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TOWARDS A SOLUTION TO THE GENERAL PROBLEM OF SELF-SELECTION

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## Abstract

Models with self-selection features are now commonplace in the economics literature. In these problems, a single agent (a monopolist, government or principal) must induce other agents (taxpayers or consumers) to reveal information about their characteristics. This need to elicit private information creates distortions in allocations relative to those obtained under full-information.

This paper considers a fairly general self-selection model and provides a characterization of these distortions due to private information. In its most general form, the model restricts agents' preferences to satisfy a single-crossing property. Using this restriction we are able to obtain a complete characterization to the self-selection problem. In addition, the role of randomization as a sorting mechanism is considered. The paper indicates that in many cases, randomization can relax otherwise binding self-selection constraints and hence allow the monopolist to extract additional consumers surplus.

## I. Introduction

The distortionary effects of imperfect information have received a considerable amount of attention in the economics literature. In a recent paper, Harris-Townsend [1981] provided a general framework for the analysis of problems with imperfect information. In particular, Harris-Townsend give a characterization of efficient allocations as well as the mechanisms to implement them in an environment with imperfect information.

In this paper, we focus on self-selection problems which constitute a sub-class of the environments encompassed in the Harris-Townsend framework. This allows us to obtain a more specific characterization of the distortionary effects of imperfect information.

In self-selection problems, a single agent (a monopolist or a government) seeks information about the characteristics of other agents as part of an optimizing problem. Since these characteristics are generally not public information, the monopolist (government) must provide incentives for the revelation of this private information. The extraction of information leads to distortions in the allocative process relative to the allocation obtained in an environment of perfect information.

Problems of this type are now commonplace. Stiglitz [1977] provided an example in which an insurance company seeks to separate agents by their probabilities of an accident. In other examples, a monopolist attempts to separate consumers by offering: different price-quality bundles as in Mussa-Rosen [1978]; different price-quantity bundles as in Spence [1979] or different price-time bundles as in Chiang-Spatt [1982]. In addition, the informational constraints on the government

in an optimal taxation problem can also be viewed as self-selection constraints as discussed first by Mirrlees [1971] and subsequently in Atkinson-Stiglitz [1976] and Stiglitz [1981]. Also, the work on optimal auctions, see Myerson [1981] or Harris-Raviv [1981], can be cast in our framework. Finally, the principal-agent problem in which the agent takes an action after observing the state of nature is also a self-selection problem (see, for example, Sappington [1981]).<sup>1</sup>

In reading the literature on self-selection models, it is apparent that each of these examples has a relatively similar structure. Despite this similarity, no attempt has been made to characterize the solution to a general model of self-selection.<sup>2</sup>

The purpose of this essay is primarily methodological. We present a general model of self-selection as a means of summarizing and unifying the existing literature. In doing so, we provide future researchers with a framework for solving the self-selection problem in other contexts.

Section II of the paper proposes a very general self-selection model and discusses many of the problems in obtaining a complete solution. Using a restriction on consumer preferences--the single-crossing property--Section III characterizes the distortions due to the sorting of agents.

Section IV of the paper discusses the role of randomization as well as the important problem of pooling versus separating solutions. This discussion includes some new results on the role of randomization as a sorting mechanism. Finally, Section V presents some additional research problems and summarizes the results.

## II. General Self-Selection Model

In this section we consider the general problem of self-selection. Consider a single-agent (hereafter called the monopolist) choosing a vector  $x_i$  for  $i = 1, 2, \dots, N$  to

$$\text{maximize } W(x_1, x_2, \dots, x_N) \quad (1)$$

$$\text{subject to } U^i(x_i) \geq \bar{U}_i \text{ for } i = 1, 2, \dots, N \quad (1.a)$$

$$U^i(x_i) \geq U^i(x_j) \text{ for all } i, j \quad (1.b)$$

$$\text{and } x_i \in X_i \text{ for all } i. \quad (1.c)$$

Here  $W(\cdot)$  is the objective function of the monopolist and  $U^i(x)$  is the utility function for agent  $i = 1, 2, \dots, N$ . The monopolist chooses a vector  $x_i$  for each agent to maximize  $W(\cdot)$  subject to three types of constraints. The first of these are individual rationality constraints (1.a) which restrict the monopolist from lowering agent's utilities below a certain lower bound,  $\bar{U}_i$ . The second type of constraints (1.b) are fundamental to sorting problems of this type. These are the self-selection constraints which ensure that agents choose the vector  $x_i$  intended for them. Finally, (1.c) are feasibility constraints on the  $x_i$ 's. Hereafter, we assume that  $N$  is finite.

The main feature of this problem is the information asymmetry: agents know their own preferences while the monopolist does not. Following Harris-Townsend and Myerson [1979] we restrict attention to direct revelation mechanisms in which agents are induced to truthfully report their characteristics to the monopolist. This is implemented by the self-selection constraints. Our main concern is to characterize the distortion created by the inclusion of these constraints in the monopolist's

problem.

Before proceeding further, it might be useful to consider some examples of self-selection models which are part of this framework.

Example 1: Price and Product Quality

As a special case of the Mussa-Rosen model, consider a monopolist who produces goods of varying quality for sale to consumers of different tastes. Consumers of type  $i$  pay a price  $p_i$  to obtain one unit of a good of quality  $q_i$  which they evaluate by  $U_i(p_i, q_i)$ . Letting  $N_i$  denote the number of agents of type  $i$  and  $c(q)$  be an increasing, convex cost function over quality, the monopolist chooses  $(p_i, q_i)$  for  $i = 1, 2, \dots, N$  to

$$\text{maximize } \sum_{i=1}^N N_i (p_i - c(q_i))$$

$$\text{subject to } U^1(p_i, q_i) \geq \bar{U}_1$$

$$U^1(p_i, q_i) \geq U^1(p_j, q_j) \text{ for all } j, i$$

$$\text{and } p_i \geq 0 ; q_i \geq 0 \text{ for all } i .$$

Example 2: Monopoly Insurance

Following Stiglitz [1977], we consider an insurance company offering risk averse agents a contract to share the risks of stochastic wealth. An agent has wealth  $w$  when no "accident" occurs and a wealth of  $(w-d)$  when an accident does occur. Agents differ in their accident probabilities,  $\pi_i$ , which are not directly observable to the firm. The insurance company offers each agent a contract specifying an indemnity  $I_i$  if an accident occurs and charges a premium  $P_i$  in the "no accident" state.

