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A STRATEGIC MARKET GAME WITH TRANSACTIONS COSTS

J. Rogawski and M. Shubik

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by

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But this barter introduced the use of money, as might be expected; for a convenient place from whence to import what you wanted, or to export what you had a surplus of, being often at a great distance, money necessarily made its way into commerce; for it is not everything which is naturally most useful that is easiest of carriage; for which reason they invented something to exchange with each other which they should mutually give and take, that being really valuable itself, should have the additional advantage of being of easy conveyance, for the purposes of life, as iron and silver, or anything else of the same nature: and this at first passed in value simply according to its weight or size; but in process of time it had a certain stamp, to save the trouble of weighing, which stamp expressed its value.

—Aristotle, Politics

1. INTRODUCTION

1.1. Money and Markets

Money is a complex institutional phenomenon with many features depending delicately upon the structure of markets, customs of society and other rules of the game which fully specify how trade may take place.

There are three broadly recognized features of money (by no means the only ones: see Shubik, 1983, Chapter 2) which are considered here.

They are money:

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- (1) as a numeraire;
- (2) as a store of wealth;
- and (3) as a means of payment.

Implicit in the above statement is that a money is somehow operationally different from other commodities. Using strategic market games, we attempt to make these properties of a money precise and easy to identify in the mathematical structures being studied.

Implicit in the idea of a modern market is that, at least to a good first approximation, the structure of a transactions technology and the price system should be anonymous. Market trade in essence, given enough traders to avoid important oligopolistic effects, ignores both names and numbers of individuals. For example, the proof of the existence theorem in Debreu (1959) for market prices is independent of the numbers of consumers or firms.

In a strategic market game price is formed by a mechanism, usually specified to have the anonymity property, i.e., that if the names of i and j are interchanged, price formation is not influenced. The exchanges which take place at a noncooperative equilibrium usually do not lead to an efficient allocation of resources.

Dubey and Rogawski (1982) have shown that for a private goods economy in strategic form, the efficiency of non-cooperative equilibria depends delicately on the market mechanism and that, generally in utility functions, the non-cooperative equilibria are inefficient.

In Section 2.3, we also introduce a notion of competitive equilibrium in the context of a strategic market game. The Walrasian budget set is replaced by the set of strategies which, though possibly infeasible, lead to a feasible outcome. The question of whether or not a

competitive equilibrium in this sense is feasible then pinpoints the relationship of efficiency to the institutional market mechanism. It is necessary to consider replicated sequences of strategic market games or to consider games with a continuum of traders in order to establish a relationship between the prices formed in the strategic market game and efficient prices.

A strategic market game is characterized by a price formation mechanism as part of the trading technology. This technology imposes constraints on the strategy sets of all individuals. The prices generated may or may not be efficient. The money or special strategic properties of a good can be examined in terms of conditions in a strategic market game.

(1) A Numeraire

Suppose the market consists of trade in $(m+1)$ commodities. It may be convenient to set one price in advance and have a mechanism determine all others. Thus, we may wish to set $p_{m+1} = 1$. In strategic models this may not be an innocent assumption. For example, in a game with n types of traders where each type is endowed with a different one of n goods if strategies involved naming prices,* the fixing of a numeraire would take away the strategy set of one type of trader (this unfortunate feature is related to problems in international central banking when one country's currency is meant to act as the international reserve currency).

*The difficulty does not arise in one period models where strategies are quantities.

(2) A Store of Wealth

We may distinguish a commodity money from a fiat money in the sense that the former has an intrinsic value independent of its use in exchange. The latter does not. In this article we confine our remarks to a commodity money. The commodity (or commodities) used as money enters into the utility function of each trader. Furthermore, we assume that the marginal utility of a money to any individual is always greater than some positive number ϵ .

The introduction of a fiat money calls for an elaboration of the laws or rules of society which enable it to be used strategically. This includes describing how it is issued and withdrawn. This is discussed elsewhere (Shubik and Wilson, 1977; Dubey and Shubik, 1979).

A money usually requires properties such that it is easily transportable, identifiable and divisible into small enough units and that it can be held in sufficient quantity by all traders. We return to this last point in Section 1.3.

(3) A Means of Payment

Consider an economy with $m+1$ goods. We define a simple market as a market in which good i may be exchanged directly for good j . We define a good to be a money if it is connected directly to all other goods via simple markets, i.e., if it can be exchanged directly for all other goods. Figure 1a shows an economy with four goods where the first is a money, and Figure 1b shows an economy with four goods where all four are monies.



FIGURE 1

Under this definition the associated strategic market game which yields the Arrow-Debreu results is one with complete simple markets, i.e., with all goods as money. When all goods are money there is always enough money, when fewer than all goods are money a shortage of money may occur. This is made more precise in Section 1.3.

1.2. A Strategic Market Game with Complete Markets

Although the bid-offer model of Dubey and Shubik (1978) had m markets and only one money, it is straightforward to extend this model to one with $m(m+1)/2$ markets in which every commodity is a money. In such an economy the price of every commodity may be quoted in terms of its exchange rate with every commodity, but it is reasonable to fix $p_{ii} = 1$ for $i = 1, \dots, m+1$. Furthermore, the relationship $p_{ij} = 1/p_{ji}$ must hold for all i, j . Thus we need consider only $m(m+1)/2$ markets and prices.

Suppose that the markets are designated by $(2,1), (3,1), (4,1), \dots, (m+1, m)$. A strategy by a trader i is a vector of dimension $m(m+1)$ of the form $(b_{21}^i, q_{21}^i, b_{31}^i, q_{31}^i, \dots, b_{m+1,m}^i, q_{m+1,m}^i)$ where:

$$\begin{aligned}
 & b_{2,1}^i + \sum_{d=3, d=2}^{m+1} \leq a_2^i \\
 (1) \quad & \sum_{d=1}^2 b_{3d}^i + \sum_{d=4}^{m+1} q_{d3}^i \leq a_3^i \\
 & \sum_{d=1}^m b_{m+1,j}^i \leq a_{m+1}^i
 \end{aligned}$$

and all $b_{jk}^i \geq 0$, $q_{jk}^i \geq 0$. (See Section 2.3 for a more detailed description of the bid-offer model.)

It is a direct extension of the proofs given in Dubey and Shubik (1978) to establish both the existence of active noncooperative equilibria (N.E.) and the convergence of the N.E.'s to competitive equilibria (C.E.'s).

As our prime purpose is to consider the effect of transactions costs on the strategic market games we do not dwell further on this game with complete markets but no transactions costs.

1.3. Pareto Optimality, Limited Markets and Enough Money

The concept of enough money is really that of enough liquidity. When as in the model in Section 1.2 every good can be directly exchanged for every good, then an individual's complete wealth and liquidity are of the same size. If there is only one money, this is not so. We may contrast the boundary conditions imposed by having $m+1$ or 1 moneys, on an individual with endowment $(a_1^i, a_2^i, \dots, a_{m+1}^i)$ facing prices $p_1, p_2, \dots, p_m, 1$ where the $m+1$ st commodity has been selected as numeraire, hence $p_{m+1} = 1$.

With all goods as money, the individual attempts to maximize

$$u^i(x_1^i, x_2^i, \dots, x_{m+1}^i)$$

subject to

$$(2) \quad \sum_{j=1}^m p_j (x_j^i - a_j^i) = a_{m+1}^i - x_{m+1}^i .$$

When only the $m+1^{\text{st}}$ good is a money, then the constraint becomes:

$$(3) \quad \sum_{j=1}^m p_j \max[(x_j^i - a_j^i), 0] \leq a_{m+1}^i .$$

This states that purchases cannot exceed cash on hand.

