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DON'T BET ON IT

by

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ABSTRACT

If two people have different probability assessments about the realization of an uncertain event, they can design a contingent agreement such as a bet or gamble that offers each of them positive expected value. Yet, in the process of formulating this kind of agreement, information about the basis for each person's probabilities may be indirectly revealed to the other. The very willingness to accept a proposed bet conveys information. This paper models a process by which private, asymmetrically-held information is progressively unveiled as a possible contingent agreement is discussed. If the two parties share priors and their information partitions are common knowledge, simple discussion of the acceptability of any proposed bet is shown to reveal enough about their private information to render the bet unacceptable.

If two people have different probability assessments about the realization of an uncertain event, they can design a bet that offers positive expected value to each person. Yet, in the process of tentatively agreeing to the bet, each may reveal information about the basis for his or her probabilities. The very willingness to accept a proposed bet conveys information. This paper models a process by which private, asymmetrically-held information is progressively unveiled. If the parties share priors and their information partitions are common knowledge, simple discussion of the acceptability of any proposed bet reveals enough about their differential information ultimately to render the bet unacceptable. This finding bears upon numerous situations in which differing probability assessments are the basis for apparent joint gains. These include contingent agreements in negotiation (Sebenius, 1981) and side bets that are proposed as part of sharing rules in syndicates (Wilson, 1968), as well as incentives for decentralized decision making (Grossman and Stiglitz, 1976).

The process of information transfer that we model is loosely related to the inferences one may draw when confronted with an unexpected proposal. If a landowner, for example, put his holdings on the market for \$10,000 and was immediately offered \$100,000, he might reconsider, suspecting that oil was on the property or that a new development was planned nearby. In another instance, the off-season vacationer considering an exotic trip at one-third the "normal" price might guess that the weather would be miserable. Closer to the betting theme of this paper would be a novice racing enthusiast who goes to the track optimistic about a particular horse, but who finds one-hundred-to-one odds against his favorite. A bet that seemed attractive on the way to the track may be refused

after exposure to the quoted odds.

Our work extends recent investigations by Aumann (1976) and Geanakoplos and Polemarchakis (1978). Aumann gives an elegant equilibrium result: If two people have the same priors and if their posteriors for an event are "common knowledge", then their posteriors must be equal. Geanakoplos and Polemarchakis supplement this equilibrium analysis by exhibiting a process for agents to revise their posteriors. Under the assumption of common priors, if both agents' information partitions are finite, they show that simple communication of posteriors back and forth will lead the agents to make revisions that converge to a common, equilibrium posterior.

While it may be difficult to imagine parties repeatedly announcing posterior distributions to each other, it is not hard to imagine parties discussing possible actions that are based on their probabilistic beliefs. Bets and gambles are examples of such actions. A bet is an agreement that Party 2 will pay 1 unit to Party 1 if an event A happens and Party 1 will pay 1 unit to Party 2 if A does not happen. A gamble is a more general agreement: a random variable G is specified and it is agreed that Party 2 will pay Party 1 G units if G is positive and Party 1 will pay Party 2 G units if G is negative. Suppose such a gamble is proposed and Party 1 is tentatively willing to take it. Knowing this, Party 2 may tentatively accept or decline. Party 1 then has the same option if Party 2 has accepted, and the dialogue continues until both are finally satisfied or a rejection is encountered. These assumptions correspond to a situation in which the parties discuss the gamble before making it and not to the situation where a firm offer is made and simply accepted or refused.

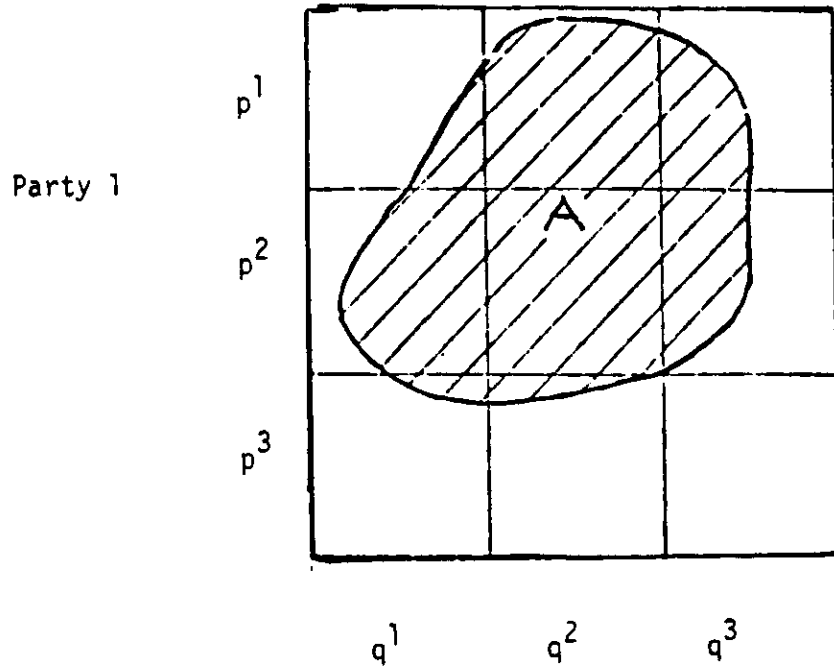
The main result of our analysis is that, if the parties have the same prior

and if they have finite information partitions, one party will ultimately refuse any such gamble even though each side may have based its posterior assessment on very different, private information. Knowing this, the parties should refuse to gamble at the outset. We first offer a simple example and then prove the result in general.

