COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut 06520

COWLES FOUNDATION DISCUSSION PAPER NO. 632

INTEGRAL POLYHEDRA IN THREE SPACE

Herbert E. Scarf

June 2, 1982

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by

Herbert E. Scarf

I. Introduction

In a series of recent papers [Scarf, 1977; 1982, Part I; 1982, Part II], I have introduced a particular class of convex polyhedra in R^n, arising in the study of integer programming problems with n variables. In the present paper a detailed analysis of these polyhedra will be given for the case in which $n = 3$; the analysis is based on an unpublished theorem demonstrated several years ago by Roger Howe, which seems to have no immediate generalization to higher values of n.

The arguments of the paper are elaborate. It is possible, however, to state an important consequence without any reference to the details of these arguments, and this will be done in order to provide the reader with some appreciation of the scope of the paper.

Let

$$A = \begin{bmatrix} a_{01} & a_{02} & a_{03} \\ a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

*The research described in this paper was supported by a grant from the National Science Foundation. I am deeply grateful to Roger Howe, Sergiu Hart, Andrew Caplin and Philip White for the many significant conversations I have had with them about the subject of this paper.

be a matrix with real entries. We shall be concerned with the set of integral vectors $h = (h_1, h_2, h_3)$ which satisfy the linear inequalities $Ah \geq b$, where $b = (b_0, b_1, b_2, b_3)'$; for each such b we assume that the cardinality of this set is finite.

The set of solutions to a system of linear inequalities is a convex set when the variables are permitted to assume arbitrary real values. The property of convexity is indispensible in developing the simplex method for the solution of linear programming problems and in demonstrating the major theorems of linear activity analysis. The absence of convexity, when the solutions to the system of linear inequalities are required to be integral, is the basic difficulty in the study of integer programming problems.

Our major conclusion will be to show that a very weak but useful version of convexity can be established for the integral solutions of $Ah \geq b$, when A has four rows and three columns. We shall demonstrate that each such matrix A has associated with it a family of parallel planes

$$\ell_1 h_1 + \ell_2 h_2 + \ell_3 h_3 = c$$

with ℓ_1, ℓ_2, ℓ_3 specific integers whose greatest common divisor is unity, and with c assuming arbitrary integral values. The planes will depend only on A and not on the right hand side b; they will possess the following property:

For any b, if the system $Ah \geq b$ has integral solutions on each of the two planes $\ell_1 h_1 + \ell_2 h_2 + \ell_3 h_3 = c$ and $\ell_1 h_1 + \ell_2 h_2 + \ell_3 h_3 = c'$, then the same system will have integral solutions on each intermediary plane $\ell_1 h_1 + \ell_2 h_2 + \ell_3 h_3 = c''$, with c'' an arbitrary integer between c and c'.

As we shall see, a knowledge of this family of parallel planes will permit us to solve the integer programming problems associated with A, by solving the corresponding two variable problems on each such plane. Specifically the solution on such a plane will permit us to say on which side of the plane the optimal solution to the original three variable problem lies. This is an unexpected form of decoupling which would be of great significance for integer programming if a suitable analogue could be found for higher values of n.

Let us now introduce the class of convex polyhedra whose analysis forms the basis for this conclusion.

1.1 [Definition]. Let Z^n be the lattice of points with integral coordinates in R^n. A bounded convex polyhedron in R^n is defined to be an integral polyhedron if its vertices are in Z^n, and if it contains no members of Z^n other than its vertices.

The basic problem is to provide a characterization of integral polyhedra, up to a unimodular transformation, i.e. a linear transformation which preserves the lattice Z^n. The following theorem provides an upper bound on the number of vertices of such a polyhedron.

1.2 [Theorem]. The number of vertices of an integral polyhedron is less than or equal to 2^n.

The proof of this theorem is based on the observation that if there are more than 2^n vertices, then there will necessarily be at least one pair, say v^1 and v^2, with $v_i^1 \equiv v_i^2 \bmod (2)$ for $i = 1, 2, \ldots, n$. The point $(v^1+v^2)/2$ will therefore be in the polyhedron, contradicting the definition.

The unit hypercube in R^n is an example of an integral polyhedron with the maximal number of vertices. As we shall see, however, when $n \geq 3$, the typical integral polyhedron is much more complex and cannot be reduced to the unit hypercube by a unimodular transformation. In particular the volume of such a figure--which is a unimodular invariant--can be arbitrarily large. Our inability to draw on theorems from the Geometry of Numbers, which relate the volume of convex polyhedra to the number of lattice points they contain, is based on the fact that integral polyhedra need have none of the symmetry properties which are indispensable in this area.

Integral polyhedra have a very simple characterization when $n = 2$ (see Scarf, 1981, Part II). The number of vertices is either three or four; in the former case a necessary and sufficient condition that a triangle with integral vertices be an integral polyhedron is that it have an area of 1/2. Moreover it can be transformed by a unimodular transformation to the triangle of Figure 1.

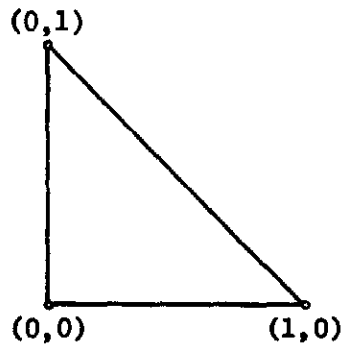

FIGURE 1

A planar polyhedron with four vertices is an integral polyhedron if and only if it is a parallelogram with integral vertices and unit area; therefore equivalent under a unimodular transformation to the unit square. Prior to such a transformation the parallelogram may take a more general

form, as illustrated in Figure 2, where the vertices are given by

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \beta \\ \gamma \end{pmatrix}, \quad \begin{pmatrix} \beta' \\ \gamma' \end{pmatrix}, \quad \begin{pmatrix} p \\ q \end{pmatrix},$$

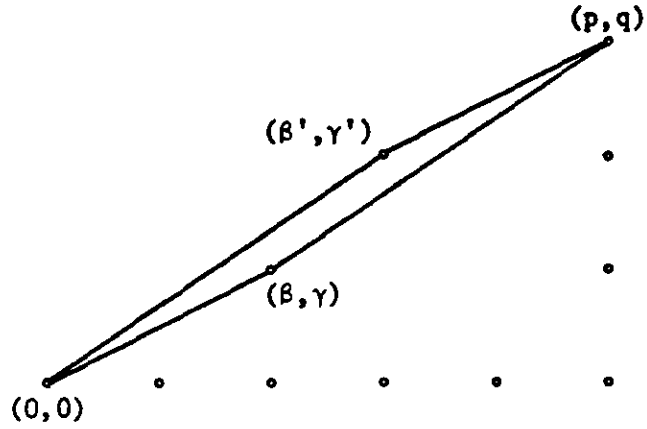

FIGURE 2

with (p,q) positive integers which are prime to each other, with (β,γ) and (β',γ') non-negative integers satisfying

$$\beta q - \gamma p = 1 ,$$

and $\beta + \beta' = p$, $\gamma + \gamma' = q$.

When $n = 3$, the number of vertices of an integral polyhedron can be 4, 5, 6, 7 or 8. Figure 3 provides an example of such a polyhedron

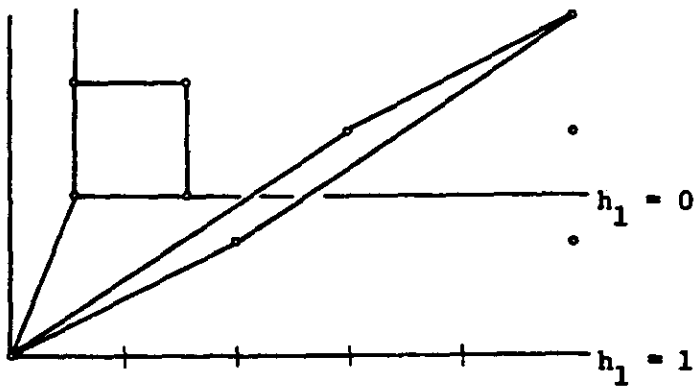

FIGURE 3

with eight vertices. Four of the vertices lie on the plane $h_1 = 0$, and form the vertices of the square; the remaining four vertices are those of an arbitrary parallelogram of unit area on the plane $h_1 = 1$, say $(1,0,0)$, $(1,\beta,\gamma)$, $(1,\beta',\gamma')$ and $(1,p,q)$, with the same notation as before. The convex hull of these eight points clearly contains no other lattice points.

An easy way to see that this polyhedron is not equivalent to the unit cube under a unimodular transformation is to notice that it contains, as a proper subset, the tetrahedron

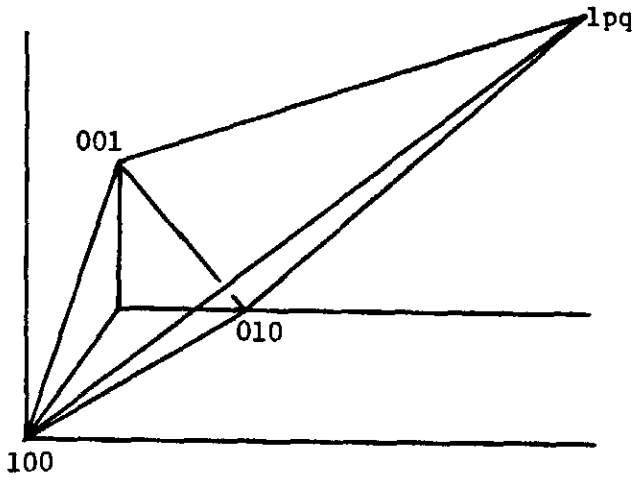

FIGURE 4

with vertices

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ p \\ q \end{pmatrix},$$

whose volume is given by $(p+q)/6$. An alternative argument is that the number of faces of this polyhedron will, in general, be larger than the number of faces of the unit cube.

Howe's remarkable result is that the polyhedra of Figure 3 are--up to a unimodular transformation--the most general integral polyhedra in 3-space.

1.3 [Howe's theorem]. An integral polyhedron with eight vertices in 3-space can, by a unimodular transformation, be brought into the form where the vertices are given by

$$\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\\beta\\\gamma\end{pmatrix}, \begin{pmatrix}1\\\beta'\\\gamma'\end{pmatrix}, \begin{pmatrix}1\\p\\q\end{pmatrix},$$

with p,q positive integers which are prime to each other, and with (β,γ) , (β',γ') non-negative integers satisfying $\beta q - \gamma p = 1$, $\beta + \beta' = p$, $\gamma + \gamma' = q$. Moreover an integral polyhedron with fewer than eight vertices is a subset of an integral polyhedron with eight vertices.

An argument for Howe's theorem will be provided in Section III of this paper. An equivalent statement of the theorem, given our description of planar integral polyhedra, is that there is a unimodular transformation which places the vertices of an integral polyhedron in 3-space on the two planes $h_1 = 0$ and $h_1 = 1$. It will be useful for us to describe Howe's theorem in a form which does not involve a unimodular transformation; we introduce the following definition:

1.3 [Definition]. A plane in 3-space will be termed a <u>lattice plane</u> if it passes through three non-colinear lattice points. Two parallel lattice planes will be said to be <u>adjacent</u> if there are no lattice points between them.

An alternative form of Howe's theorem is therefore

1.4 [Alternative form of Howe's theorem]. The vertices of an integral polyhedron in 3-space lie on two adjacent lattice planes.

The term characteristic plane will be used to describe the lattice plane (or one of its parallel translates) associated with a given integral polyhedron. Typically such a polyhedron will have a unique characteristic plane associated with it, though there are examples, such as the unit cube, with several characteristic planes. Howe's theorem gives rise to an obvious conjecture about the form of integral polyhedra with 2^n vertices in n-space, i.e. that all of the vertices lie on two adjacent lattice planes of dimension $n-1$. Unfortunately there are simple examples which show that this conjecture is false when $n = 4$, and the true nature of integral polyhedra in higher dimensions is not known.

II. The Collection of Integral Polyhedra Associated with an Integer Program

Integral polyhedra arise in a very natural way in the study of integer programming problems. Let

$$A = \begin{bmatrix} a_{01} & \cdots & a_{0n} \\ a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

be an $(m+1) \times n$ matrix and $h = (h_1, \ldots, h_n)$ a typical lattice point in n-space. We make the following simplifying assumptions about the matrix A .

2.1 [Assumptions]. The entries in each row of A are independent over the integers, in the sense that the origin is the only lattice point satisfying

$$\sum_j a_{ij} h_j = 0 , \quad \text{for any } i .$$

Moreover the set of lattice points satisfying the inequalities

$$\begin{bmatrix} a_{01} & \cdots & a_{0n} \\ a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} h \geq \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_m \end{pmatrix}$$

is assumed to be finite for any choice of the right hand side.

Consider a placement of these inequalities, given by a particular value of the right side, so that the region defined by the inequalities is free of lattice points. Enlarge the region by relaxing the inequalities until no further relaxation is possible without introducing a lattice point. In this process some of the constraint planes may be relaxed to infinity; the remaining planes will be relaxed so that they contain a single lattice point. Clearly, the convex hull of the lattice points obtained by this process will be an integral polyhedron. Figure 5 illustrates the construction with five inequalities in 2-space. The resulting

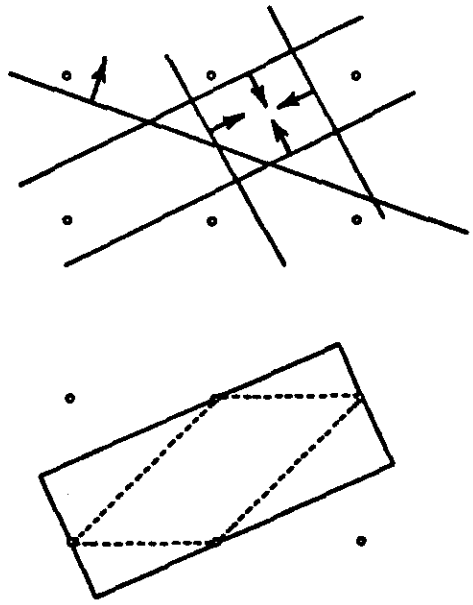

FIGURE 5

integral polyhedron is given by the parallelogram of unit area defined by the dashed lines.

The reader may easily verify that a variety of parallelograms arise when these inequalities are relaxed in a different order, or if the process begins with a different lattice free region. In n-space as well, the constraint matrix A will have associated with it, a collection of integral polyhedra obtained by relaxation of the constraint planes from an arbitrary lattice free region. Many of these polyhedra will be equivalent to each other under translation; the number of equivalence classes will typically be finite but large.

In order to describe the role played by the collection of integral polyhedra associated with the matrix A in the solution of integer programming problems, we introduce the following definition of neighboring lattice points.

2.2 [Definition]. Let the matrix A be given. Two lattice points h and k are defined to be <u>neighbors</u> if they are vertices of an integer polyhedron obtained by relaxing the constraint planes from a lattice free region.

Consider an integer program of the form

$$\begin{aligned} \max \quad & a_{01}h_1 + \ldots + a_{0n}h_n \\ & a_{11}h_1 + \ldots + a_{1n}h_n \geq b_1 \\ & \qquad\qquad\qquad\quad \vdots \\ & a_{m1}h_1 + \ldots + a_{mn}h_n \geq b_m \,, \end{aligned}$$

with $h = (h_1, \ldots, h_n)$ integral. An integral point h which satisfies the inequalities of the programming problem is said to be a <u>local maximum</u>

if every neighbor of h either violates one of the inequalities, or yields a lower value of the objective function than does h. The following theorem is demonstrated in [Scarf, 1982, Part I].

2.3 [Theorem]. For any value of the right hand side, a local maximum to the integer program is global.

It is of interest to inquire whether there are alternative definitions of neighborhoods, based on the matrix A, for which a local maximum is global. For each lattice point h, let $N(h)$ be a finite set of lattice points called the neighborhood of h. We require the neighborhood system to have the following two properties:

1. $N(h) \equiv N(0) + h$,
2. The neighborhoods are symmetric in the sense that $h \in N(0)$ implies that $-h \in N(0)$.

The first property states that neighborhoods associated with two different lattice points are translates of each other, and the second implies that if $h \in N(k)$, then $k \in N(h)$. Figure 6 illustrates a typical neighborhood of the origin for a matrix A with two columns.

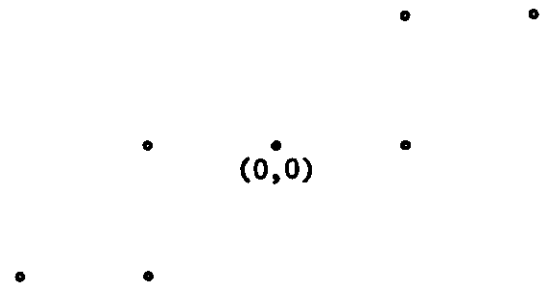

FIGURE 6

The following converse to Theorem 2.3 is also demonstrated in [Scarf, 1982, Part I].

2.4 [Theorem]. Let $N(h)$ be a neighborhood system with the property that a local maximum to the integer programming problem is global, for all right hand sides. Then $N(h)$ contains all of the neighbors of h given by Definition 2.2.

These two theorems imply that the collection of integral polyhedra obtaining by relaxing the constraint planes from a lattice free region provides the unique minimal neighborhood system for which a local maximum is global for all integer programs obtained by specifying the right hand side. This result motivates the study of the particular class of integral polyhedra which are associated with the specific matrix A.

The second major result of the present paper--in addition to Howe's theorem--will be the demonstration that an important property is shared by all of the integral tetrahedra associated with a matrix A with four rows and three columns. This property will suggest a rapid computational procedure for integer programs with three variables and three inequalities. Before describing this result, however, it is useful to discuss the two variable problem drawing on the material presented in [Scarf, 1982, Part II].

If the matrix A has three rows and two columns the associated integral polyhedra are planar triangles of area 1/2. It may be shown that, up to translation, only two triangles arise, which may, by a unimodular transformation be brought into the form displayed in Figure 7, i.e. the two triangles obtained by slicing the unit square along one of its diagonals. For such a problem the minimal neighborhood system contains six lattice points.

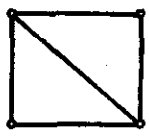

FIGURE 7

When A has four rows and two columns the associated integral polyhedra are triangles of area 1/2 and parallelograms of area 1. The collection of parallelograms exhibits a very specific structure. Consider a particular parallelogram with vertices

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \beta \\ \gamma \end{pmatrix}, \quad \begin{pmatrix} \beta' \\ \gamma' \end{pmatrix}, \quad \begin{pmatrix} p \\ q \end{pmatrix},$$

with $\beta + \beta' = p$, $\gamma + \gamma' = q$, and $\beta q - \gamma p = 1$.

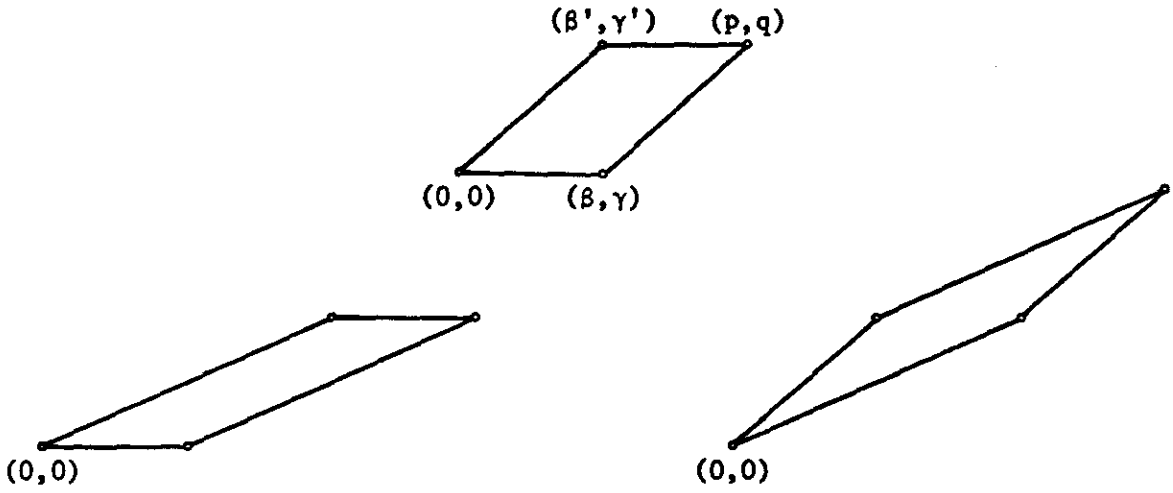

FIGURE 8

The two parallelograms obtained by replacing either (β,γ) or (β',γ') by their reflections through (p,q) will be called <u>successors</u> of the original parallelogram, as in Figure 8. The two parallelograms obtained by replacing (p,q) by its reflection either through (β,γ) or (β',γ') will be called <u>predecessors</u> of the original parallelogram. A <u>chain</u> of parallelograms is a linearly ordered finite sequence of parallelograms, with each parallelogram followed by one of its two possible successors. There are, of course, $2^{\ell-1}$ different chains of length ℓ, starting with a specific initial parallelogram. Figure 9 illustrates a particular

chain beginning with the unit square. Since the chain is linearly ordered

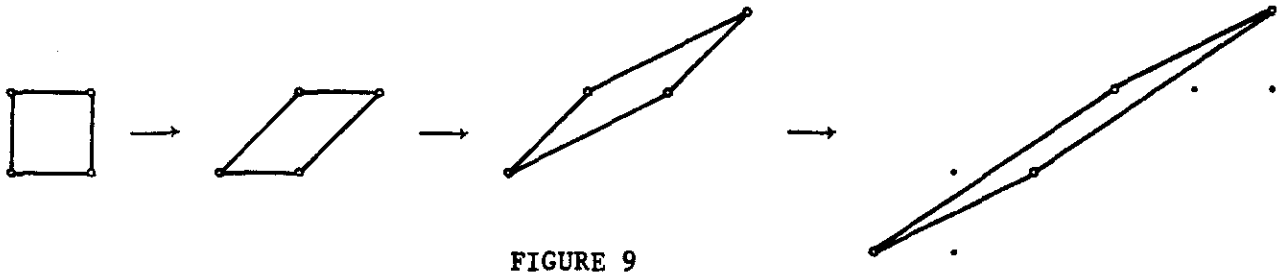

FIGURE 9

there will be no ambiguity in using the terms right and left to refer to successors and predecessors.

