+2n}\,\Gamma_m\left(\frac{K_2}{2}\right)\Gamma_m\left(\frac{T-K}{2}\right)\Gamma_n\left(\frac{n}{2}\right)}$$

$$\cdot \mathrm{etr}\left[-\frac{1}{2}(I+rr')D\right](\det D)^{(T-K_1-m)/2} |e_{11} - e_{12}'r|^{T-K_1-m}$$

$$\cdot \sum_{f=0}^{\infty} \sum_{\varphi} \frac{1}{\left(\frac{K_2}{2}\right)_{\varphi}} \sum_{\substack{J[2] \\ (\varphi \in J[2])}} \frac{\theta_{\varphi}^{J[2]}}{j_1!\, j_2!} \omega_n(j_1, J_2)$$

$$\cdot C_{\varphi}^{J[2]}\left(\frac{T}{4}\overline{\Pi}_{22}(e_{11}\beta + e_{12})(e_{11}\beta' + e_{12}')\overline{\Pi}_{22}', \frac{T}{4}\overline{\Pi}_{22}(I+\beta r')D(I+r\beta')\overline{\Pi}_{22}'\right)$$

where the additional factor $(1/2^n)$ arises because of the original sign restriction on the final column of E_{22} which is now relaxed. To integrate out D we use the following result (where x is an $n \times 1$ vector and X is an $n \times n$ matrix):

$$\text{(33)} \quad \int_{D>0} \mathrm{etr}\left\{-\frac{1}{2}(I+rr')D\right\}(\det D)^{(T-K_1-m)/2} C_{\varphi}^{J_1, J_2}(xx', XDX')dD$$

$$= \Gamma_n\left(\frac{T-K_1}{2}, J_2\right)\left[\det\left\{\frac{1}{2}(I+rr')\right\}\right]^{-(T-K_1)/2} C_{\varphi}^{J_1, J_2}(xx', 2X(I+rr')^{-1}X')$$

which can be deduced from Davis (1980), equation (5.14). In (33) the explicit notation J_1, J_2 replaces $J[2]$ in (32). Additionally, since xx' has only one nonzero latent root, the right side of (33) may be written in the form

$$(34)\quad 2^{n(T-K_1)/2}\Gamma_n\left(\frac{T-K_1}{2}, J_2\right)[\det(I+rr')]^{-(T-K_1)/2}C_{\varphi}^{[j_1],J_2}(xx', 2X(I+rr')^{-1}X')$$

where $[j_1]$ denotes the partition of j_1 with leading part j_1 i.e. $\{j_1, 0, \ldots, 0\}$. We now deduce

$$(35)\quad \mathrm{pdf}(e_{11}, e_{12}, r)$$

$$= \frac{\pi^{n^2/2}\mathrm{etr}\left\{-\frac{T}{2}(I+\beta\beta')\bar{\Pi}_{22}'\bar{\Pi}_{22}\right\}e^{-(e_{11}^2+e_{12}'e_{12})/2}}{2^{(T-K_1)/2-1}\Gamma_m\left(\frac{K_2}{2}\right)\Gamma_m\left(\frac{T-K}{2}\right)[\det(I+rr')]^{(T-K_1)/2}}$$

$$\cdot|e_{11}-e_{12}'r|^{T-K_1-m}\sum_{f=0}^{\infty}\frac{1}{\varphi\left(\frac{K_2}{2}\right)_{\varphi}}\sum_{\substack{j_1,J_2\\(\varphi\in j_1\cdot J_2)}}\frac{\theta_{\varphi}^{[j_1],J_2}}{j_1!J_2!}\omega_n(j_1,J_2)\Gamma_n\left(\frac{T-K_1}{2}, J_2\right)$$

$$\cdot C_{\varphi}^{[j_1],J_2}\left(\frac{T}{4}\bar{\Pi}_{22}(e_{11}\beta+e_{12})(e_{11}\beta'+e_{12}')\bar{\Pi}_{22}', \frac{T}{2}\bar{\Pi}_{22}(I+\beta r')(I+rr')^{-1}(I+r\beta')\bar{\Pi}_{22}'\right) .$$