Agents evaluate these contracts by  $V^i(P_i, I_i) \equiv V(P_i, I_i; \pi_i)$  where

$$V(P_i, I_i; \pi_i) = \pi_i U(w - d + I_i) + (1 - \pi_i) U(w - P_i)$$

and  $U(\cdot)$  is a strictly increasing, concave function of wealth. With  $N_i$  agents of each type, the monopolist chooses  $(P_i, I_i)$  for  $i = 1, 2, \dots, N$  to

$$\text{maximize } \sum_{i=1}^N N_i ((1 - \pi_i) P_i - \pi_i I_i)$$

$$\text{subject to } V(P_i, I_i; \pi_i) \geq V(0, 0; \pi_i)$$

$$V(P_i, I_i; \pi_i) \geq V(P_j, I_j; \pi_i)$$

and

$$P_i \leq w, \quad I_i \geq d - w.$$

### Example 3: Optimal Taxation

As discussed in Stiglitz [1981], the optimal taxation problem fits into the self-selection framework as well. Due to differences in ability, agents receive different wages and hence have different preferences over consumption,  $c_i$ , and income,  $y_i$ . We can parameterize preferences by the wage,  $w_i$ , and denote agent  $i$ 's utility as  $U(c_i, y_i; w_i)$ . Both  $y_i$  and  $c_i$  are publicly observable while  $w_i$  is private. The government's revenue from the  $N_i$  type  $i$  agents is  $N_i(y_i - c_i)$ . Hence the government chooses  $(y_i, c_i)$  for  $i = 1, 2, \dots, N$  to

$$\begin{aligned} & \text{maximize} \quad \sum N_i (y_i - c_i) \\ & \text{subject to} \quad U(y_i, c_i; w_i) \geq \bar{U}_i \quad \text{for } i = 1, \dots, N \\ & \quad \quad \quad U(y_i, c_i; w_i) \geq U(y_j, c_j; w_j) \quad \text{for all } i, j \\ & \text{and} \quad \quad \quad y_i \geq 0, \quad c_i \geq 0 \quad \text{for } i = 1, 2, \dots, N. \end{aligned}$$

These examples clearly fit into the general framework of self-selection models. The interested reader can verify that the other models discussed in the introduction are encompassed in the general set-up as well.

In the solution to these, and other, self-selection problems, a particular pattern emerges. First, as we discuss in the following section, agents are generally ordered by "types." This ordering may refer to tastes in Example 1, to accident probabilities in Example 2 and to abilities (and hence wages) in the final example. Given an ordering of agents, the solutions to these problems have the following general characteristics:

- Individual rationality constraints are binding for only one type of agent;
- All agents except one type receive a distorted allocation in which marginal rates of substitutions are not equalized;
- The self-selection constraints are binding in a particular fashion. Agents are indifferent between their bundle and that intended for an adjacent type.

At this level of generality, it is difficult to be more specific about this pattern. In terms of the examples, the monopolist offers perfect insurance to high-risk agents and imperfect insurance to other risk classes. In addition, only the lowest risk class have a binding individual rationality constraint and agents are indifferent between their

insurance contract and that provided to agents of the next lowest risk class. A similar distortion of low quality to all but the highest class of buyers is shown by Mussa-Rosen. The purpose of this paper is to show that the pattern outlined above is a fundamental characteristic of self-selection problems.

To show this, it is necessary to put additional structure on the programming problem outlined above. First of all, (1) is not necessarily a standard concave programming problem.<sup>3</sup> A similar problem arises in the principal-agent literature (see Grossman-Hart [1983]) and is discussed by Mirrlees [1981] for the optimal taxation problem. We comment on the convexifying role of randomization in Section IV.

A second problem with (1) is the multitude of self-selection constraints. With  $N$  agents, there will be  $N(N-1)$  constraints which obviously complicates any attempt to characterize the solution to (1). This difficulty is overcome by placing additional structure on the preferences of the agents.

Finally, we constrain the  $x_i$ 's to be 2-dimensional. This is an assumption used throughout the sorting literature and will be discussed further in the conclusion. We now turn to the more structured version of (1).

### III. The Single-Crossing Property and Self-Selection Models

In this section we characterize the solution to (1) with some additional assumptions. First we restrict  $x_i \in \mathbb{R}^2$  for all  $i$  and denote the first component of  $x_i$  by  $q_i$  and the second by  $p_i$ . Secondly, we assume there is a single parameter,  $\phi_i$ , which characterizes the differences in agents' preferences over  $x$ . Together these assumptions imply that  $U^i(x_i) = U(p_i, q_i, \phi_i)$ . The essence of the informational asymmetry is that an individual agent's  $\phi_i$  is not directly observable to the monopolist.

With respect to preferences, we assume that  $W(p_1, \dots, p_N, q_1, \dots, q_N)$  is increasing and concave in  $p_i$  and decreasing and concave in  $q_i$ . Furthermore, given  $(p_j, q_j)$  for  $j \neq i$ , the monopolist's preferred sets over  $(p_i, q_i)$  are convex. Consumers view  $q_i$  as a "good" (quality) so that utility is increasing and concave in  $q$  while  $p$  is a "bad" (price) so that utility is decreasing and concave in  $p$ . We assume  $U(\cdot)$  is quasi-concave as well. For motivational purposes, we have used the notation  $(p_i, q_i)$  and may refer to these elements of  $x_i$  as price and quality respectively. However, the central point of the paper is to show that the solution to (1) is actually independent of the labels placed on the elements of  $x_i$ .

Finally, we assume that agents' preferences satisfy conditions which imply the single-crossing property defined below. This assumption allows us to order agents by their preferences and enables us to exclude a large number of the self-selection constraints.

Specifically, we assume that  $U_{q\phi}(p, q, \phi) \geq 0$  and  $U_{p\phi}(p, q, \phi) \geq 0$  with at least one strict inequality. These conditions imply that the marginal rate of substitution between price and quality,  $MRS(p, q, \phi_i)$ ,

is an increasing function of  $\phi$  for all  $(p,q)$  where

$$\text{MRS}(p, q, \phi_1) \equiv \frac{-U_q(p, q, \phi_1)}{U_p(p, q, \phi_1)} .$$

It is straightforward to demonstrate that the monotonicity of the MRS in  $\phi$  implies that indifference curves of agents with different tastes cross only once in  $(p,q)$  space. Hence the term single-crossing property (SCP) .

The SCP has been used elsewhere in the sorting literature.<sup>4</sup>

Many of the self-selection models have structures which are special cases of this property. For example, in the insurance problem, the accident probability is the taste parameter  $\phi$  and it is clear that an agent's MRS between premia and indemnities is monotone in the accident probability. Similarly, in the price and product quality model and in the taxation problem, preferences satisfy these properties. In addition, in his investigation of equilibria in competitive markets with information asymmetries, Riley [1979] made use of the SCP . As discussed by Hart [1982] a similar assumption on technology plays an important role in the optimal labor contracts literature with asymmetric information.

With this assumption, it is natural to view agents with larger  $\phi$ 's as "higher taste-types." These agents are willing to pay more for a given increment in "quality."