A proof of the following theorem may be found in [Scarf, 1982, Part II].

2.5 [Theorem]. The integral polyhedra associated with a 4×2 matrix consist of triangles and parallelograms. Up to translations, the parallelograms form a chain. There are two pairs of triangles, the first pair obtained by slicing the initial (or leftmost) parallelogram through the diagonal not containing the origin, and the second pair obtained by slicing the final (or rightmost) parallelogram through the diagonal which does contain the origin.

If the chain is given as in Figure 9, the two pairs of triangular polyhedra are as in Figure 10. Each of the triangles is obtained by relaxing the constraint lines from a lattice free region, with a particular line being relaxed to infinity.

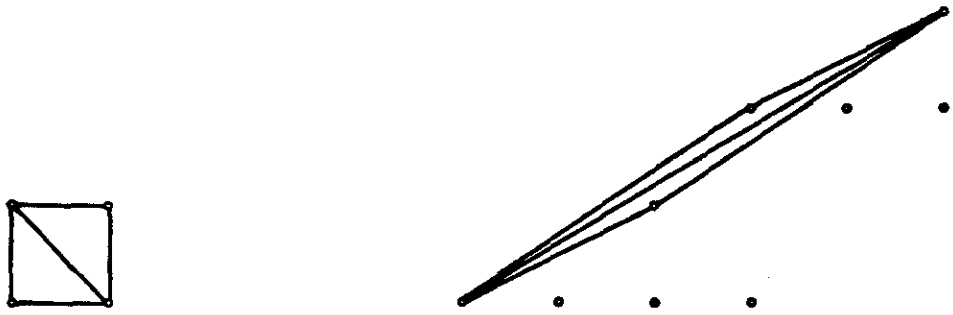

FIGURE 10

Now let us turn our attention to a matrix A of size 4×3. Each integral polyhedron associated with A will be a tetrahedron, and by Howe's theorem will have a characteristic plane so that the four vertices are contained on this plane and an adjacent plane. Two tetrahedra which are translates of each other will, of course, have parallel characteristic planes which are identified as being the same. There may however be an substantial number of non-translation equivalent tetrahedra arising from the same matrix A, and there is no apparent reason to think that they share a common characteristic plane. But as the following theorem indicates they do indeed.

2.6 [Theorem]. The integral tetrahedra arising from a 4×3 matrix have a common characteristic plane.

The proof of theorem 2.6 is quite difficult, and will be given after Howe's theorem is demonstrated in the next section. The latter theorem, of course, implies Howe's theorem for tetrahedra, since any particular integral tetrahedron is the relaxation from a lattice free region of some system of four inequalities in three space. No generalization of this theorem to higher dimensions is known.

The theorem may be interpreted in terms of the minimal neighborhood system for which a local maximum is global when the integer program

consists of three variables and three inequalities. It states that, after a unimodular transformation, the neighbors of any lattice point (h_1, h_2, h_3) will have their first coordinates equal to $h_1 - 1$, h_1, or $h_1 + 1$. If the three variable problem is solved as a two variable problem on the plane $h_1 = a$, a sufficient condition for optimality is that no improvement be possible on the two planes $h_1 = a \pm 1$.

III. Howe's Theorem

We begin by demonstrating Howe's theorem for tetrahedra. We have the following preliminary lemma:

3.1 [Lemma]. An integral tetrahedron can, by a unimodular transformation, be brought to the form in which the four vertices are

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

with $x \geq 0$, $y \geq 0$, $z \geq 1$.

Take an arbitrary face of the tetrahedron, and by a unimodular transformation, bring it to the plane $z = 0$. The three vertices on this plane form an integral triangle which can therefore be put in the form

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

and without loss of generality we may assume that the fourth vertex

(x,y,z) has $z \geq 1$. For any integers **a** and **b** the transformation

$$\begin{aligned} x' &= x - az \\ y' &= y - bz \\ z' &= z \end{aligned}$$

is a unimodular transformation which leaves the first three vertices unchanged. By an appropriate choice of **a** and **b** , we can make $0 \leq x' < z$ and $0 \leq y' < z$. Without loss of generality we may therefore assume that the fourth vertex satisfies

$$0 \leq x < z \quad \text{and} \quad 0 \leq y < z \ .$$

If $z < x + y$, then

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \alpha_1 \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \alpha_4 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

with $\alpha_4 = 1/z$, $\alpha_3 = 1 - y/z > 0$, $\alpha_2 = 1 - x/z > 0$, and $\alpha_1 = 1 - \alpha_2 - \alpha_3 - \alpha_4 = (x+y-z-1)/z \geq 0$, contradicting the assumption that the tetrahedron contains no lattice points other than its vertices. It follows that $z \geq x+y$.

The unimodular transformation

$$\begin{aligned} x' &= x \\ y' &= y \\ z' &= -x - y + z + 1 \end{aligned}$$

brings the four vertices to the form

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z' \end{pmatrix},$$

with $x \geq 0$, $y \geq 0$, $z' \geq 1$. This demonstrates 3.1.

Let the four vertices now be as in Lemma 3.1. These vertices will clearly lie on two adjacent lattice planes if $x \leq 1$, or $y \leq 1$, or $z \leq 1$. In order to demonstrate Howe's theorem it is therefore sufficient to show that there is a fifth lattice point in the tetrahedron whenever x, y, z are all strictly larger than unity.

3.2 [Lemma]. Let the integral tetrahedron have the four vertices of Lemma 3.1, with $x, y, z \geq 1$. Let $D = x+y+z-1$, and for $h = 1, 2, \ldots, D-1$, let $f(h) = \left\lceil \frac{xh}{D} \right\rceil + \left\lceil \frac{yh}{D} \right\rceil + \left\lceil \frac{zh}{D} \right\rceil$, where $\lceil t \rceil$ is the least integer $\geq t$. Then $f(h) \equiv h+2$ for all $h = 1, 2, \ldots, D-1$.

In order to demonstrate Lemma 3.2 we assume first that $f(h) \leq h+1$ for some $h = 1, 2, \ldots, D-1$, and define a lattice point (a,b,c) by

$$a \geq \left\lceil \frac{xh}{D} \right\rceil, \quad b \geq \left\lceil \frac{yh}{D} \right\rceil, \quad c \geq \left\lceil \frac{zh}{D} \right\rceil,$$

and $a+b+c = h+1$. Then

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \end{bmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}$$

with

$$\alpha_1 = a - \frac{xh}{D} \geq 0$$

$$\alpha_2 = b - \frac{yh}{D} \geq 0$$

$$\alpha_3 = c - \frac{zh}{D} \geq 0 ,$$

$$\alpha_4 = \frac{h}{D} \geq 0 ,$$

and $\sum_1^4 \alpha_j = 1$. It follows that if the tetrahedron is indeed an integral polyhedron, we must have $f(h) \geq h+2$.

To complete the proof of Lemma 3.2 we distinguish two subcases:

1. x , y , and z are all prime to D . In this case it is trivial to verify that $f(h) + f(D-h) \equiv D+4$. The simultaneous inequalities $f(h) \geq h+2$, and $f(D-h) \geq D-h+2$, therefore imply that each is an equality.

2. One of the integers, say x , has a common factor with D . But then xh/D is an integer for some integral h with $1 < h < D-1$, and for that value of h we must have

$$f(h) + f(D-h) \leq D+3 .$$

It follows that either $f(h) \leq h+1$ or $f(D-h) \leq D-h+1$, and either one of these inequalities is sufficient to produce a fifth lattice point in the tetrahedron. Lemma 3.2 has therefore been demonstrated.

We complete the proof of Howe's theorem for tetrahedra by showing that the conditions $f(h) \equiv h+2$, and x , y , z each prime to D , are inconsistent with $x \geq 2$, $y \geq 2$, $z \geq 2$. Several arguments are available; we adopt one using elementary number theoretic considerations. Begin by constructing the table:

$$\begin{array}{c|ccc} h & & & \\ \hline 1 & x & y & z \\ 2 & 2x & 2y & 2z \\ 3 & 3x & 3y & 3z \\ \vdots & \vdots & \vdots & \vdots \\ D-1 & (D-1)x & (D-1)y & (D-1)z \end{array}$$

with all entries reduced modulo D. Observe that the sum of two rows this table, or an integral multiple of any row in the table, when reduced modulo D, is either identically zero or equal to another row of the table.

Since $\left\lceil \frac{hx}{D} \right\rceil = \left\lceil \frac{(h-1)x}{D} \right\rceil + 1$ if $hx \bmod(D) < x$, and $\left\lceil \frac{hx}{D} \right\rceil = \left\lceil \frac{(h-1)x}{D} \right\rceil$ otherwise, it follows that in each row of the table (other than the first) precisely one of the three conditions $(hx)\bmod(D) < x$, $(hy)\bmod(D) < y$, $(hz)\bmod(D) < z$ must hold if the function $f(h)$ is to be linear.

For example when $(x,y,z) = (2,3,7)$, $D = 11$ and the table is given by

h				f(h)
1	2	3	7	3
2	4	6	③	4
3	6	9	10	4
4	8	①	⑥	6
5	10	4	②	7
6	①	7	9	8
7	3	10	⑤	9
8	5	②	①	11
9	7	5	8	11
10	9	8	④	12

In each row an entry is encircled if it is less than the corresponding entry at the top of its column. As may be seen $f(h) - f(h-1)$ is equal to the number of encircled elements in row h. There are no encircled entries in row 3; this implies that f is not linear and the tetrahedron whose vertices are the three unit vectors and $(2,3,7)$ contains an additional lattice point.

Since x, y, z are prime to D each column of the table must be a permutation of $1, 2, \ldots, D-1$. It follows that there are 3 rows of the table of the form

$$\begin{matrix} D-1 & a_2 & a_3 \\ b_1 & D-1 & b_3 \\ c_1 & c_2 & D-1 \end{matrix}$$

Observation 1. If $x, y, z \geq 2$, the three rows are distinct. If this were not so, there would be an h with, say, $hx \equiv D-1 \bmod(D)$ and $hy \equiv D-1 \bmod(D)$. But then $(D-h)x \equiv 1 \bmod(D)$, $(D-h)y \equiv 1 \bmod(D)$, and row $D-h$ of the table would contain at least two encircled elements.

Observation 2. If $x, y, z \geq 2$, the entries in the above three rows are all different from 1. Suppose to the contrary that $a_2 = 1$. But then

$$y(D-1), \quad y, \quad ya_3 \bmod(D)$$

is a row in the table. Since the entry y appears in the second column in the first row only, we must have $y(D-1) \equiv x \bmod(D)$, and therefore $(x+y) \equiv 0 \bmod(D)$. But $D = x+y+z-1$ and this implies $z = 1$.

Observation 3. If $a_2 < y$ then $b_3 < z$ and $c_1 < x$. Assume to the contrary that $a_2 < y$ and $b_1 < x$. Then

$$D-1+b_1\ ,\quad D-1+a_2\ ,\quad a_3+b_3 \bmod(D)\ ,$$

or b_1-1 , a_2-1 , $a_3+b_3 \bmod(D)$ is equal to hx , hy , $hz \bmod(D)$ for some h . Since $a_2-1 \neq 0$, this is an actual row in the table with at least two encircled entries. This contradiction implies $b_3 < z$, and a similar argument shows that $c_1 < x$.

Without loss of generality we may therefore assume that the encircled entries in these three rows are

$$\begin{array}{ccc} D-1 & \textcircled{a_2} & a_3 \\ b_1 & D-1 & \textcircled{b_3} \\ \textcircled{c_1} & c_2 & D-1 \end{array}\ .$$

Observation 4. $b_1 + c_1 < D$, $a_2 + c_2 < D$, $a_3 + b_3 < D$. To see this we again add the first of these two rows $\bmod(D)$, and obtain

$$b_1-1\ ,\quad \textcircled{a_2-1}\ ,\quad a_3+b_3 \bmod(D)\ .$$

But $a_3 + b_3 < D-1+z$. We cannot therefore have $a_3 + b_3 > D$, since this implies that $a_3 + b_3 \bmod(D) < z$. The other two inequalities are verified by similar arguments.

The final argument which contradicts $x, y, z \geq 2$ is obtained by adding the three rows together $\bmod(D)$ and obtaining the row

$$b_1+c_1-1\ ,\quad a_2+c_2-1\ ,\quad a_3+b_3-1\ .$$

But $b_1+c_1-1 > x$, $a_2+c_2-1 > y$, $a_3+b_3-1 > z$, a contradiction which demonstrates that at least one of the three coordinates x , y , z must be 0 or 1, and the four vertices of the integral tetrahedron lie on two adjacent lattice planes. This verifies Howe's theorem when the integral polyhedron has four vertices.

One final observation before proceeding to the case in which the integral polyhedron contains more than four vertices: Let one of the coordinates, say x, of the fourth vertex, be equal to 0. Then

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} y \\ z \end{pmatrix}$$

constitute an integral triangle with $y \geq 0$, $z \geq 1$. We must have $y, z = (1,1)$ and the four vertices are

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

a tetrahedron of volume 1/6. It follows that if the integral tetrahedron has volume $> 1/6$, all three of the coordinates x, y, z are ≥ 1, and at least one of them is equal to unity. If, say, $x = 1$, then y and z must be relatively prime and the tetrahedron is as in Figure 4.

We begin our analysis of integral polyhedra with five vertices:

3.3 [Lemma]. If four of the vertices of an integral polyhedron with five vertices lie in a single plane, then the fifth vertex lies in an adjacent plane.

Without loss of generality we may assume that the four co-planar vertices are given by

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

and that the fifth vertex is given by x , y , z with $x \geq 2$. By subtracting suitable multiples of the first coordinate from the second and third coordinates we may assume that $1 \leq y \leq x$, and $1 \leq z \leq x$. But if $x \leq y+z-1$, then

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix},$$

with $\alpha_1 = (y+z-x-1)/x$, $\alpha_2 = 1 - y/x$, $\alpha_3 = 1 - z/x$, $\alpha_4 = 1/x$, and $\sum_1^4 \alpha_j = 1$. If, on the other hand $x \geq y+z-1$ then

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & x \\ 1 & 1 & 0 & y \\ 1 & 0 & 1 & z \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}.$$

with $\alpha_1 = (x-y-z+1)/x$, $\alpha_2 = (z-1)/x$, $\alpha_3 = (y-1)/x$, $\alpha_4 = 1/x$, and $\sum_1^4 \alpha_j = 1$. This contradiction demonstrates Lemma 3.3.

We now assume that no four vertices of the five vertexed integral polyhedron are co-planar, and demonstrate a lemma which permits us to draw upon our earlier analysis of tetrahedron.

3.4 [Lemma]. Consider an integral polyhedron with five vertices, no four of which are co-planar. Then there is a subset of four vertices, say v^1 , v^2 , v^3 , v^4 with the following two properties:

1. The volume of the tetrahedron generated by these four vertices is $\geq 2/6$, and

2. The line joining the fifth vertex and one of these four vertices passes through the interior of the triangle formed by the remaining three vertices.

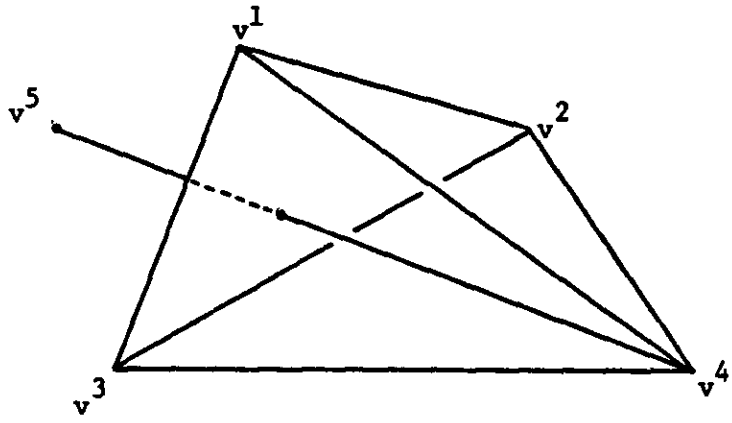

FIGURE 11

In order to demonstrate lemma 3.4 we begin by finding α_1 , α_2 , α_3 , α_4 , α_5 , not all zero, with $\sum_1^5 \alpha_j = 0$, and $\sum_1^5 \alpha_j v^j = 0$. None of the α's are equal to 0, since otherwise the remaining four vertices are co-planar. If four of the α's have the same sign, say $\alpha_2, \ldots, \alpha_5 < 0$ and $\alpha_1 < 0$, then

$$v^1 = -\frac{\alpha_2}{\alpha_1}v^2 - \frac{\alpha_3}{\alpha_1}v^3 - \frac{\alpha_4}{\alpha_1}v^4 - \frac{\alpha_5}{\alpha_1}v^3 ,$$

and v^1 would be a convex combination of the remaining four vertices, contradicting the assumption that the five-vertexed figure is an integral polyhedron. It follows that three of the α's are of one sign, and two of the other are of opposite sign, say $\alpha_1, \alpha_2, \alpha_3 > 0$, and $\alpha_4, \alpha_5 < 0$. But then

$$\frac{\alpha_4 v^4 + \alpha_5 v^5}{\alpha_4 + \alpha_5} = \frac{\alpha_1 v^1 + \alpha_2 v^2 + \alpha_3 v^3}{\alpha_1 + \alpha_2 + \alpha_3}$$

which demonstrates the second part of the lemma.

In order to demonstrate the first part let us assume that the tetrahedra with vertices (v^1, v^2, v^3, v^4), and (v^1, v^2, v^3, v^5) both have volume 1/6. We may then employ the arguments of lemma 3.1 to bring the first of these tetrahedra to the form

$$\begin{pmatrix} 1 & 0 & 0 & v_1^4 \\ 0 & 1 & 0 & v_2^4 \\ 0 & 0 & 1 & v_3^4 \end{pmatrix},$$

with $v_1^4, v_2^4 \geq 0$, and $v_3^4 \geq 1$. Since the volume of the tetrahedron is $|v_1^4 + v_2^4 + v_3^4 - 1|/6$ it follows that $v_1^4 + v_2^4 + v_3^4 = 2$, and v^4 must be one of the following three vectors

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix},$$

The volume of the tetrahedron with vertices (v^1, v^2, v^3, v^5) is also 1/6, and therefore $v_1^5 + v_2^5 + v_3^5 = 0$ or 2. But if the latter alternative were to hold, a convex combination of v^4 and v^5 could not pass through the triangle generated by the remaining three vertices. It follows that $v_1^5 + v_2^5 + v_3^5 = 0$, and

$$(v_1^4 + v_1^5)/2 > 0$$

$$(v_2^4 + v_2^5)/2 > 0$$

$$(v_3^4 + v_3^5)/2 > 0 \ .$$

If $v^4 = (1,0,1)$ these inequalities imply $v_1^5 \geq 0$, $v_2^5 \geq 1$, $v_3^5 \geq 0$, contradicting $v_1^5 + v_2^5 + v_3^5 = 0$. A similar argument holds if $v^4 = (0,1,1)$, and finally if $v^4 = (0,0,2)$, we arrive at a contradiction because $v_1^5 \geq 1$, $v_2^5 \geq 1$, $v_3^5 \geq -1$. It follows that one of the two tetrahedra has volume $\geq 2/6$, and lemma 3.4 has been demonstrated.

We are now prepared to prove the following theorem which provides a canonical form for an integral polyhedron with five vertices.

3.5 [Theorem]. An integral polyhedron with five vertices, no four of which are co-planar, can be brought by a unimodular transformation, to the form

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & y & 0 \\ 0 & 0 & 1 & z & 0 \end{pmatrix},$$

with y and z positive and relatively prime.

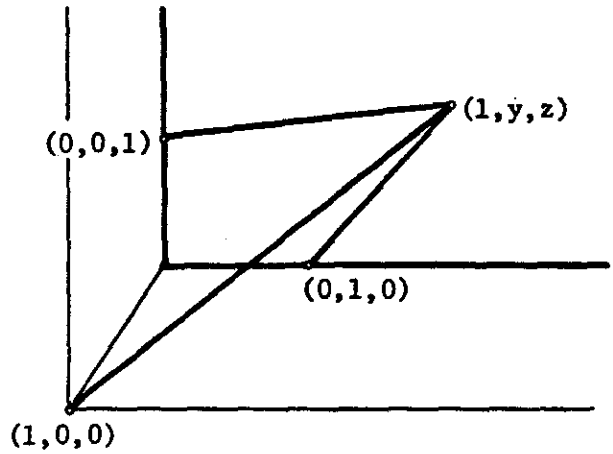

FIGURE 12

By lemma 3.4, there is a unimodular transformation bringing four of the vertices to those of Figure 12, and with the fifth vertex, with coordinates a, b, c lying strictly in the cone with vertex $(1,y,z)$, i.e.

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ y \\ z \end{pmatrix} + \alpha_1 \begin{pmatrix} 0 \\ -y \\ -z \end{pmatrix} + \alpha_2 \begin{pmatrix} -1 \\ 1-y \\ -z \end{pmatrix} + \alpha_3 \begin{pmatrix} -1 \\ -y \\ 1-z \end{pmatrix},$$

with $\alpha_1, \alpha_2, \alpha_3 > 0$. Solving these equations we obtain

$$\alpha_1 = a + (1-a-b-c)/(y+z)$$

$$\alpha_2 = b + y(1-a-b-c)/(y+z)$$

$$\alpha_3 = c + z(1-a-b-c)/(y+z),$$

and therefore

$$\alpha_1 \geq \mathrm{fr}(A/(y+z)),$$

$$\alpha_2 \geq \mathrm{fr}(yA/(y+z)),$$

$$\alpha_3 \geq \mathrm{fr}(zA/(y+z)),$$

with $A = 1-a-b-c$, and $fr(t)$ the fractional part of t .

