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GLOBAL STABILITY IN A CLASS OF MARKETS WITH THREE COMMODITIES AND THREE CONSUMERS

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GLOBAL STABILITY IN A CLASS OF MARKETS WITH THREE COMMODITIES AND THREE CONSUMERS

by

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1. In this paper we study the problem of a classical tâtonnement process. We focus our attention on the distribution of initial holdings and examine the extent to which a process will be globally stable.

It is widely held that in general one cannot expect the tatonnement process to be globally stable unless the class of market excess demand functions satisfies some stringent conditions.[1] There are two sources of this result. First, the counterexamples provided by Scarf [6] show that the tâtonnement process may be <u>globally unstable</u>. Second, a series of the analyses, originated by Sonnenschein [7] and put forward by Mantel [5], Debreu [3] and others, imply that solution paths of a tâtonnement process are potentially arbitrary.

In this paper we shall generalize the recent analysis given by Hirota [4] who examines the set of initial holdings for which Scarf's example is <u>globally stable</u>. His type of example is characterized by

$$\{u^j(x_{ij}, x_{kj}) = \min[x_{ij}, x_{kj}], \; a^j = (a_{1j}, a_{2j}, a_{3j})\}$$

$$(i \neq j \; , \quad k \neq j \; ; \quad j = 1, 2, 3)$$

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[1]See Arrow and Hurwicz [1] and Arrow, Block and Hurwicz [2].

where $u^j(\cdot)$ and a^j denote the j^{th} person's utility function and initial endowment vector. Let C be the parameter space defined by $\sum_{i=1}^{3} a_{ij} = 1$ $(j = 1, 2, 3)$ and $\sum_{j=1}^{3} a_{ij} = 1$ $(i = 1, 2, 3)$, each point of which yields the same equilibrium price. Then the tatonnement process can be properly settled into the framework of a parametric differential equation. Scarf's $a^1 = (0,1,0)$, $a^2 = (0,0,1)$, $a^3 = (1,0,0)$ that leads to unisolated limit cycles is a special point of C. Hirota has shown that a subset G of C for which a tatonnement process is guaranteed to converge to an equilibrium price is quite large; more precisely, the ratio of measure G to measure C is equal to $\left(\frac{3}{2} - \log 2\right) \doteqdot 0.806$, indicating the extent to which a process can be globally stable when Scarf's utility function dominates the system.[2]

The aim of this paper is to extend the above result to a more general case irrespective of choice of class of utility function. In order to do so we consider a space of utility functions that includes Scarf's function $\min[x_{ij}, x_{kj}]$ as a limiting case, and also take a similar parameter space C of initial holdings as in the above. What we attempt to show is that whatever utility function we choose from our space, the ratio of measure G to measure C is necessarily larger than or equal to $\left(\frac{3}{2} - \log 2\right) \doteqdot 0.806$.

2. Let us consider a pure exchange economy that is characterized by:

$$\{u^j(x_{ij}, x_{kj}),\ a^j = (a_{1j}, a_{2j}, a_{3j})\} \quad (i \neq j\ ,\ k \neq j\ ,\ j = 1, 2, 3)$$

where $u^j(\cdot)$ and a^j denote the j^{th} person's utility function and initial endowment vector.

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[2] See Section 6 of Hirota [4].

Each person's utility function is assumed to satisfy:

(A.1) $u^j(\cdot)$ is defined on the nonnegative orthant of R^2, is continuously differentiable, and has positive partial derivatives.

(A.2) $u^j(\cdot)$ is strictly quasi concave, i.e., for each real number α, the set defined by $u^j(\cdot) \geqq \alpha$ is strictly convex.

(A.3) The marginal rate of substitution is unity if the amounts of two goods are identical, i.e.,

$$(\partial u^j/\partial x_{ij})/(\partial u^j/\partial x_{kj}) = 1 \quad \text{if} \quad x_{ij} = x_{kj} .$$

The space of all utility functions satisfying (A.1) to (A.3) includes Scarf's function $\min[x_{ij}, x_{kj}]$ as a limiting case. For instance, a C.E.S. function converges to $\min[x_{ij}, x_{kj}]$ as the elasticity of substitution goes to zero.

(A.1) and (A.2) are the conventional postulates to yield a diminishing marginal rate of substitution. (A.3) together with a restriction on initial stock vectors that appears soon, serves to make it possible for each good to have an identical price at equilibrium, and also is utilized to obtain our theorem. Since no homothetic assumption is made except along a ray, the model developed here covers a situation in which there are inferior goods.