It is trivially easy to observe that the imposition of condition (3) instead of (2) can limit the ability of the individual to maximize. In particular, if an individual has no money, he cannot buy anything! (This condition is extremely strong and possibly unreasonable. In most societies those who have assets but are illiquid are able to obtain credit--but the granting of credit brings in a host of new problems such as bankruptcy and insolvency, which are not dealt with here.)

An economy which uses a money may be said to have enough money when the strategic market game has an interior solution, i.e., when condition (3) is no more binding than (2). In essence, this means that the individuals all have enough money to finance a float that may exist between when they sell resources and when they are paid.

Let us denote a strategic market game where the traders have initial resources a_1, \dots, a_{m+1} for the first m goods and ka_{m+1} for the $m+1^{\text{st}}$ good, by $(a_1, \dots, a_m, a_{m+1})$. We may consider a sequence of games in which all individuals have more of all commodities in their initial endowments by multiplying through by k .

A question commonly asked of exchange structures is: why do certain markets not exist. The usual, commonsense, and to a great extent accurate answers are transactions costs, trust and a variety of other factors concerning uncertainty. But it is useful to separate difficulties and observe that we may answer questions concerning the implications of restricted numbers of moneys and markets separately from questions about why certain markets do not exist.

We may assume axiomatically that certain markets do not exist, without offering an explanation of why. Utilizing the bid-offer mechanism for forming prices in any market we can define a strategic market game for every network of simple markets. Optimality cannot be achieved with fewer than m simple markets for $m+1$ goods, but with only one money either there has to be enough money or credit must be introduced. As the number of markets is enlarged, the strategy sets of the individuals are enlarged until a strategy attains the dimension of $m(m+1)$ ($m(m+1)/2$ bids and $m(m+1)/2$ offers).

As the number of moneys is increased, the feasible set of exchanges is increased, but it is only guaranteed that the feasible set of the strategic market game with $m+1$ moneys will touch the Pareto optimal surface of the unconstrained exchange at a competitive equilibrium.

2. ON TRANSACTIONS COSTS AND EFFICIENCY

2.1. On Modeling Transactions Costs

It is suggested that there are at least six or seven qualitatively different factors involving transactions costs. They can be divided into two classes: (1) the production technology of transactions, and (2) the information and organization structure.

There are at least four ways to characterize the physical effects of transactions costs. They are:

- (1) Simple constraints on the existence of markets; i.e., assume that certain markets do not exist. This has been discussed in Section 1.3.
- (2) Associate with each transaction a technology which consumes resources, assuming that the transactions technology can be described by a convex production set.
- (3) Assume that the transactions technology is described by an "approximately convex" set, i.e., a set which has only small nonconvexities such as those caused by setup costs large to the individual, but small to the economy as a whole. Upper and lower bounding convex sets can be used to replace and approximate the actual set.
- (4) The transactions technology may display large increasing returns to scale which have to be considered directly.

The information aspects of transactions pose problems in:

- (5) The costs of search, data gathering and processing.
- (6) The appropriability, purchase and sale of information.
- (7) The public good aspects of communication and organizational networks.

In the remainder of this article, we concentrate on the second item noted in this listing. We consider that transactions utilize resources, but we do not tackle increasing returns. It is known that small

nonconvexities such as those caused by firms having both small setup costs and capacity constraints, can be handled and offer useful insights into transactions costs and the structure of industry and the number of competitors. However, as our concern is with the existence of markets and the implications of simple transactions costs, we limit our investigation to the simplest case.

2.2. Pareto Optimality and Institutional Efficiency

The introduction of transactions costs even in their simplest form causes an important modification to the attainable set of final distributions. Pareto optimality, or efficiency cannot be described without first specifying the set of feasible outcomes. Without transactions costs, Pareto optimality is defined independently from the distribution of resources. With transactions costs the feasible set of outcomes and, hence, the Pareto optimal set depends upon the initial distribution (see Arrow, 1979; Dubey-Rogawski, 1981).

Although the definition of Pareto optimality is applicable to any set of outcomes which may or may not be dependent upon initial conditions, it appears to be desirable both for clarity and emphasis to refer to the Pareto set of a distribution dependent feasible set as the institutionally efficient set.

Once distribution costs are regarded as a fact of life, then the relevant criterion of efficiency, taking institutions as given, must take these costs into account. If we are permitted to vary institutions (game theoretically change the rules of the game), then we may compare two or more sets of institutions to investigate their relative efficiency.

2.3. A Strategic Market Game with Transactions Costs

In this section we take the strategic market game defined and analyzed by Dubey and Shubik and introduce transactions costs in a one period model. We establish the general existence of a noncooperative equilibrium with active trade if transactions costs are not too prohibitive. We then observe that if the commodity used as a money in the m market economy has low transactions costs and is in sufficient supply and adequately distributed, whereas the others all have positive transactions costs, that commodity will emerge as a money in the game with $m(m+1)/2$ markets if two fairly simple triangular inequalities hold between it and all other goods. Specifically, if $T^k(i,j)$ stands for the transactions cost of trading i for j by individual k , then if for every triad $i, j, m+1$

$$(4) \quad T^k(i,j) \geq T^k(i, m+1) + T^k(m+1, j)$$

and
$$T^k(j,i) \geq T^k(j, m+1) + T^k(m+1, j)$$

then $m+1$ will emerge as a money. This is made more precise in Section 2.4, and the relationship between transactions costs, markets, increasing returns to scale and institutional change is discussed in Section 3.

The Model

Assume that there are n traders and $(m+1)$ commodities, where the $(m+1)^{\text{st}}$ commodity is singled out as a money. Let

$$\mathbb{R}_+^{\ell} = \{(x_1, \dots, x_{\ell}) \in \mathbb{R}^{\ell} : x_j \geq 0 \text{ for all } j\}$$

and let

$$\mathbb{R}_{++}^{\ell} = \{(x_1, \dots, x_{\ell}) \in \mathbb{R}^{\ell} : x_j > 0 \text{ for all } j\}.$$

Each trader j is given a vector $a^j = (a_1^j, \dots, a_{m+1}^j) \in \mathbb{R}_{++}^{m+1}$ of commodities as an initial endowment. For each trader j , let $Y_j = \mathbb{R}_+^{m+1}$; we view Y_j as the j^{th} trader's outcome space and the utility function u_j of j is defined on Y_j . We assume throughout the paper that u_j is a restriction to \mathbb{R}_+^{m+1} of a c^2 -function on \mathbb{R}^{m+1} which is strictly concave and strictly increasing in all variables, i.e.,

$$\partial u_j / \partial x_k > 0 \text{ for all } k .$$

Set

$$Y = \left\{ (y_1, \dots, y_n) \in Y_1 \times \dots \times Y_n : \sum_{j=1}^n y_j = \sum_{j=1}^n a^j \right\} .$$

Thus Y is the space of all reallocations of the initial endowments among the n traders. Given strategy sets S_j for $j = 1, \dots, n$, let $S = S_1 \times \dots \times S_n$. A map $\varphi : S \rightarrow Y$ is of the form $\varphi(\bar{s}) = (\varphi_1(\bar{s}), \dots, \varphi_n(\bar{s}))$ with $\varphi_j(\bar{s}) \in Y_j$ for $\bar{s} \in S$. Such a map is called a market mechanism and defines a strategic market game: $\varphi_j(\bar{s})$ is the j^{th} trader's outcome determined by the strategy choice \bar{s} and $u_j(\varphi_j(\bar{s}))$ is the utility of the outcome. A strategy choice $\bar{s} = (s_1, \dots, s_n)$ is called a Nash equilibrium (abbreviated N.E.) if for all j :

$$u_j(\varphi_j(\bar{s})) \geq \max_{s'_j \in S_j} u_j(\bar{s} | s'_j)$$

where $(\bar{s} | s'_j)$ denotes the strategy choice obtained by replacing the j^{th} coordinate of \bar{s} with s'_j .