Henceforth, we assume $T-K_1-m$ to be an even integer (as in LIML: I), facilitating the reduction of (35). In particular, let $T-K_1-m = 2k$ for some integer k . Define $q' = (e_{11}, e_{12}')$ and $a' = (1, r')$. To reduce (35) we need to evaluate the integral

$$(36)\quad \int e^{-q'q/2}(a'q)^{2k}C_{\varphi}^{[j_1],J_2}\left(\frac{T}{4}\bar{\Pi}_{22}[\beta\,\vdots\, I]qq'\begin{bmatrix}\beta'\\I\end{bmatrix}\bar{\Pi}_{22}', \frac{T}{2}\bar{\Pi}_{22}(I+\beta r')(I+rr')^{-1}(I+r\beta')\bar{\Pi}_{22}'\right)dq$$

where the integral is over $0 < q_m < \infty$, $-\infty < q_i < \infty$, $(i \neq m)$. We first rewrite $(a'q)^{2k}$ in the form

$$(37) \qquad (a'q)^{2k} = C_{[k]}(aa'qq') \ .$$

The argument matrix in this zonal polynomial is $m \times m$. To multiply the polynomials that arise in the integrand of (36) it is convenient to write

$$(38) \quad C_{\varphi}^{[j_1],J_2}(xx',B) = C_{[\varphi]_n}^{[j_1],[J_2]_n}(E_{10}xx'E_{10}',E_{10}BE_{10}')$$

where $E_{10}' = [I_n \vdots 0]$ is an $n \times m$ matrix and the notation $[\ ]_n$ signifies that the ordered partitions retain $\leq n$ non-zero parts while the argument matrices are now $m \times m$ $(m = n+1)$. We use the following representation of a product of polynomials with multiple argument matrices that is due to Chikuse (1980), equation (3.11):

$$(39) \quad C_{[k]}(aa'qq')C_{[\varphi]_n}^{[j_1],[J_2]_n}\Big(\tfrac{T}{4}E_{10}\overline{\Pi}_{22}[\beta \vdots I]qq'\Big[\tfrac{\beta'}{I}\Big]\overline{\Pi}_{22}'E_{10}',$$

$$\tfrac{T}{2}E_{10}\overline{\Pi}_{22}(I+\beta r')(I+rr')^{-1}(I+r\beta')\overline{\Pi}_{22}'E_{10}'\Big)$$

$$= \sum_{\psi \in [k]\cdot[j_1]\cdot[J_2]_n} \left\{\gamma_k^{k,j_1,J_2;\psi}\alpha_{\varphi}^{k,j_1,J_2;\psi} C_{[k]}(I)C_{[\varphi]_n}(I)/C_{\psi}(I)\right\}$$

$$C_{\psi}^{[k],[j_1],[J_2]_n}\Big(aa'qq',\tfrac{T}{4}E_{10}\overline{\Pi}_{22}[\beta \vdots I]qq'\Big[\tfrac{\beta'}{I}\Big]\overline{\Pi}_{22}'E_{10}',$$

$$\tfrac{T}{2}E_{10}\overline{\Pi}_{22}(I+\beta r')(I+rr')^{-1}(I+r\beta')\overline{\Pi}_{22}'E_{10}'\Big)$$

where the indexed constants γ and α are defined by Chikuse (1980) in her equations (3.1) and (3.13).

To evaluate (36) we transform $q \to H'q = p$ for H orthogonal and integrate over the orthogonal group, normalized so that the measure over the whole group is unity. From Chikuse (1980) equation (3.12) we deduce that

$$(40)\quad \int_{O(m)} C_{\psi}^{[k],[j_1],[J_2]_n}\Big(aa'Hpp'H',\ \tfrac{T}{4}E_{10}\overline{\Pi}_{22}[\beta\vdots I]Hpp'H\begin{bmatrix}\beta'\\ I\end{bmatrix}\overline{\Pi}'_{22}E'_{10},$$

$$\tfrac{T}{2}E_{10}\overline{\Pi}_{22}(I+\beta r')(I+rr')^{-1}(I+r\beta')\overline{\Pi}'_{22}E'_{10}\Big)(\underline{dH})$$

$$= \sum_{\sigma\in[k]\cdot[j_1]} \gamma_{\sigma}^{k,j_1,J_2;\psi} C_{\psi}^{\sigma,[J_2]_n}\Big(pp',\ \tfrac{T}{2}E_{10}\overline{\Pi}_{22}(I+\beta r')(I+rr')^{-1}(I+r\beta')\overline{\Pi}'_{22}E'_{10}\Big)$$

$$\cdot C_{\sigma}^{[k],[j_1]}\Big(aa',\ \tfrac{T}{4}\begin{bmatrix}\beta'\\ I\end{bmatrix}\overline{\Pi}'_{22}\overline{\Pi}_{22}[\beta\vdots I]\Big)\ .$$