We now use the SCP to simplify (1). The first step is to demonstrate that  $p_i$  and  $q_i$  will be increasing in  $\phi$  —i.e.  $p_{i+1} \geq p_i$  and  $q_{i+1} \geq q_i$  . This will be used to reduce the number of constraints in (1) from  $N(N-1)$  to  $2N-2$  through the adjacency condition. To prove these results, we make use of a specific property of preferences satisfying

the SCP.

Lemma 1: If preferences satisfy SCP, then for any  $(\hat{p}, \hat{q})$  and any taste-types  $\phi_j$  and  $\phi_1$  with  $\phi_j > \phi_1$ ,

$$(i) \quad U(p, q, \phi_1) \geq U(\hat{p}, \hat{q}, \phi_1) \text{ implies}$$

$$U(p, q, \phi_j) \geq U(\hat{p}, \hat{q}, \phi_j) \text{ if } (p, q) \geq (\hat{p}, \hat{q})$$

and (ii)  $U(p, q, \phi_j) \geq U(\hat{p}, \hat{q}, \phi_j)$  implies

$$U(p, q, \phi_1) \geq U(\hat{p}, \hat{q}, \phi_1) \text{ if } (p, q) \leq (\hat{p}, \hat{q}) .$$

Proof:<sup>5</sup> We prove (ii) of the lemma, and leave the proof of (i) to the reader. Figure 1 depicts the situation. We want to show that for any  $(p, q)$  less than  $(\hat{p}, \hat{q})$  in both components (i.e. southwest of  $(\hat{p}, \hat{q})$ ) if agent  $j$  prefers  $(p, q)$  to  $(\hat{p}, \hat{q})$ , so will agent 1.

From SCP, we know that  $MRS_j(\hat{p}, \hat{q}) > MRS_1(\hat{p}, \hat{q})$  and hence that in a neighborhood of  $(\hat{p}, \hat{q})$  agent  $j$ 's preferred set lies in agent 1's for  $(p, q)$  less than  $(\hat{p}, \hat{q})$ . (See Figure 1.) By the quasi-concavity of  $U(\cdot)$ , for all  $t \in [0, 1]$  and any  $(p^*, q^*)$  such that  $U(p^*, q^*, \phi_j) \geq U(\hat{p}, \hat{q}, \phi_j)$ , then

$$U(tp^* + (1-t)\hat{p}, tq^* + (1-t)\hat{q}, \phi_j) \geq U(\hat{p}, \hat{q}, \phi_j) .$$

There will exist a  $t^*$  for every  $(p^*, q^*)$  which will place  $(tq^* + (1-t)\hat{q}, tp^* + (1-t)\hat{p})$  in the neighborhood of  $(\hat{p}, \hat{q})$  described above such that

$$U(t^*p^* + (1-t^*)\hat{p}, t^*q^* + (1-t^*)\hat{q}, \phi_1) > U(\hat{p}, \hat{q}, \phi_1) .$$

Hence, from the curvature of  $U(\cdot)$ ,

$$t^*U(p^*, q^*, \phi_1) + (1-t^*)U(\hat{p}, \hat{q}, \phi_1) > U(\hat{p}, \hat{q}, \phi_1) .$$

Therefore,  $U(p^*, q^*, \phi_i) > U(\hat{p}, \hat{q}, \phi_i)$  as we wished to show. A similar proof for part i of the lemma is left to the reader.  $\square$

Intuitively, as indicated in Figure 1, once we fix a  $(\hat{p}, \hat{q})$  point and divide the space into four quadrants then for all points north-east of  $(\hat{p}, \hat{q})$ , i's preferred set lies in j's and vice-versa for points southwest of  $(\hat{p}, \hat{q})$ .

Lemma 2: Both price and quality must be non-decreasing functions of  $\phi$ .

Proof: Assume otherwise, so that for some  $i$ ,  $p(\phi_i) > p(\phi_{i+1})$  and  $q(\phi_i) > q(\phi_{i+1})$ .<sup>6</sup> By the monotonicity of  $U(\cdot)$ , the other cases of  $p(\phi_i) > p(\phi_{i+1})$  and  $q(\phi_i) < q(\phi_{i+1})$  or  $p(\phi_i) < p(\phi_{i+1})$  and  $q(\phi_i) > q(\phi_{i+1})$  are obviously inconsistent with self-selection.

We use Lemma 1 to show that  $p(\phi_i) > p(\phi_{i+1})$  and  $q(\phi_i) > q(\phi_{i+1})$  will contradict self-selection. For this pair to satisfy the self-selection constraints,

$$U(p(\phi_{i+1}), q(\phi_{i+1}), \phi_{i+1}) \geq U(p(\phi_i), q(\phi_i), \phi_{i+1}) .$$

From Lemma 1 and  $p(\phi_i) > p(\phi_{i+1})$  and  $q(\phi_i) > q(\phi_{i+1})$ ,

$$U(p(\phi_{i+1}), q(\phi_{i+1}), \phi_i) \geq U(p(\phi_i), q(\phi_i), \phi_i)$$

which contradicts self-selection.  $\square$

We are now ready to state an important property of self-selection models when preferences satisfy the SCP. The adjacency condition (AC) guarantees that a sufficient condition for self-selection is that agents prefer their bundle to those intended for their neighbors. That is, under AC, if