In this section we consider a bid-offer model with transactions costs. The j^{th} player submits a strategy $s_j = (b_1^j, q_1^j, \dots, b_m^j, q_m^j) \in \mathbb{R}_+^{2m}$

where b_k^j is a bid of money to buy commodity k and q_k^j is an offer of an amount of commodity k for sale. A choice $\bar{s} = (s_1, \dots, s_n)$ defines a price vector $p = (p_1, \dots, p_k)$ with the price p_k of the k^{th} commodity given by:

$$p_k = \begin{cases} \frac{\sum_{d=1}^n b_k^d / \sum_{j=1}^n q_k^j}{\sum_{j=1}^n q_k^j} & \text{if } \sum_{j=1}^n q_k^j \neq 0 \\ 0 & \text{if } \sum_{j=1}^n q_k^j = 0 \end{cases}$$

A transactions cost function is a vector-valued function $L : \mathbb{R}_+^{2m} \in \mathbb{R}_+^{m+1}$ and $L((b_1, q_1, \dots, b_j, q_j))$ represents the cost in commodities of transacting the bids b_j and offers q_j at the markets. We shall assume for ease in exposition that the function L is linear, i.e., is given by a $(2m) \times (m+1)$ matrix with non-negative entries. With somewhat more labor we could let the transactions costs sets be convex.

For $j = 1, \dots, n$ and $\bar{s} = (s_1, \dots, s_n)$ a vector of bids and offers for each of the traders, let $\tilde{\varphi}_j(\bar{s})$ be the vectors in \mathbb{R}^{m+1} whose components $(x_1^j, \dots, x_{m+1}^j)$ are:

$$x_k^j = a_k^j - q_k^j + \frac{b_k^j}{p_k} \quad \text{for } 1 \leq k \leq m$$

$$x_{m+1}^j = a_{m+1}^j - \sum_{k=1}^m b_k^j + \sum_{k=1}^m q_k^j p_k .$$

Thus $\tilde{\varphi}_j$ is the j^{th} payoff function for the bid-offer model without transactions costs. Set

$$\varphi_j(\bar{s}) = \tilde{\varphi}_j(\bar{s}) - L(s_j) .$$

The functions φ_j define the bid-offer model with transactions costs and in this game, the j^{th} trader's strategy set is

$$S_j = \left\{ s_j = (b_1, q_1, \dots, b_m, q_m) \in \mathbb{R}_+^{2m} : \sum_{k=1}^m b_k + L_{m+1}(s_j) \leq a_{m+1}^j \right. \\ \left. \text{and } q_k^j + L_k(s_j) \leq a_k^j \text{ for } k=1, \dots, m \right\}$$

where $L_k(s_j)$ is the k^{th} component of the vector $L(s_j)$. Note that the linearity of L implies that s_j is convex.

We call the game defined above Γ . An N.E. \bar{s} of Γ is active if trade takes place, i.e., if \bar{s} is not the zero vector. It is clear that $\bar{s} = 0$ is an N.E. However, as in (5), we wish to investigate the existence of nontrivial N.E. and the convergence of N.E. to competitive equilibria (abbreviated C.E.) supported by a price system under replication of the economy. We first establish some preliminary lemmas.

For $j = 1, \dots, n$ and $k = 1, \dots, m$, let $B_k^j = \sum_{\ell \neq j} b_k^\ell$ and $Q_k^j = \sum_{\ell \neq j} q_k^\ell$. In dealing with the strategies of a single trader, we will drop the superscript j when the meaning is clear.

Lemma 1. Fix j and assume strategies $s_k \in S_k$ are fixed for $k \neq j$. Let $x = (x_1, \dots, x_{m+1}) \in Y_j$ and let $\psi_j(x) = \{s_j \in S_j : \varphi_j(s_1, \dots, s_n) = x\}$. Then the set $\psi_j(x)$ is a convex subset of S_j .

Proof. Define two functions of variables b and q by:

$$f_1(b, q) = c - q + b \left(\frac{Q+q}{B+b} \right)$$

$$f_2(b, q) = c - b + q \left(\frac{B+b}{Q+q} \right)$$

where c , B and Q are constants. It is easy to check that if $0 \leq \lambda \leq 1$ and if $f_\alpha(b, q) = f_\alpha(b', q')$ for $\alpha = 1$ or 2 , then

$$f_\alpha(\lambda b + (1-\lambda)b', \lambda q + (1-\lambda)q') = f_\alpha(b, q) = f_\alpha(b', q') .$$

Since the transactions cost function $L(s_j)$ is linear, the lemma follows immediately from the form of the payoff function φ_j .

We recall some observations made in (5) regarding the structure of the strategy sets in the bid-offer model without transactions costs. Consider a market with one commodity being bought or sold in a commodity money. In Figure 2, the point with coordinates (A, M) indicates the initial endowment of an individual.

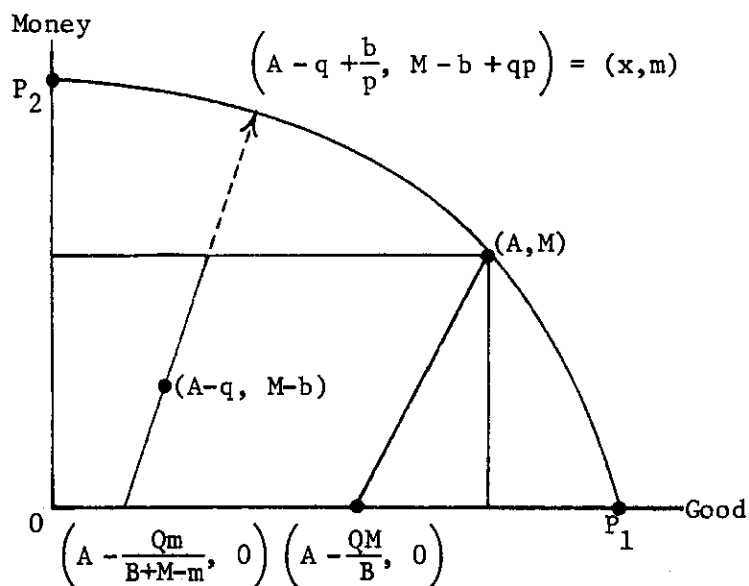


FIGURE 2

The strategy set is the rectangle (q, b) with $0 \leq q \leq A$ and $0 \leq b \leq M$. In Figure 2, we represent a bid (q, b) by the point $(A - q, M - b)$. The equation of the curve P_1P_2 is

$$m = M + \frac{(A-x)B}{(Q+a-x)}$$

and all final holdings lie on this curve, where (x,m) is the point on the curve. Let B (resp. Q) be the total bid (resp. offer) of the other traders. Then a bid of (q,b) takes the initial holding of (A,M) to a final holding of $(A - q + \frac{b}{p}, M - b + qp)$ where $p = (b+B)/(q+Q)$. The set of strategies in Figure 2 which leads to the outcome (x,m) is the intersection of the rectangle with the line joining (x,m) to $(A - \frac{Am}{B+M-m}, 0)$. In particular, the curve P_1P_2 is concave and if s_1, s_2 are two strategies in the rectangle, then the strategies $s(\lambda) = \lambda s_1 + (1-\lambda)s_2$ with $0 \leq \lambda \leq 1$ map onto the portion of the curve joining $f(s_1)$ to $f(s_2)$, where $f(s)$ denotes the outcome associated to a strategy s in the rectangle. The concavity of the curve P_1P_2 implies that $f(s(\lambda)) = \lambda f(s_1) + (1-\lambda)f(s_2) + \xi(\lambda)$ where $\xi(\lambda)$ is a vector (ξ_1, ξ_2) with $\xi_j > 0$ for $j = 1, 2$.

