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AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut 06520

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INFORMATION ABOUT MOVES IN EXTENSIVE GAMES: II

P. Dubey and M. Kaneko

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by

Pradeep Dubey and Mamoru Kaneko

1. INTRODUCTION

Our first concern in this sequel[1] is to develop a finite version of the non-atomic result (propositions 5, 5*) of [1]. The problem is not completely routine because information is discontinuous with moves. Indeed the example in Section 2 of this paper openly flouts the non-atomic result. It consists of two sequences of games $n\Gamma \underset{n}{\to} \Gamma$, $n\Gamma_c \underset{n}{\to} \Gamma_c$. Here $n\Gamma_c$, Γ_c are obtained from $n\Gamma$, Γ by coarsening information sets; and the finite-player games $n\Gamma$, $n\Gamma_c$ "converge" with n to the non-atomic games Γ, Γ_c. But when the Nash plays of $n\Gamma$, $n\Gamma_c$ are computed they are found to diverge. And thus the non-atomic result of [1], which says in particular that the Nash plays of Γ and Γ_c coincide, is called into question.

The conundrum becomes clear if we notice that, in point of fact, the games $n\Gamma$ do <u>not</u> converge to Γ. This is because of the information conditions. In Γ no single player can affect the integral (average) of bids by the very assumption that he is a point in a non-atomic continuum. In sharp contrast, in every n --no matter how large n is--any unilateral deviation of bids by a player does change the average and can be

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[1]The notation and terminology of [1] is carried over into this paper.

precisely observed by others. The large n case therefore does not reflect the non-atomic assumption as far as information is concerned. Intuitively one would think that that assumption should translate into a finite game to mean: very small changes in the average cannot be observed by anyone. This, in turn, may be imagined to stem from an intrinsic grid on the scales of measurement; or it may simply be thought of as a behavioral postulate of inertia in players' reactions (there is a positive lower bound on the change that must occur for anyone to react). Once we make such a postulate everything falls into place. But, in doing so, we are forced to break away from the standard notion of a Nash Equilibrium (N.E.) and to alter it in order to take this lower bound into account. So we introduce "ε-N.E.'s" in Section 3 (where $\varepsilon \equiv$ the lower bound). Then a finite version of the non-atomic result of [1] becomes available (Proposition 1). It can be transformed into a formal convergence theorem (Proposition 2). The problem here is to construct a natural model of a sequence of extensive games that is "convergent." One such is suggested in Section 4, and it may be of some interest beyond the use to which it is put in this paper.

With the introduction of this bound, however, a cat is now let out of the bag. Consider the standard "convex case," i.e., one in which the set of moves is convex at any position and each player's payoff is concave in his own moves. It includes a large class of "normal form" games, and _a fortiori_ the games obtained by repeating them, provided suitable payoffs are chosen (e.g. discounted sums, lim inf of the average). Then a curious result (Proposition 3) occurs: the set of plays achieved at ε-N.E.'s is independent of the information conditions and of $\varepsilon > 0$. Moreover, this is true regardless of the number of players. The upshot

(Section 5.2) is an "anti-folk theorem" for finite-player games: for any $\varepsilon > 0$, the ε-N.E. plays of the repeated game Γ^∞ coincide with the N.E. plays of the minimal information variant of Γ^∞. Thus a certain delicateness in the folk theorem is brought to light. It dramatically breaks down with the slightest coarsening of information, i.e., making $\varepsilon > 0$, no matter how small.

The bound on a player's capacity of observation so far pertained only to others' moves. It is impelling to extend it to all observation. In Section 6 we impose bounds also on what a player can see of his own strategies and payoffs. Propositions 1 and 2 can essentially be retrieved with the obvious modifications. This time the anti-folk theorem breaks down. But it breaks in a manner which is continuous with these two additional bounds (Proposition 4). If their magnitude is small, we still get a diluted version of it. And, in any case, the folk theorem is far from getting reinstated.