We transform $p \to H'p = w$ for H orthogonal and note that

$$(41)\quad \int_{O(m)} C_{\psi}^{\sigma,[J_2]_n}\Big(Hww'H',\tfrac{T}{2}E_{10}\overline{\Pi}_{22}(I+\beta r')(I+rr')^{-1}(I+r\beta')\overline{\Pi}'_{22}E'_{10}\Big)(\underline{dH})$$

$$= C_{\psi}^{\sigma,[J_2]_n}\Big(I,\ \tfrac{T}{2}E_{10}\overline{\Pi}_{22}(I+\beta r')(I+rr')^{-1}(I+r\beta')\overline{\Pi}'_{22}E'_{10}\Big)C_{\sigma}(ww')/C_{\sigma}(I)$$

$$= (w'w)^{k+j_1}\ \frac{\theta_{\psi}^{\sigma,[J_2]_n} C_{\psi}(I)}{C_{\sigma}(I)C_{[J_2]_n}(I)}C_{J_2}\Big(\tfrac{T}{2}\overline{\Pi}_{22}(I+\beta r')(I+rr')^{-1}(I+r\beta')\overline{\Pi}'_{22}\Big)$$

where the integral is evaluated using equation (5.13) of Davis (1980) and the final line follows from equation (5.2) *ibidem*.

The manipulations from (37) to (41) separate the variables that enter the argument matrices of the polynomials in the integrand of (36). They leave us with the integral

$$(42)\qquad 2^{-1}\int e^{-w'w/2}(w'w)^{k+j_1}dw$$

where the factor 1/2 arises because we take the domain of integration in (42) to be unrestricted (i.e. $-\infty < w_i < \infty$) while that of q in (36) satisfies the restriction $0 < q_m < \infty$. The integral (42) may be evaluated by the same argument as that given in LIML: I (equation (41)ff). We find that (42) equals

$$2^{m/2+k+j_1-1}\pi^{m/2}\Gamma\left(\frac{m}{2}+k+j_1\right)/\Gamma\left(\frac{m}{2}\right) . \tag{43}$$

From (35), (39), (40), (41) and (43) we deduce the following general form for the joint density of β_{LIML} :

$$\text{pdf}(r) = \frac{\pi^{(nm+1)/2}\text{etr}\left\{-\frac{T}{2}(I+\beta\beta')\bar{\Pi}_{22}'\bar{\Pi}_{22}\right\}}{\Gamma\left(\frac{m}{2}\right)\Gamma_n\left(\frac{n}{2}\right)\Gamma_m\left(\frac{K_2}{2}\right)\Gamma_m\left(\frac{T-K}{2}\right)(1+r'r)^{(T-K_1)/2}} \tag{44}$$

$$\sum_{f=0}^{\infty}\sum_{\varphi}\frac{1}{\left(\frac{K_2}{2}\right)_{\varphi}}\sum_{\substack{[j_1],J_2\\(\varphi\in[j_1]\cdot J_2)}}\frac{2^{j_1}\theta_{\varphi}^{[j_1],J_2}}{j_1!j_2!}\omega_n(j_1,J_2)\Gamma\left(\frac{T-K_1}{2}+j_1\right)\Gamma_n\left(\frac{T-K_1}{2}, J_2\right)$$

$$\sum_{\psi\in[k]\cdot[j_1]\cdot[J_2]_n}\left\{\gamma_k^{k,j_1,J_2;\psi}\alpha_{\varphi}^{k,j_1,J_2;\psi}C_{[k]}(I)C_{[\varphi]_n}(I)\right\}$$

$$\sum_{\sigma\in[k]\cdot[j_1]}\gamma_{\sigma}^{k,j_1,J_2;\psi}C_{\sigma}^{[k],[j_1]}\left(aa',\ \frac{T}{4}\begin{bmatrix}\beta'\\ I\end{bmatrix}\bar{\Pi}_{22}'\bar{\Pi}_{22}[\beta\vdots I]\right)$$

$$\cdot\left\{\theta_{\psi}^{\sigma,[J_2]_n}/C_{\sigma}(I)C_{[J_2]_n}(I)\right\}C_{J_2}\left(\frac{T}{2}\bar{\Pi}_{22}(I+\beta r')(I+rr')^{-1}(I+r\beta')\bar{\Pi}_{22}'\right)$$