We now return to our bid-offer model with transaction costs.

For $\bar{s} \in S$, let

$$Y_j(\bar{s}) = \{\varphi_j(\bar{s}|s'_j) : s'_j \in S_j\}.$$

Thus $Y_j(\bar{s})$ is the set of outcomes that trader j can obtain by changing his own strategy if the other traders remain fixed at s_k ($k \neq j$).

Let $Y''_j(\bar{s})$ be the set of outcomes $x \in Y_j(\bar{s})$ such that

$$u_j(x) = \max_{x' \in Y_j(\bar{s})} u_j(x').$$

For $k = 0, \dots, m+1$ define $Y_j^k(\bar{s})$ as follows. Set $Y_j^0(\bar{s}) = Y''_j(\bar{s})$

and for $k = 1, \dots, m+1$ let

$$Y_j^k(\bar{s}) = \left\{ x \in Y_j^{k-1}(\bar{s}) : \min_{s_j \in \psi_j(x)} L_k(s_j) = \min_{x' \in Y_j^{k-1}(\bar{s})} \left(\min_{s_j \in \psi_j(x')} L_k(s_j) \right) \right\}.$$

Let $Y_j'(\bar{s}) = Y_j^{m+1}(\bar{s})$. Then for all $x_1, x_2 \in Y_j'(\bar{s})$, there exist $s_1 \in \psi(x_1)$ and $s_2 \in \psi(x_2)$ such that $L(s_1) = L(s_2)$. In fact, we have the following lemma.

Lemma 2. $Y_j'(\bar{s})$ consists of a single point.

Proof. Suppose $x_1, x_2 \in Y_j'(\bar{s})$ are distinct and let $s_j \in \psi(x_j)$ for $j = 1, 2$ be such that $L(s_1) = L(s_2)$. For $0 \leq \lambda \leq 1$ let $s(\lambda) = \lambda s_1 + (1-\lambda)s_2$. Then $s(\lambda) \in S_j$ since S_j is convex and $L(s(\lambda)) = L(s_1) = L(s_2)$ since L is linear. Let $\ell = L(s_j)$. Define the vector $\xi(\lambda)$ by

$$\tilde{\varphi}_j(s(\lambda)) = \lambda \tilde{\varphi}_j(s_1) + (1-\lambda)\tilde{\varphi}_j(s_2) + \xi(\lambda)$$

so that $\varphi_j(s(\lambda)) = \lambda \tilde{\varphi}_j(s_1) + (1-\lambda)\tilde{\varphi}_j(s_2) + \xi(\lambda) - \ell$. It follows from our previous remarks that $\xi(\lambda) \geq 0$ and that at least one component of $\xi(\lambda)$ is strictly positive. Since u_j is strictly concave and strictly increasing in all variables,

$$\begin{aligned} u_j(\varphi_j(s(\lambda))) &> u_j(\lambda \tilde{\varphi}_j(s_1) + (1-\lambda)\tilde{\varphi}_j(s_2) - \ell) = u_j(\lambda \varphi_j(s_1) + (1-\lambda)\varphi_j(s_2)) \\ &> \lambda u_j(\varphi_j(s_1)) + (1-\lambda)u_j(\varphi_j(s_2)) = u_j(\varphi_j(s_1)) \end{aligned}$$

and this is a contradiction. This proves the lemma.

Following (5), we define an ϵ -modified game Γ_ϵ in which an $(n+1)^{\text{st}}$ player places a fixed bid of $\epsilon > 0$ and a fixed offer of $\epsilon > 0$ in each of the m markets. The strategy sets of the original n traders remain the same but the outcome functions for Γ_ϵ , which we denote by

φ_j^ε , are different.

Proposition 3. For all $\varepsilon > 0$, an N.E. of Γ_ε exists.

Proof. Consider the map $\phi : S \rightarrow S$ defined by $\phi(\bar{s}) = \psi_1(x_1) \times \dots \times \psi_n(x_n) \subseteq S$ where x_j is the unique point in $Y_j'(\bar{s})$ for $j = 1, \dots, n$. Then $\phi(\bar{s})$ is a non-empty convex-valued correspondence by Lemma 1 and is easily seen to be upper semicontinuous. Hence, a fixed point exists by Kakutani's theorem and a fixed point is an N.E.

Convergence

We wish to consider the convergence of Nash equilibria of the game Γ_ε as ε tends to zero. Let $p(\varepsilon) = (p_1(\varepsilon), \dots, p_m(\varepsilon))$ be the price vector at an N.E. of Γ_ε . Although it may happen that $p_j(\varepsilon)$ tends to zero as ε tends to zero, we consider two approaches to transactions costs which lead to the conclusion that there exists positive constraints C and D such that $C < p_j(\varepsilon) < D$ for all $j = 1, \dots, m$. An economic discussion of these approaches will then be given. It is clear immediately, however, that a bound on prices cannot exist if the transactions costs are too high, e.g., if they make bidding or offering unfeasible.

In the first approach we suppose that there is a zeroth commodity (to be thought of as the trader's personal time) for which there is no market but which is used to pay transactions costs. We assume that the transactions cost function $L(b, q)$ is of the form $L_0(b, q)e_0 + L_{m+1}(b, q)e_{m+1}$ (where e_i denotes i^{th} unit basis vector)—thus transactions costs involve only time and money. The outcome spaces are then $R_+^{m+1} = \{(x_0, x_1, \dots, x_{m+1}) : x_j \geq 0\}$. As before, we assume that L_0 and L_{m+1} are linear. We shall say that transactions costs are not

too high if for all traders k ,

$$a_{m+1}^k e_{m+1} + a_0^k e_0 > (m+1)L_{m+1}(b, q)e_m + L_0(b, q)e_0$$

for all bids and offers (b, q) in the k^{th} trader's strategy set.

Let $A = (m+1)^{-1} \min_k(a_{m+1}^k)$. Fix a commodity j and let k be such that $b_j^k \leq \frac{1}{2} \sum_{i=1}^n b_j^i$. Obviously one of the following two inequalities holds:

$$(i) \quad a_{m+1}^k - \sum_{i=1}^m b_i^k \geq A - \tilde{L}_{m+1}$$

$$(ii) \quad a_{m+1}^k - \sum_{i=1}^m b_i^k < A - \tilde{L}_{m+1}$$

where \tilde{L}_{m+1} is the maximum of $L_{m+1}(b, q)$ over all (b, q) in k 's strategy set. By our assumption $A - \tilde{L}_{m+1} > \tau > 0$ for some positive τ which is independent of k . We also have $x_0^k > \tau'$ for some τ' which is independent of k when transactions costs are not too high.