The restriction we put on initial stock vectors is

(1) $$\sum_{i=1}^{3} a_{ij} = \theta , \quad j = 1, 2, 3$$

(2) $$\sum_{j=1}^{3} a_{ij} = \theta , \quad i = 1, 2, 3 .$$

In the above, θ is an arbitrary positive value. (1) yields that each trader has an identical income when the price of each good is the same. (2) is a condition that the total supply of each good is θ. (1) and (2) together imply that the endowment matrix $A = [a_{ij}]$ equals $\theta \cdot \bar{A}$ where $\bar{A}$ is a double stochastic matrix. Thus, if the four elements of A, say, $a = (a_{11}, a_{21}, a_{12}, a_{22})$, are chosen so as to satisfy (1) and (2), the others can be determined by them; $a_{3i} = \theta - a_{1i} - a_{2i}$ $(i = 1, 2)$, $a_{i3} = \theta - a_{i1} - a_{i2}$ $(i = 1, 2)$, and $a_{33} = a_{11} + a_{12} + a_{21} + a_{22} - \theta$. This allows us to consider the following parameter space

$$C(\theta) = \{a = (a_{11}, a_{21}, a_{12}, a_{22}) \geqq 0 \mid \theta \geqq a_{i1} + a_{i2} \quad (i = 1, 2)$$
$$\theta \geqq a_{1i} + a_{2i} \quad (i = 1, 2)$$
$$a_{11} + a_{21} + a_{12} + a_{22} \geqq \theta\} .$$

Next we consider the j^{th} person's demand functions which are obtained by maximizing $u^j(\cdot)$ subject to his budget equation

(3) $$p_i x_{ij} + p_k x_{kj} = M_j$$

where p_i and p_k are nominal prices of the i^{th} and k^{th} goods, and $M_j = \sum p_i a_{ij}$ the nominal income of the j^{th} person.

The necessary and sufficient conditions for the above problem are

(4) $$\partial u^j / \partial x_{\tau j} - \lambda_j p_\tau \leqq 0 , \quad \tau = i, k$$

(5) $$x_{\tau j} \cdot [\partial u^j / \partial x_{\tau j} - \lambda_j p_\tau] = 0 , \quad \tau = i, k$$

(6) $$x_{\tau j} \geqq 0 , \quad \tau = i, k$$

where λ_j is the Lagrangian multiplier that is positive in our context. Let the solution above be denoted by $x_{\tau j}(p_i, p_k; M_j)$'s whose properties are well known.

Summing up all the person's excess demand functions we have the following market excess demand functions with a parameter $a \in C$

$$\begin{aligned} f_1(p;a) &= x_{12}(p_1, p_3; M_2) + x_{13}(p_1, p_2; M_3) - \theta \\ f_2(p;a) &= x_{21}(p_2, p_3; M_1) + x_{23}(p_1, p_2; M_3) - \theta \\ f_3(p;a) &= x_{31}(p_2, p_3; M_1) + x_{32}(p_1, p_3; M_2) - \theta \end{aligned} \tag{7}$$

where $p = (p_1, p_2, p_3)$ is a nominal price vector.

Here we consider the set of general equilibrium price vectors that are positive, i.e., $E(a) = \{p > 0 \mid f_i(p;a) = 0 \text{ for all } i\}$. $p^* = (1,1,1)$ is always an element of $E(a)$ for any $a \in C$ because we can have $x_{\tau j}(1,1;\theta) = \theta/2$ by (A.3). It is noted that some $\tilde{p} \neq \lambda p^*$ might be another element of $E(a)$.

Let us now consider a tatonnement process represented by the following parametric differential equation

$$\dot{p}_i = f_i(p;a) \ , \quad i = 1, 2, 3 \tag{8}$$

where $\dot{p}_i$ is the time derivative of p_i .

Let us take an arbitrary initial point $p(0)$ satisfying

$$(p_1)^2 + (p_2)^2 + (p_3)^2 = 3 \tag{9}$$

$$p_i > 0 \ , \quad i = 1, 2, 3 \ . \tag{10}$$

By Walras' law $\sum p_i f_i(p;a) \equiv 0$, the solution $p[t; p(0); a]$ to (8) satisfies (9).