These inequalities permit us to demonstrate that $(a,b,c) \leq 0$. If $A \equiv 0 \bmod(y+z)$, then α_1 , α_2 , α_3 are positive integers, therefore ≥ 1 , and

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \leq \begin{pmatrix} -1 \\ 1-2y \\ 1-2z \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} .$$

On the other hand if $A \not\equiv 0 \bmod(y+z)$, then we may easily show that

$$fr(yA/(y+z)) + fr(zA/(y+z)) = 1 ,$$

and therefore

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \leq \begin{pmatrix} 0 \\ -y\ fr(A/(y+z)) + fr(yA/(y+z)) \\ -z\ fr(A/(y+z)) + fr(zA/(y+z)) \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} .$$

But if (a,b,c) is not equal to $(0,0,0)$, then

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} ,$$

with $\beta_1 = -a/A \geq 0$, $\beta_2 = -b/A \geq 0$, $\beta_3 = -c/A \geq 0$, $\beta_4 = 1/A \geq 0$, with $\sum_1^4 \beta_j = 1$. This contradiction demonstrates Howe's theorem for an integral polyhedron with five vertices.

In order to complete the proof of Howe's theorem it is necessary

to consider integral polyhedra with 6, 7 and 8 vertices. The arguments are quite elementary extensions of our previous considerations; it will be sufficient to illustrate the basic ideas when the integral polyhedron contains six vertices.

We consider, first, the case in which no four of the six vertices are co-planar. By Theorem 3.5 such a polyhedron can be put in the form

$$\begin{array}{c} v^1 \quad v^2 \quad v^3 \quad v^4 \quad v^5 \quad v^6 \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & v_1^6 \\ 0 & 0 & 1 & 0 & y & v_2^6 \\ 0 & 0 & 0 & 1 & z & v_3^6 \end{bmatrix} \end{array},$$

with y and z positive integers prime to each other. By an alternative application of this same theorem the vertices v^2, v^3, v^4, v^5, v^6 lie in two adjacent lattice planes. If, however, y and z are both strictly larger than one, the only adjacent lattice planes containing v^2, v^3, v^4, v^5 are $x = 0, 1$, and therefore v^6 must be on one of these planes as well. On the other hand if $y = 1$ and $z > 1$, the four vertices also lie on the adjacent lattice planes $y = 0, 1$, but not on a third pair of adjacent lattice planes. It follows that v_6 lies either on one of the pair $x = 0, 1$ or on one of the pair $y = 0, 1$; in both cases the six vertices lie on two adjacent lattice planes. Finally if both y and $z = 1$, three pairs of adjacent lattice planes are possible for v^6.

The argument is also quite simple if four of the six vertices are co-planar. By a unimodular transformation the co-planar vertices can be brought to the four vertices of the unit square in the plane $x = 0$.

By Lemma 3.3 the fifth and sixth vertices lie on the planes $x = \pm 1$. If both of them are on the same plane then all six vertices lie on two adjacent lattice planes. Assume therefore to the contrary that the fifth vertex is on the plane $x = 1$ (without loss of generality we may take it to be 100), and that the sixth vertex lies on the plane $x = -1$, and is given by $-1, y, z$.

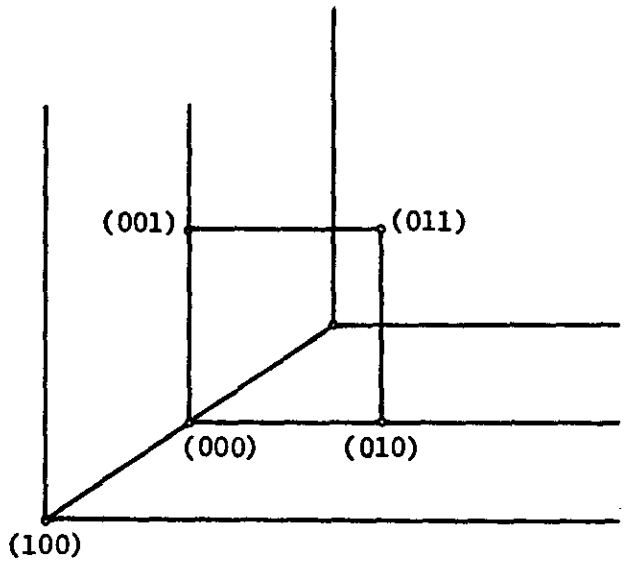

FIGURE 13

The four vertices of the integral polyhedron lying on $x = 0$, are extreme points of the polyhedron. By drawing supporting hyperplanes to the integral polyhedron through these four vertices, and examining the intersections of these hyperplanes on $x = -1$ we see that either y or $z = 1$. In either case the six vertices lie on two adjacent lattice planes.

IV. A Sufficient Condition

We now turn our attention to a 4×3 matrix A satisfying assumptions 2.1, and begin our demonstration that all of the integral tetrahedra obtained by relaxing the constraint planes from a lattice free region share a common characteristic lattice plane. The argument is complex, and it will be useful to start with a description of a condition which permits us to recognize when a given lattice plane is indeed the common characteristic plane.

Let the lattice plane be given by $x = 0$, and consider the collection of triangles and parallelograms obtained by relaxing the four inequalities from a lattice free region on this plane. Theorem 2.5 tells us that the parallelograms will, up to translation, form a chain, as illustrated in Figure 9. There will be an initial, left-most parallelogram, and each parallelogram in the chain will be followed by one of its two possible successors. One pair of triangles will be obtained by slicing the initial parallelogram along the diagonal not containing $(0,0)$, and the other pair by slicing the final parallelogram along the diagonal which does contain $(0,0)$.

Consider a particular parallelogram in the chain with the four constraint planes placed at their respective vertices. This involves a

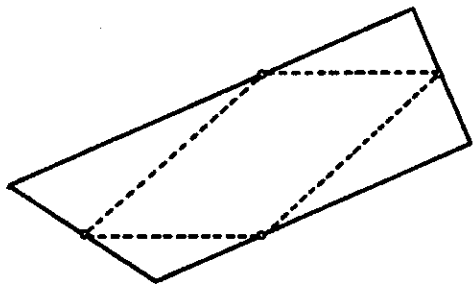

FIGURE 14

particular specification of the right hand side of the inequalities $Ah \geq b$, so that each vertex of the parallelogram satisfies all four of the inequalities, and a particular one of them with equality. We say that the parallelogram has a lattice point <u>in front</u>, if there is a lattice point (h_1, h_2, h_3) satisfying these inequalities with $h_1 \geq 1$, and has a lattice point <u>in back</u> if $h_1 \leq 1$. The parallelogram is said to be <u>doubled</u> if there are lattice points in front and in back.

If the relaxation is a triangle, as in Figure 15, three of the constraint inequalities are placed at the vertices of the triangle, and one of them is relaxed to infinity. Again if there is a lattice point

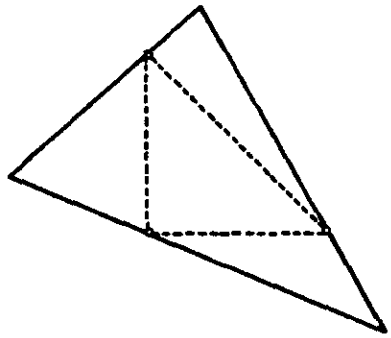

FIGURE 15

(h_1, h_2, h_3) satisfying the three inequalities it will be said to be <u>in front</u> if $h_1 \geq 1$, <u>in back</u> if $h_1 \leq -1$, and the triangle will be said to be <u>doubled</u> if it has lattice points both in front and in back. If the relaxation is a parallelogram, lemma 3.3 tells us that there will be a lattice point in front if and only if there is a lattice point satisfying the inequalities with $h_1 = 1$, and similarly for lattice points in back. The situation is somewhat more complex for triangles.

As we shall see, if a given lattice plane has the property that none of the relaxations on that plane are doubled, then it is, in fact,

a characteristic lattice plane for all of the tetrahedra obtained by relaxing the four inequalities from a lattice free region in three space.

4.1 [Theorem]. Let the lattice plane $x = 0$ have no doubled relaxations. Let $b = (b_0, b_1, b_2, b_3)'$ be such that the inequalities $Ah \geq b$ have a pair of integral solutions (h_1, h_2, h_3), and (h_1', h_2', h_3') with $h_1' \geq h_1 + 2$. Then there are integral solutions satisfying the inequalities strictly for every $x = h_1 + 1, \ldots, h_1' - 1$.

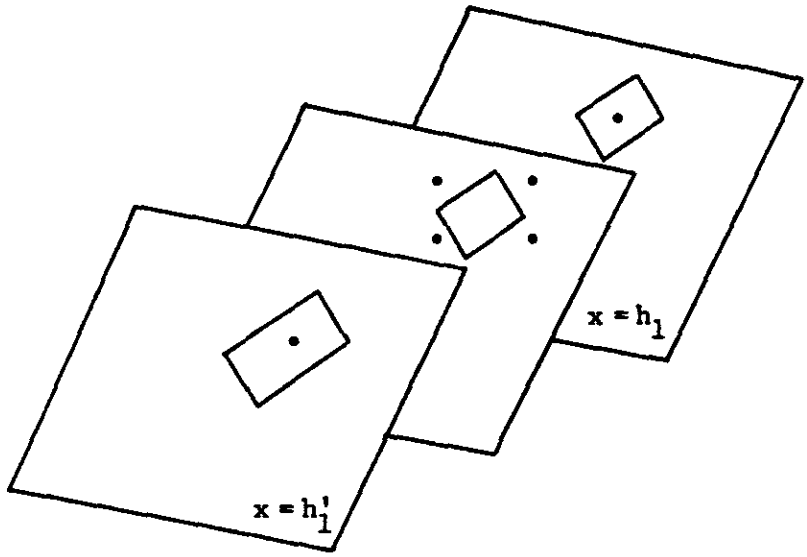

FIGURE 16

In order to demonstrate 4.1 we draw the intersections of the constaint equalities on the planes $x = h_1'$, $x = h_1$, and on any intermediary plane. If no lattice points satisfy the inequalities strictly on the intermediary plane, then there will be a relaxation on that plane which is doubled, contradicting our assumption.

Theorem 4.1 implies that $x = 0$ is a common characteristic plane for all of the tetrahedron arising from the relaxation of the four inequalities, starting with a lattice free region in three-space. If two

vertices of such a tetrahedron had their first coordinates differing by more than 1, the relaxation would already have encountered a lattice point on some intermediary plane. We see that the search for a common characteristic plane may be accomplished by finding a lattice plane on which no relaxation is doubled. It will therefore be useful for us to analyze in greater detail the relaxations appearing on an arbitrary lattice plane, allowing for the possibility that some of the relaxations are doubled.

4.2 [Lemma]. Consider a parallelogram P in the chain of relaxations on the plane $x = 0$, which contains no lattice points on the plane $x = a$. Assume that a relaxation of this lattice free region on the plane $x = a$ is to the left of P. Then every parallelogram to the left of P on $x = 0$ is also free of lattice points on $x = a$, and has a relaxation on the plane $x = a$, which is to its own left.

Let the parallelogram P on $x = 0$, have the vertices

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \beta \\ \gamma \end{pmatrix}, \quad \begin{pmatrix} \beta' \\ \gamma' \end{pmatrix}, \quad \begin{pmatrix} p \\ q \end{pmatrix},$$

with p, q positive and relatively prime, and with β, γ, β', γ' non-negative integers satisfying $\beta + \beta' = p$, $\gamma + \gamma' = q$, and $\beta q - \gamma p = 1$.

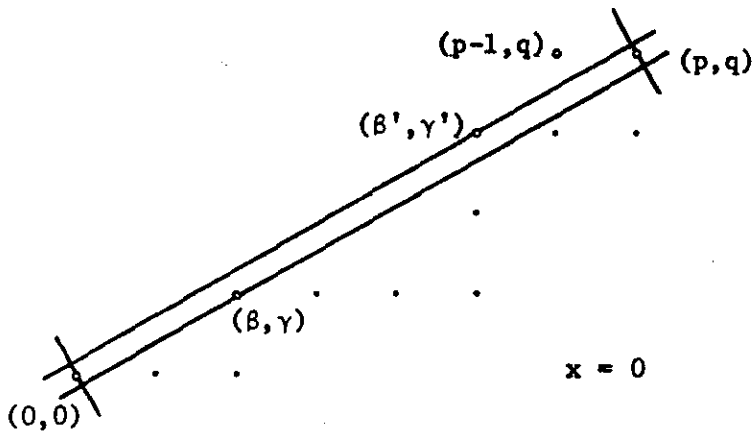

FIGURE 17

When the four inequalities are placed as in Figure 17, there are assumed to be no lattice points on the plane $x = a$, and the relaxation on this lattice free region is assumed to be to the left of P. Let us first consider the case in which this relaxation is a parallelogram--rather than a triangle--which, without loss of generality, we take to be the unit square. When the four constraint planes are placed on the vertices of

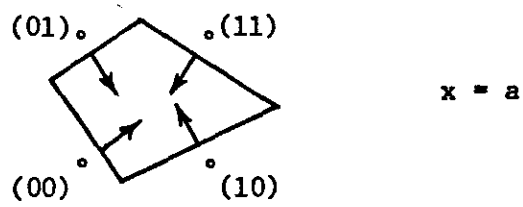

FIGURE 18

P, their intersections on $x = a$ are as in Figure 18.

The immediate predecessor of P is obtained by reflecting (p,q) either through (β,γ) or (β',γ'). There is no loss in generality in assuming that, as in Figure 17, $(\beta',\gamma') \geq (\beta,\gamma)$; this assumption in conjunction with the inequalities on the slopes of the four lines implied

by the fact that the unit square is a relaxation tells us that the predecessor of P has the vertices

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \beta \\ \gamma \end{pmatrix} \quad \begin{pmatrix} \beta' - \beta \\ \gamma' - \gamma \end{pmatrix} \quad \begin{pmatrix} \beta' \\ \gamma' \end{pmatrix}, \quad \text{as in Figure 19.}$$

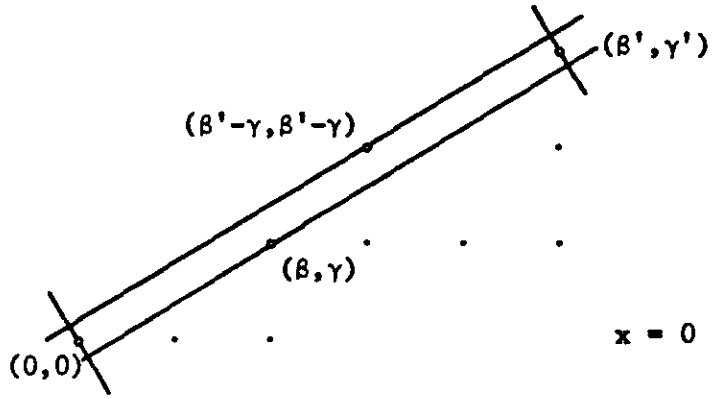

FIGURE 19

We must show that this predecessor is free of lattice points on $x = a$, and that the relaxation on $x = a$ is to the left of the unit square.

The predecessors of the unit square, if they are not triangles, are obtained by reflecting the vertex opposite to $(0,0)$ through one of the vertices adjacent to $(0,0)$. Let us continue the sequence of predecessors until we first reflect through a vertex <u>above</u> $(0,0)$, as in Figure 20. We shall demonstrate that when the constraint planes are placed at the vertices of the immediate predecessor of P, on the plane $x = 0$, the region on the plane $x = a$ is free of lattice points

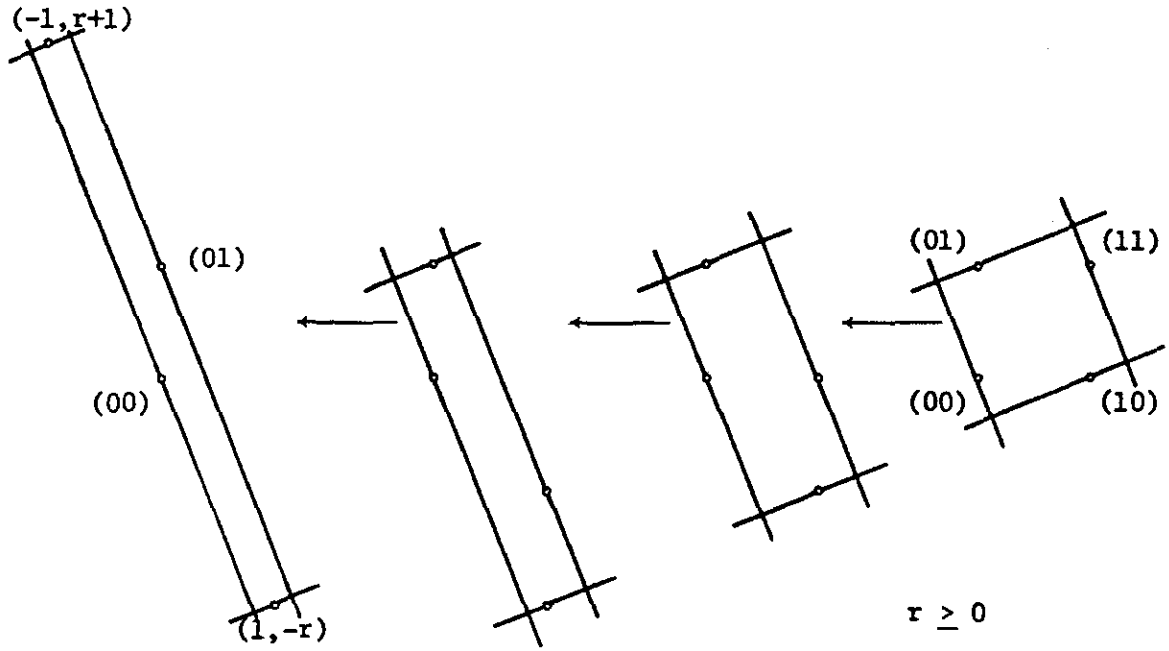

FIGURE 20

and has a relaxation with vertices

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -r \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ r+1 \end{pmatrix}.$$

We make the following observations about the transition from Figure 17 to Figure 19.

1. The constraint plane through $(0,0,0)$ has not been moved and it rejects the point $(a,0,0)$.

2. The constraint plane through $(0,\beta,\gamma)$ has not been moved. Since it rejects $(a,1,0)$ it will certainly reject $(a,1,-r)$.

3. The constraint plane through $(0,p,q)$ originally rejected $(a,1,1)$. When it is shifted to $(0,\beta',\gamma')$ it will reject $(a,1-\beta,1-\gamma)$, and therefore $(a,0,1)$ since $\beta \geq 1$, $\gamma \geq 0$.

4. The constraint plane through $(0,\beta',\gamma')$ originally rejected $(a,0,1)$. When shifted to $(0,\beta'-\beta,\gamma'-\gamma)$ it will reject $(a,-\beta,1-\gamma)$. In order to show that it rejects $(a,-1,r+1)$ it is sufficient to show that it rejects $(a,-1,1)$; this follows from the observation that this

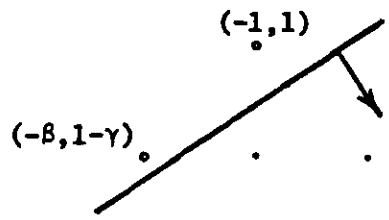

FIGURE 21

constraint plane, at its original position of $(0,\beta',\gamma')$, certainly rejects $(0,p-1,q)$ (see Figure 17).

The above argument is predicated on the assumption that the chain of predecessors of the unit square continues until a parallelogram is reached by reflecting through a vertex above $(0,0)$. One alternative possibility is illustrated in Figure 22, in which the chain ends with the pair of

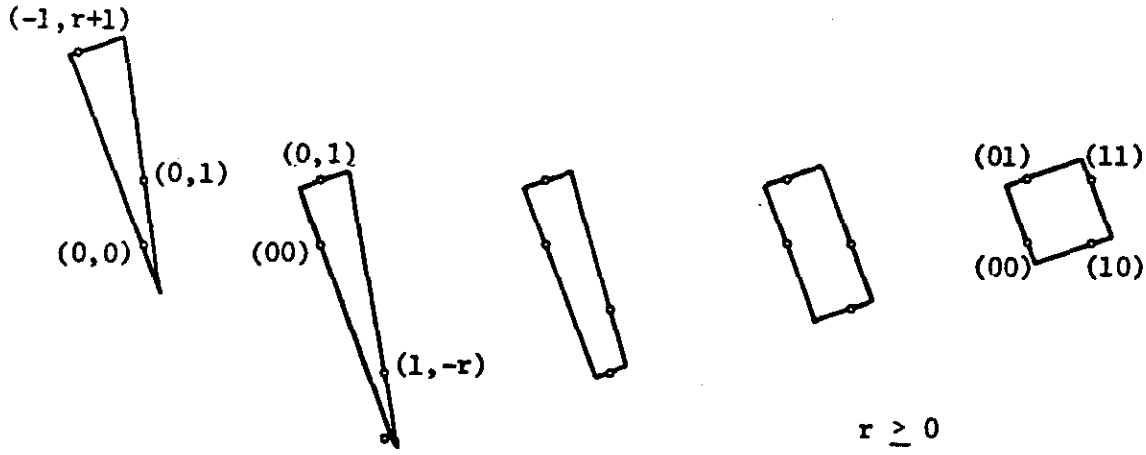

FIGURE 22

triangles with vertices

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ r+1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -r \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Arguments identical to those just given show that when the constraint planes are placed on the vertices of the predecessor of P, the region on the plane $x = a$ is free of lattice points and has a relaxation given

by the first of these triangles.

The chain may also end with the pair of triangles with vertices

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -r \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -r-1 \end{pmatrix}, \begin{pmatrix} 1 \\ -r \end{pmatrix},$$

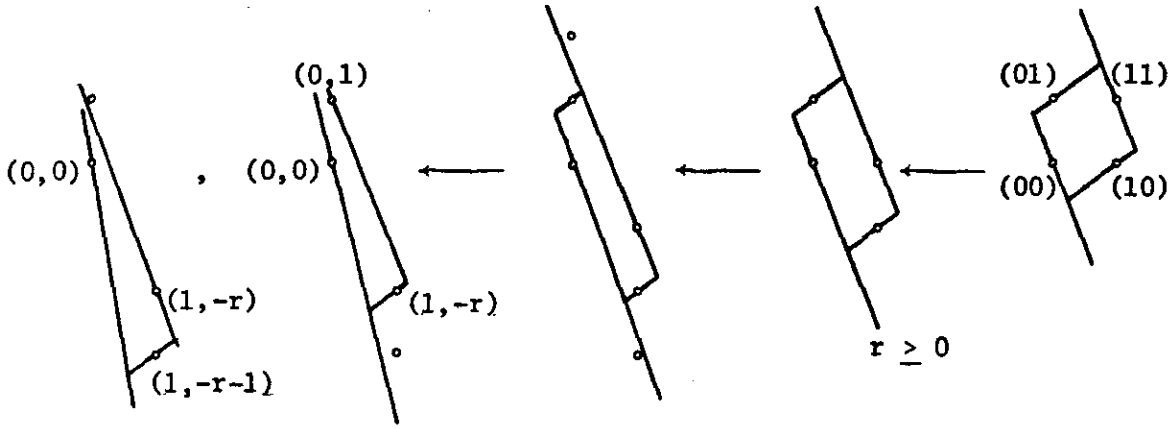

FIGURE 23

as in Figure 23. When the constraint planes are placed at the vertices of the immediate predecessor of P, the region on the plane $x = a$ has a relaxation given by the first of these triangles.