Suppose that inequality (i) holds. Let x^k be k 's final holding at the N.E. of Γ_ϵ with price vector $p(\epsilon) = (p_1(\epsilon), \dots, p_m(\epsilon))$ and let $x^k(\Delta)$ be k 's final holding under an increased bid of Δ on the j^{th} commodity. For Δ sufficiently small, this is feasible if (i) holds. Let ℓ_0^Δ and ℓ_{m+1}^Δ be the transactions costs in the 0^{th} and $(m+1)^{\text{st}}$ commodities of the increased bid of Δ . For $i = 1, \dots, m$, let \bar{b}_i and \bar{q}_i denote the sum of the bids and offers on commodity i , respectively. We have

$$x_0^k(\Delta) - x_0^k = -\ell_0 \Delta$$

$$x_i^k(\Delta) - x_i^k = 0 \quad \text{for } i = 0, j, m+1$$

$$x_j^k(\Delta) - x_j^k = \Delta \frac{(\bar{q}_j + \epsilon)(\bar{b}_j + \epsilon - b_j^k)}{(\bar{b}_j + \epsilon)(\bar{b}_j + \epsilon - \Delta)} - \frac{\Delta}{2p_j(\epsilon)}$$

$$x_{m+1}^k(\Delta) - x_{m+1}^k = \left(\frac{q_j^k}{(\bar{q}_j + \epsilon)} - 1 - \ell_{m+1} \right) \Delta \geq -\Delta(1 + \ell_{m+1}) .$$

Since u_k is strictly increasing in all variables and the range of φ_j is bounded, it is not hard to see that there is a constant $h > 0$ which depends only on u_k such that for all outcomes $x \in R_+^{m+1}$ and vectors $y \in R_+^{m+1}$, if $\|x-y\| < h$, then $u_k(y + e_t) > u_k(x)$ for all $t = 0, \dots, m+1$. (See Lemma 5 for a precise statement.)

Let $z = -2p(\epsilon_j)(\ell_0 e_0 + (1 + \ell_{m+1})e_{m+1})$. The above inequalities show that

$$x^k(\Delta) \geq x^k + \frac{\Delta}{2p_j(\epsilon)}(z + e_j) .$$

If $x^k + z + e_j \geq 0$ and $\|z\| \leq h$, then $u_k(x^k + z + e_j) > u^k(x^k)$.

Since u^k is strictly concave, this would imply that $u^k\left(x^k + \frac{\Delta}{2p_j(\epsilon)}(z + e_j)\right) > u^k(x^k)$, contradicting the assumption that we are at an N.E. Hence

either $\|z\| \geq h$, in which case $p_j(\epsilon) \geq \frac{1}{2h} \|\ell_0 e_0 + (1 + \ell_{m+1})e_{m+1}\|^{-1}$,

or some component of $x^k + z + e_j$ is negative, that is, either

$x_{m+1}^k - 2p(\epsilon_j)(1 + \ell_{m+1}) < 0$ or $x^k - 2p(\epsilon_j)\ell_0 < 0$. In the first case,

we obtain

$$2p(\epsilon_j) > (1 + \ell_{m+1})^{-1}(A - \tilde{L}_{m+1}) > (1 + \ell_{m+1})^{-1}\tau$$

since $x_{m+1}^k > A - L_{m+1}$ when (i) holds, and in the second case, we have $2p(\varepsilon_j) > \ell_0^{-1} \tau'$. Thus if (i) holds, $p(\varepsilon_j)$ is bounded below by a constant which depends only on the utility functions, the initial endowments of the traders, and the transactions cost function.

Now suppose that (ii) holds. Then $\sum_{i=1}^m b_i^k > a_{m+1}^k - A + \tilde{L}_{m+1} \geq mA$

and for some i , $b_i^k \geq A$. If $i = j$, then we obviously have

$p_j(\varepsilon) \geq \frac{\Delta}{\sum_{j=1}^n a_j^k}$. Otherwise trader k can decrease b_i^k by Δ and in-

crease b_j^k by an amount Δ for Δ sufficiently small. Let

$\lambda \Delta = \ell_0 e_0 + \ell_{m+1} \Delta e_{m+1}$ be the net transactions cost for this change of strategy. Let $x^k(\Delta)$ denote k 's new final holding. Then

$$x_0^k(\Delta) - x_0^k = -\ell_0 \Delta$$

$$x^k(\Delta) - x^k = 0 \quad \text{for } \ell \neq 0, i, j, m+1$$

$$x_j^k(\Delta) - x_j^k \geq \frac{\Delta}{2p(\varepsilon_j)}$$

$$x_i^k(\Delta) - x_i^k = \frac{(\bar{q}_i + \varepsilon)(b_i^k - \varepsilon)}{\bar{b}_i - \Delta + \varepsilon} - \frac{(\bar{q}_i + \varepsilon)b_i^k}{\bar{b}_i + \varepsilon} - \frac{-(\bar{q}_i + \varepsilon)}{\bar{b}_i + \varepsilon}$$

$$x_{m+1}^k(\Delta) - x_{m+1}^k = \left(\frac{q_j^k}{q_j + \varepsilon} - \frac{q_i^k}{q_i + \varepsilon} \right) \Delta - \ell_{m+1} \Delta.$$

Set
$$z = -2p(\varepsilon_j) \left(\ell_0 e_0 + \frac{\bar{q}_i + \varepsilon}{b_i + \varepsilon} e_i + \left(\frac{q_i^k}{q_i + \varepsilon} + \ell_{m+1} \right) e_{m+1} \right).$$

Then $x^k(\Delta) \geq x^k + \frac{\Delta}{2p(\varepsilon_j)}(z + e_j)$ and, as before, we see that either

$\|z\| > h$ or some component of $x^k + z$ is negative. If $\|z\| < h$,

then, since $b_i^k + \varepsilon > b_i^k > A$ and $q_i^k \leq \bar{q}_i + \varepsilon \leq \sum_{\ell=1}^n a_1^\ell + \varepsilon$, we easily

obtain a lower bound on $p_j(\epsilon)$ which depends, as before, only on the initial endowments, utility functions, and transactions costs. If some component of $x^k + z$ is negative, we have one of the following three inequalities:

$$x_0^k - 2p(\epsilon_j)\ell_0 < 0 \Rightarrow 2p(\epsilon_j) > \tau^1 \ell_0^{-1}$$

$$x_i^k - 2p(\epsilon_j) \frac{(\bar{q}_i + \epsilon)}{(b_i + \epsilon)} < 0$$

$$x_{m+1}^k - 2p(\epsilon_j) \left(\frac{q_i^k}{\bar{q}_i + \epsilon} + \ell_{m+1} \right) < 0$$

Since
$$x_i^k \geq \frac{(\bar{q}_i + \epsilon)b_i^k}{(b_i + \epsilon)} \geq \frac{(\bar{q}_i + \epsilon)A}{(b_i + \epsilon)} \quad \text{and} \quad x_{m+1}^k \geq \frac{q_i^k b_i^k}{(\bar{q}_i + \epsilon)} \geq \frac{Aq_i^k}{(\bar{q}_i + \epsilon)},$$

in all cases we obtain, as before, a lower bound on $p_j(\epsilon)$.

We must now show that the prices $p_j(\epsilon)$ are also bounded above. Fix a commodity j and choose a trader k such that $g_j^k \leq \frac{1}{2}g_j$. Then, if B_j denotes $\frac{1}{2} \min_{\ell} (a_j^\ell)$, then either (i) $a_j^k - q_j^k \geq B_j$ or (ii) $a_j^k - q_j^k < B_j$. Suppose first that (i) holds and let $x^k(\Delta)$ denote the final holding of k if k increases q_j^k by Δ . This is feasible if Δ is sufficiently small and we have

$$x_0^k(\Delta) - x_0^k = -\ell_0 \Delta$$

$$x_i^k(\Delta) - x_i^k = 0 \quad \text{if } i \neq 0, j, m+1$$

$$x_j^k(\Delta) - x_j^k \geq -\Delta$$

$$x_{m+1}^k(\Delta) - x_{m+1}^k \geq \frac{1}{2}p_j(\epsilon) - \ell_{m+1} \Delta.$$

If $p_j(\varepsilon) \leq 2\ell_{m+1}$, we are done. Otherwise $x_{m+1}^k(\Delta) - x_{m+1}^k > 0$ and if we set $z = \frac{-2}{p_j(\varepsilon) - 2\ell_{m+1}}(\ell_0 e_0 + e_j)$, we obtain

$$x^k(\Delta) \geq x^k + \Delta \left(\frac{p_j(\varepsilon) - 2\ell_{m+1}}{2} \right) (z + e_{m+1}) .$$