where

$$k = (T-K_1-m)/2\ ; \quad \text{and} \quad a' = (1,\ r') .$$

4. THE LEADING CASE

When $\bar{\Pi}_{22} = 0$ each term in the series (44) is zero with the exception of the leading term, for which $f = j_1 = j_2 = 0$. The general expression for the joint density reduces after some simplification to:

$$(45)\quad \mathrm{pdf}(r) = \frac{\pi^{nm/2}\Gamma\left(\frac{T-K_1}{2}\right)\Gamma_n\left(\frac{T-K_1}{2}\right)\omega_n(0,0)}{\Gamma\left(\frac{m}{2}\right)\Gamma_n\left(\frac{n}{2}\right)\Gamma_m\left(\frac{K_2}{2}\right)\Gamma_m\left(\frac{T-K}{2}\right)(1+r'r)^{(T-K_1)/2}} \frac{(a'a)^k}{C_{[k]}(I)} .$$

Now

$$(46)\quad (a'a)^k = (1+r'r)^{(T-K_1)/2-m/2}$$

and

$$(47)\quad C_{[k]}(I) = \frac{2^{2k}k!\left(\frac{m}{2}\right)_k}{(2k)!} = \frac{\left(\frac{m}{2}\right)_k}{\left(\frac{1}{2}\right)_k} = \frac{\pi^{1/2}\Gamma\left(\frac{T-K_1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{T-K_1}{2}-\frac{n}{2}\right)} .$$

We deduce that

$$(48)\quad \mathrm{pdf}(r) = \frac{\pi^{nm/2}\,\Gamma_n\left(\frac{T-K_1}{2}\right)\Gamma\left(\frac{T-K_1}{2}-\frac{n}{2}\right)\omega_n(0,0)}{\Gamma_n\left(\frac{n}{2}\right)\Gamma_m\left(\frac{K_2}{2}\right)\Gamma_m\left(\frac{T-K}{2}\right)(1+r'r)^{(n+1)/2}} \propto \frac{1}{(1+r'r)^{(n+1)/2}} .$$

Thus, β_{LIML} has a multivariate Cauchy distribution in this leading case as shown by direct methods in LIML: I.

The reduction of the constant coefficient in (48) presents some difficulties when $n > 1$ in view of the multiple series expression for $\omega_n(0,0)$, viz from (28):

$$(49)\quad \omega_n(0,0) = \pi^{-n^2/2}\Gamma\left(\frac{m}{2}+1\right)\Gamma_n\left(\frac{n}{2}\right)\sum_{j_3}\sum_{J_3}\frac{\left(\frac{-K_2+m+1}{2}\right)_{J_3}}{j_3!}B\left(m\frac{T-K}{2}+j_3,\ \frac{K_2}{2}-\frac{n}{2}\right)$$

$$\cdot\frac{\Gamma_n\left(\frac{T-K-1}{2};\ J_3\right)}{\Gamma_n\left(\frac{T-K+n}{2}+1;\ J_3\right)}C_{J_3}(I)$$

When $n = 1$ direct evaluation of the series can be achieved as follows:

$$(50)\quad \omega_1(0,0) = \pi^{-1/2}\frac{\Gamma(2)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{T-K}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{T-K}{2}+\frac{3}{2}\right)}\sum_{j_3}\frac{\left(\frac{-K_2+3}{2}\right)_{j_3}}{j_3!}$$

$$\cdot\frac{\left(\frac{T-K}{2}-\frac{1}{2}\right)_{j_3}}{\left(\frac{T-K}{2}-\frac{3}{2}\right)_{j_3}}B\left(T-K+j_3,\ \frac{K_2}{2}-\frac{1}{2}\right)$$

$$= \frac{\Gamma\left(\frac{T-K}{2}-\frac{1}{2}\right)\Gamma\left(\frac{K_2}{2}-\frac{1}{2}\right)\Gamma(T-K)}{\Gamma\left(\frac{T-K}{2}+\frac{3}{2}\right)\Gamma\left(T-K+\frac{K_2}{2}-\frac{1}{2}\right)}\sum_{j_3}\frac{\left(\frac{-K_2+3}{2}\right)_{j_3}}{j_3!}\frac{(T-K)_{j_3}\left(\frac{T-K}{2}-\frac{1}{2}\right)_{j_3}}{\left(T-K+\frac{K_2}{2}-\frac{1}{2}\right)_{j_3}\left(\frac{T-K}{2}+\frac{3}{2}\right)_{j_3}}$$