According to the traditional Arrow-Block-Hurwicz approach, one has to put a stringent condition on the $f_i(p;a)$'s in order to ensure the global stability of (8). Our approach here is quite different. Our concern is to find a subset of the parameter space $C(\theta)$, for which the solution to (8) is guaranteed to converge to $p^* = (1,1,1)$, i.e.,

$$G(\theta) = \{a \in C(\theta) \mid \lim_{t\to\infty} p[t;\, p(0);\, a] = p^*\} .$$

From now on we will call G the stable set. $G(\theta)$ is non-empty: Take $a^* = \left(0, \frac{\theta}{2}, \frac{\theta}{2}, 0\right) \in C(\theta)$. Then a special assignment's $a^1 = \left(0, \frac{\theta}{2}, \frac{\theta}{2}\right)$, $a^2 = \left(\frac{\theta}{2}, 0, \frac{\theta}{2}\right)$, $a^3 = \left(\frac{\theta}{2}, \frac{\theta}{2}, 0\right)$ constitutes no trade equilibrium. As is well known,[3] $p[t;\, p(0);\, a^*]$ converges to p^*. Our concern is to determine the size of $G(\theta)$.

What we show is

<u>Theorem</u>: For arbitrary utility functions satisfying (A.1) to (A.3) and any positive value of θ,

$$\text{(11)} \qquad \frac{\text{measure } G(\theta)}{\text{measure } C(\theta)} = \frac{\int_{a\in G} da}{\int_{a\in C} da} \geqq \frac{3}{2} - \log 2 \doteqdot 0.806 .$$

Since the stable set can be this large, the phenomenon of global stability dominates the case of instability. The ratio of (11) indicates the probability of (8) being globally stable when each point of the parameter space C is assumed to occur with equal frequency. This suggests a probabilistic way to grasp the possible phenomena behind the system.

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[3] See p. 530 in Arrow and Hurwicz [1].

As is analyzed in Hirota [4], the right hand side value of (11) exactly equals the ratio of measure G to measure C when Scarf's utility function $\min[x_{ij}, x_{kj}]$ is assumed. We see, therefore, that substitutability among goods has the effect of increasing the extent of global stability of the tâtonnement process.

3. Let us prove our theorem by contrasting our economy here and a limiting case of ours, i.e., the j^{th} person's utility function is given by $\overline{u}^j(x_{ij}, x_{kj}) = \min[x_{ij}, x_{kj}]$ $(i \neq j\ , k \neq j\ ; j = 1, 2, 3)$.

The demand functions obtained by maximizing $\overline{u}^j(\cdot)$ subject to (3) are

$$(12) \qquad \overline{x}_{\tau j}(p_i, p_k; M_j) = \frac{M_j}{p_i + p_k}, \quad \tau = i, k$$

Below we write $[\overline{f}_1(p;a), \overline{f}_2(p;a), \overline{f}_3(p;a)]$ for the market excess demand functions constructed from (12). In the above, a is an element of the parameter space $C(\theta)$.

We define

$$(13) \qquad g_{\tau j}(p_i, p_k; M_j) = x_{\tau j}(p_i, p_k; M_j) - \overline{x}_{\tau j}(p_i, p_k; M_j)$$

where $x_{\tau j}(p_i, p_k; M_j)$ is the demand function of our economy.

Since $x_{\tau j}(\cdot)$ and $\overline{x}_{\tau j}(\cdot)$ are defined on the same budget equation (3), we have

$$(14) \qquad p_i g_{ij}(p_i, p_k; M_j) + p_k g_{kj}(p_i, p_k; M_j) \equiv 0 .$$

By using (13) our market excess demand functions (7) can be decomposed into two parts as follows

$$f_1(p;a) = \bar{f}_1(p;a) + [g_{12}(\cdot) + g_{13}(\cdot)]$$

(15) $$f_2(p;a) = \bar{f}_2(p;a) + [g_{21}(\cdot) + g_{23}(\cdot)]$$

$$f_3(p;a) = \bar{f}_3(p;a) + [g_{31}(\cdot) + g_{32}(\cdot)] \ .$$

Let us take a Liapunov function

(16) $$v(p) = [q-1]^2$$

(where $q = \prod_{i=1}^{3} p_i$), the time derivative of which is given by $\dot{v} = 2[q-1]\cdot\dot{q}$. Since the solution to (8) with an initial point $p(0)$ satisfying (9) and (10) always satisfies (9), $(q-1)$ is negative for all $p \neq p^*$. On the other hand, simple calculation yields