As before, we conclude that either $\|z\| > h$, in which case

$$p_j(\varepsilon) \geq \frac{1}{2}h \|\ell_0 e_0 + e_j\|^{-1} + 2\ell_{m+1}$$

or some component of $x^k + z$ is negative. In this second case, we have either

$$x_0^k - \frac{2\ell_0}{p_j(\varepsilon) - 2\ell_{m+1}} < 0$$

or

$$x_j^k - \frac{2}{p_j(\varepsilon) - 2\ell_{m+1}} < 0$$

and since $x_0^k > \tau'$ and $x_j^k \geq a_j^k - q_j^k \geq B_j$, it is clear that we obtain an upper bound on $p_j(\varepsilon)$. Finally, if (ii) holds instead of (i)

$q_j^k > a_j^k - B_j \geq \frac{1}{2}a_j^k \geq B_j$ and thus

$$p_j(\varepsilon) \leq \frac{\bar{b}_j + \varepsilon}{q_j^k + \varepsilon} \leq \frac{\sum_{\ell=1}^n a_{m+1}^{\ell} + \varepsilon}{B_j + \varepsilon} .$$

We summarize the above discussion.

Proposition 4. If transactions costs are not too high and involve only the zeroeth and $(m+1)^{st}$ commodity, there are positive constants $M > N$

such that for all $\varepsilon > 0$ and all N.E. of Γ_ε with price system $(p_1(\varepsilon), \dots, p_m(\varepsilon))$, we have:

$$N < p_j(\varepsilon) < M \text{ for all } j = 1, \dots, m.$$

In our second approach to interior solutions of the strategic market game with transactions costs, we consider a sequence of games Γ_ℓ which differ from the game Γ only in that for all $k = 1, \dots, n$, the initial endowment of trader k is $\ell a^k = (\ell a_1^k, \dots, \ell a_{m+1}^k)$. We will show the existence of interior N.E. of Γ_ℓ as ℓ tends to infinity under the following assumptions:

- 1) There is a positive constant $y > 0$ such that for all $k = 1, \dots, n$,

$$\lim_{x_j^k \rightarrow \infty} \frac{\partial u^k}{\partial x_j} = 0 \quad \text{and} \quad \frac{\partial u^k}{\partial x_j} / \frac{\partial u^k}{\partial x_i} > y \text{ for all commodities } i \text{ and } j.$$

- 2) Transactions costs are not too high.

Assumption 1) is a satiation condition on the utility functions together with the requirement that the relative marginal utilities of any two commodities does not tend to zero (and hence also not to infinity). The second condition is precisely as follows: There is a vector $\tau \in \mathbb{R}_{++}^{m+1}$ such that $a^k - (m+1)L(b,q) > \tau$ for all (b,q) in k 's strategy set for the game Γ . We are now assuming that L is given by an arbitrary $(m+1) \times 2m$ matrix.

Consider an N.E. of Γ_ℓ with equilibrium price vector $p(\ell) = (p_1(\ell), \dots, p_m(\ell))$ and let $x^k(\ell)$ be the final holding of trader k at this N.E. We note first that for all i , $x_i^k(\ell)$ tends to infinity with ℓ . To prove this, suppose that $x_i^k(\ell) < M$ for all ℓ , for some

constant M . Let $\{b_i^k(\ell), q_i^k(\ell)\}$ denote the strategy of trader k .

Then assumption 2) implies that $q_i^k(\ell) \geq \ell\tau_i - M$ and hence that

$$p_i(\ell) \leq \frac{\ell A}{\ell\tau_i - M} \leq \frac{A}{\tau_i}, \quad \text{where } A = \sum_{k=1}^n a_{m+1}^k. \quad \text{We may conclude that}$$

$x_{m+1}^k(\ell) \geq q_i^k(\ell)p_i(\ell) \geq (\ell\tau_i - M)p_i(\ell)$. Let $x_i^k(\ell, \Delta)$ be k 's outcome if $q_i^k(\ell) - \Delta$ is offered instead of $q_i^k(\ell)$. This is feasible and we

have:

$$x_j^k(\ell, \Delta) - x_j^k(\ell) \geq 0 \quad \text{for } j \neq i, m+1$$

$$x_i^k(\ell, \Delta) - x_i^k(\ell) = \Delta \left(1 - \frac{b_i^k(\ell)}{\bar{b}_i(\ell)} \right)$$

$$x_{m+1}^k(\ell, \Delta) - x_{m+1}^k(\ell) = -\Delta \frac{\bar{b}_i(\ell)}{q_i^k(\ell)} \left(\frac{\bar{q}_i(\ell) - q_i^k(\ell)}{\bar{q}_i(\ell) - \Delta} \right) \geq -\Delta \frac{A}{\tau_i} \left(\frac{\bar{q}_i(\ell) - q_i^k(\ell)}{\bar{q}_i(\ell) - \Delta} \right).$$

Note that if $x_i^k(\ell) < M$ for all ℓ , then $b_i^k(\ell)/\bar{b}_i(\ell)$ tends to zero since in this case k 's find percentage share of $\bar{q}_i(\ell)$ tends to zero (because $\bar{q}_i(\ell) \geq q_i^k(\ell) \geq \ell\tau_i - M$). There are two possibilities:

a) $p_i(\ell) > \omega$ for some fixed $\omega > 0$ for all ℓ , or b) $p_i(\ell)$ tends to zero as $\ell \rightarrow \infty$. In case a), $x_{m+1}^k(\ell)$ and hence also $x_{m+1}^k(\ell, \Delta)$ tend to infinity with ℓ for fixed Δ and assumption 1) immediately implies that for ℓ large, $u^k(x^k(\ell, \Delta)) > u^k(x^k(\ell))$, contradicting the assumption that we are at an N.E. (we omit the obvious details).

In case b), $x_{m+1}^k(\ell, \Delta)$ approaches $x_{m+1}^k(\ell)$ while $x_i^k(\ell, \Delta)$ approaches $x_i^k(\ell) + \Delta$, as $\ell \rightarrow \infty$ and again it is clear that $u^k(x^k(\ell, \Delta)) > u^k(x^k(\ell))$ for ℓ sufficiently large. Therefore $x_i^k(\ell)$ must tend to infinity as $\ell \rightarrow \infty$.

Next we remark that there exist positive constants N and M such that $N < p_j(\ell) < M$ for all ℓ sufficiently large. We sketch

the proof but omit the details, as the argument follows closely the argument given for the games Γ_ξ . It is enough to observe that since $x_i^k(\ell)$ tends to infinity with ℓ , the assumptions 1) and 2) permit the same argument to go through if one uses the following lemma, which generalizes Lemma C of (5).

Lemma 5. Let $f(x)$ be a strictly increasing differentiable function on \mathbb{R}_+^{m+1} such that for all i and j , $\frac{\partial f}{\partial x_j} / \frac{\partial f}{\partial x_i} > \eta$ for some positive constant η . Then there is a constant h such that for all $j = 1, \dots, m+1$ and all $x, y \in \mathbb{R}_+^{m+1}$, if $\|x-y\| \leq h$ then $f(y+e_j) > f(x)$.

Proof. Let $y = x+\delta$ and set $\varphi(t) = f(x+t(\delta+e_j))$. By the mean-value theorem, there is a t_0 such that $0 \leq t_0 \leq 1$ and:

$$\begin{aligned} f(y+e_j) - f(x) &= \varphi(1) - \varphi(0) = \varphi'(t_0) \\ &= (\nabla f(x+t_0(\delta+e_j))) \cdot (\delta+e_j) \\ &= \sum_{i=1}^{m+1} \frac{\partial f}{\partial x_i}(x+t_0(\delta+e_j)) \cdot \delta_i + \frac{\partial f}{\partial x_j}(x+t_0(\delta+e_j)) \end{aligned}$$

and this is positive if $|\delta_i| < \frac{\eta}{(m+1)}$.