1. Example

Suppose that Ω , the set of possible states of the world, is represented by the large rectangle in Figure 1. Parties 1 and 2 both adopt a uniform prior distribution over the rectangle. Party 1 partitions the rectangle into (p^1, p^2, p^3) by the horizontal lines, and receives the private information that the true state of the world ω is in p^2 . Party 2, whose information partition (q^1, q^2, q^3) is indicated by vertical lines, finds out privately that $\omega \in q^3$. Each side knows the other's partition, and that information has been received, but does not know what the information is.

Suppose that event A , known to both parties and represented by the shaded subset of the large rectangle, is of interest. The following bet is proposed: if $\omega \in A$, Party 2 pays one unit to Party 1, and if $\omega \notin A$, Party 1 pays one unit to Party 2. This implies that the bet is attractive to Party 1 if the probability of A is greater than half and that the bet is attractive to Party 2 if the probability of A is less than one-half. We assume the risk-neutrality of both parties. The parties successively announce their willingness or unwillingness to take the bet. Denote by c_i the knowledge of Parties 1 and 2 about each other's information. Each party can deduce c_i . Before discussion (0th iteration) both parties know only $c_0 = (p^1, p^2, p^3; q^1, q^2, q^3)$.



Party 2

Figure 1

The bet is offered to Party 1 who says yes (denoted Y) since the conditional probability of $\omega \in A$ (given knowledge that $\omega \in p^2$) exceeds one half. Party 2 thereby knows that Party 1's information could not have been that $\omega \in p^3$, or Party 1 would have said no (N). However, the possibility that $\omega \in p^1$ or $\omega \in p^2$ cannot be ruled out. Thus $c_1 = (p^1, p^2; q^1, q^2, q^3)$, a fact now known to each potential bettor. Party 2 then calculates the conditional probability that $\omega \in A$ given that $\omega \in q^3$ and that either $\omega \in p^1$ or $\omega \in p^2$, finds it to be less than half, and also tentatively says Y. Party 1 deduces that Party 2's information could not have been that $\omega \in q^2$, but must have been that $\omega \in q^1$ or $\omega \in q^3$. Thus, $c_2 = (p^1, p^2; q^1, q^3)$. Since the conditional probability that $\omega \in p^2$ and that $\omega \in q^1$ or $\omega \in q^3$ exceeds half, Party 1 again says Y, and both sides realize that $c_3 = (p^2; q^1, q^3)$. Party 2 now says Y, implying that $c_3 = (p^2; q^3)$. Since the conditional probability of A given that $\omega \in p^2$ and $\omega \in q^3$ is less than half, Party 1 now refuses the bet. The sequence of responses to the proposed bet is YYYYN.

2. General Result

Now consider a general probability space (Ω, S, π) where Ω is the space of states of the world, S is the collection of all possible events (sets) that are made up of elements of Ω , and π is common prior of Parties 1 and 2. Party 1's partition of the space is P with elements p^i ($i \in I_0$, a set of m integers). Party 2's partition is Q with elements q^j ($j \in J_0$, a set of n integers). We assume that the coarsest common refinement of P and Q consists of events whose probability does not vanish. If the true state of the world is ω , then Party 1 is privately informed of that element of P which contains ω (denoted $p^i(\omega)$). Similarly, Party 2 privately learns which element of Q contains ω (denoted $q^j(\omega)$). Each side knows the other's partition. Suppose a gamble G is proposed. The two parties take turns saying Y

or N to G, beginning with Party 1. We assume that a party says Y if and only if his current expectation from the gamble is strictly positive.

Proposition 1. The gamble G will be refused by one of the parties after a finite number of repetitions of this process. The number of Y's before an N is at most $\min(2m-2, 2n-1)$.

To prove this proposition, set

$$P_1 = \{p^i | i \in I_0, E(G|p^i) > 0\},$$

and

$$Q_1 = \{q^j | j \in J_0, E(G|q^j \cap (UP_1)) < 0\},$$

and define P_k and Q_k for $k > 1$ by

$$P_k = \{p^i | i \in I_0, E(G|p^i \cap (U(Q_1 \cap \dots \cap Q_{k-1}))) > 0\},$$

and

$$Q_k = \{q^j | j \in J_0, E(G|q^j \cap (U(P_1 \cap \dots \cap P_k))) < 0\}.$$

Notice that P_k and Q_k , for $k \geq 1$, do not depend on the true state of nature and hence are known to both Parties at the outset.