2. AN EXAMPLE

To highlight the problem of the discontinuity of information it might be best to examine an example in detail. There are two types of traders who exchange two commodities through a trading-post. The initial endowments are (1,0) and (0,1) for the two types, and all of them have the same utility function $u(x_1, x_2) = \sqrt{x_1 x_2}$. A move of a trader is to bid a quantity of his commodity for sale in the trading-post. Those of the first type move simultaneously at the start of the game. The second type of traders can find out the average quantity bid of commodity 1 before they make their moves, also simultaneously, i.e., without knowledge of the move of anyone of their own type. The total amount received of commodity 1(2) is then disbursed to traders of type 2(1) in proportion to their bids. If any one type bids a total of zero then no trade takes place and the bids are returned. We examine a replication sequence $\{n\Gamma\}_{n=1}^{\infty}$ where the game $n\Gamma$ has n traders of each type. The limit of this seuqence appears to be the game Γ in which there is a continuum of each type. But this is not quite true because information is discontinuous in going from $n\Gamma$ to Γ, with the result that a direct asymptotic version of Propositions 5 and 5* of [1] cannot be obtained.

Formally the player-set in $n\Gamma$ is $nN = nL \cup nM$, where $nL = \{1, \ldots, n\}$ and $nM = \{n+1, \ldots, 2n\}$. The set of positions is

$$X = \{x_0\} \cup [0,1]^{nL} \cup ([0,1]^{nL} \times [0,1]^{nM}) ;$$

and

$$\pi(x_0) = nL , \quad \pi(x) = nM \quad \text{for all} \quad x \in [0,1]^{nL} ;$$

$$\pi(x) = \phi \quad \text{for} \quad x \in [0,1]^{nL} \times [0,1]^{nM} ;$$

$$S_i^{x_0} = [0,1] \quad \text{for} \quad i \in nL ;$$

$$S_i^{x} = [0,1] \quad \text{for} \quad i \in nM \quad \text{and} \quad x \in [0,1]^{nL} ;$$

$$I_i = \{x_0\} \quad \text{for} \quad i \in nL ;$$

$$I_i = \{\{s^{x_0} \in [0,1]^{nL} : \frac{1}{n}\sum_{i \in nL} s_i^{x_0} = \alpha\} : \alpha \in [0,1]\} \quad \text{for} \quad i \in nM$$

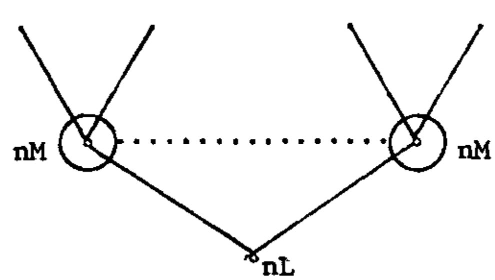

Figure 1. The game $n\Gamma$

For any $z = (s^{x_0}, \{s^x\}_{x\in[0,1]} nL)$ in $Z_F(n\Gamma)$, the final holding ξ_i of player i is determined by $p(z)$ ($= (s^{x_0}, s^{x_1})$ with $x_1 = (s^{x_0})$) by the rule:

$$\xi_i(z) = \begin{cases} \left(1 - s_i^{x_0}, \dfrac{s_i^{x_0}}{\sum\limits_{\ell\in nL} s_\ell^{x_0}} \cdot \sum\limits_{\ell\in nM} s_\ell^{x_1}\right) & \text{if } \sum\limits_{\ell\in nL} > 0 \text{ and } \sum\limits_{\ell\in nM} > 0 ; \\ (1,0) & \text{otherwise} \end{cases}$$

if $i \in nL$; and

$$\xi_i(z) = \begin{cases} \left(\dfrac{s_i^{x_1}}{\sum\limits_{\ell\in nM} s_\ell^{x_1}} \cdot \sum\limits_{\ell\in nL} s_\ell^{x_0}, 1 - s_i^{x_1}\right) & \text{if } \sum\limits_{\ell\in nL} > 0 \text{ and } \sum\limits_{\ell\in nM} > 0 : \\ (0,1) & \text{otherwise} \end{cases}$$

if $i \in nM$. The payoffs to the players are of course the utilities of their final holdings.

$n\Gamma$ has some trivial inactive N.E.'s at which any one type bids nothing and the other type bids arbitrarily. All these lead to the same final allocation (1,0), (0,1) as the initial one.