$$= \frac{\Gamma\left(\frac{T-K}{2}-\frac{1}{2}\right)\Gamma\left(\frac{K_2}{2}-\frac{1}{2}\right)\Gamma(T-K)}{\Gamma\left(\frac{T-K}{2}+\frac{3}{2}\right)\Gamma\left(T-K+\frac{K_2}{2}-\frac{1}{2}\right)}{}_3F_2\left(T-K,\ \frac{T-K}{2}-\frac{1}{2},\ \frac{-K_2+3}{2};\ \frac{T-K+3}{2},\ T-K+\frac{K_2}{2}-\frac{1}{2};\ 1\right).$$

The ${}_3F_2$ series is well poised (see, for example, Rainville (1963) page 92) and therefore sums to:

$$(51)\quad \frac{\Gamma\left(\frac{T-K}{2}+1\right)\Gamma\left(\frac{T-K}{2}+\frac{3}{2}\right)\Gamma\left(T-K+\frac{K_2}{2}-\frac{1}{2}\right)\Gamma\left(\frac{K_2}{2}\right)}{\Gamma(T-K+1)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{T-K_1}{2}-\frac{1}{2}\right)\Gamma\left(\frac{T-K_1}{2}\right)}.$$

We deduce that

$$(52)\quad \omega_1(0,0) = \frac{\Gamma\left(\frac{T-K}{2}-\frac{1}{2}\right)\Gamma\left(\frac{K_2}{2}-\frac{1}{2}\right)\Gamma(T-K)\Gamma\left(\frac{T-K}{2}+1\right)}{\Gamma\left(\frac{T-K}{2}+\frac{3}{2}\right)\Gamma(T-K+1)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{T-K_1}{2}-\frac{1}{2}\right)} \; \frac{\Gamma\left(\frac{T-K}{2}-\frac{3}{2}\right)\Gamma\left(\frac{K_2}{2}\right)}{\Gamma\left(\frac{T-K_1}{2}\right)} .$$

It now follows from (48) and (52) that when $n = 1$

$$(53)\quad \mathrm{pdf}(r) = \frac{\pi\Gamma\left(\frac{T-K}{2}-\frac{1}{2}\right)\Gamma\left(\frac{K_2}{2}-\frac{1}{2}\right)\Gamma(T-K)\Gamma\left(\frac{T-K}{2}+1\right)\Gamma\left(\frac{K_2}{2}\right)}{(1/2)\pi\Gamma_2\left(\frac{K_2}{2}\right)\Gamma_2\left(\frac{T-K}{2}\right)\Gamma(T-K+1)(1+r^2)} .$$

Note that

$$\Gamma_2\left(\frac{K_2}{2}\right) = \pi^{1/2}\Gamma\left(\frac{K_2}{2}\right)\Gamma\left(\frac{K_2}{2}-\frac{1}{2}\right)$$

$$\Gamma_2\left(\frac{T-K}{2}\right) = \pi^{1/2}\Gamma\left(\frac{T-K}{2}\right)\Gamma\left(\frac{T-K}{2}-\frac{1}{2}\right)$$

so that

$$\mathrm{pdf}(r) = \frac{2\Gamma(T-K)\Gamma\left(\frac{T-K}{2}+1\right)}{\pi\Gamma(T-K+1)\Gamma\left(\frac{T-K}{2}\right)(1+r^2)} .$$

Finally, using the duplication formula for $\Gamma(z)$ we deduce that

$$\frac{\Gamma(T-K)}{\Gamma(T-K+1)} = \frac{1}{2}\,\frac{\Gamma\left(\frac{T-K}{2}\right)}{\Gamma\left(\frac{T-K}{2}+1\right)} .$$

Hence,

$$\mathrm{pdf}(r) = \frac{1}{\pi(1+r^2)} ,$$

the univariate Cauchy distribution which we obtained by other means in LIML: I.