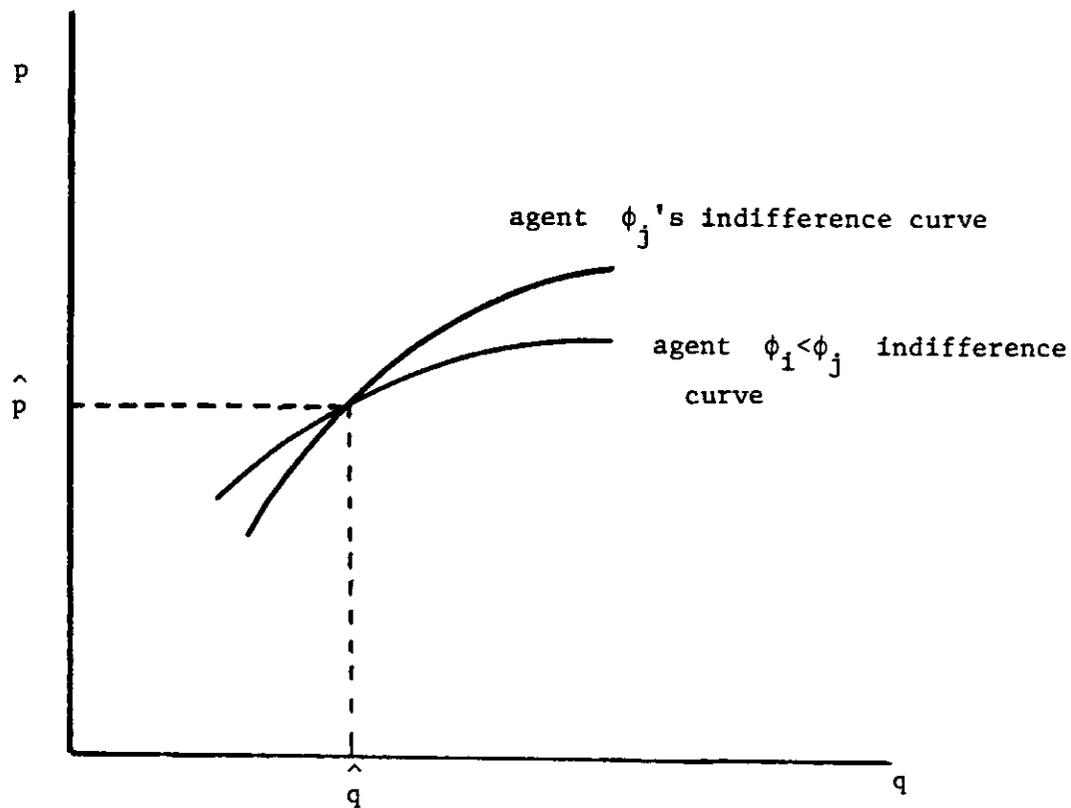


Figure 1.

$$U(p_i, q_i, \phi_i) \geq U(p_{i-1}, q_{i-1}, \phi_i)$$

and  $U(p_i, q_i, \phi_i) \geq U(p_{i+1}, q_{i+1}, \phi_i)$  for all  $i$ ,

then  $U(p_i, q_i, \phi_i) \geq U(p_j, q_j, \phi_i)$  for all  $i, j$ .

Lemma 3: If preferences satisfy SCP, then AC holds.

Proof: We assume that  $\forall i \ U(p_i, q_i, \phi_i) \geq U(p_k, q_k, \phi_i)$  for  $k = i+1$  and  $k = i-1$ . We wish to show that this and the SCP imply that  $U(p_i, q_i, \phi_i) \geq U(p_j, q_j, \phi_i)$  for all  $j$ . Suppose that for some  $j > i$ ,  $U(p_j, q_j, \phi_i) > U(p_i, q_i, \phi_i)$ . We show that this leads to a contradiction and leave the case of  $j < i$  to the reader.

We have assumed that

$$U(p_i, q_i, \phi_i) \geq U(p_{i+1}, q_{i+1}, \phi_i)$$

and  $U(p_{i+1}, q_{i+1}, \phi_{i+1}) \geq U(p_{i+2}, q_{i+2}, \phi_{i+1})$ .

Using Lemmas 1 and 2, this implies that

$$U(p_i, q_i, \phi_i) \geq U(p_{i+2}, q_{i+2}, \phi_i).$$

Continuing in this manner, it is straightforward to demonstrate that for  $j > i$

$$U(p_i, q_i, \phi_i) \geq U(p_j, q_j, \phi_i). \quad \square$$

Using the SCP, we see that the number of constraints in the monopolist's problem can be reduced considerably. We now proceed to the solution of that problem.

Using the results obtained thus far, the monopolist chooses  $(p_i, q_i)$  for  $i = 1, 2, \dots, N$  to

$$\text{maximize } W(p_1, q_1, \dots, p_N, q_N) \quad (2)$$

$$\text{subject to } U(p_i, q_i, \phi_i) \geq \bar{U}_i \text{ for } i = 1, 2, \dots, N \quad (2.a)$$

and, for all  $i$ ,

$$U(p_i, q_i, \phi_i) \geq U(p_{i-1}, q_{i-1}, \phi_i) \quad (2.b)$$

$$U(p_i, q_i, \phi_i) \geq U(p_{i+1}, q_{i+1}, \phi_i) . \quad (2.c)$$

The solution to this problem when agents are separated (i.e.

$p_{i+1} > p_i > p_{i-1}$  and  $q_{i+1} > q_i > q_{i-1}$  for all  $i$ ) is summarized by the following proposition. We consider the pooling solution in the next section.

Proposition 1: In an interior separating solution to (2),

- (i)  $U(p_1, q_1, \phi_1) = \bar{U}_1$  and  $U(p_i, q_i, \phi_i) > \bar{U}_i$  for  $i \neq 1$ .
- (ii)  $U(p_i, q_i, \phi_i) = U(p_{i-1}, q_{i-1}, \phi_i)$  for  $i = 2, 3, \dots, N$ ,  
and  $U(p_i, q_i, \phi_i) > U(p_{i+1}, q_{i+1}, \phi_i)$  for  $i = 1, 2, \dots, N-1$ .
- (iii)  $-\frac{U_q(p_N, q_N, \phi_N)}{U_p(p_N, q_N, \phi_N)} = -\frac{W_{q_N}(p_1, q_1, \dots, p_N, q_N)}{W_{p_N}(p_1, q_1, \dots, p_N, q_N)}$
- (iv)  $-\frac{U_q(p_i, q_i, \phi_i)}{U_p(p_i, q_i, \phi_i)} > -\frac{W_{q_i}(p_1, q_1, \dots, p_N, q_N)}{W_{p_i}(p_1, q_1, \dots, p_N, q_N)}$  for  $i \neq N$ .

Proof: Condition (i) is an implication of the self-selection constraints.

Assume, to the contrary that  $U(p_i, q_i, \phi_i) = \bar{U}_i$  for  $i \neq 1$ . So

$(p_i, q_i)$  is on  $i$ 's indifference curve through the origin.<sup>7</sup> Applying

Lemma 2 implies that  $U(p_{i-1}, q_{i-1}, \phi_i) > U(p_i, q_i, \phi_i)$  which violates self-selection.

To show that (2.a) must be binding for  $i = 1$ , assume that it was not. Then the monopolist could profitably raise  $p_1$  until the individual rationality constraint was binding for agent  $\phi_1$  and then raise the other prices to meet the self-selection constraints.

To see (iii), we note that the first-order conditions with respect to  $q_N$  and  $p_N$  are

$$W_{q_N}(p_1, q_1, \dots, p_N, q_N) + \delta_N U_{q_N}(p_N, q_N, \phi_N) - \beta_{N-1} U_{q_N}(p_N, q_N, \phi_{N-1}) = 0 \quad (3)$$

and

$$W_{p_N}(p_1, q_1, \dots, p_N, q_N) + \delta_N U_{p_N}(p_N, q_N, \phi_N) - \beta_{N-1} U_{p_N}(p_N, q_N, \phi_{N-1}) = 0 \quad (4)$$

In these expressions,  $\delta_i$  is the multiplier for (2.b) and  $\beta_i$  is the multiplier for (2.c). We have made use of the fact that  $\beta_N = \delta_{N+1} = 0$  and that (2.a) is not binding for  $i = N$ . These conditions imply that  $\delta_N > 0$  and hence  $\beta_{N-1} = 0$  in a separating solution.<sup>8</sup> The ratio of (3) and (4) implies (iii).