All other Nash plays of $n\Gamma$ are given by:

s^{x_0} is arbitrary;

$$s_i^x = \begin{cases} \dfrac{n-1}{2n-1} & \text{if } x \in I_i(x_1) , \quad x_1 = (s^{x_0}) \\ 0 & \text{otherwise} \end{cases}$$

if $i \in nM$. Thus the set of Nash allocations (i.e. those produced at some N.E.) of $n\Gamma$ is:

$$\left\{ (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n}) : x_i = \left(1 - \alpha_i, \frac{\alpha_i}{\sum_{i \in nL} \alpha_\ell} \cdot \frac{n(n-1)}{2n-1}\right) \right.$$

$$\text{for } i \in nL \text{ , } x_i = \left(\frac{1}{n} \sum_{i \in nL} \alpha_i, 1 - \frac{n-1}{2n-1}\right) \text{ for } i \in nM \text{ , where}$$

$$\left. \text{the } \alpha_i \in [0,1] \text{ are arbitrary and } \sum_{i \in nL} \alpha_i > 0 \right\} \cup \{\text{initial endowment}$$

Now consider the game $n\Gamma_c$ obtained by assuming that players of type 2 observe nothing, i.e., $I_i = [0,1]^{nL}$ for $i \in nM$.

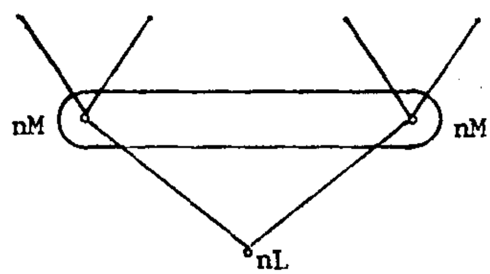

Figure 2. The game $n\Gamma_c$

Besides the inactive N.E.'s, $n\Gamma_c$ has a unique active N.E.:

$$s_i^{x_0} = \frac{n-1}{2n-1} \text{ if } i \in nL \text{ ;}$$

$$s_i^{x} = \frac{n-1}{2n-1} \text{ for all } x \in I_i \text{ , if } i \in nM \text{ .}$$

Thus the Nash allocations of $n\Gamma_c$ are either given by the initial allocation or by:

$$x_i = \begin{cases} \left(1 - \frac{n-1}{2n-1}, \frac{n-1}{2n-1}\right) & \text{if} \quad i \in nL \\ \left(\frac{n-1}{2n-1}, 1 - \frac{n-1}{2n-1}\right) & \text{if} \quad i \in nM \end{cases}$$

Thus it is clear that the N.E.'s of $n\Gamma$ and $n\Gamma_c$ do not converge as $n \to \infty$.

On the other hand, if we look directly at the limit game[1] Γ (in which there is a continuum of each type) then, by Proposition 5 (or 5*) of [1], Γ and Γ_c have the same N.E.'s (in terms of the plays, i.e., allocations, produced). Indeed one can easily compute that these consist of the inactive N.E.'s and the unique active N.E. in which all traders bid 1/2 to wind up with the final holding (1/2, 1/2) (which is also the "competitive allocation" of the underlying economy).

3. BOUNDED CAPACITY OF OBSERVATION

(with respect to others' moves)

The example of Section 2 shows us that we need to do some "doctoring" to the Nash plays of $n\Gamma$ in order to develop an asymptotic version of the non-atomic result. In Γ an arbitrary unilateral change in strategy by a player does not affect the integral of moves anywhere and therefore it cannot be observed by others. We translate this condition into a finite setting quite directly to mean: very small changes made by a single player (say of "size" less than ε for some positive ε), cannot be seen by others. Then we are lead to the notion of an "ε-N.E.," which is simply an N.E. modified by this constraint on observation.

To formulate this precisely let Γ now stand for an arbitrary extensive game as in Section 2 of [1]:

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[1]Which can be formalized in the obvious way.

$$\Gamma = (N, X, \pi, \{S^x\}_{x \in X}, \Phi, \{h_i\}_{i \in N}, \{I_i\}_{i \in N}) .$$

Throughout we assume that N is finite. W.l.o.g. (see Section 2.4 of [1]), there are no ending positions in Γ. There is a metric space D with distance denoted d, and each S_i^x $(x \in X, i \in \pi(x))$ is a non-empty subset of D. Furthermore the fictitious paths attached to ending positions (see again Section 2.4 of [1]) are assumed to be identical, with the same move at each step, whose distance from any "real" move in Γ is ∞. Define the metric δ on X by the rule:

if $x = (s^{x_0}, s^{x_1}, \ldots, s^{x_m})$ and $y = (r^{y_0}, r^{y_1}, \ldots, r^{y_k})$ (with $x_0 \equiv y_0$), then:

$$\delta(x,y) = \begin{cases} \max\limits_{0 \leq \ell \leq m} \sum\limits_{i \in \pi(x_\ell)} d(s_i^{x_\ell}, r_i^{y_\ell}) & \text{if } m = k \text{ and } \pi(x_\ell) = \pi(y_\ell) \\ & \text{for } \ell = 0, 1, \ldots, m ; \\ \infty & \text{otherwise} \end{cases}$$

Let $p = (s^{x_0}, s^{x_1}, \ldots, s^{x_m}, \ldots)$ and $\tilde{p} = (r^{y_0}, r^{y_1}, \ldots, r^{y_m}, \ldots)$ be two plays. Then

$$\delta^*(p,\tilde{p}) = \sup_{\ell=0,1\ldots} \delta(x_\ell, y_\ell)$$

constitutes a metric on $P(\Gamma)$.