We summarize the above discussion in the following proposition.

Proposition 6. There are constants N, M, P and a function $f(\ell)$ which tends to infinity as $\ell \rightarrow \infty$ depending only on $\{a^k\}$ and $\{u^k\}$ such that for all $\ell \geq P$ and all N.E. of Γ_ℓ with price vector $p(\ell)$ and final holding vectors $x^1(\ell), \dots, x^n(\ell)$, the following inequalities hold:

$$N < p_j(\ell) < M \text{ for } j = 1, \dots, m$$

$$x_j^k(\ell) > f(\ell) \text{ for } j+1, \dots, m \text{ and } k+1, \dots, n .$$

We will denote by a subscript ϵ (for $\epsilon > 0$) the game obtained from a game Γ by adding an $(n+1)^{\text{st}}$ trader who places a fixed bid of ϵ and supply of ϵ in each market. By Proposition 3, N.E. of these ϵ -modified games exist. An N.E. of the game Γ will be called a G.N.E. (good Nash equilibrium) if the prices p and strategy choices $s \in S$ are obtained as a limit as $\epsilon \rightarrow 0$ of prices $p(\epsilon)$ and strategy choices $s(\epsilon)$ of N.E. of the game Γ_ϵ . Since S is compact, by picking convergent subsequences of $p(\epsilon)$ and $s(\epsilon)$, the next Proposition follows immediately from Propositions 4 and 6.

Proposition 7. In the following two cases, a G.N.E. exists:

- a) If transactions costs are not too high and involve only the zero and $(m+1)^{\text{st}}$ commodity, a G.N.E. for Γ exists.
- b) If transactions costs are not too high and assumption 2) on $\Gamma(\ell)$ is satisfied, then a G.N.E. exists for Γ_ℓ for all sufficiently large ℓ .

Replication. The type of a trader is characterized by the utility function and the initial endowment. Let $\Gamma(\ell)$ be the strategic market game with kn traders in which there are ℓ traders of type (u^k, a^k) for $k = 1, \dots, n$. An N.E. of $\Gamma(\ell)$ will be called symmetric (abbreviated S.N.E.) if the strategies of traders of the same type are identical. It is clear that the proof of Proposition 3 yields an S.N.E. for $\Gamma_\epsilon(\ell)$ for all $\epsilon > 0$ because players of the same type must solve the same optimization problem and hence restrict the map ϕ in the proof of

Proposition 3 to the subset of symmetric strategies in S . Furthermore the statement of Proposition 7 remains valid if G.N.E. is replaced by S.G.N.E. (symmetric good Nash equilibrium).

A basic property of the bid-offer market game without transactions costs is that S.G.N.E.'s converge to competitive (Walrasian) equilibrium under replication. We show that this remains true after the introduction of transactions costs (which are not too high) provided that the notion of budget sets is modified. The standard definition of budget set makes no sense in this context since the price of a final allocation depends on the strategy used to obtain it. Thus we define for each trader k and price vector p :

$$B^k(p) = \{s_k \in \mathbb{R}^{2m} : \tilde{p} \cdot \varphi'_k(s_k) + L(s_k) \leq \tilde{p} \cdot a^k; \varphi'_k(s_k) + L(s_k) \in \mathbb{R}_+^{m+1}\}$$

where $\tilde{p} = (p, 1)$ is the price vector with the price of money normalized and $\varphi'_k(s_k)$ is the final holding of k if the prices are fixed at p .

It is the set of strategies such that the final holding is feasible, even if the strategy leading to it is not. A competitive equilibrium (abbreviated C.E.) then consists of a price vector and for each k a strategy choice $s_k \in B^k(p)$ such that $u^k(\varphi_k(s_k))$ is a maximum for u^k on the set of $\{\varphi_k(s_k) : s_k \in B^k(p)\}$. A strategy choice is called

interior if $(\sum_{i=1}^m b_i^k + L(s_k)) < a_{m+1}^k$ for all k , where

$s_k = (b_1^k, q_1^k, \dots, b_m^k, q_m^k)$. A C.E. is called symmetric (abbreviated

S.C.E.) if the strategy choices of traders of the same type are identical.

In the replicated game $\Gamma(\ell)$, an S.C.E. will be denoted by

(p, s_1, \dots, s_n) where s_k is the strategy used by all traders of

type k .

Proposition 8. Let $s(\ell) \in S$ be a sequence of S.G.N.E.'s with price vectors $p(\ell)$ such that $s(\ell) \rightarrow s$ and $p(\ell) \rightarrow p$ as $\ell \rightarrow \infty$. Then (p, s_1, \dots, s_n) is an S.C.E. for $T(\ell)$ for all ℓ .

We first note that the prices $p_j(\ell)$ of the j^{th} commodity satisfy $N < p_j(\ell) < M$ for some fixed constants N, M which are independent of ℓ because the inequalities proved in Propositions 4 and 6 for a fixed ℓ do not depend in any way on ℓ . The proof of Proposition 8 is based on the following two observations.

Let $Q_i^k(\ell, \epsilon)$ and (resp. $B_i^k(\ell, \epsilon)$) denote the total offer (resp. bid) on commodity i by the traders other than k at the N.E. of the game $\Gamma(\ell)_\epsilon$ with price vector $p(\ell, \epsilon)$ (where $p(\ell, \epsilon) \rightarrow p(\ell)$ for a suitable sequence of $\epsilon \rightarrow 0$ defines the G.N.E. $s(\ell)$). Set

$$p_i^k(\ell, \epsilon) = p_i(\ell, \epsilon)^2 (Q_i^k(\ell, \epsilon) / B_i^k(\ell, \epsilon)) .$$

and let $p^k(\ell, \epsilon) = (p_1^k(\ell, \epsilon), \dots, p_m^k(\ell, \epsilon))$.

The first observation is that at the N.E. $s(\ell, \epsilon)$ with price vector $p(\ell, \epsilon)$ (converging to $(p(\ell), s(\ell))$), the associated final holding $x^k(\ell, \epsilon)$ of trader k satisfies the condition:

$$u^k(x^k(\ell, \epsilon)) \geq u^k(s_k) \text{ for } s_k \in S_k \cap B^k(p_1^k(\ell, \epsilon)) .$$

If there are no transactions costs, this is proved in Lemma 4 of (5). That argument is valid, mutatis mutandis, in the present context if it is remarked that for any two strategies $s_k, s'_k \in B^k(p_1^k(\ell, \epsilon))$ and all $0 \leq \lambda \leq 1$,

$$L(\lambda s_k + (1-\lambda)s_k) = \lambda L(s_k) + (1-\lambda)L(s_k)$$

by the linearity of transactions costs.

The second observation is that as $\ell \rightarrow \infty$, $Q_i^k(\ell, \epsilon)$ and $B_k(\ell, \epsilon)$ approach $Q_i(\ell, \epsilon)$ and $B_i(\ell, \epsilon)$ respectively, where $Q_i(\ell, \epsilon)$ and $B_i(\ell, \epsilon)$ are the total offers and bids on commodity i , and hence that $p^k(\ell, \epsilon) \rightarrow p(\ell, \epsilon)$ as $\ell \rightarrow \infty$. This is a trivial calculation which we omit.