To generate (ii) and (iv), using  $\delta_N > 0$  implies that  $\delta_{N-1} > 0$  from the first-order conditions for  $p_{N-1}$  and  $q_{N-1}$ . Continuing in this manner, we obtain  $\delta_i > 0$  for  $i = 2, 3, \dots, N$  and  $\beta_i = 0$  for all  $i$  as in (ii). From our assumptions that  $U_{p\phi}$  and  $U_{q\phi}$  are non-negative, (iv) follows directly from the first-order conditions.  $\square$

This proposition shows that the properties of the self-selection examples discussed in Section II hold for a fairly wide class of problems. Properties (i) and (ii) of the proposition tell us which of the individual rationality and self-selection constraints must be binding in the optimal solution. Intuitively, it is easy to see why the self-selection constraints are binding on the bundle provided for the next

lowest taste type. Consider the solution to the monopolist's problem when agents' characteristics are publicly known. That is, solve (2) subject to (2.a) but not (2.b) and (2.c). The result--called the full-information solution--will have agents served efficiently (i.e. equalized marginal rates of substitution between the firm and each agent) and all individual rationality constraints will be binding. Once individual characteristics are private information, the full-information solution will not be implementable: Agents will strictly prefer the bundle of the lower taste agents to their own. Hence the self-selection constraints are binding in the manner described in (ii) to prevent this type of misrepresentation of preferences.

The distortions due to self-selection are summarized by (iii) and (iv). The highest-taste agents ( $\phi_N$ ) are served efficiently. For all other types, in the optimal solution the consumer's marginal rate of substitution exceeds that of the firm. Both prices and qualities are too low as shown in Figure 2. In the examples of the self-selection literature this distortion takes the form of incomplete insurance for low risk agents, low quality goods for low taste agents, etc.

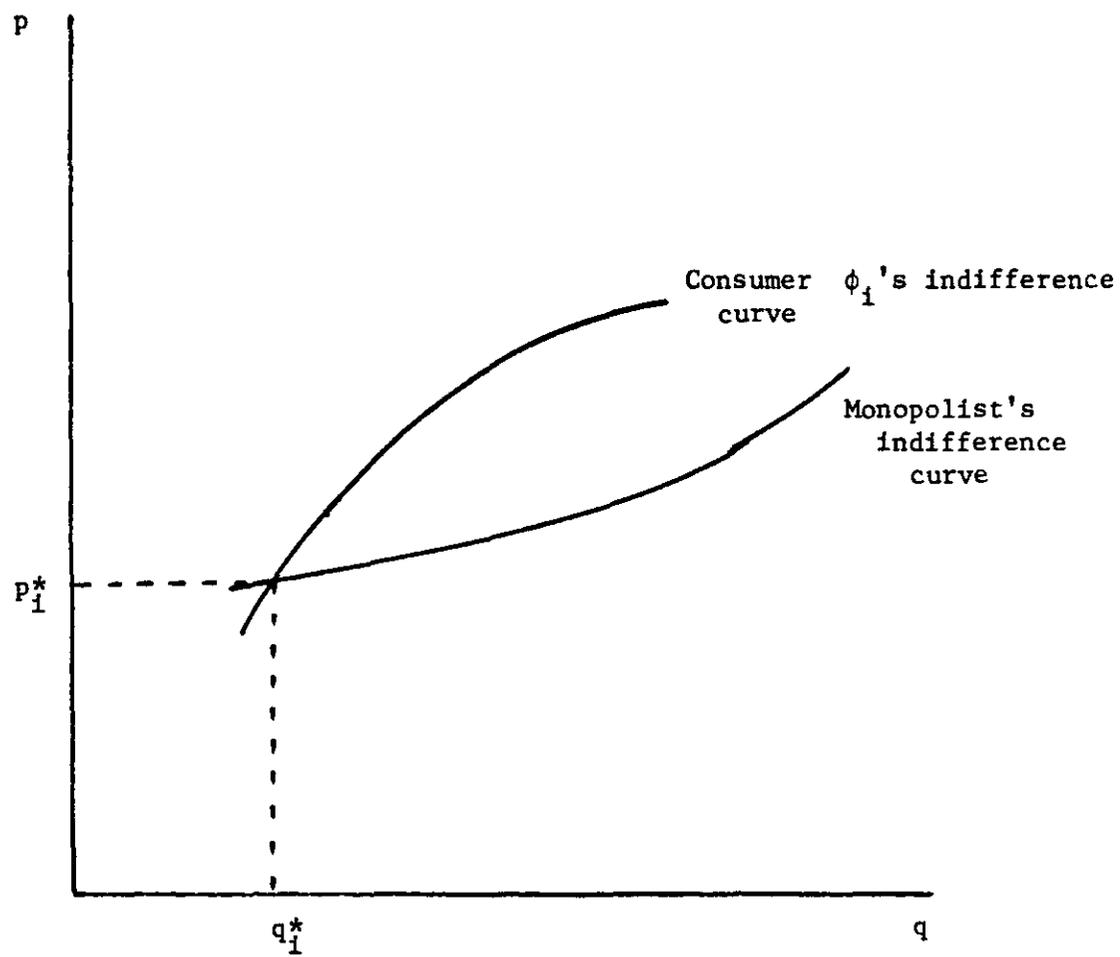


Figure 2.

#### IV. Comments on Separation versus Pooling of Taste Types and the Role of Randomization

##### A. A Characterization of Pooled Solutions

In the previous section, we focused on the optimal solution to (2) when complete separation of taste-types occurred. That is, agents of different tastes received distinct  $(p,q)$  bundles. In all cases, however, this will not be the optimal solution to the monopolist's problem. As discussed, for example, in Stiglitz [1977] and Mussa-Rosen [1978], the monopolist may choose to bunch agents with different tastes at a common  $(p,q)$  offering. Using this strategy, larger consumer's surplus can be extracted from agents of "higher tastes" than those who are pooled. The conditions for pooling are described by Stiglitz and Mussa-Rosen and will not be repeated here. Intuitively, the monopolist may choose to pool when the number of agents with "higher tastes" is large relative to the number of agents being pooled.

Our interest is in the distortionary effects of pooling in the solution to (2). For the purposes of this discussion, we continue to assume that preferences satisfy the SCP and the cross-partial restrictions discussed in the previous section. Before characterizing the solution to (2) with pooling, we obtain some restrictions on the type of pooling that is implementable. These same restrictions hold in the case of a continuum of tastes as discussed in Mussa-Rosen.