Let $z = \{s^x\}_{x \in X} \in Z_F(\Gamma)$ and let $p(z) = (s^{x_0}, s^{x_1}, \ldots, s^{x_m}, \ldots)$. For $\varepsilon > 0$ and $t_i \in S^i$, $p^\varepsilon(z|t_i) = (r^{y_0}, r^{y_1}, \ldots, r^{y_m}, \ldots)$ is, intuitively speaking, the play that results from player i's deviation to t_i, <u>after</u> accounting for the fact that the others cannot observe any change (in positions) of distance less than ε. Precisely, $(r^{y_0}, \ldots, r^{y_m}, \ldots)$ is defined as follows (with $x_0 \equiv y_0$):

(2.1) If $i \in \pi(y_\ell)$ then $r_i^{y_\ell} = t_i(y_\ell)$, for all $\ell = 0, 1, \ldots$

(2.2) For all $j \neq i$, $j \in \pi(y_k)$ and any $k = 0, 1, \ldots$:

$$r_j^{y_k} = \begin{cases} s_j^{x_k} & \text{if (a) } \delta(x^k, y^k) < \varepsilon \\ & \text{(b) for } 0 \leq \ell \leq k,\ s_j^{x_\ell} \in S_j^{y_\ell} \text{ whenever} \\ & \qquad j \in \pi(x_\ell) \\ s_j(y_k) & \text{otherwise} \end{cases}$$

Note that (2.2) is compatible with the assumption that player j has perfect recall.

Consider any $\varepsilon > 0$. A $z = \{s^x\}_{x \in X} \in Z_F(\Gamma)$ will be called an <u>ε-modified Nash Equilibrium</u> of Γ, and denoted <u>ε-N.E.</u>, if:

$$h_i(p^\varepsilon(z|t_i)) \leq h_i(p(z))$$

for all $t_i \in S^i$ and all $i \in N$.

Next we need to define the notion of an <u>ε-inner play</u> of Γ for $\varepsilon > 0$. This is a play $p = (s^{x_0}, s^{x_1}, \ldots, s^{x_m}, \ldots)$ in $P(\Gamma)$ with the property that for all $\ell = 0, 1, \ldots$

$$\left.\begin{cases} (x_\ell, y) < \varepsilon \\ i \in \pi(x_\ell) \end{cases}\right\} \Rightarrow s_i^{x_\ell} \in S_i^y .$$

In other words, the move employed by any player in p remains feasible for small changes in positions around p.

If z is an ε-N.E., and if $p(z)$ is an ε-inner play, then we shall say that z is a <u>(ε)-N.E.</u>

<u>Proposition 1</u>. Assume (a) z is an (ε)-N.E. of Γ; (b) $\delta^*(p^\varepsilon(z|t_i), p(z)) < \varepsilon$ for all $t_i \in S^i$ and $i \in N$. Then $p(z)$ is a (ε)-N.E. play of $\tilde{\Gamma}$ for every[1] $\tilde{\Gamma} \in \Delta(\Gamma)$.

<u>Proof</u>. Let $z = \{s^x\}_{x \in X}$. Since N is finite, $F(\tilde{\Gamma}) = P(\Gamma\_)$, i.e., there exists $z = \{\tilde{s}^x\}_{x \in X} \in F(\tilde{\Gamma})$ such that $p(\tilde{z}) = p(z)$. It suffices to show that $\tilde{z}$ is an (ε)-N.E. of $\tilde{\Gamma}$. If this were not so then for some $\tilde{t}_i \in \tilde{S}^i$ (where $\tilde{S}^i$ denotes the strategy set of i in $\tilde{\Gamma}$) we would have $h_i(p^\varepsilon(\tilde{z}|\tilde{t}_i)) > h_i(p(\tilde{z}))$. Let $\{I_i^j\}_{j=1}^K$, where K could be ∞, be the information sets of i through which the play $p^\varepsilon(\tilde{z}|\tilde{t}_i)$ passes. Define $t_i \in S^i$ by

$$t_i(x) = \begin{cases} \tilde{t}_i(x) & \text{if } x \in I_i^j \text{ for some } j \\ \text{arbitrary otherwise.} \end{cases}$$