One further argument is required to complete the demonstration of lemma 4.2. We have assumed that when the inequalities are placed at the vertices of P, the lattice free region on $x = a$ has a relaxation given by a parallelogram which we have taken to be the unit square. It is possible, however, that the relaxation is a triangle at the left hand side of the chain. A similar analysis to that given above demonstrates that when the constraint planes are placed at the immediate predecessor of A (be it a parallelogram or triangle), the region on $x = a$ is free of lattice points and has a relaxation given by a translate of that same triangle on the left hand side of the chain. This completes the proof

of lemma 4.2, which has the following important implication for the structure of the relaxations appearing on a given lattice plane.

4.3 [Theorem]. Subject to a possible reinterpretation of the words "back" and "front":

1. If a relaxation is free of lattice points in front, then every relaxation to its left is free of lattice points in front, and

2. If a relaxation is free of lattice points in back then every relaxation to its right is free of lattice points in back.

Let us assume that there is a relaxation on $x = 0$, say the unit square, which has a lattice point in front, say $(1,0,0)$, and such that the predecessor of the unit square has no lattice points in front, as in Figure 24. In order to demonstrate the first part of Theorem 4.3 we

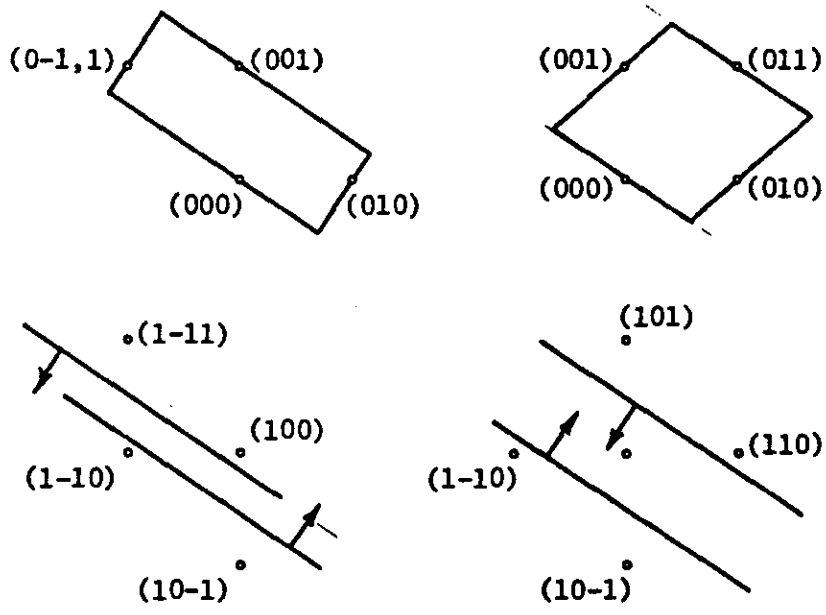

FIGURE 24

must establish two facts: first that all predecessors of the unit square are free of lattice points in front, and secondly that all successors of

the unit square do contain lattice points in front. The first argument is quite easy. From the fact that the immediate predecessor of the unit square has no lattice points in front, we can argue that the constraint plane through (0,1,1) must eliminate (1,0,1) and (1,1,0), and the constraint plane through (0,0,0) must eliminate (1,-1,0) and (1,0,-1). The predecessor of the unit square must therefore have a relaxation on $x = 1$ to its own left. From lemma 4.2 all predecessors of the unit square are free of lattice points on the front plane.

It is somewhat more subtle to establish that every successor to the unit square contains lattice points in front. What is clear is that as we move to the right of the unit square we cannot encounter a relaxation which is free of lattice points in front followed by a relaxation which does contain a lattice point in front; the argument we have already given rules out this possibility. But it does seem possible, without any additional argument, that several consecutive successors of the unit square do contain lattice points in front, and the remaining successors are indeed free of lattice points in front. To eliminate this possibility I will show that every successor of the unit square has its own successor which contains lattice points in front.

In order to demonstrate this last fact we must examine the planes <u>behind</u> the plane $x = 0$. From assumption 2.1 the immediate predecessor of the unit square on $x = 0$ will be free of lattice points on the plane $x = -a$ for some sufficiently large value of a. The relaxation on the plane $x = -a$ will contain lattice points on the plane $x = 0$, and therefore must be to the right of the immediate predecessor of the unit square. It follows from lemma 4.2 with <u>left</u> replaced by <u>right</u>, that the unit square and all of its successors on $x = 0$ are free of lattice

points on $x = -a$, and have relaxations to their own right on this plane. Any such relaxation will, of necessity, have lattice points in front. This demonstrates that any successor of the unit square has its own successor with lattice points in front and concludes the proof of the first statement of theorem 4.3. The second statement follows from a similar argument.

Theorem 4.3 suggests some very substantial simplifications in testing whether a given lattice plane is the characteristic plane for the collection of tetrahedra obtained by relaxing the four inequalities from a lattice free region in three-space. For example, if a single parallelogram on a given lattice plane has no lattice points either in front or in back, then there can be no doubled relaxations on that lattice plane and it is indeed the characteristic plane. Another way to state this is that if one of the tetrahedra is degenerate, in the sense of having all four vertices in the same plane, then that plane must be the characteristic plane. In the next section I shall describe a condition which implies the existence of a degenerate tetrahedron.

Another sufficient condition for a given lattice plane to be the characteristic plane is the existence of a pair of adjacent parallelograms in the chain, one of which is free of lattice points in front, and the other free of lattice points in back. This observation will be used repeatedly in demonstrating the existence of a characteristic plane associated with the matrix A.

V. A Special Case

In order to display the quality of the arguments developed in Section IV, let us examine a special case in which information about a single tetrahedron is sufficient to yield a characteristic plane for all of the integral tetrahedra associated with the matrix A. Let us assume that the tetrahedron has the four vertices

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ p \\ q \end{pmatrix},$$

of the type guaranteed by Howe's theorem, with the additional assumption that p and q are both ≥ 2. Moreover when the constraint planes are drawn through the four vertices they will be assumed to intersect the planes $x = 0, 1$ as in Figure 25.

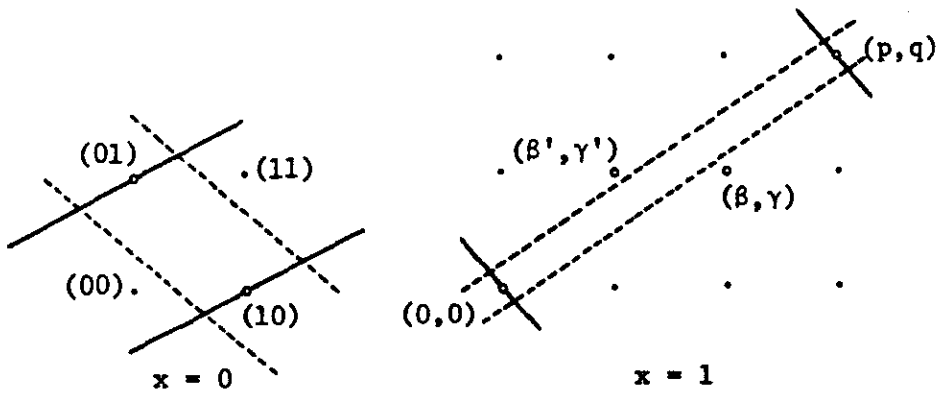

FIGURE 25

On the plane $x = 0$, the inequalities relax to the unit square; on $x = 1$, they relax to the parallelogram with vertices

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \beta \\ \gamma \end{pmatrix}, \begin{pmatrix} \beta' \\ \gamma' \end{pmatrix}, \begin{pmatrix} p \\ q \end{pmatrix}.$$

The immediate predecessor of this latter parallelogram is obtained by reflecting (p,q) either through (β,γ) or (β',γ'). Without loss of generality we assume, as in Figure 25, that $(\beta,\gamma) \geq (\beta',\gamma')$ and that the predecessor is given by

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \beta-\beta' \\ \gamma-\gamma' \end{pmatrix}, \begin{pmatrix} \beta \\ \gamma \end{pmatrix}, \begin{pmatrix} \beta' \\ \gamma' \end{pmatrix}.$$

Let us show that when the constraint planes are placed at the vertices of this latter parallelogram on $x = 1$, there will be no lattice points satisfying the inequalities on $x = 0$.

Observe first of all that in Figure 25, the point with coordinates $(1, \beta-\beta', \gamma-\gamma'+1)$ lies above the line connecting $(1,0,0)$ and $(1,p,q)$ since

$$\gamma - \gamma' + 1 \geq \frac{q}{p}(\beta - \beta')$$

follows from

$$p\gamma - p\gamma' + p - q\beta + q\beta' = p - 2 \geq 0 .$$

This point is therefore accepted by the constraint plane through $(0,1,0)$. When this constraint plane is translated to $(1, \beta-\beta', \gamma-\gamma')$ it must therefore reject the point $(0,1,-1)$, and any lattice points below and to the right of this latter point. In a similar fashion the constraint plane through $(0,0,1)$ when translated to $(1,\beta',\gamma')$ must reject $(0,-1,1)$ and any lattice points above and to the left of $(0,-1,1)$.

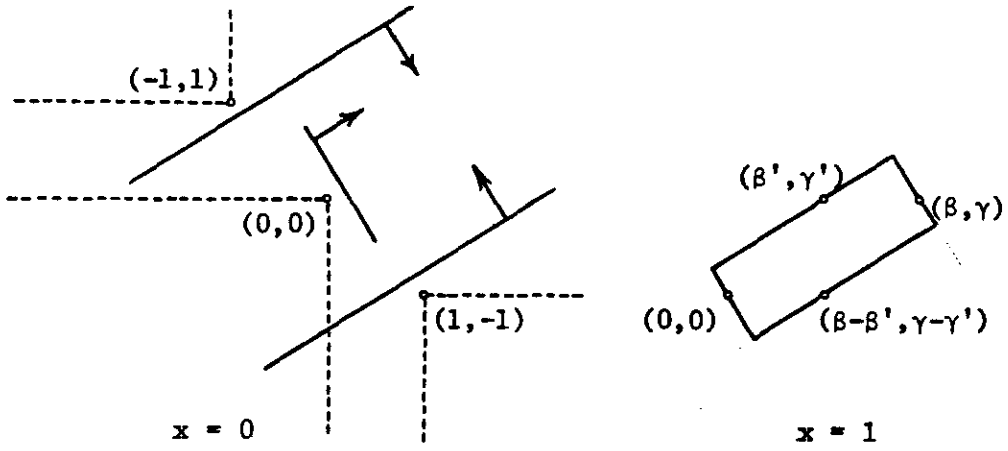

FIGURE 26

The plane through $(1,0,0)$ rejects $(0,0,0)$, and any lattice points below and to the left of $(0,0,0)$. And finally the plane through $(1,p,q)$ when translated to $(1,\beta,\gamma)$ must reject $(0,0,0)$ and any lattice points above and to the right of $(0,0,0)$. This demonstrates that while the parallelogram with vertices

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \beta \\ \gamma \end{pmatrix}, \quad \begin{pmatrix} p \\ q \end{pmatrix}, \quad \begin{pmatrix} \beta' \\ \gamma' \end{pmatrix}$$

does have lattice points in back, its immediate predecessor--and therefore

all predecessors--do not.

In the same fashion the successors of the unit square are free of lattice points in front, and from the arguments of Section IV, the lattice plane $x = 0$ is a characteristic plane for all tetrahedron obtained by relaxing the four inequalities from a lattice free region.

At first glance this argument seems very promising. It does, however, depend on the special assumption that there is at least one relaxation which yields a tetrahedron with a unique characteristic plane. The argument becomes extremely tedious when this assumption is not satisfied, and while it probably can be carried out I have chosen to select the alternative approach followed in the remainder of this paper.

VI. Perturbations of the Matrix A

Let the matrix A satisfying assumption 2.1 have the characteristic plane $x = 0$. This will be revealed by the fact that none of the relaxations on that plane--either parallelograms or triangles--has lattice points both in front and in back. If the matrix is perturbed slightly the same property will persist, and the characteristic plane is therefore locally constant. The disappearance of $x = 0$ as the characteristic plane will be revealed by the appearance of a doubled object, which may either be a new relaxation on $x = 0$, or one of the relaxations which previously appeared. In either case such a change can only occur when a constraint plane passes through more than one lattice point, an event which we call a singularity of the perturbation.

Our argument for the existence of a characteristic plane associated with a matrix A satisfying 1. 2, will be to select a different matrix A' which does have a characteristic plane, and perturb it until the original

matrix A is reached. We shall show that, along the path of perturbations, a new characteristic plane appears whenever the previous characteristic plane is lost.

The collection of lattice planes in three-space, i.e. planes passing through three non-collinear lattice points, is denumerable; we can therefore assume that in this series of perturbations, none of the constraint planes is ever parallel to a lattice plane. We need only be concerned, therefore, with the change in characteristic planes that occurs when one of the constraint planes passes through a 1-dimensional line of lattice points. Moreover we may assume that this violation of assumption 1.2 does not occur simultaneously for more than one constraint plane. Such a line of lattice points will be called a singular line.

6.1 [Lemma]. Let the characteristic plane immediately prior to a singularity be $x = 0$, and assume that the singular line is contained in this plane. Then $x = 0$ persists as a characteristic plane after the singularity.

Let us assume, to the contrary, that after the singularity there is a tetrahedron with vertices h^1, h^2, h^3, h^4 --obtained by relaxing from a lattice free region--two of whose vertices have first coordinates differing by more than one unit. This tetrahedron must be lost, as a relaxation, when we reverse the perturbation, and that can only occur in one of the following two ways:

1. The constraint plane passing through one of the vertices, say h^1, eliminates another vertex, say h^2, or

2. The constraint plane passing through one of the vertices, say h^1, admits a number of lattice points which are already accepted by the constraint planes passing through the remaining vertices.

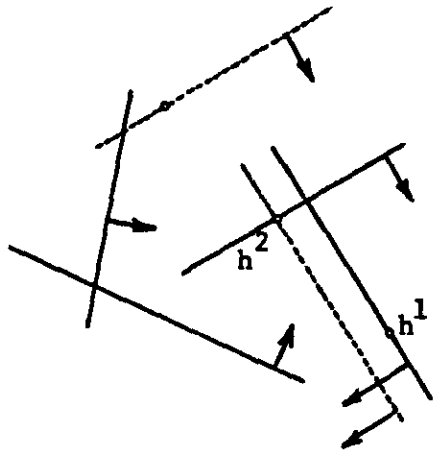

FIGURE 27

The first case is illustrated schematically in Figure 27; by assumption h^1 and h^2 have the same first coordinate. We obtain a different relaxation (after the perturbation), by pressing the constraint plane through h^1 to h^2, leaving the constraint planes through h^3 and h^4 as they were, and relaxing the constraint plane through h^2 until a new lattice point is reached, which is accepted by the other three constraint planes. This new tetrahedron will be of the second type, and of course, two of its vertices will have first coordinates differing by more than one unit.

The second case is illustrated in Figure 28, which is drawn

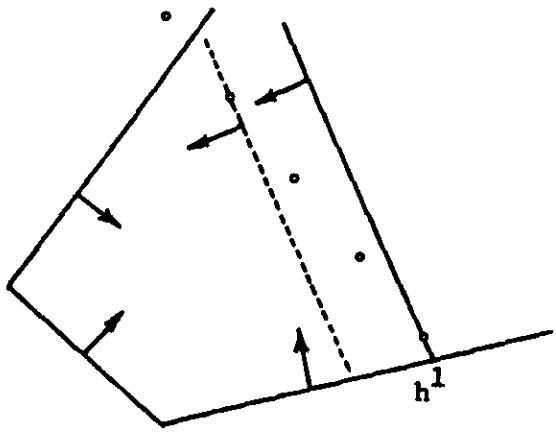

FIGURE 28

immediately after the singularity under the reverse perturbations. If the constraint plane through h^1 is pressed in to the last point on the half-line of lattice points accepted by the remaining constraint planes (a point whose first coordinate is identical with that of h^1), we obtain a tetrahedron which contradicts the assumption that $x = 0$ is a characteristic plane. This demonstrates lemma 6.1.

The lemma tells us that we need not be concerned with a change in the chain of triangles and parallelograms on the characteristic plane in examining the consequences of a perturbation. The only relevant concern is whether one of the previously undoubled objects becomes doubled in passing through a singularity. At such a singularity one of the constraint planes will be pivoting around a particular vertex of each relaxation and will suddenly admit a half line of lattice points either in front or behind the original characteristic plane; to be specific let this occur behind the original plane. It follows that some of the objects may gain lattice points in back and some may loose lattice points in front.

We shall assume that the terms "back" and "front" are interpreted as in theorem 4.3, and we consider the left-most object in the chain which is doubled after the singularity. From the above argument we can say that a lattice point has just been introduced behind the original plane, and that the predecessor of this object has no lattice points in front. In the next section we shall show that a new characteristic plane is available after the singularity when the left most doubled object is a parallelogram, and then consider the case in which the object is a triangle.

VII. A Doubled Parallelogram

Let us assume that the left most doubled object after the singularity is the unit square on the plane $x = 0$, that it contains the lattice point $(1,0,0)$, and that its immediate predecessor contains no lattice points in front. A particular lattice point will just have been introduced on the plane $x = -1$. From Figure 29 we see immediately that

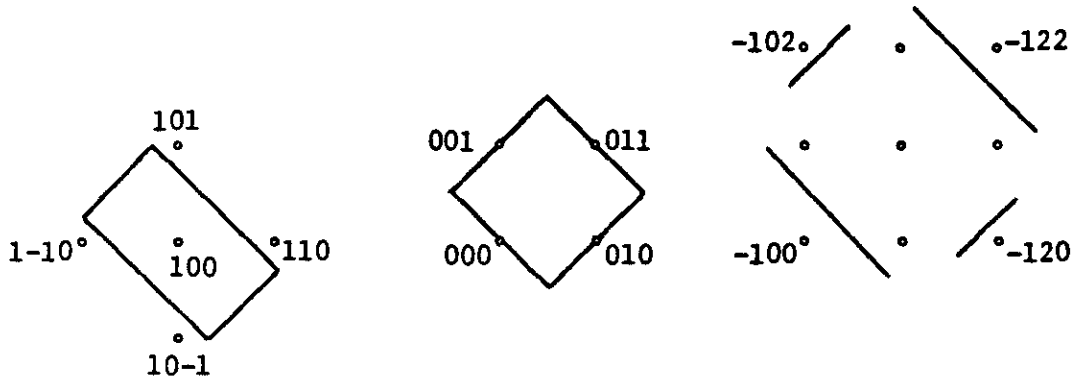

FIGURE 29

the new lattice point must be of the form $(-1,p,1)$ or $(-1,1,q)$. Without loss of generality we may assume that the point is given by $(-1,p,1)$ with $p \geq 1$.

From the assumption that the immediate predecessor of the unit square contains no lattice point in front we can easily see that the constraint plane through $(0,0,0)$ must eliminate both $(1,-1,0)$ and $(1,0,-1)$; it must therefore accept $(-1,1,0)$ and $(-1,0,1)$. In the same way the constraint plane through $(0,1,1)$ must accept $(-1,2,1)$ and $(-1,1,2)$. We are now prepared to exhibit the new characteristic plane, by a series of arguments which breaks into three major cases depending on whether $p = 1$, $p = 2$, or $p \geq 3$.

Case 1. $p = 1$. We shall demonstrate that the new characteristic plane is given by $z = \text{const}$. Let us begin by remarking that Figure 29 can be drawn with greater specificity, since the plane through $(0,0,1)$

must eliminate (-1,0,1) and (-1,1,2), and the plane through (0,1,0) must eliminate (-1,1,0) and (-1,2,1). The information in this figure can

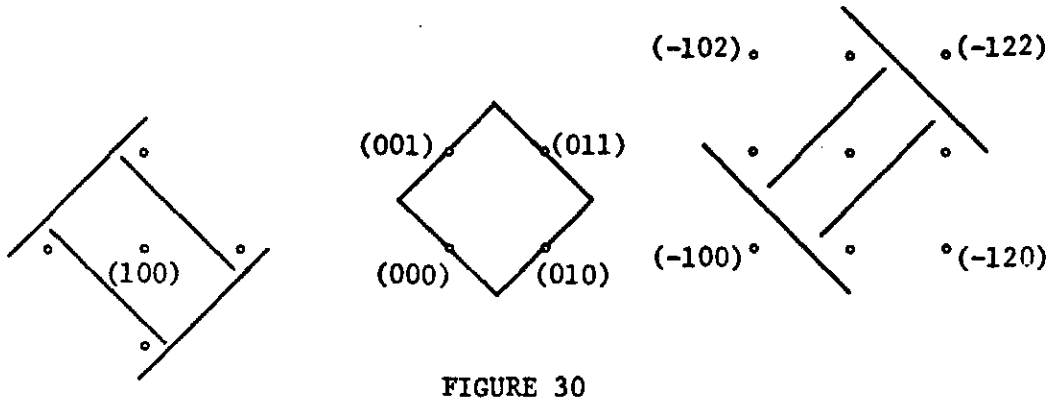

FIGURE 30

be translated to the two planes z = 0, 1, as in Figure 31.