Lemma 4: If taste-types  $i$  and  $i+k$  are pooled, then all agents between are pooled as well.

Proof: If  $q_i = q_{i+k}$  and  $p_i = p_{i+k}$ , then in order for the price and quality schedules to be monotone (as shown in Lemma 2),

$q_j = q_i = q_{i+k}$  and  $p_j = p_i = p_{i+k}$  for  $j$  satisfying  $i \leq j \leq i+k$ .  $\square$

Lemma 5: In the solution to (2),  $q_n > q_{n-1}$  and  $p_n > p_{n-1}$  —i.e. the highest taste-types are never pooled.

Proof: Assume to the contrary that agent  $\phi_N$  is pooled with agent  $\phi_j$  and hence, by Lemma 4, will all  $i \in [j, N]$ . Call the bundle received by this group  $(\hat{p}, \hat{q})$ . Solving (2) subject to a restriction that these agents are pooled,<sup>9</sup> one obtains

$$-\frac{U_q(\hat{p}, \hat{q}, \phi_j)}{U_p(\hat{p}, \hat{q}, \phi_j)} = -\frac{W_{\hat{q}}(p_1, q_1, \dots, \hat{p}, \hat{q})}{W_{\hat{p}}(p_1, q_1, \dots, \hat{p}, \hat{q})}.$$

This is the counterpart to condition (iii) of Proposition 1 where the lowest taste-type in the pooled segment containing  $\phi_N$  is served efficiently. By the SCP,

$$-\frac{U_q(\hat{p}, \hat{q}, \phi_N)}{U_p(\hat{p}, \hat{q}, \phi_N)} > -\frac{W_{\hat{q}}(p_1, q_1, \dots, \hat{p}, \hat{q})}{W_{\hat{p}}(p_1, q_1, \dots, \hat{p}, \hat{q})}.$$

That is, the marginal rate of substitution for the highest taste-type exceeds that of the firm. As is obvious from Figure 3, the monopolist can profitably separate the  $\phi_N$  types from the rest of the pooled group by moving along the indifference curve of type  $\phi_N$  through  $(\hat{p}, \hat{q})$ . By the SCP, this adjustment will not violate any self-selection constraints.  $\square$

Proposition 2: The solution to (2), if pooling occurs, is characterized by the conditions stipulated in Proposition 1.

Proof: Following the steps in the proof of Proposition 1, it is obvious that conditions (i) and (ii) of the proposition will continue to hold. From Lemma 5, we know that high taste agents are never pooled. Hence,

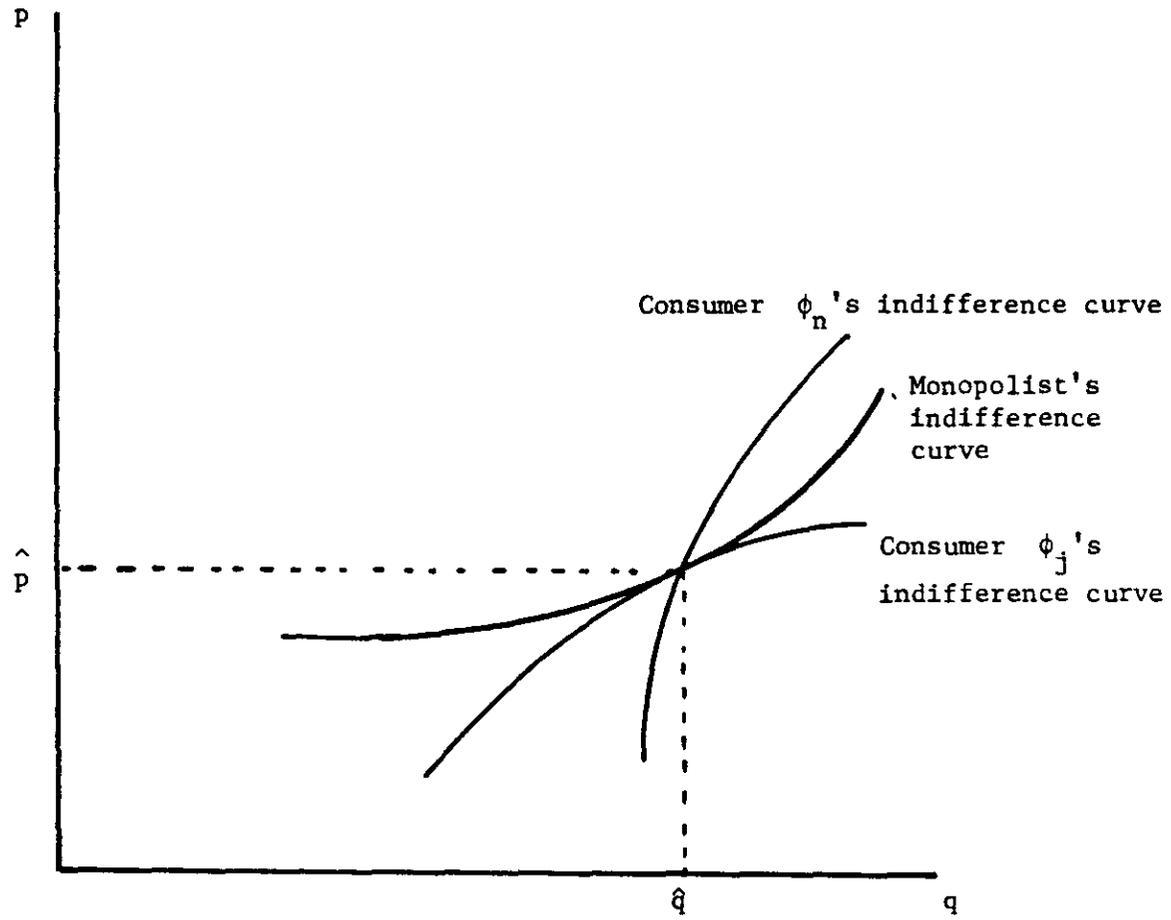


Figure 3.

they continue to receive an efficient allocation as described by condition (iii) of Proposition 1.

To show the distortions for the other agents, partition the set of agents into two groups. The first receives a separate bundle--i.e. are not pooled--and hence obtain a distorted bundle described in Proposition 1. The other group contains the intervals of taste-types pooled with one another. Choose an arbitrary group of pooled agents and call the lowest taste-type in this interval agent  $i$  and the highest, agent  $i+k$ . From Lemma 4, only taste-types between  $i$  and  $i+k$  belong to this group of pooled agents.