(17) $$\dot{q} = p_2p_3f_1(p;a) + p_1p_3f_2(p;a) + p_1p_2f_3(p;a) \ ; \ \text{by (8)}$$

$$= \varphi(p;a) + \psi(p;a) \ ; \ \text{by (15)}$$

where

(18) $$\varphi(p;a) = p_2p_3\bar{f}_1(p;a) + p_1p_3\bar{f}_2(p;a) + p_1p_2\bar{f}_3(p;a)$$

and

(19) $$\psi(p;a) = p_1[p_2g_{31}(\cdot) + p_3g_{21}(\cdot)]$$

$$+ p_2[p_1g_{32}(\cdot) + p_3g_{12}(\cdot)]$$

$$+ p_3[p_2g_{13}(\cdot) + p_1g_{23}(\cdot)] \ .$$

Next we find a subset of C such that $\dot{q}$ is positive for all $p \neq p^*$; such a subset leads to the global stability of (8).

(18) actually becomes a quadratic form

$$(20) \qquad \varphi(p;a) = \left(\frac{1}{2}\right)\cdot p[A + A' - \theta\cdot I^c]p'$$

$$= \left(\frac{1}{2}\right)[p_1 - p_3,\ p_2 - p_3][\tilde{A} + \tilde{A}' - \theta\cdot\tilde{I}^c]\begin{bmatrix} p_1 - p_3 \\ p_2 - p_3 \end{bmatrix}$$

where $\tilde{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, A' and $\tilde{A}'$ denote the transposed matrices of A and $\tilde{A}$, and I^c and $\tilde{I}^c$ denote matrices such that all the diagonal elements are nil and all the off-diagonal elements are unity.

(20) tells us that $\varphi(p;a)$ is nonnegative for all p if and only if the right hand side of (20) is semi positive definite, i.e.,

$$(21)^4 \qquad 4a_{11}a_{22} \geqq (\theta - a_{12} - a_{21})^2 .$$

Next we show the positivity of (19) for all $p \neq p^*$. First consider the first parenthesis of (19).

(14) leads to

$$(22) \qquad p_2 g_{31}(\cdot) + p_3 g_{21}(\cdot) = [(p_2 + p_3)/p_3](p_3 - p_2)\cdot g_{21}(\cdot) .$$

Let us show (22) is positive unless $p_2 = p_3$. Suppose $x_{21}(\cdot) \geqq x_{31}(\cdot) \geqq 0$ for $p_2 > p_3$. Then from the conditions (4)-(6) we have

$$\partial u^1/\partial x_{21} - \lambda_1\cdot p_2 = 0 , \quad \partial u^1/\partial x_{31} - \lambda_1\cdot p_3 \leqq 0$$

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[4] (21) with inequality is equivalent to Boundary condition (#): $\overline{f}_i(p;a) > 0$ if $p_i = 0$, $p_j > 0$ and $p_k > 0$ for differing i, j, k. (21) with equality is delicate for Scarf's type example: the system $\dot{p}_i = \overline{f}_i(p;a)$ $(i = 1, 2, 3)$ is subject to a limit cycle when $a_{11} = a_{22} = 0 = (\theta - a_{21} - a_{21})$, and is globally stable when $a_{11} + a_{22} > 0$ and $4a_{11}a_{22} = (\theta - a_{12} - a_{21})^2$. See propositions 2 and 3 of Hirota [4].

which yield $(\partial u^1/\partial x_{21})/(\partial u^1/\partial x_{31}) \geq p_2/p_3 > 1$. On the other hand, the diminishing marginal rate of substitution (by (A.1) and (A.2)) and (A.3) together say that if $x_{21}(\cdot) \geq x_{31}(\cdot)$, then $(\partial u^1/\partial x_{21})/(\partial u^1/\partial x_{31})$ must be less than or equal to unity. Therefore $0 \leq x_{21}(\cdot) < x_{31}(\cdot)$ has to hold for $p_2 > p_3$.