5. FINAL REMARKS

The exact theory for the LIML estimator developed in this paper has application beyond the realm of the simultaneous equations model. There is a well known formal mathematical equivalence between the model studied here and the linear functional relationship in mathematical statistics. This equivalence has been explored in detail by Anderson, who applied the distribution theory for the LIML estimator (in the two endogenous variable case) within the setting of a linear functional relationship involving only two variables. Upon appropriate symbolic translation our distribution theory for the LIML estimator in the general case may be applied directly to the maximum likelihood estimator in the multivariate linear functional relationship. Thus, our results generalize the presently known exact theory in the latter setting as well as the simultaneous equations model.

The numerical implementation of the general expression for the joint density (44) is hampered by computational difficulties at present. First, multiple series involving matrix argument polynomials like (44) are often very slow to converge, particularly when the argument matrices have some large latent roots. Second, available tabulations of the polynomials and constants that appear in (44) are currently limited to low orders (involving only single digits in most cases); and algorithms for their generation are only in the incipient stages of development. (A recent discussion of these issues has been given by the author elsewhere (1983b).) Computational work with the series (44) in its general form must therefore await improvements in technology which enhance computational speed and the development of general purpose software for the algorithmic generation of the required polynomials and constants. In the meantime, these practical shortcomings of the general exact theory increase the present value of the leading case analyses developed in LIML: I and the refinements of asymptotic theory derived by Anderson (1974) and others.

REFERENCES

Anderson, T. W. (1974), "An asymptotic expansion of the distribution of the limited information maximum likelihoodestimate of a coefficient in a simultaneous equation system," Journal of the American Statistical Association, 69, 565-573.

__________ (1976), "Estimation of linear functional relationships: approximate distributions and connections with simultaneous equations in econometrics," Journal of the Royal Statistical Society, Series B, 38, 1-36.

__________ (1982), "Some recent developments on the distributions of single equation estimators," Chapter 4 in W. Hildenbrand (ed.), Advances in Econometrics, Cambridge University Press.

Basmann, R. L. (1974), "Exact finite sample distribution for some econometric estimators and test statistics: A survey and appraisal," Chapter 4 in M. D. Intriligator and D. A. Kendrick (eds.), Frontiers of Quantitative Economics, Vol. 2, North-Holland.

Chikuse, Y. (1980), "Invariant polynomials with matrix arguments and their applications," in R. E. Gupta (ed.), Multivariate Statistical Analysis, North-Holland.

Constantine, A. G. (1963), "Some noncentral distribution problems in multivariate analysis," Annals of Mathematical Statistics, 34, 1270-1285.

__________ (1966), "The distribution of Hotelling's generalised T_0^2," Annals of Mathematical Statistics, 37, 215-225.

Davis, A. W. (1980), "Invariant polynomials with two matrix arguments extending the zonal polynomials," in P. R. Krishnaiah (ed.), Multivariate Analysis, North-Holland.

__________ (1981), "On the construction of a class of invariate polynomials in several matrices, extending zonal polynomials," Annals of the Institute of Statistical Mathematics, 33, 297-313.

Fuller, W. A. (1977), "Some properties of a modification of the limited information estimator," Econometrica, 45, 939-953.

James, A. T. (1954), "Normal multivariate analysis and the orthogonal group," Annals of Mathematical Statistics, 25, 40-75.

Kunitomo, N. (1981), "On a third order optimum property of the LIML estimator when the sample size is large," Discussion Paper #502, Economics Department, Northwestern University.

Mariano, R. S. and T. Sawa (1972), "The exact finite-sample distribution of the limited information maximum likelihood estimator in the case of two included endogenous variables," Journal of American Statistical Association, 67, 159-163.

Morimune, K. (1981), "Approximate distributions of the k-class estimators when the degree of overidentifiability is large compared with the sample size," Discussion Paper #159, Institute of Economic Research, Kyoto University.

Phillips, P. C. B. (1980b), "The exact finite sample density of instrumental variable estimators in an equation with n+1 endogenous variables," Econometrica, 48:4, 861-868.

__________ (1982), "The exact distribution of LIML: I," Cowles Foundation Discussion Paper #658.

__________ (1983a), "Exact small sample theory in the simultaneous equations model," in M. D. Intriligator and Z. Griliches (eds.), Handbook of Econometrics, North-Holland.

__________ (1983b), "Marginal densities of instrumental variable estimators in the general single equation case," Advances in Econometrica (forthcoming).

Rainville, E. D. (1963), Special Functions, Macmillan, New York.

Sargan, J. D. (1976), "Econometric estimators and the Edgeworth approximation," Econometrica, 44, 421-428, and (1977), "Erratum," Econometrica, 15, 272.