It is clear by continuity $s^k(\ell) = \lim_{\epsilon \rightarrow 0} s^k(\ell, \epsilon)$ is optimal in $B^k(p(\ell, \epsilon))$ for all k . If $s^k(\ell)$ is interior, then $s^k(\ell)$ is still an optimal strategy of prices $p(\ell)$ even if no strategic budget constraint is imposed (that is, if k maximizes over $B^k(p(\ell))$ instead of $B^k(p(\ell)) \cap S_r$).

2.4. A Strategic Market Game with Complete Markets and Transactions Costs

In Section 2.3 we derive prices for all goods and hence can evaluate the resources utilized in transactions. As we specified only m markets we noted only the physical costs (i.e., consumption of commodities) required for trade in these markets. If we consider the possibility that all $m(m+1)/2$ markets exist then if at prices $(p_1^*, \dots, p_m^*, 1)$ the equilibrium prices associated with the game with m markets, for all individuals and any two commodities i and j it is cheaper or as cheap to exchange i for j via money than directly, the N.E.'s of the game with m markets remain N.E.'s of the game with $m(m+1)/2$ markets.

The above condition does not rule out the possibility of the existence of other N.E.'s in the larger game with more than m markets active. The interplay between costs and the thickness of a market provide for the possibility of many equilibria.

Dubey and Shubik (1978) have already noted that there was the possibility of a multiplicity of equilibria caused by the effect of wash sales (i.e., buying and selling in the same market thereby making the market "thicker"). This source of multiplicity is somewhat cut down by the presence of transactions costs.

3. SOME ECONOMIC INTERPRETATIONS

3.1. General Equilibrium or Strategic Market Game Analysis

Foley (1970), Hahn (1971), Kurz (1974a, b) and others have approached the problem of transactions costs by direct modifications of the general equilibrium model. In contrast here we are explicit in the construction of a game in strategic form. In terms of the existence of different buying and selling prices to different agents as noted in the modeling of Foley and Hahn or in the introduction of a vector of real resources consumed in exchange as modeled by Kurz, our model is related to the earlier work. However, our emphasis is somewhat different. In particular we are concerned with models with price formation mechanisms (even though they are relatively simplistic); furthermore we are concerned with features such as the interactions among finite numbers of traders strategically trading off the possibilities of influencing the thickness of markets versus paying extra transactions costs. We are also concerned with the strategic meaning of enough money and enough of other resources to avoid unreasonable restrictions on trade. Finally we are concerned with efficiency properties of the equilibria of strategic market games with transactions costs.

It should be noted that although like both Foley and Hahn we have "own" prices for buying and selling our model generates only F.O.B. prices

at each market. As we describe physical processes for each market the only prices formed are market prices.

3.2. Barter, Complete or Limited Markets

In the two models noted here we contrast an economy with m markets with one which may have as many as $m(m+1)/2$ markets active. We do not attempt to characterize barter, although Kurz (1974b, p. 419) simultaneously refers to the model with $m(m+1)/2$ markets as a barter exchange economy and as a natural extension of Arrow-Debreu, we wish to make a further distinction. An economy with complete markets is, if it is anything, a model of the ideal of modern economics and finance where all commodities are perfectly liquid. The properties of mass, anonymous, aggregative markets forming prices remain.

Barter in contrast with the mass market may in general be neither mass nor anonymous. Fewness of individuals, and who they are all count. The number of different barter arrangements is far larger than the number of anonymous markets. It involves the combinations of individual groupings as well as the groupings of goods.

3.3. Enough Goods, Time and Money

Kurz (1974b, p. 423) noted that it is possible that even if an individual has positive wealth the transactions costs of exchanging goods may be so high that trade is prevented and a zero price exists for a good of positive worth.

In the formulation and analysis of the strategic market game we encounter the difficulties noted by Kurz. In Section 2.3 we propose two ways to avoid them. Both are mathematically adequate, but must be interpreted as crude first order approximations for a complex phenomenon.

In essence in an ongoing society much of transactions costs are paid for in the use of the individual's own time, whether walking or driving to the store or standing in line for a bargain. Among the other more common inputs utilized in transactions are transportation, packaging and paper-work. It is unlikely that the lack of paper bags or virtually any other input would stop trade; in actuality substitutes are virtually always available.

Our first way to avoid the difficulties attempts to operationalize the idea that in essence "one eats up time and money" in most transactions activities. This is similar to Kurz's "leisure services" (Kurz, 1974b, p. 423).

A different set of conditions sufficient (but not necessary) to sidestep the difficulties is obtained by inflating the size of the initial holdings of all traders, assuming all traders have strictly concave utility functions which approach saturation and that the ratio of marginal utilities between any two commodities is bounded.

The precise conditions for there to be enough commodities to facilitate trade involve relationships among the distribution of commodities, their marginal utility and explicit roles in the technology of transactions. Societies work out many different institutional and technological arrangements to satisfy them. The detailed description of these conditions is not of prime concern here.

3.4. Efficiency, Transactions Costs and Numbers

In Section 2.3 we defined efficiency in the context of the game with transactions costs. We were able to suggest an analogue to the competitive equilibrium and to show that for a finite number of players

the N.E. were not efficient but under replication institutional efficiency is achieved.

3.5. Increasing Returns, Setup Costs, Institutions and Customs

It is well known that transactions technologies may involve set up costs and increasing returns to scale. Although mathematical techniques exist which can "handle nonconvexities which are not too severe," we wish to sketch at a less formal level a possible dynamics of increasing returns which accounts for a nonsymmetry in the growth and decline of market institutions. This nonsymmetry has its interpretation to some extent in terms of social custom, professional courtesy and threat non-cooperative equilibrium in even large games.

Suppose, for example, a profession such as medicine has a custom of professional courtesy where A will treat B without charge. In essence the members of the class do not need to know each other as individuals, but merely as members of the class. They may need an identification only of violators of the norm. But if all know that a violator of the code will be identified and "punished" by even an anonymous upholder of the code then stable threat equilibrium can be established. If A is treated by B this does not mean that B must be treated by A. In essence a code of behavior works on a clearing house basis.

In terms of markets and the need for money the existence of customs, codes or professional courtesy provides for pockets of exchange outside of explicit markets. This in turn makes the problem of measuring trade and the required amount of money in an economy somewhat more difficult.

The relationship between custom, threat equilibria and increasing returns comes in the dynamics of the formation of social networks. A chance event such as a war can provide an exogenous event bringing a

great many individuals together. This may provide the basis for a veteran's magazine which in turn makes it economically feasible to form a discount or swap club.

It is suggested that to a great extent the dynamics of the formulation of financial institutions may depend in an important way upon one or more events which enable a few organizations to overcome the barriers of new institution building in a fashion characterized by increasing returns to communication network size and volume of trade.

The type of exogenous event that provides enough impetus may be a war, an innovation, a new tax or a change in the law. The growth of the money market funds in the United States in the 1970's provides an example. The constraints on banking and high interest rates provided an incentive large enough for a new institution to go through the zone of high costs and low volume. In 1982 banking laws changed and at first sight it might appear that the money market funds could be wiped out as fast as they appeared. But in the course of a decade the institutions have built up an equity or a value in and of themselves associated with their customer lists, interfacing with brokers and other aspects of transactions convenience.

3.6. Taxation and the Quantity of Money

We suggest that in a modern economy with m actual economic goods, k financial instruments and a fiat money, a little more than $m+k$, but considerably less than $(m+k)(m+k+1)/2$ markets will exist. If a tax changes or law or technology influences the economy appropriately both the number of financial instruments and markets will change.

When attempts are made to control an economy by taxation or legal

restrictions there is a high probability that conditions may be created for the formation of new instruments and markets. Thus even the definition of optimal supply of money cannot be given in a satisfactory manner without specifying the transactions technology and estimating how it may change as taxes or other controls or technological developments influence the number of markets and instruments.

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