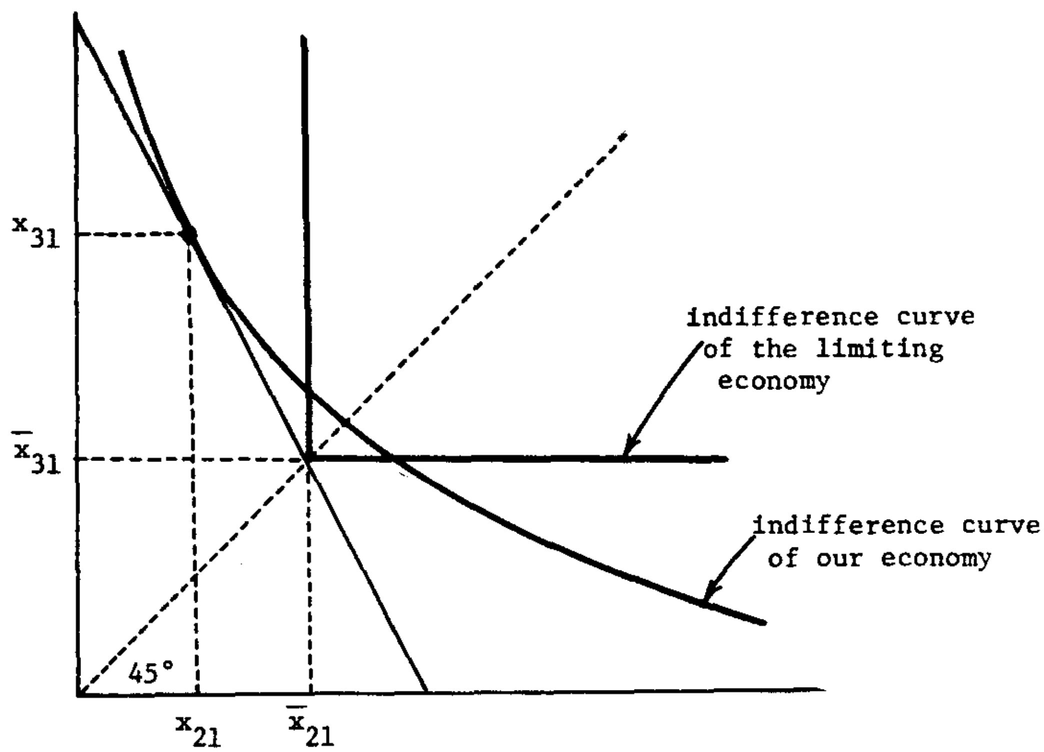

FIGURE 1

It is also clear that $x_{21}(\cdot) < \bar{x}_{21}(\cdot) = \bar{x}_{31}(\cdot) < x_{31}(\cdot)$ holds for $p_2 > p_3$, so that $g_{21}(\cdot) = x_{21}(\cdot) - \bar{x}_{21}(\cdot) < 0$. Similarly we have $g_{21}(\cdot) > 0$ for $p_2 < p_3$. Thus, (22) is positive unless $p_2 = p_3$.

The positivity of the second and third parentheses of (19) can be obtained similarly unless $p_1 = p_3$ and $p_1 = p_2$, so that we may summarize these as

$$\psi(p;a) \text{ is positive for all } p \neq p^* .$$

Let $\overline{G}(\theta)$ be the set defined by

$$\overline{G}(\theta) = \{a \in C(\theta) \mid (21) \text{ is met}\} .$$

Hence we know that for any $\tilde{a} \in \overline{G}(\theta)$, the solution $p[t; p(0); \tilde{a}]$ to (8) converges to $p^* = (1,1,1)$. So our stable set $G(\theta)$ defined in Section 2, contains $\overline{G}(\theta)$, i.e., $G(\theta) \supseteq \overline{G}(\theta)$.

When $\theta = 1$, the measure of $C(1)$ and $\overline{G}(1)$, as shown in Hirota [4], equals

$$\text{measure } C(1) = \int_{a \in C(1)} da = \frac{1}{8}$$

$$\text{measure } \overline{G}(1) = \int_{a \in \overline{G}(1)} da = \frac{1}{8}\left(\frac{3}{2} - \log 2\right) .$$

Since there are linear maps $\theta \cdot I : C(1) \to C(\theta)$ and $\theta \cdot I : \overline{G}(1) \to \overline{G}(\theta)$ where I is the identity matrix, we have

$$\text{measure } C(\theta) = \theta^4 \cdot \text{measure } C(1), \text{ measure } \overline{G}(\theta) = \theta^4 \cdot \text{measure } \overline{G}(1) ,$$

so that

$$\frac{\text{measure } G(\theta)}{\text{measure } C(\theta)} \geqq \frac{\text{measure } \overline{G}(\theta)}{\text{measure } C(\theta)} = \frac{3}{2} - \log 2 \doteqdot 0.806 .$$

This completes the proof.

We have clarified the extent to which a tâtonnement process is guaranteed to converge to an equilibrium price. But our model setting has been fairly restrictive. A subsequent question is to examine to what extent our result will still hold under more general settings; for instance, when a different equilibrium price from p^* is the case, the utility function involves three kinds of commodities, and there are more commodities than three. Now we cannot provide a definite answer to them.

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