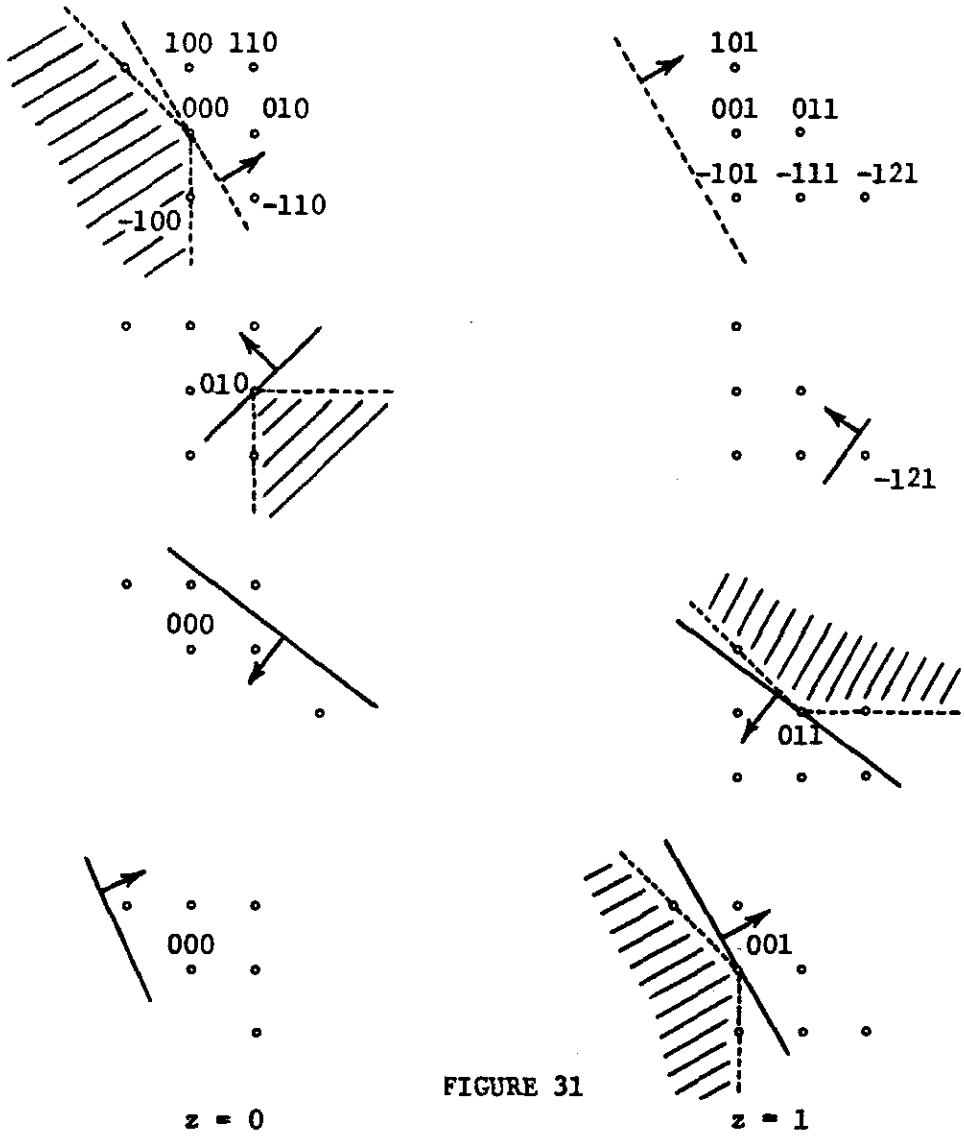

FIGURE 31

The constraint plane through (0,0,0) hasbeen drawn with a dashed line to distinguish it from the plane through (0,0,1) whichhas a similar slope. The shaded regions indicate lattice points which are eliminated by the various constraint planes.

It is easy to see that all of the relaxations in this plane are

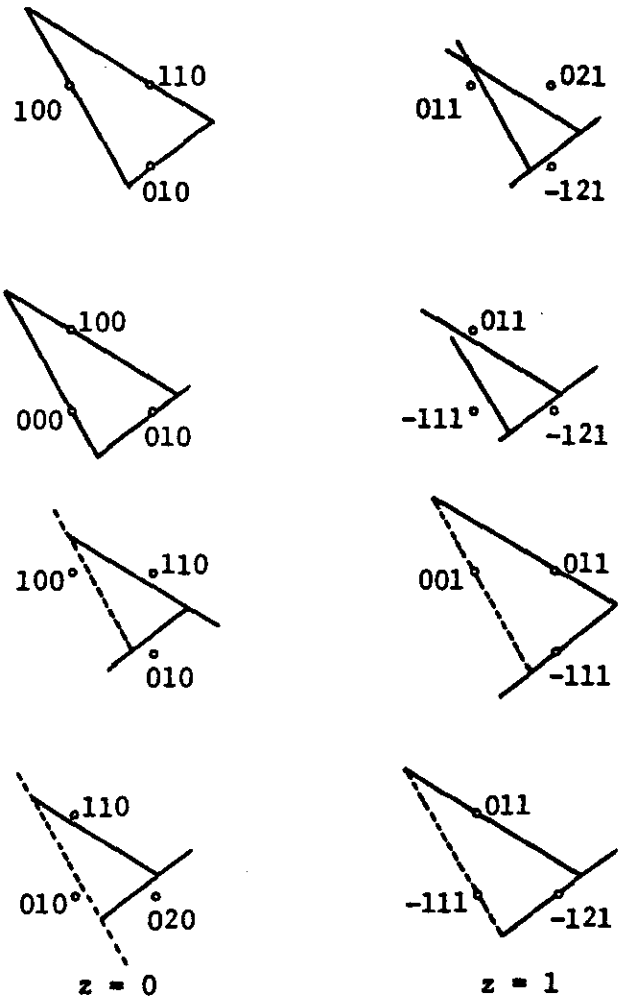

FIGURE 32

triangles. The two triangles on $z = 0$ are free of lattice points on $z = 1$, and the two triangles on $z = 1$ are free of lattice points on $z = 0$. This concludes our argument for the case $p = 1$.

Case 2. $p = 2$.

We shall demonstrate that the new characteristic plane is either $x + y = \text{const.}$ or $z = \text{const.}$ Let us redraw Figure 29 with the additional information that $(-1,2,1)$ has just been accepted, and translate this

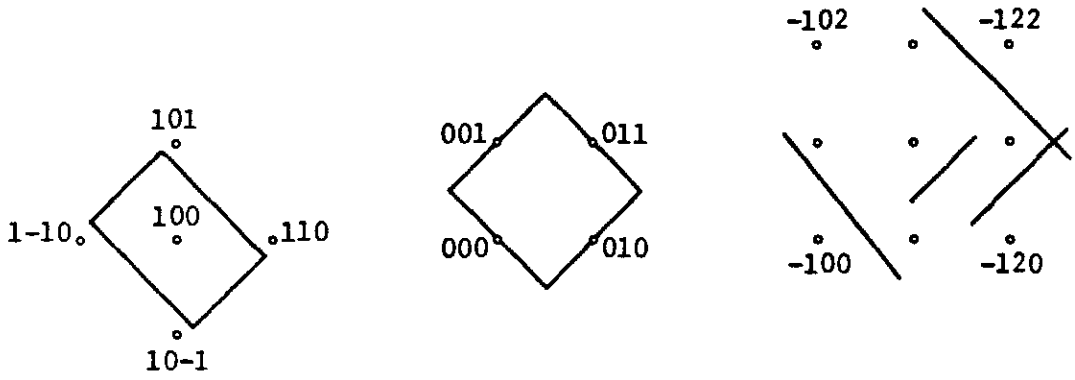

FIGURE 33

information to the planes $x + y = 0, 1$ in Figure 34. The slopes of the constraint planes permit us to select two adjacent parallelograms which are relaxations on this plane. Let us draw one of these on the plane $x + y = 1$, as in Figure 35.

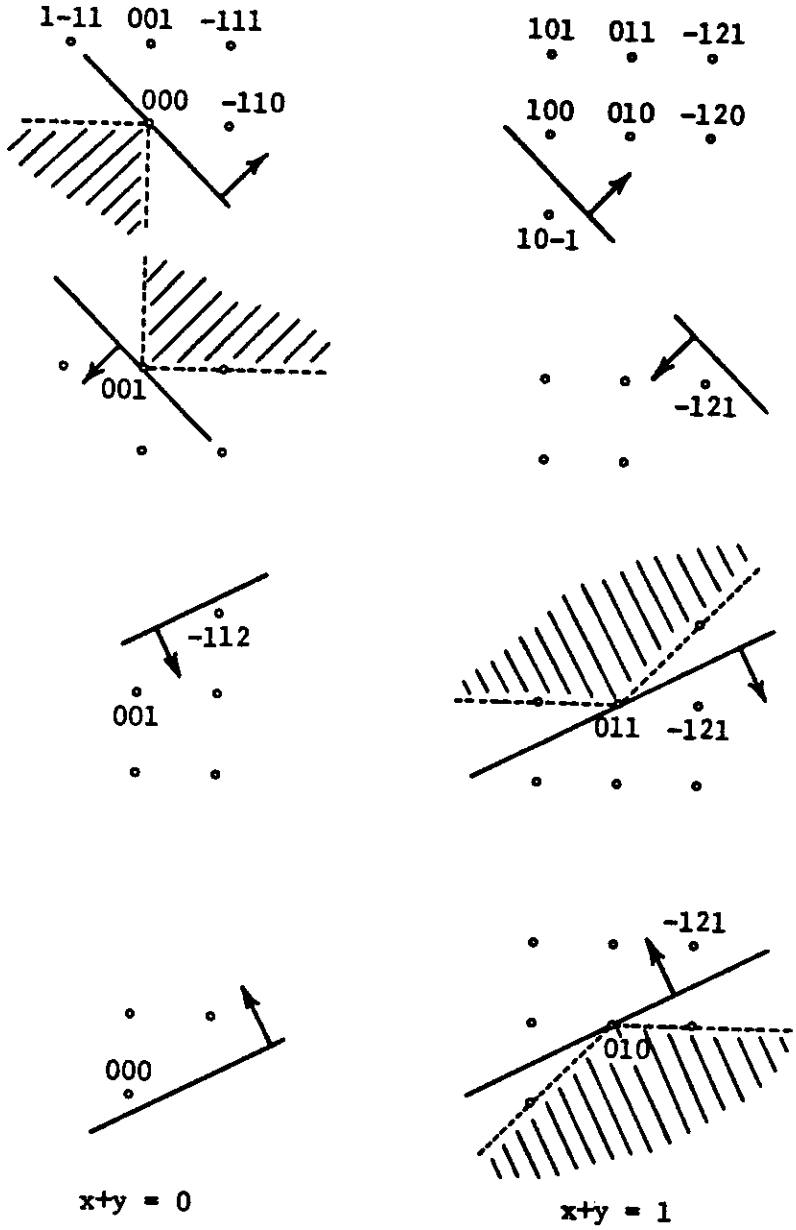

FIGURE 34

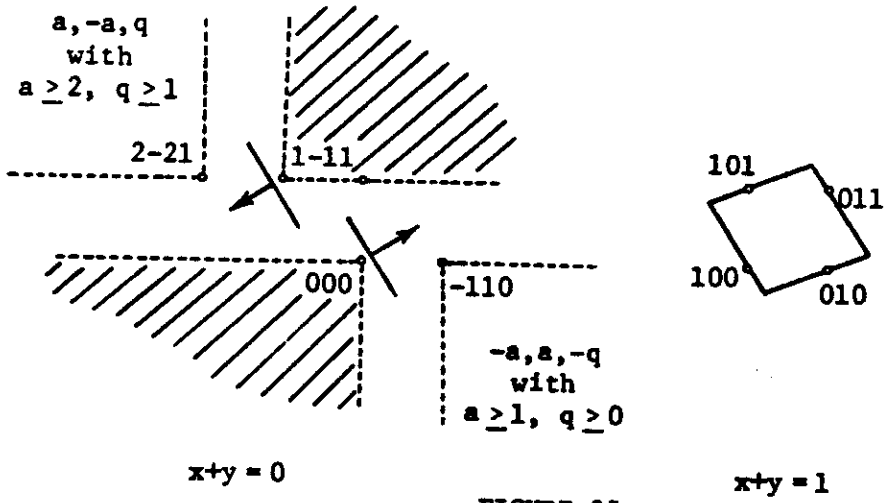

FIGURE 35

Consider the region to the lower right on the plane $x+y = 0$, consisting of lattice points $(-a,a,-q)$ with $a \geq 1$, $q \geq 0$. All of these points must be eliminated by the original inequalities, as in Figure 34, since none of them lies on a possible line of singularities containing $(-1,2,1)$. The constraint plane through $(0,1,1)$ accepted all of these lattice points, each of which must therefore have been rejected by one of the other three constraint planes in their position in Figure 34. But these other three constraint planes have been compressed, or remained where they were, in making the transition from Figure 34 to Figure 35. Therefore all lattice points in the region to the lower right are rejected.

Now consider a lattice point $(a,-a,q)$ with $a \geq 2$, $q \geq 1$ in the region to the upper left on the plane $x+y = 0$. If this point is accepted by the plane through $(0,0,0)$ translated to $(1,0,0)$, and by the plane through $(0,0,1)$ translated to $(0,1,1)$, then

1. the plane through $(0,0,0)$ accepts $(a-1,-a,q)$

2. the plane through $(0,0,1)$ accepts $(a,-a-1,q)$, and therefore $(a-1,-a+1,q)$

3. the plane through $(0,1,0)$ accepts $(1,0,0)$ and therefore accepts $(a-1,-a+2,0)$.

Let us draw this information on the plane $x = a-1$. It follows

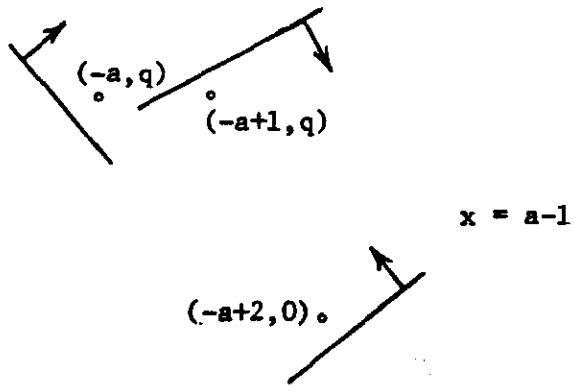

FIGURE 36

that the plane through (0,1,1) must reject (a-1,-a+1,q) , since otherwise the predecessor of the unit square, obtained by pressing in the plane through (0,0,0) , will contain the point (a-1,-a+1,q) in front. But then the plane through (0,1,1) translated to (1,0,1) must reject (a,-a,q) . This permits us to conclude that the parallelogram on x+y = 1

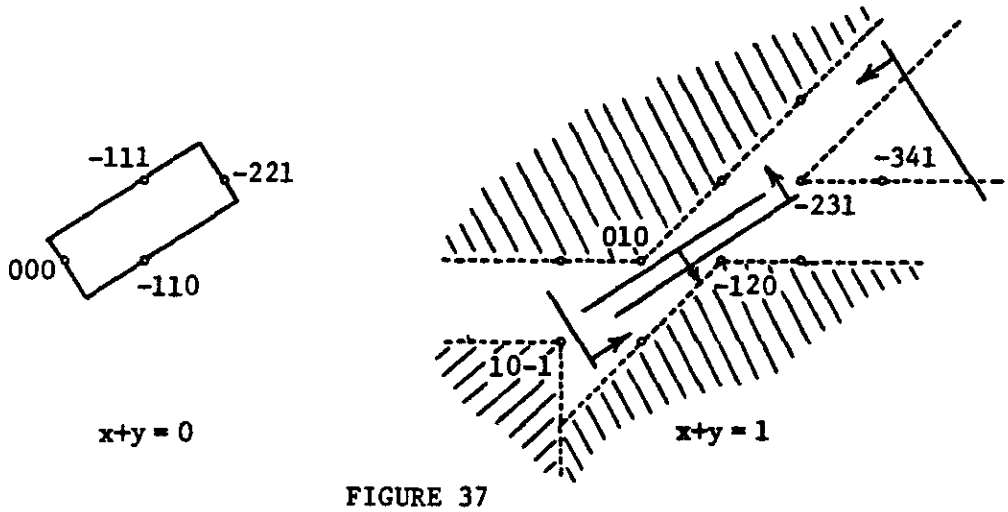

FIGURE 37

is free of lattice points on x+y = 0 .

Now let us draw the adjacent parallelogram on the plane x+y = 0 . The only possible lattice points satisfying these inequalities on the plane x+y = 1 lie in the wedge with vertex (-2,3,1) . Aside from one special case these can be ruled out by considering which of the original constraint planes has just accepted (-1,2,1) . For example if the plane through (0,1,1) has just accepted (-1,2,1) , the only points in the wedge are very far from the vertex and they will be eliminated by the fourth inequality to the right of (-3,4,1) . A similar remark is valid if (-1,2,1) has just been accepted by the plane through (0,1,0) . In both of these cases we have therefore demonstrated that the new characteristic plane is given by x+y = const.

If the plane through (0,0,1) has just accepted (-1,2,1) , then

when translated to $(-2,2,1)$, it will just have accepted $(-3,4,1)$. The only possible points on $x+y = 1$ which might satisfy the inequalities are therefore $(-2,3,1)$ and $(-3,4,1)$. The first of these points can be eliminated, since if it were accepted by the translated planes, it would follow that the plane through $(0,1,1)$, and the plane through $(0,1,0)$ both accept $(-1,3,1)$, which is impossible. The second possibility can, however, actually occur when the planes through $(0,1,1)$ and $(0,1,0)$ both accept $(-2,4,1)$, and the plane through $(0,0,1)$ has just accepted $(-1,2,1)$. Aside from this special case we have demonstrated that $x+y = \text{const.}$ is the new characteristic plane when $p = 2$. We shall show that in this special case the new characteristic plane is given by $z = \text{const.}$, making use of a different argument which does not require a drawing on the new plane.

Consider the three planes $x = 1, 0, -1$ as in Figure 38, drawn

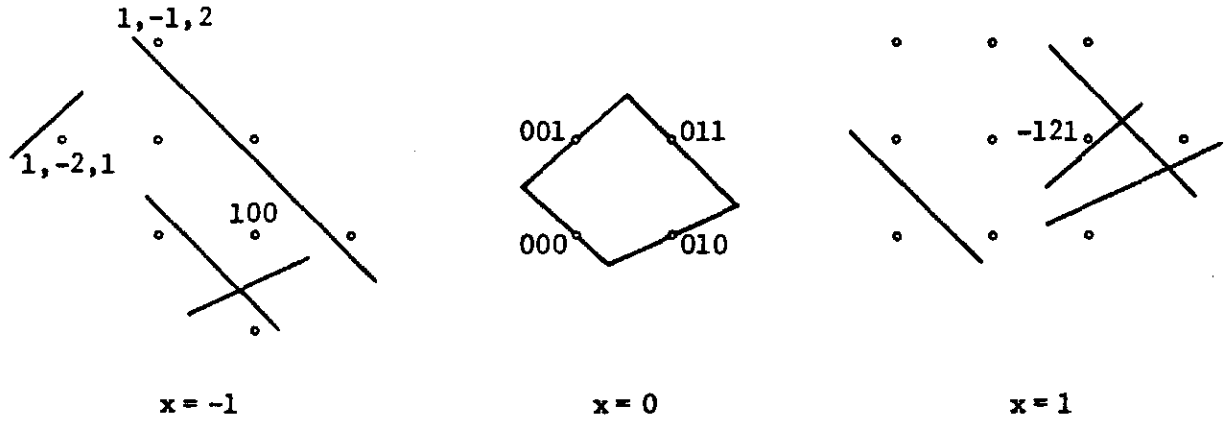

FIGURE 38

immediately before the singularity. From Figure 37 we see that the plane through $(0,1,0)$ translated to $(-1,1,0)$ must accept $(-3,4,1)$ as well as $(-2,3,1)$. In its original position it must accept $(-2,4,1)$, and therefore rejects $(2,-2,-1)$. It must also accept $(-1,3,1)$. The plane through $(0,1,1)$ must then reject $(-1,3,1)$ and therefore accept $(1,-1,1)$. Moreover since this latter plane accepts $(-2,4,1)$ it must accept $(-1,3,0)$

and consequently it must also reject $(1,-1,2)$.

Now let us translate these planes as in Figure 39. Since the plane through $(0,1,0)$ rejects $(1,-1,0)$ when translated to $(-1,2,1)$ there are no lattice points strictly satisfying these inequalities on either

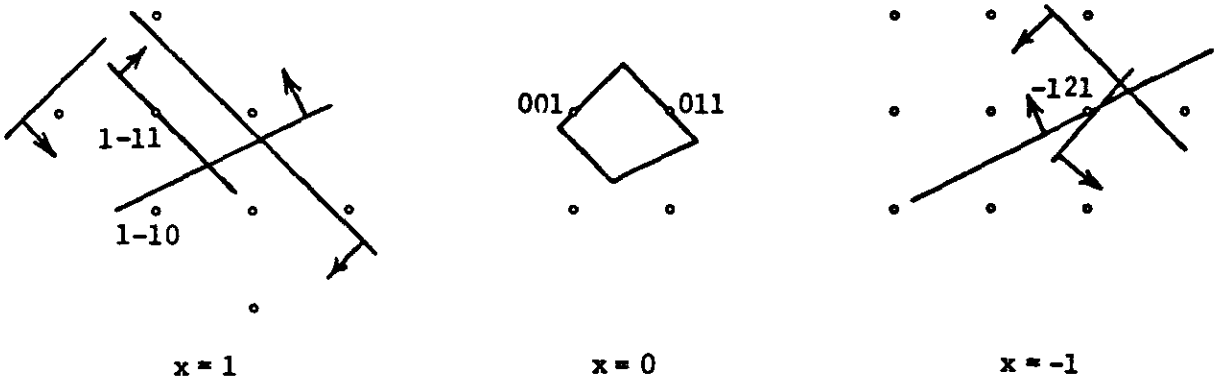

FIGURE 39

of the three planes. Since $x = \text{const.}$ is a characteristic plane there are no lattice points satisfying these inequalities on any other plane $x = \text{const.}$, just prior to the singularity. Immediately after the singularity the only possible lattice points satisfying the inequalities must lie on $z = 1$. But there are none on this plane and it follows that the parallelogram with vertices

$$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

has no lattice points either in front or in back, after the singularity. The plane $z = \text{const.}$ must therefore be a new characteristic plane. This concludes our argument for $p = 2$.

Case 3. $p \geq 3$.