Since agents  $i$  and  $i-1$  are not pooled, in the solution to (1)  $\delta_i > 0$  and  $\beta_{i-1} = 0$ . Hence,

$$MRS_i(p_i, q_i, \phi_i) = - \frac{Uq(p_i, q_i, \phi_i)}{Up(p_i, q_i, \phi_i)} > - \frac{Wq_i}{Up_i}.$$

By the SCP, all other members of this pooled group have  $MRS_j(p_i, q_i, \phi_j) > - \frac{Wq_i}{Wp_i}$  for  $j = i, i+1, \dots, i+k$ . That is, agents of the pooled group have an MRS exceeding that of the firm as in condition (iv) of Proposition 1. A similar argument holds for members of the other pooled groups as well.  $\square$

Hence, as in the solution with complete separation, agents are willing to pay the cost of higher quality but the monopolist restricts quality to capture more surplus from higher taste agents. The amount of this distortion depends, in part, on the magnitude of the pooling. The larger are the pooled segments, the more distorted will be the allocations of pooled agents.

An issue closely related to that of separation versus pooling is exclusion. The monopolist may choose to not serve certain taste-types at all as a means of extracting greater surplus from higher taste agents. We have ignored this issue by assuming that the monopolist serves all  $N$  types in the market. One can view (2) as the second stage of a problem where, in the first stage, the monopolist chooses which types to serve. We have chosen to ignore this first-stage and concentrate on the distortions created in the second stage.

#### B. Randomization as a Sorting Mechanism

One means of sorting agents not yet considered here is through randomization. If agents differ in tastes and degrees of risk aversion, then randomization is a potentially useful device for extracting additional surplus from agents. Cooper-Ross [1982], Matthews [1981], Stiglitz [1981] and Prescott-Townsend [1982] provide examples which illustrate the potential gains through randomization. Intuitively, if degrees of risk aversion are correlated in the right way with tastes, then randomization may be profitable. In addition, allowing randomization by the monopolist will ensure that the constraint set in the general self-selection problem will be convex.

To see the gains from randomization consider a monopolist offering prices and product qualities to agents of two taste-types. From our discussion to this point, the non-randomized solution will leave the low-taste agents with zero consumer's surplus and the high-taste agents indifferent between their bundle and that intended for the other agents. The high-taste agents receive positive surplus which, due to imperfect information, the monopolist is unable to capture.

Now suppose, as in Cooper-Ross, that the monopolist's preferences are linear with respect to quality--i.e. the monopolist is risk neutral. In addition, assume that the low-taste agents are risk neutral while the high-taste agents are risk averse with respect to quality.

In this extreme setting, the monopolist can extract all the consumer's surplus of the high-taste agents by randomizing quality at the low price. Due to the risk neutrality of the low-taste types and the monopolist, randomizing quality at the low price has no effects as long as the average quality equals that from the non-randomized solution. However, due to the risk aversion of high-taste agents the previously binding self-selection constraint has been loosened by the randomization. Hence, the price charged to the high-taste agents can be increased. In fact, the monopolist can extract all the surplus of the high-taste agents by increasing the variability of quality at the low price and simultaneously increasing the high price.

Obviously, this is an extreme case where there are only "benefits" to randomization. If the monopolist and/or the low-taste agents are risk-averse, then randomization has "costs" as well. Intuitively, whether randomization is a profitable sorting mechanism will depend on the relative numbers of agents and their degrees of risk aversion.

To illustrate this tradeoff, we consider an example which extends the work of Cooper-Ross. A monopolist faces agents of two taste-types. There are  $N_1$  "low taste" agents with preferences  $bq-p$  and  $N_2$  "high-taste" agents with preferences  $U(q) - p$ . We assume that  $U'(q) > b$  for all  $q$  and  $U''(q) < 0$ . The monopolist is risk neutral with respect to profits but risk averse with respect to quality since unit profits are

$$\pi = p - C(q) \text{ with } C'(\cdot) > 0 \text{ and } C''(\cdot) > 0 .$$

That is, the convexity of the cost schedule implies the risk aversion of the monopolist.

Following the results established in the earlier section, the non-randomized solution will satisfy

$$bq_1^* = p_1^* ,$$

$$U(q_2^*) - p_2^* = U(q_1^*) - p_1^*$$

$$U'(q_2^*) = C'(q_2^*)$$

$$C'(q_1^*) < b .$$

As before the "high-taste" agents are indifferent between their own bundle  $(p^*, q^*)$  and the intended for the "low-taste" agents  $(p_1^*, q_1^*)$  . The point of randomization is to relax this self-selection constraint.

In the most general form of randomization, the monopolist could offer lotteries over quality in both the high and low price markets. However, there are only costs and no benefits to randomizing at the high price--i.e. there are no self-selection constraints to relax. Hence we concentrate on randomization in the low-price market.

One convenient form of randomization is to allow the monopolist to control a mean-preserving spread over quality in the low price market. We denote by  $\tilde{q}$  this random quality where

$$\tilde{q}_1 = q_1 + \lambda \varepsilon .$$

Here  $\varepsilon$  equals 1 and -1 equiprobably. The monopolist, as before, chooses  $q_1$  and also controls the variability of quality by its choice of  $\lambda$  .

Formally, the monopolist solves

$$\begin{aligned} & \text{maximize } N_2(p_2 - C(q_2)) + N_1(p_1 - E_\epsilon C(q_1 + \lambda\epsilon)) \\ & p_1, q_1, \lambda \end{aligned} \quad (5)$$

$$\text{subject to } bq_1 - p_1 \geq 0 \quad (5.a)$$

$$U(q_2) - p_2 \geq E_\epsilon U(q_1 + \lambda\epsilon) - p_1 \quad (5.b)$$

$$U(q_2) - p_2 \geq 0 \quad (5.c)$$

$$\lambda \geq 0 .$$

In this problem, constraints (5.a) and (5.b) are the individual rationality and self-selection constraints which were binding in the non-randomized solution. With randomization possible, we need to ensure that "high-taste" agents obtain non-negative surplus as well--hence we include (5.c). Finally,  $\lambda$  must also be non-negative as in (5.d). Using  $\phi$ ,  $\delta$ ,  $\mu$ ,  $\gamma$  as multipliers for (5.a), (5.b), (5.c) and (5.d) respectively, the first-order conditions for (5) imply

$$N_1 E_\epsilon C'(q_1 + \lambda\epsilon) = \phi b - N_2 E_\epsilon U'(q_1 + \lambda\epsilon) \quad (6.a)$$

and

$$N_1 E_\epsilon (\epsilon C'(q_1 + \lambda\epsilon)) = -N_2 E_\epsilon (\epsilon U'(q_1 + \lambda\epsilon)) + \gamma . \quad (6.b)$$

These are the derivatives with respect to  $q_1$  (6.a) and  $\lambda$  (6.b) under the assumption that (5.c) is not binding.<sup>10</sup> Our interest is in whether  $\lambda > 0$  in the optimal solution to (5)--i.e. whether randomization is profitable.