Lemma 1. Suppose all the announcements preceding Party 1's (Party 2's) turn in round k are Y's. Then Party 1's (Party 2's) announcement in round k will also be Y if and only if $\omega \in U P_k$ ($\omega \in U Q_k$). (Notice that $\omega \in U P_k$ is equivalent to $p^i(\omega) \in P_k$ and $\omega \in U Q_k$ is equivalent to $q^j(\omega) \in Q_k$.)

To prove Lemma 1, consider the first round of announcements. Party 1 who makes the first announcement, will announce Y if and only if

$$E(G|p^i(\omega)) > 0,$$

that is, if and only if $p^i(\omega) \in P_1$. Party 2 knows this, and so after Party 1's Y,

Party 2's expectation for G is

$$E(G|q^j(\omega) \cap (U P_1)) .$$

Party 2 will now say Y if and only if this expectation is less than zero, that is, if and only if $q^j(\omega) \in Q_1$. The proof of Lemma 1 can be completed by induction.

Lemma 2. (a) Suppose Party 1's announcement in the first round is Y . If Party 2's announcement in the first round is also Y , then $P_1 \cap P_2$ is a proper subset of P_1 . (b) Suppose all the announcements preceding Party 1's (Party 2's) in round k are Y 's, where $k > 1$. If Party 1's (Party 2's) announcement in round k is also Y , then $Q_1 \cap \dots \cap Q_k$ is a proper subset of $Q_1 \cap \dots \cap Q_{k-1}$ ($P_1 \cap \dots \cap P_{k+1}$ is a proper subset of $P_1 \cap \dots \cap P_k$).

To prove (a), suppose that both Parties say Y in round 1. Then $p^i(\omega) \in P_1$ and $q^j(\omega) \in Q_1$, or $\omega \in (U P_1) \cap (U Q_1)$. Hence $(U P_1) \cap (U Q_1) \neq \emptyset$. Now if $P_1 \cap P_2$ is not a proper subset of P_1 , then every element of p^i of P_1 is in P_2 and thus satisfies

$$E(G|p^i \cap (U Q_1)) > 0 . \tag{1}$$

Now every element q^j of Q_1 satisfies

$$E(G|q^j \cap (U P_1)) < 0 . \tag{2}$$

Equations (1) and (2) are contradictory. To see this, note that the sets $p^i \cap (U Q_1)$ for $p^i \in P_1$ and the sets $q^j \cap (U P_1)$ for $q^j \in Q_1$ are simply two different partitions for the non-empty set $(U P_1) \cap (U Q_1)$, and that it is impossible for all the conditional expectations in one partition to be negative while all the conditional expectations in another partition are positive. Hence $P_1 \cap P_2$ is a proper subset of P_1 .

The proof of (b) is similar.

If the first $2k$ announcements are Y 's, then Lemma 1 implies that $p^i(\omega) \in P_1 \cap \dots \cap P_k$ and $q^j(\omega) \in Q_1 \cap \dots \cap Q_k$. Similarly, if the first $2k+1$ announcements are Y 's, then $q^j(\omega) \in Q_1 \cap \dots \cap Q_{k+1}$. But Lemma 2 implies that $P_1 \cap \dots \cap P_{k+1}$ is always a proper subset of $P_1 \cap \dots \cap P_k$, and similarly for the Q 's. The stated limit on the number of consecutive Y 's follows directly from these observations.

3. Relation to Aumann's Proposition

The foregoing analysis suggests a straightforward generalization of Aumann's proposition. Given two partitions $P \equiv \{p^i | i \in I_0\}$ and $Q \equiv \{q^j | j \in J_0\}$, we define their meet $R \equiv PAQ \equiv \{r^k | k \in K_0\}$ to be the finest partition of Ω that is refined by both P and Q . In our example, R consists of the single set Ω itself. Given any $\omega \in \Omega$, note that $p^i(\omega)$ and $q^j(\omega)$ are both contained in $r^k(\omega)$. We say that an event A is common knowledge at ω if $r^k(\omega) \subset A$. (For an exposition, see Milgrom (1981)). Let G be an arbitrary real-valued random variable defined on (Ω, S, π) . We define the random variables $e_p(\omega) = E(G | p^i(\omega))$ and $e_q(\omega) = E(G | q^j(\omega))$. Let a be a real number. We then have

Proposition 2. If it is common knowledge at ω that $e_p \leq a$ and $e_q \geq a$, then it is common knowledge at ω that $e_p = e_q = a$.

The proof is similar to that of Lemma 2. Aumann established Proposition 2 in the case where G takes on the values 0 and 1 and where it is common knowledge at ω that $e_p = b$ and $e_q = c$ by showing then that $b = c$. (A related result is given in Milgrom and Stokey (1980)).

Proposition 2 implies that for any "bet" or "gamble" as described before, it cannot be common knowledge that both sides wish to take it. More precisely, if it is common knowledge that the expectation of each side given its information is non-negative, then it is common knowledge that this expectation is zero. It is not necessary to go through the stylized process of offer and acceptances that underlies the explicit convergence result of Proposition 1. Instead, at the moment when it becomes common knowledge that both sides wish to gamble -- as they extend their hands, so to speak, to "shake" on the deal -- there is a contradiction and at least one side will withdraw.

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