We shall demonstrate that the new characteristic plane is given by $z = \text{const.}$ or $x+y = \text{const.}$ Let us redraw Figure 29 to show that the point $(-1,p,1)$ has just been accepted. The point $(-1,p-1,1)$ must

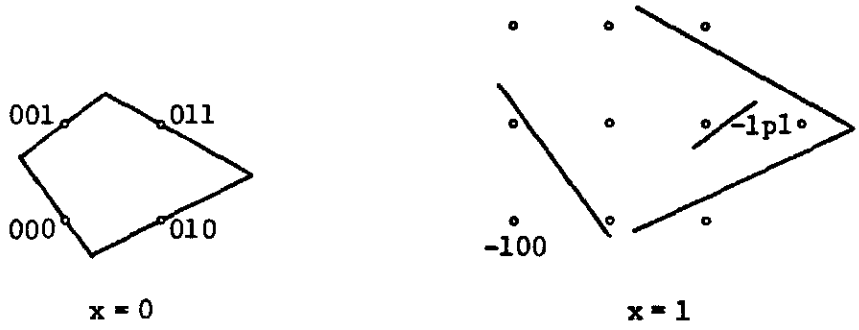

FIGURE 40

be rejected by the plane through $(0,0,1)$, the point $(-1,1,0)$ either by the plane through $(0,0,1)$ or the plane through $(0,1,0)$, and the point $(-1,p+1,1)$ either by the plane through $(0,1,1)$ or the plane through $(0,1,0)$. Consider the two configurations of Figure 41.

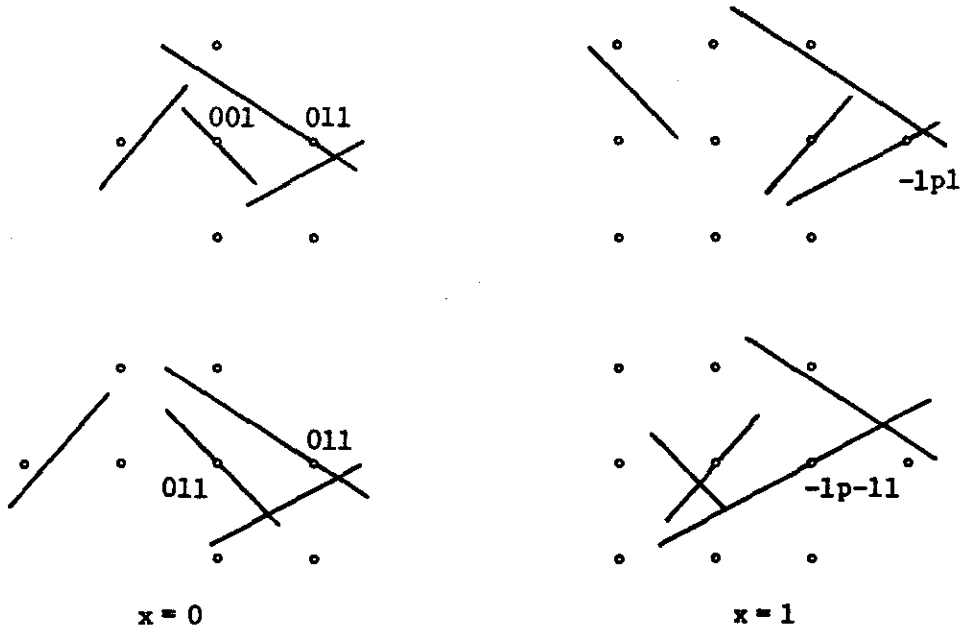

FIGURE 41

Depending on the slope of the plane originally at $(0,0,0)$ and now at $(0,0,1)$, at least one of these two configurations contains no lattice points strictly on the two planes $x = 0, 1$. Since $x = \text{const.}$ was a characteristic plane prior to the singularity, the only lattice points satisfying the inequalities, immediately after the singularity, lie on the line of singularities. If $(-1,p,1)$ has just been accepted either by the plane through $(0,0,1)$ or the plane through $(0,1,1)$ the line of singularities is in the plane $z = 1$ itself and the corresponding parallelogram is free of lattice points in front and in back. In either of these two cases $z = \text{const.}$ is the new characteristic plane.

If $(-1,p,1)$ has just been accepted by the line through $(0,1,0)$ the possible lattice points accepted by the inequalities are $(-2,2p-1,2)$, $(-3,3p-2,3)$, ... in the first figure, and $(-2,2p-2,2)$, $(-3,3p-3,3)$, ... in the second figure. In order to complete the argument it is necessary to consider several subcases.

3.1. The plane originally through $(0,0,1)$ rejects $(-1,p,2)$. In the first figure the point $(-2,2p-1,2)$, and all subsequent points on the line of singularities, are eliminated by this plane translated to $(-1,p-1,1)$. In the second figure $(-2,2p-2,2)$, and all subsequent points, are eliminated by this plane translated to $(-1,p-2,1)$. The new characteristic plane is therefore $z = \text{const.}$

3.2. The plane originally through $(0,0,1)$ accepts $(-1,p,2)$ and the plane through $(0,1,1)$ rejects $(-1,p+1,1)$. In this case the first figure is free of other lattice points on $x = 0,1$. The plane through $(0,1,1)$ eliminates $(-2,p+2,2)$ and therefore $(-2,2p-1,2)$ if $2p-1 \geq p+2$ or $p \geq 3$. The new characteristic plane is therefore $z = \text{const.}$

3.3. The plane originally through (0,0,1) accepts (-1,p,2) and the plane through (0,1,1) accepts (-1,p+1,1) . We then draw Figure 42. Since the line of singularities lies in the plane x+z = 0 ,

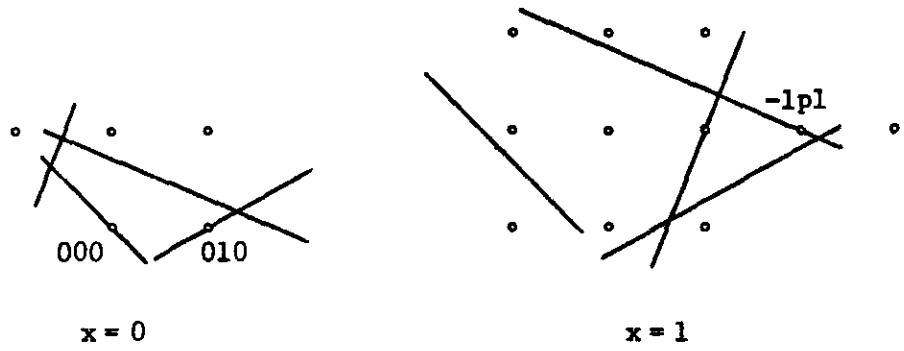

FIGURE 42

we can conclude that the parallelogram with vertices

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ p \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ p-1 \\ 1 \end{pmatrix}$$

contains no lattice points on either side, immediately after the singularity. It follows that x+z = const. is the new characteristic plane.

VIII. A Doubled Triangle

In this section we consider the case in which the left-most doubled object immediately after the singularity is a triangle, again making the assumption that lattice points have been introduced behind the characteristic plane $x = 0$, but not in front, and adopting the convention of theorem 4.3. The following lemma will be useful in demonstrating that this doubled triangle must be one of the pair appearing at the right end of the chain.

8.1 [Lemma]. Consider the two triangles at either end of the chain on the plane $x = 0$. If precisely one of the pair has lattice points on the plane $x = -a$, with $a \neq 0$, then it is doubled.

Let the two triangles be as in Figure 43, and assume that the first

FIGURE 43

of these triangles contains a point on the plane $x = -a$, say $(-a,0,0)$. If the other triangle is to contain no lattice points on this same plane, then the configuration of Figure 44 must obtain, and the first triangle must contain at least three lattice points on $x = a$.

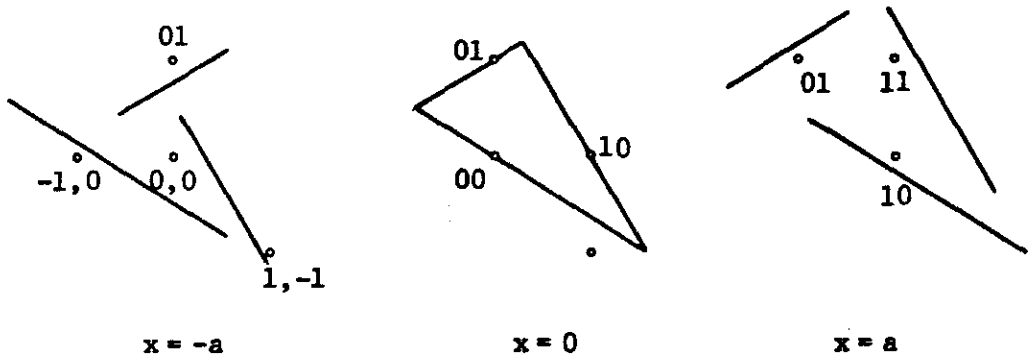

FIGURE 44

Now let us assume that one of the triangles to the left of the chain on $x = 0$ is doubled immediately after the singularity. Since new lattice points are only introduced behind $x = 0$, that triangle must contain a lattice point in front prior to the singularity. If $x = 0$ is a characteristic plane, prior to the singularity, lemma 8.1 implies that both triangles to the left have lattice points in front, and from theorem 4.3 all relaxations on $x = 0$ have lattice points in front. Therefore none of them have lattice points in back, prior to the singularity. But it is easy to see that assumption 2.1 is violated if none of the relaxations on $x = 0$ have lattice points in back. In such a case if the four constraint planes are placed so as to yield a lattice free region on $x = 0$, there will be no lattice points on any plane behind $x = 0$. Conversely if there is a lattice point satisfying the inequalities on any plane $x = a$, there will be lattice points on every parallel plane in front. This contradiction to assumption 2.1 implies that if the left-most doubled object, after the singularity, is a triangle it must be one of the two at the right end of the chain.

Let us begin our analysis by assuming that the chain of relaxations on the plane $x = 0$ does contain some relaxations which are parallelogrmas, deferring to the next section the case in which all of the relaxations on $x = 0$ are triangles. Without loss of generality we may take

the last of these parallelograms to be the unit square, and assume that the pair of triangles to the right are given by Figure 45. In each of

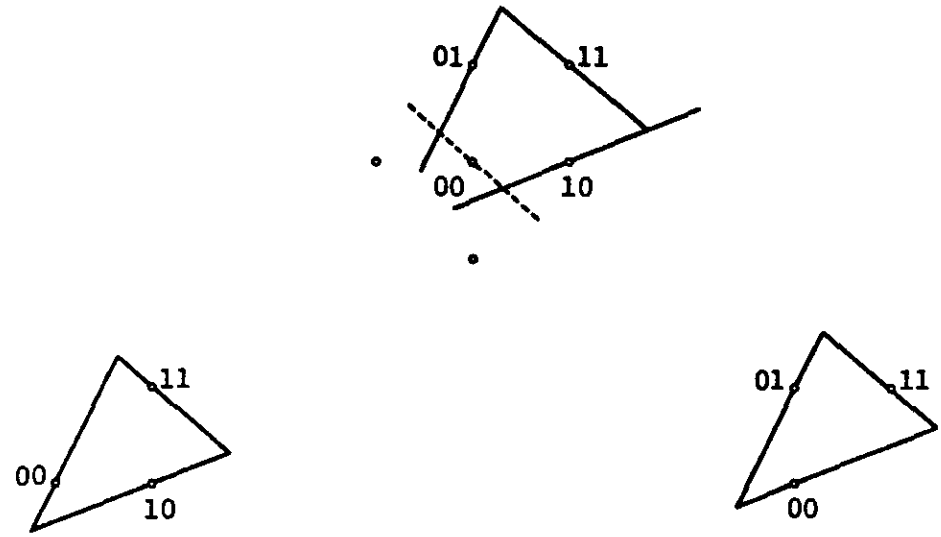

FIGURE 45

these triangles the plane originally through $(0,0,0)$ has been relaxed to $-\infty$. We assume that the unit square contains no lattice points in front, and to be specific let us assume that the point $(-1,1,1)$ has just been admitted in the first of the above triangles as we pass through the singularity. Immediately after the singularity the configuration of Figure 46 must obtain. It follows that when the dashed line is placed

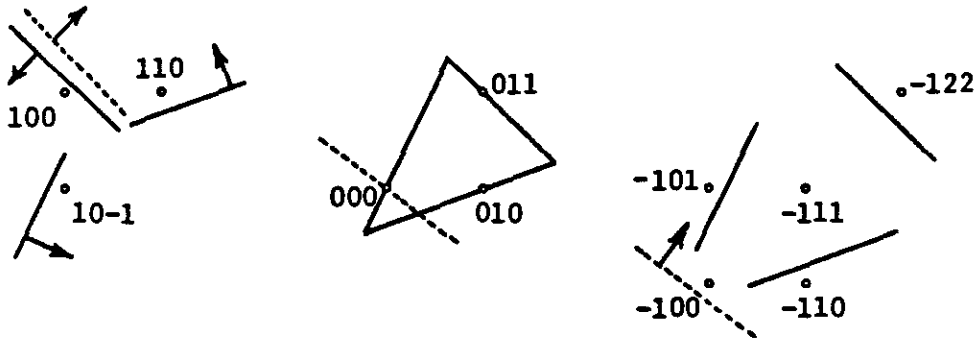

FIGURE 46

at $(0,0,0)$ it must eliminate $(1,0,0)$ and therefore accept $(-1,0,0)$.

The argument which provides a new characteristic plane immediately

after the singularity depends on which of the three constraint planes has just admitted the point $(-1,1,1)$.

Case 1. The plane through $(0,1,1)$ has just admitted $(-1,1,1)$. Observe first of all that the plane through $(0,0,0)$ in Figure 46 must eliminate $(-1,0,0)$ since otherwise the other triangle has a lattice point in back prior to the singularity, and from lemma 8.1, it would already have been doubled.

1.1. The plane through $(0,1,1)$ eliminates $(0,0,2)$. We then have the configuration of Figure 47. In this figure the line of singu-

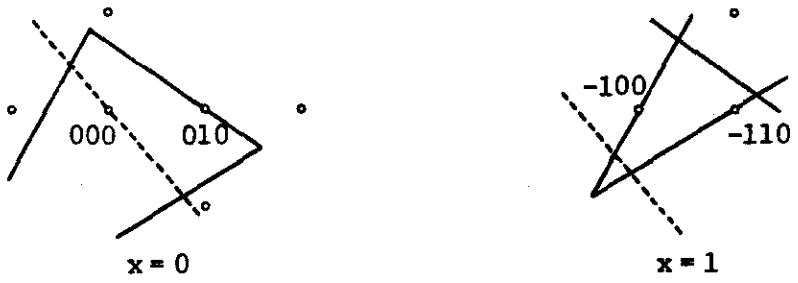

FIGURE 47

larities lies in the plane $z = 0$, which is therefore the new characteristic plane.

1.2. The plane through $(0,1,1)$ accepts $(0,0,2)$. In this case Figure 48 permits us to argue that $y = 0$ is the new characteristic plane.

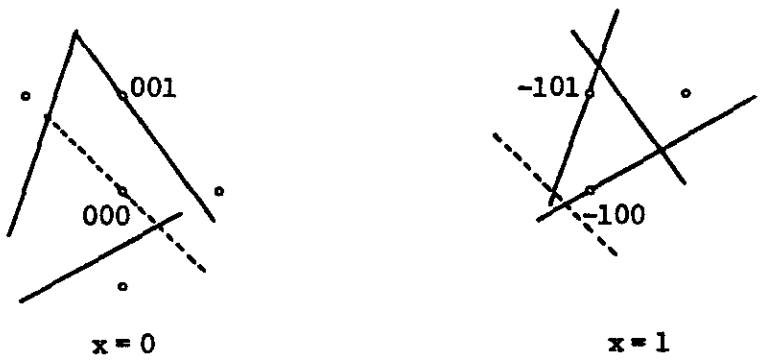

FIGURE 48

Case 2. The plane through (0,1,0) in Figure 46 just admits (-1,1,1) . Observe that the plane through (0,1,1) must reject (-1,1,2) since otherwise the other triangle contains a lattice point in back prior to the singularity and from lemma 8.1 it would already have been doubled.

2.1. The plane through (0,0,0) accepts (-1,0,0) . Consider Figure 49, drawn immediately after the singularity. Since the line of

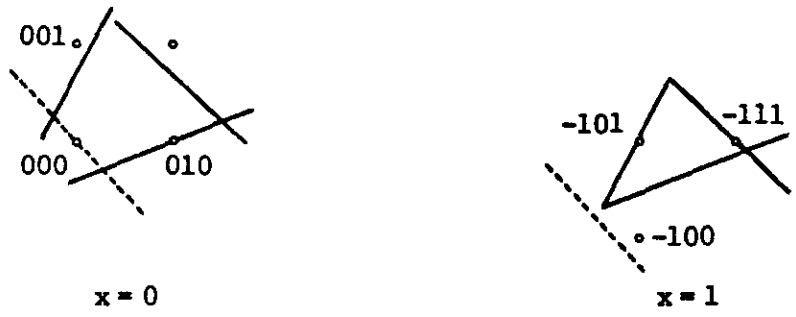

FIGURE 49

singularities lies in the plane $x+z = 0$, it follows that this plane is the new characteristic plane.

2.2. The plane through (0,1,1) accepts (-1,2,1) , and therefore rejects (1,0,1) . When this plane is translated to (-1,1,1) , as in Figure 49, it will reject (0,0,1) and the parallelogram with vertices

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

will again be free of lattice points on the planes $x = 0, 1$. The line of singularities lies in the plane of this parallelogram and therefore $x+z = \text{const.}$ is the new characteristic plane.

2.3. The plane through (0,0,0) rejects (-1,0,0) , and the plane through (0,1,1) rejects (-1,2,1) and accepts (-1,0,2) . Figure 50 illustrates this configuration immediately after the singularity. We shall demonstrate that y = const. is the new characteristic plane.

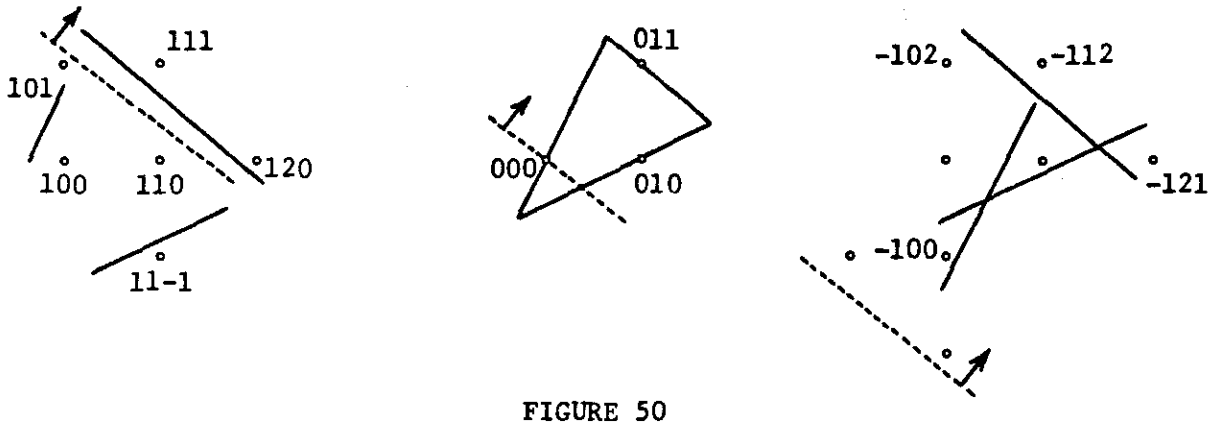

FIGURE 50

Let us translate the above information to the two planes y = 0,1 , as in Figure 51. The slopes of the constraint planes permit us to recognize two adjacent parallelograms which are relaxations on the planes y = const. We shall demonstrate that one of them is free of lattice points in front and the other in back.

In Figure 52 one of these relaxations is drawn on the plane $\bar{y} = 1$. The only possible lattice points satisfying these inequalities on y = 0 lie in the wedge with vertex (1,0,2) , and are of the form (a,0,c) with $a \geq 1$, $c \geq 2$. But such a lattice point must be eliminated by one of the four inequalities in their position in Figure 50. It cannot be eliminated by the plane through (0,1,0) , and is therefore eliminated by one of the remaining planes. These planes, however, are compressed, in the transition from Figure 50 to Figure 52, and there are, therefore no lattice points on y = 0 .