Proposition 3: A sufficient condition for randomization is

$$N_1 A^M(q_1^*) \leq N_2 A^C(q_1^*) \quad (7)$$

where  $A^M(q_1^*)$  is the monopolist's absolute degree of risk aversion at  $q_1^*$  and  $A^C(q_1^*)$  is that for the high-taste consumer.

Proof: To show that (7) is a sufficient condition for randomization, we rewrite (6.b) as

$$N_1 (C'(q_1 + \lambda) - C'(q_1 - \lambda)) = -N_2 (U'(q_1 + \lambda) - U'(q_1 - \lambda)) + \gamma \quad (8)$$

since  $\lambda$  takes on the value of  $-1$  and  $1$  equiprobably. A Taylor series expansion of  $C'(q_1 + \lambda)$  and  $U'(q_1 + \lambda)$  around  $C'(q_1 - \lambda)$  and  $U'(q_1 - \lambda)$  respectively (dropping terms above the second degree) leaves

$$N_1 C''(q_1 - \lambda) = -N_2 U''(q_1 - \lambda) + \gamma \geq -N_2 U''(q_1 - \lambda) . \quad (9)$$

Now, we investigate whether  $\lambda = 0$  satisfies (6.a) and (9). Setting  $\lambda = 0$ , (6.a),  $U'(q) > b$  for all  $q$  and  $\phi = N_1 + N_2$  implies

$$C'(q_1^*) < U'(q_1^*) .$$

From this and (9) with  $\lambda = 0$ , we have the ratio

$$N_1 \frac{C''(q_1^*)}{C'(q_1^*)} > \frac{-N_2 U''(q_1^*)}{U'(q_1^*)} \quad (10)$$

Using the Arrow-Pratt measure of risk-aversion,

$$N_1 A^M(q_1^*) > N_2 A^C(q_1^*)$$

is a necessary condition for  $\lambda = 0$ . If (7) holds, then obviously  $\lambda = 0$  will not satisfy (6.a) and (6.b).  $\square$

Expression (7) has a very intuitive appeal.  $N_1^M(q_1^*)$  measures the cost to the monopolist of randomization in the low-price market. Alternatively  $N_2^C(q_1^*)$  measures the local gains to the monopolist in terms of price increases in the "high-taste" market that randomization will yield. If the monopolist is locally more risk averse and low-taste agents more numerous, randomization will not pay.

While Proposition 3 gives a sufficient condition for randomization, it does not characterize the optimal  $\lambda^*$ . Intuitively, if (7) is satisfied at  $\lambda = 0$ , randomization will continue until either the monopolist becomes more risk averse than the consumers or until (5.c) becomes binding.

It is much more difficult to discuss randomization with the generality of earlier sections of this paper. First, it is necessary to have separability of preferences to make use of risk aversion measures defined over a single variable. Given this restriction, one can generalize this discussion to many taste-types. Randomization never occurs at  $(p_N, q_N)$  as there are no higher-taste agents to extract surplus from. Otherwise the monopolist weights the costs of randomization--the sum of his risk aversion and type  $\phi_i$ 's risk aversion at  $q_i$  --against the gains--measured by agent  $i+1$ 's risk aversion at  $q_i$ . Randomization becomes more likely at  $(p_i, q_i)$  as the risk aversion of the  $\phi_{i+1}$  types rises relative to that of the  $\phi_i$  types.

## V. Conclusion

The purpose of this paper was to characterize the solution to a general model of self-selection. Under certain conditions on preferences, we characterized the distortions created by the need for allocations to satisfy self-selection constraints. From a methodological viewpoint, this paper highlights the general nature of this distortion as well as the common properties of the solutions to self-selection problems.

The obvious extension of this paper is to consider preferences which do not permit the ordering of tastes. As noted in the text, this would imply that the simplifications of the monopolist's optimization problem via the adjacency condition would not hold any longer.

Another generalization of the problem would be to allow the monopolist to set each  $x_i$  conditional on the announced taste-types of all agents--i.e.,  $x_i(\phi_1, \phi_2, \dots, \phi_N)$ . In addition, the relative numbers of agents of each type may be random from the point of view of the monopolist. Under these conditions, the problem of the monopolist would resemble that considered in the public goods literature (see, for example, Laffont-Maskin [1980]) since this introduces a game between agents. With this generalization, the structure of (2) would be slightly altered to take into account the strategic relation across consumers. In particular, the truth-telling constraints would require the use of a Bayesian-Nash equilibrium concept. Nonetheless, at least for separable preferences, the problem appears to be isomorphic to that considered in this paper. Further research in this area would be fruitful.

Finally, the analysis restricted  $x_i \in \mathbb{R}^2$  and  $\phi_i \in \mathbb{R}$ . Generalizing these models to more general circumstances and then investigating the revelation properties of the model would be of considerable interest. <sup>11</sup>

## FOOTNOTES

1. This, of course, does not constitute a complete list. Recent additions include the problem of regulation under asymmetric information discussed by Baron-Myerson [1982] and Sappington [1982]. Also, see the evolving literature on implicit contracts under asymmetric information (see, for example, Azariadis [1983], Chari [1983], Grossman-Hart [1981] or Hart [1982]).
2. Since beginning this research I have found that Stiglitz (Information and Economic Analysis) discusses the similarity of these models as well. Also Antle and Dye [1981] consider a general self-selection model when bundles are exchangeable. More recently, Maskin and Riley [1982] have investigated the general class of sorting problems as well.
3. In a standard concave programming problem, the objection function is concave and the constraint set is convex. Without further restrictions on preferences, the set of  $x_i$  for  $i = 1, 2, \dots, N$  satisfying (1.b) will not necessarily be convex.
4. See also the discussion in Maskin-Riley [1982].
5. I am grateful to Richard McLean for helpful comments on this version of the proof.
6. Here we use the notation  $p(\phi_i)$  and  $q(\phi_i)$  instead of  $p_i$  and  $q_i$  to emphasize the dependence on these variables on  $\phi_i$ .
7.  $U_i \equiv U(0, 0, \phi_i)$  for all  $i$ . That is, obtaining zero price and quality is equivalent to not being served at all.
8. It is straightforward to show that both  $\delta_i$  and  $\beta_{i-1}$  positive would be inconsistent with the SCP when  $(p_i, q_i)$  and  $(p_{i-1}, q_{i-1})$  are not identical.
9. When agents are pooled, the SCP implies that we can drop the self-selection constraints for all members of the pooled group except for the  $\phi_j$  types.
10.  $\phi = N_1 + N_2$  and  $\delta = N_2$  come from the first-order conditions with respect to  $p_1$  and  $p_2$ .
11. See the recent paper by Holmstrom-Weiss [1982] for a start in this direction.

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