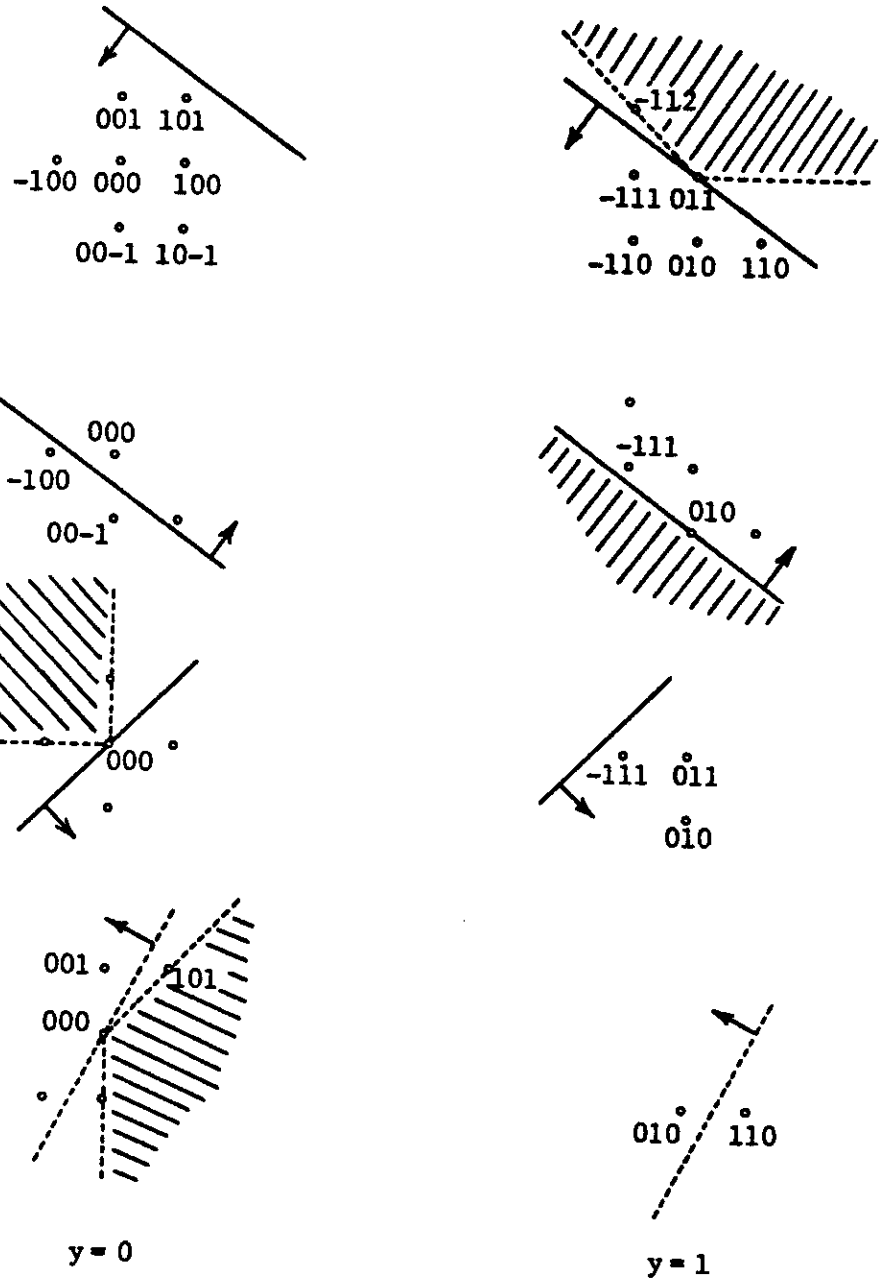

FIGURE 51

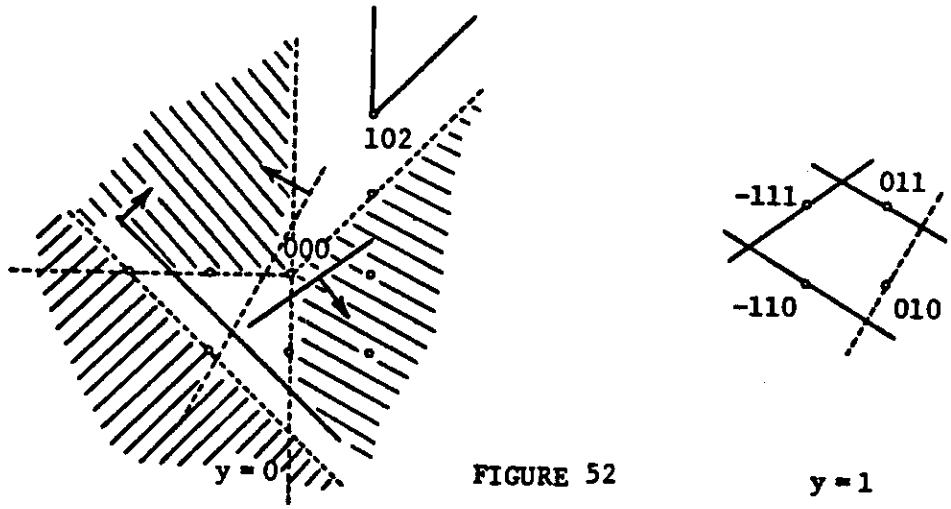

FIGURE 52

In Figure 53 an adjacent relaxation is drawn on the plane $y = 0$.

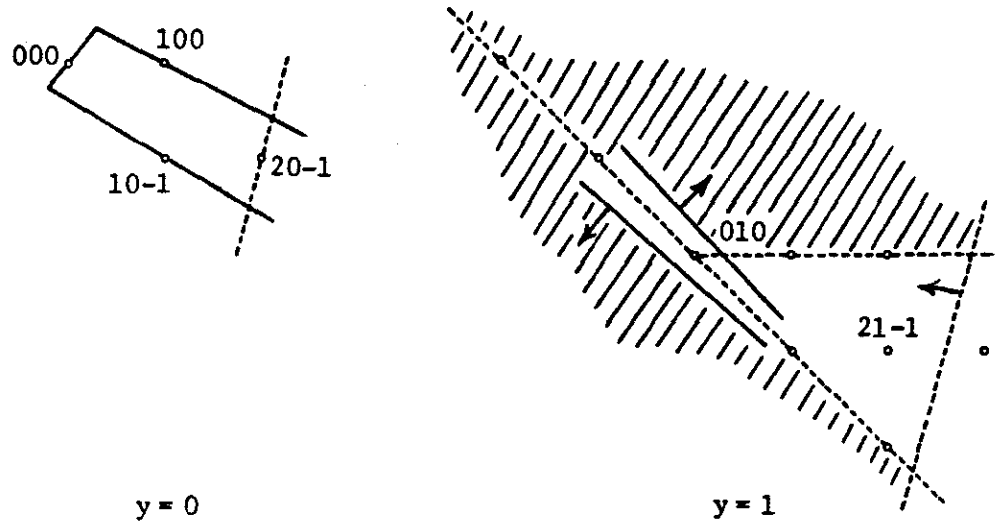

FIGURE 53

The only possible lattice point satisfying the inequalities on $y = 1$ is $(2,1,-1)$. But the plane through $(0,1,1)$ accepts $(-1,0,2)$, and therefore rejects $(1,2,0)$; when translated to $(1,0,0)$ it rejects $(2,1,-1)$. This demonstrates that $y = \text{const.}$ is the new characteristic plane.

2.4. The plane through $(0,0,0)$ rejects $(-1,0,0)$ and the plane through $(0,1,1)$ rejects both $(-1,2,1)$ and $(-1,0,2)$. In this final subcase we consider the configuration of Figure 54, drawn immediately after the singularity. There are no lattice points on either of the planes

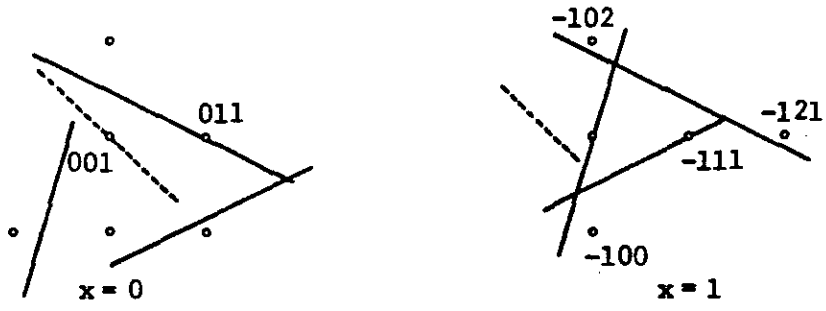

FIGURE 54

$x = 0,1$, and therefore the only possible lattice points satisfying the inequalities are on the line of singularities $(-1,1,1)$, $(-2,1,2)$, ... But the plane through $(0,1,1)$ eliminates $(-1,2,1)$ and $(-1,0,2)$; it therefore eliminates $(-2,1,2)$ and we conclude that the new characteristic plane is given by $z = \text{const}$.

Case 3. The plane through $(0,0,0)$ has just admitted $(-1,1,1)$. Observe, first of all, that the plane through $(0,1,0)$ must reject $(-1,2,1)$ since if this were not true the other triangle at this end of the chain would contain a lattice point prior to the singularity, and from lemma 8.1 it would already have been doubled. We also remark that prior to the singularity the unit square contains the point $(-1,1,1)$ on the back plane and therefore none of the quadrilaterals in the chain are free of lattice points both in front and in back. We shall, following a suggestion by Philip White, consider three subcases, the first of which yields a new characteristic plane after the singularity, and the second and third of which yield a new characteristic plane prior to the singularity which contains a relaxation free of lattice points on both sides. If this latter plane is lost as we pass through the singularity we enter a case different from the present one.

3.1. The plane through $(0,1,1)$ accepts $(-1,1,2)$. Consider the configuration of Figure 55 drawn immediately after the singularity. Since the line of singularities lies in the plane $x+y = 0$, this plane is the new characteristic plane after the singularity.

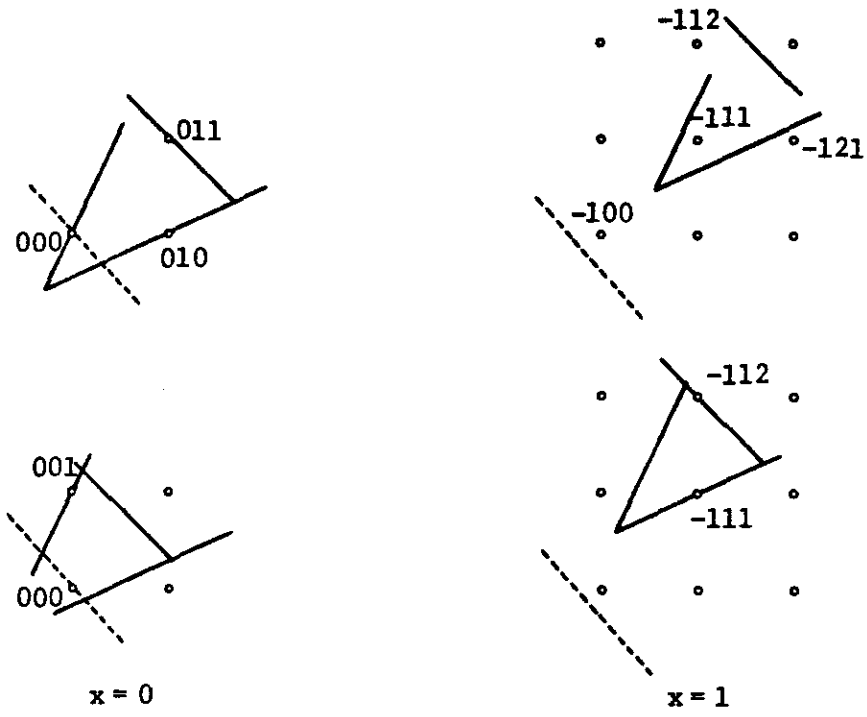

FIGURE 55

3.2. The plane through (0,1,1) rejects (-1,1,2) but accepts (0,0,2) . We consider Figure 56 drawn immediately prior to the singularity.

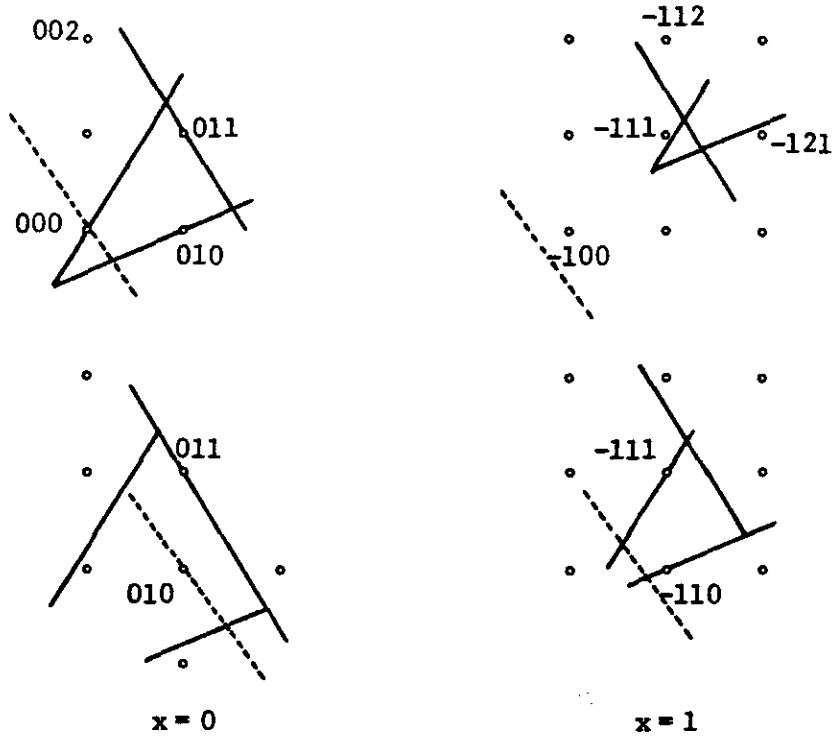

FIGURE 56

We see that the parallelogram on the plane $y = 1$, with vertices

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

contains no lattice points on back or on front. The plane $y = \text{const.}$ is also a characteristic plane prior to the singularity which, if it is lost, leads to a case other than the one currently being considered.

3.3. The plane through $(0,1,1)$ rejects both $(-1,1,2)$ and $(0,0,2)$. We consider Figure 57, drawn immediately prior to the singu-

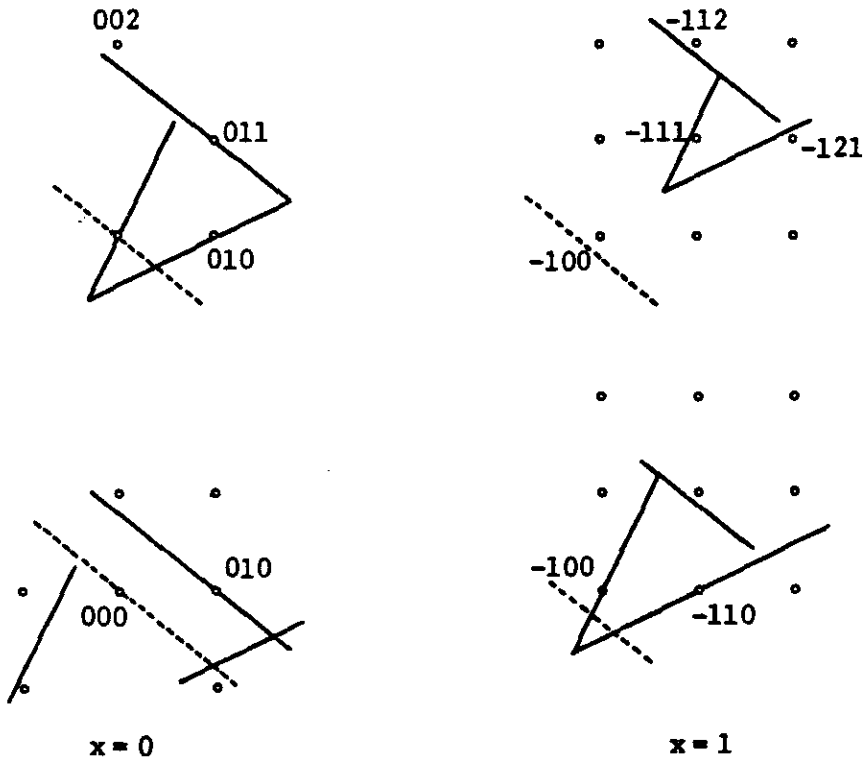

FIGURE 57

larity, and conclude that $z = 0$ is also a characteristic plane prior to the singularity, but contains a parallelogram free of lattice points on both sides.

Our argument for determining a new characteristic plane when the left-most doubled object is a triangle is now complete, in the case where the chain contains at least one relaxation which is a parallelogram. The final case, to be considered in the next section, involves the special case of a chain all of whose relaxations are triangles.

IX. The Conclusion of the Argument

As before we take $x = \text{const.}$ to be the characteristic plane prior to the singularity, but address the final case in which all of the relaxations on this plane are triangles. Without loss of generality we may take them to be the triangles of Figure 58. In the first pair of triangles the

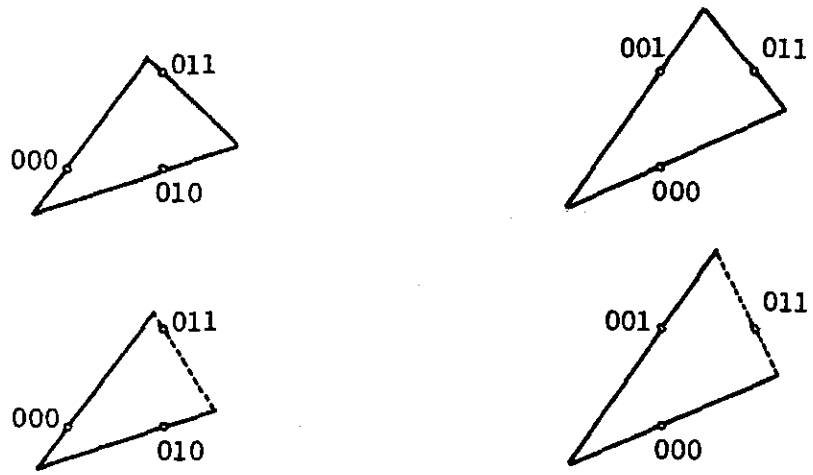

FIGURE 58

dashed line is relaxed to infinity, and in the second pair the solid line originally through $(0,1,1)$ is relaxed to infinity. We assume that prior to the singularity there are no lattice points behind the first two triangles, and no lattice points in front of the second pair. We also assume that in passing through the singularity the triangle in the upper left of Figure 58 admits the point $(-1,1,1)$ on the back plane. Figure 59 describes the configuration immediately after the singularity. The dashed

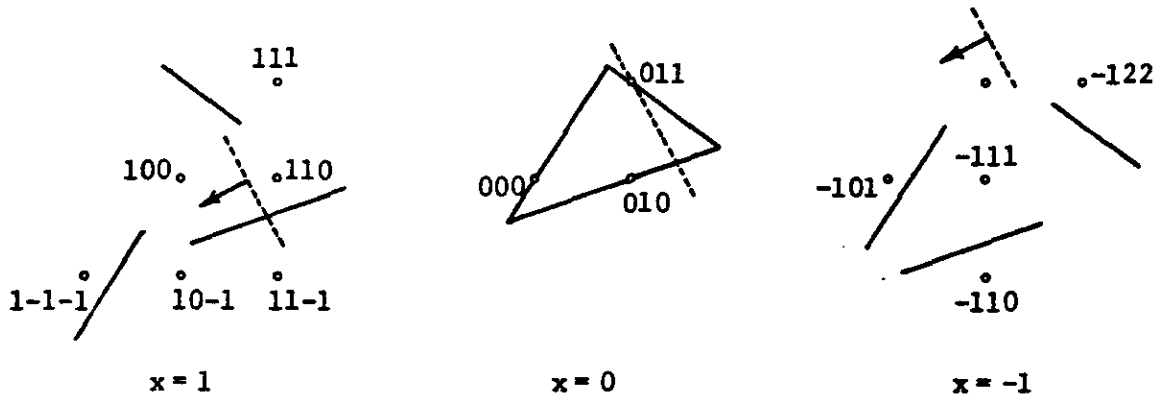

FIGURE 59

line through (0,1,1) must reject (1,1,0) since otherwise the lower left triangle of Figure 58 would contain this point in front.

As before the argument which produces the new characteristic plane depends on which of the constraint planes has just admitted (-1,1,1) in passing through the singularity.

Case 1. The plane through (0,0,0) just admits (-1,1,1) . In this case the plane through (0,1,0) must reject (-1,2,1) since otherwise the triangle in the upper right would contain a lattice point in back prior to the singularity. Moreover the dashed line through (0,1,1) must eliminate (1,0,0) , and therefore accept (-1,2,2) . Consider the configuration of Figure 60, drawn immediately after the singularity.

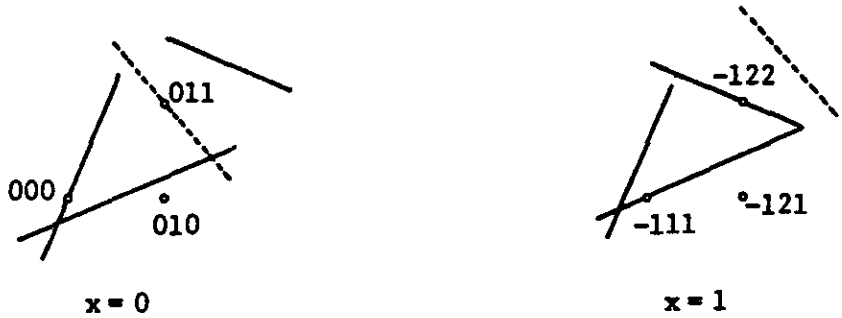

FIGURE 60

Since the line of singularities lies in the plane containing the parallelogram with vertices

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

we conclude that this parallelogram is free of lattice points on both sides after the singularity, and that $y-z = \text{const.}$ is the new characteristic plane.

Case 2. The plane through $(0,1,0)$ just admits $(-1,1,1)$. In this case the plane through $(0,1,1)$ must eliminate $(-1,1,2)$ since otherwise the triangle in the upper right contains a lattice point in front prior to the singularity. Moreover the dashed line through $(0,1,1)$ must eliminate $(1,0,-1)$ and therefore accept $(-1,2,3)$. The information in Figure 59 may be translated to the two planes $x-y+z = 0, -1$ as in Figure 61.

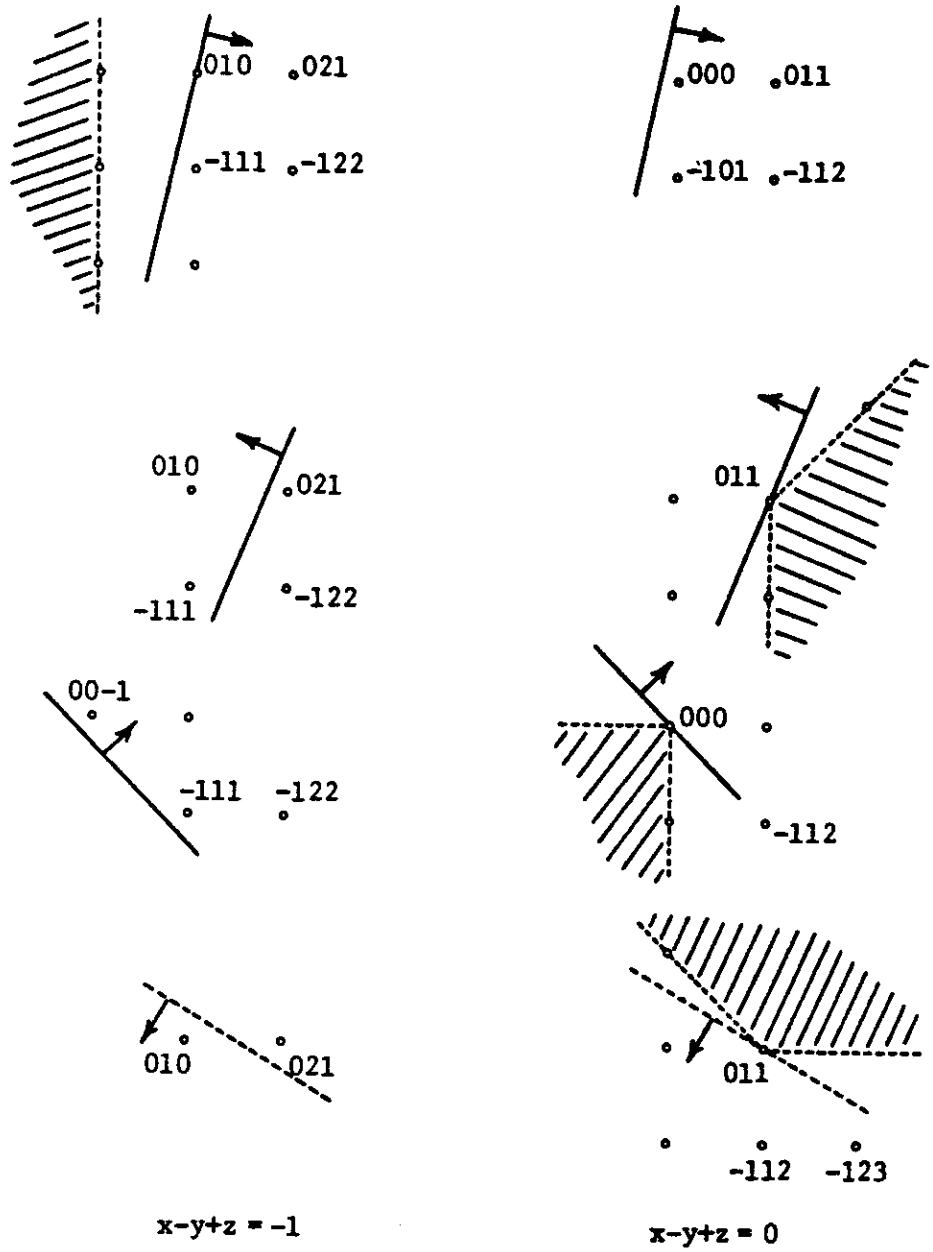

FIGURE 61

The slopes in this Figure permit us to recognize a relaxation which is the parallelogram on the plane $x-y+z = 0$ as in Figure 62. This

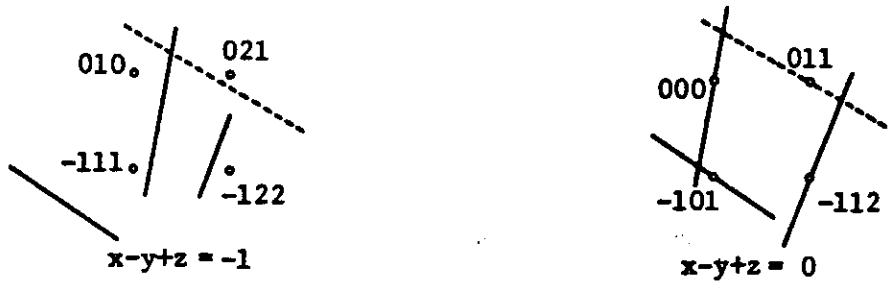

FIGURE 62

parallelogram contains no lattice points on the plane $x-y+z = -1$. In order to verify that this plane is the new characteristic plane we must verify that an adjacent relaxation on $x-y+z = -1$ is free of lattice points on $x-y+z = 0$. It is necessary to distinguish two subcases.

2.1. The plane through $(0,0,0)$ rejects $(-1,1,2)$. In this case there is a parallelogram to the left of the one previously drawn.

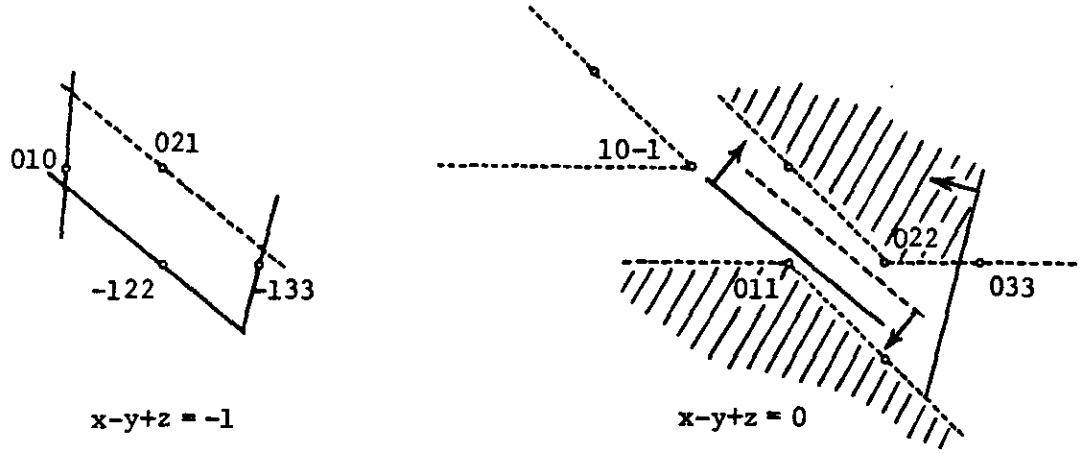

FIGURE 63

We see that the only possible lattice points satisfying these inequalities on the plane $x-y+z = 0$ lie in the wedge with vertex $(1,0,-1)$, i.e. points of the form $(a,-b,-a-b)$ with $a \geq 1$, $b \geq 0$. The following argument shows that this is impossible when the four planes are translated to their position in Figure 63.

a. If the plane through $(0,0,0)$ translated to $(-1,2,2)$ accepts $(a,-b,-a-b)$, then in its original position it accepts $(a+1,-b-2, -a-b-2)$ and therefore accepts $(a,-b-1, -a-b-1)$.

b. If the dashed line through $(0,1,1)$ translated to $(0,2,1)$ accepts $(a,-b,-a-b)$ then in its original position it accepts $(a,-b-1,-a-b)$.

c. Let the plane through $(0,1,0)$ accept $(a,-b,-a-b)$.

These three statements imply that if the constraint planes are drawn in their original position as in Figure 59, then the configuration of Figure 64 will arise on the plane $x = a$. The triangle to the lower

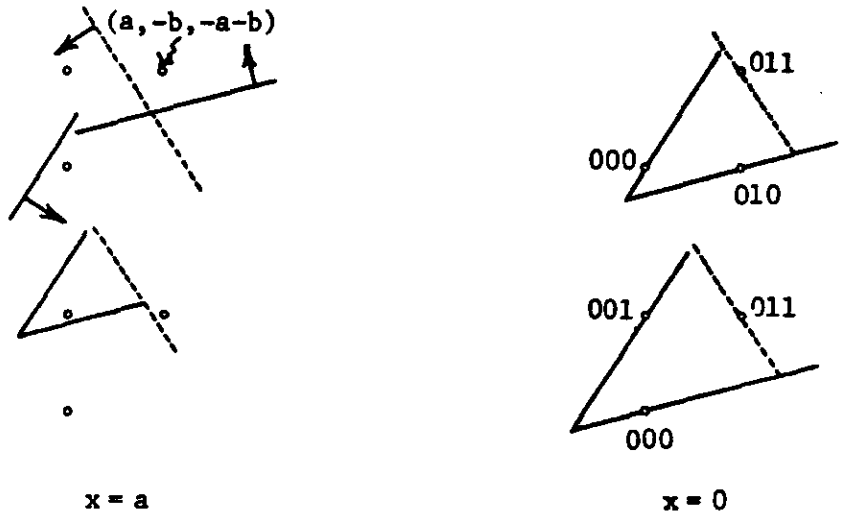

FIGURE 64

right will therefore contain the lattice point $(a,-b-1,-a-b)$ in front immediately after the singularity. Since lattice points are not admitted in front of $x = 0$ in passing through the singularity, we have a contradiction to our assumption that this triangle contains no lattice points in front prior to the singularity. We conclude that $x-y+z = \text{const.}$ is the new characteristic plane in case 2.1.

2.2. The plane through $(0,0,0)$ accepts $(-1,1,2)$. In this case the relaxations to the left of the parallelogram previously drawn are the pair of triangles of Figure 65, which when drawn on the plane $x-y+z = -1$ are free of lattice points on $x-y+z = 0$. This demonstrates that $x-y+z = \text{const.}$ is the new characteristic plane after the singularity.

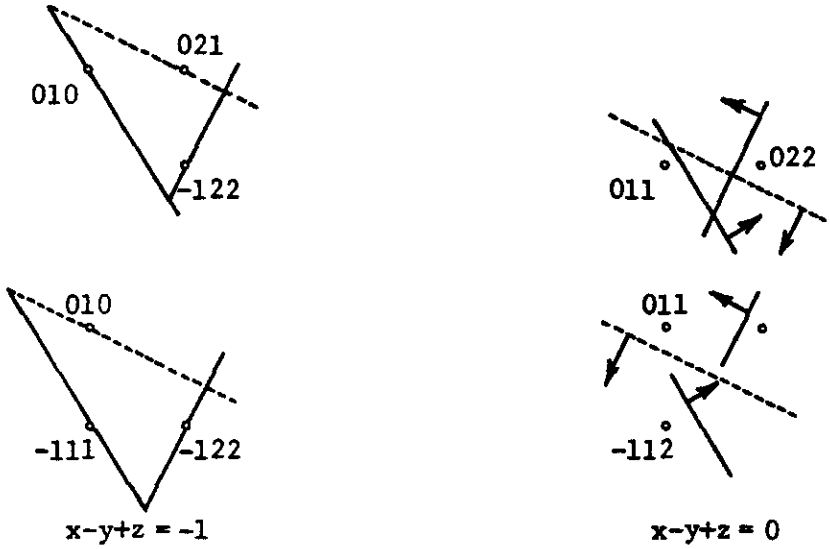

FIGURE 65

Case 3. The plane through (0,1,1) just admits (-1,1,1) . In this case the plane through (0,0,0) must reject (-1,0,0) since if this were not so the other triangle of the pair would contain a lattice point in back prior to the singularity. Moreover the dashed line through (0,1,1) in Figure 59 must reject (1,0,0) and therefore accept (-1,2,2) . We translate this information to the planes y-z = 0, 1 , drawn immediately prior to the singularity. In this figure the plane through (0,1,0) has been drawn with a dashed line since otherwise it may be difficult to distinguish from the plane through (0,0,0) .

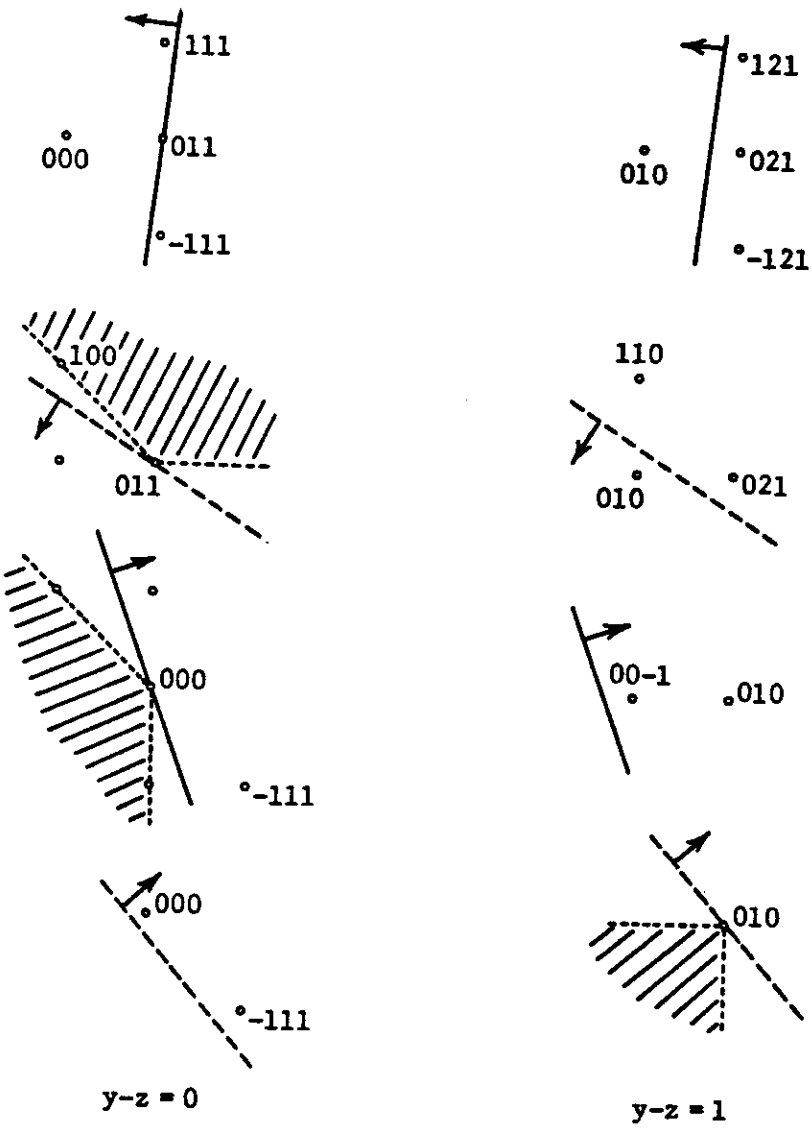

FIGURE 66

A pair of triangular relaxations can immediately be recognized and are drawn on the plane $y-z = 1$ in Figure 67. Neither of these triangles contain a lattice point on the plane $y-z = 0$. In order to conclude that $y-z = \text{const.}$ is a characteristic plane prior to the singularity we distinguish two subcases.

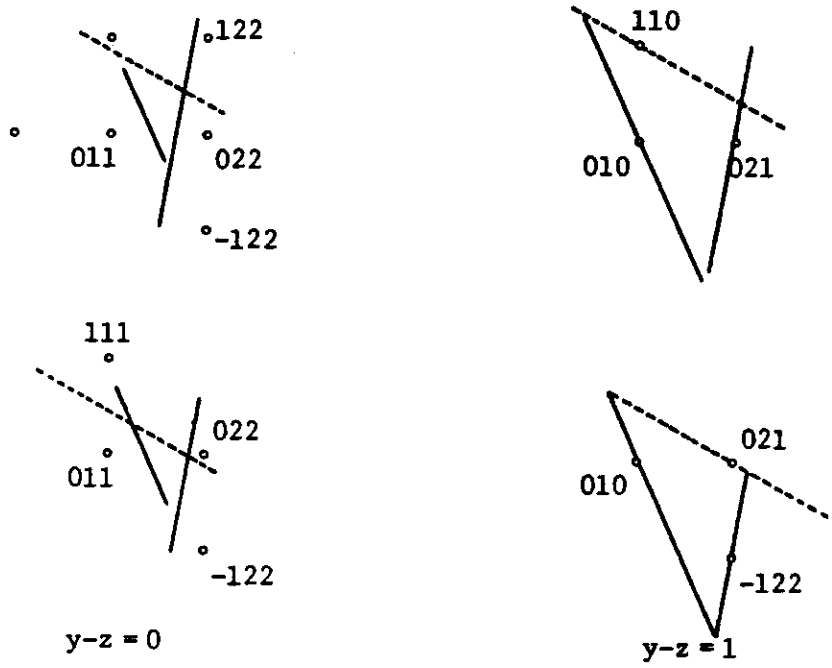

FIGURE 67

3.1. The plane through (0,1,0) admits (1,0,-1) . In this case the relaxation on y-z = 0 , given by the parallelogram of Figure 68 is free of lattice points on the plane y-z = 1 , except possibly in the wedge with vertex (1,0,-1) , i.e. points of the form (a,-b,-b-1)

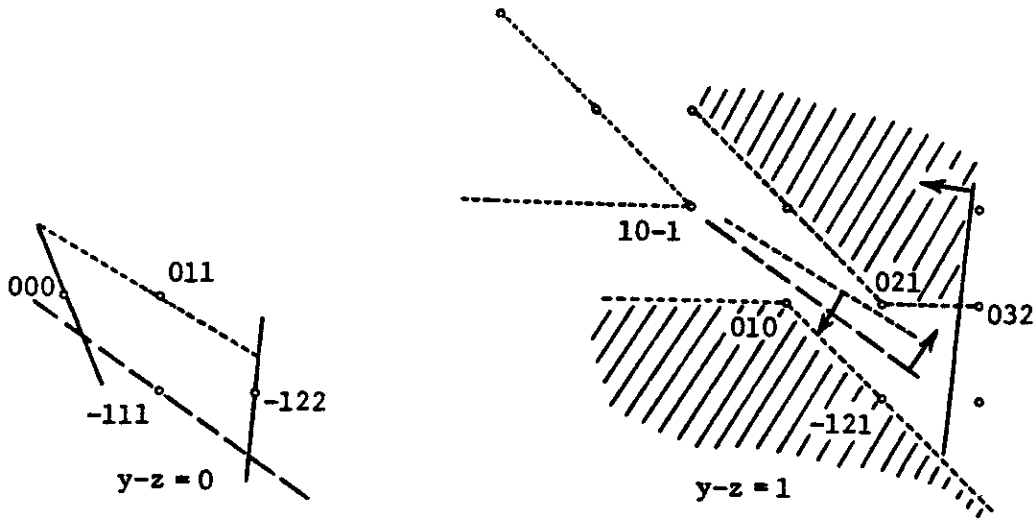

FIGURE 68

with $a \geq 1$, $b \geq 0$. But any such point must be rejected by the plane through (0,0,0) , the plane through (0,1,0) or the dashed plane through (0,1,1) in Figure 59. In making the transition from the position of Figure 59 to that of Figure 68, two of these planes are unchanged and one of them has been pressed in. It follows that $y-z = \text{const.}$ is a characteristic plane prior to the singularity. Since the line of singularities is contained in this plane, it persists as a characteristic plane after the singularity.

3.2. The plane through (0,1,0) rejects (1,0,-1) . In this case there are no parallelograms which appear as relaxations on the plane $y-z = \text{const.}$ In addition to those of Figure 67, there are the pair of trinagles drawn on the plane $y-z = 0$ in Figure 69, which are free of

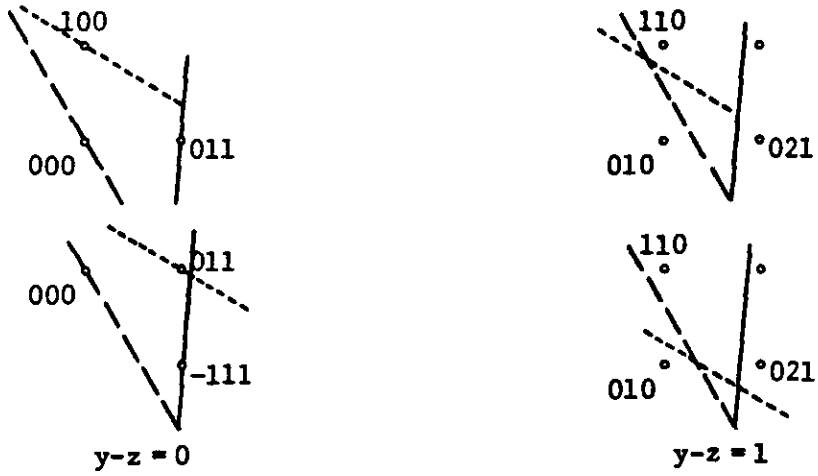

FIGURE 69

lattice points on the plane $y-z = 1$. The plane $y-z = \text{const.}$ is therefore a characteristic plane both before and immediately after the singularity.

We have finally reached the conclusion of this extremely lengthy argument and demonstrated that if the matrix A is perturbed in such a way as to lose its associated characteristic plane, there always will be an alternative

characteristic plane which is available after the perturbation. To conclude that an arbitrary matrix satisfying 2.1 has a characteristic plane it is therefore sufficient to exhibit a specific matrix with this property in order to initiate the perturbations. The reader may wish to construct such an example, or make use of the following simple observation:

Let A have the sign pattern

$$\begin{bmatrix} - & - & - \\ + & - & - \\ - & + & - \\ - & - & + \end{bmatrix},$$

and assume that $\sum_{j=1}^{3} \alpha_{ij} > 0$ for $i = 1, 2, 3$. Then $x = \text{const.}$, $y = \text{const.}$, and $z = \text{const.}$ are all characteristic planes.

X. An Application to Integer Programming

Let us assume that the matrix A has been transformed so that $x = \text{const.}$ is the characteristic plane, and consider the integer program

$$\begin{aligned} \max \quad & a_{01}h_1 + a_{02}h_2 + a_{03}h_3 \\ & a_{11}h_1 + a_{12}h_2 + a_{13}h_3 \geq b_1 \\ & a_{21}h_1 + a_{22}h_2 + a_{23}h_3 \geq b_2 \\ & a_{31}h_1 + a_{32}h_2 + a_{33}h_3 \geq b_3 \\ & h_j \text{ integral.} \end{aligned}$$

We consider the two-variable problem in which the first coordinate has been fixed at a particular value, say $h_1 = a$. When the objective function

is placed at the optimal solution to this problem, and the constraints are drawn on the plane, the resulting region will contain no lattice points in its interior. The inequalities may be relaxed to yield a parallelogram

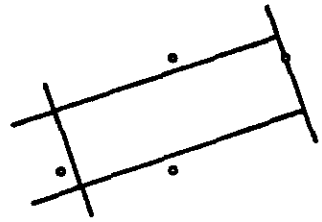

FIGURE 70

in the chain or one of the triangles appearing at either end of the chain.

Since $x = \text{const.}$ is a characteristic plane none of these objects will contain lattice points both in front and in back. If the relaxation is a parallelogram which is free of lattice points on both sides, the solution on the plane $h_1 = a$ is, in fact, the optimal solution to the three variable problem, since there are no lattice points which satisfy the inequalities and yield a higher value of the objective function. If the relaxation has lattice points in front, the optimal solution must satisfy $h_1 \geq a$, since there are no lattice points with $h_1 < a$ which satisfy the inequalities and yield a higher value of the objective. And similarly if the relaxation has lattice points in back the optimal solution must satisfy $h_1 \leq a$.

We conclude that solving the two variable problem on $h_1 = a$ provides us with information as to whether the first coordinate should be increased or decreased in moving to the optimal solution. Perhaps the simplest way to translate this observation into a working algorithm for the three variable problem is by repeated bisection of the range of h_1.

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