We will prove that $p^\varepsilon(\tilde{z}|\tilde{t}_i) = p^\varepsilon(z|t_i)$. Put $p^\varepsilon(\tilde{z}|\tilde{t}_i) = (s^{x_0}, s^{x_1}, \ldots, s^{x_m}, \ldots)$, $p^\varepsilon(z|t_i) = (r^{y_0}, r^{y_1}, \ldots, r^{y_m}, \ldots)$ and $p(z) = (w^{z_0}, w^{z_1}, \ldots, w^{z_m}, \ldots)$ where $x_0 \equiv y_0 \equiv z_0$. Make the inductive assumption that $x_\ell = y_\ell$ for $\ell \le k$. (For $\ell = 1$ this is obvious.) By (b) $\delta(z_k, y_k) < \varepsilon$, hence also $\delta(z_k, x_k) < \varepsilon$. Also by (b), $w_j^{z_\ell} \in S_j^{y_\ell}$ $(= S_j^{x_\ell})$ for all $\ell \le k$ and $j \in \pi(z_\ell)$ $(= \pi(y_\ell) = \pi(x_\ell))$. By the definition of $p^\varepsilon(\ldots)$ we have $s_j^{x_k} = w_j^{z_k}$ and $r_j^{y_k} = w_j^{z_k}$,

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[1] Recall that $\Delta(\Gamma)$ is the class of all games obtained by varying only the information sets of Γ (Section 3 of [1]).

therefore $s_j^{x_k} = r_j^{y_k}$, for $j \in \pi(z_k) \setminus \{i\}$. If $i \in \pi(z_k) = \pi(y_k) = \pi(x_k)$ then $\tilde{t}_i(x_k) = t_i(x_k) = t_i(y_k)$. Thus $s^{x_k} = r^{y_k}$ which proves that $x_{k+1} = y_{k+1}$.

But then we have $h_i(p^\varepsilon(z|t_i)) = h_i(p^\varepsilon(\tilde{z}|\tilde{t}_i)) > h_i(p(\tilde{z})) = h_i(p(z))$ contradicting that z is a (ε)-N.E. of Γ.

Q.E.D.

It is often the case that an ε-N.E. is also automatically a (ε)-N.E. In particular this is so for <u>layered games</u> $\Gamma = (N, X, \ldots)$, i.e. those in which, for any fixed ℓ, $\pi(x)$ is constant and so is S_i^x $(i \in \pi(x))$ for $x \in X_\ell$ = the <u>ℓ^{th}-layer</u> of $\Gamma = \{x \in X : x$ is at a distance[1] ℓ from the start x_0 of $\Gamma\}$. Clearly in a layered game every play is ε-inner for any ε, thus its ε-N.E.'s and (ε)-N.E.'s coincide.

<u>Denote the (ε)-N.E. plays of Γ by</u> $\eta_\varepsilon(\Gamma)$. Then, mimicking the proof of Proposition 2 in [1], we see that:

(2.3) If[2] $\Gamma \dashv \tilde{\Gamma}$ then $\eta_\varepsilon(\Gamma) \subset \eta_\varepsilon(\tilde{\Gamma})$.

Thus, under refinement of information, the (ε)-N.E. plays grow in a nested manner. Proposition 1 gives conditions under which the growth is stopped.

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[1] If $y = (s^{x_0}, \ldots, s^{x_\ell})$ it is said to be <u>at a distance</u> $\ell+1$ from x_0.

[2] I.e., if $\tilde{\Gamma}$ is obtained by refining information sets of Γ (see Section 2 of [1]).

4. CONVERGENCE

Proposition 1 already has within it the germ of a convergence result. Indeed the only problem is to precisely formulate the notion of a "convergent sequence" of extensive games. For simplicity we will focus on the case of replication, i.e., when the r^{th} game has r clones of each of the players of the first, original game. Also players will be assumed to observe the average of others' moves. (See, however, end of Section.)

Fix $\hat{\Gamma} = (N, \hat{X}, \pi, \{\hat{S}^x\}_{x\in X}, \Phi, \{h_i\}_{i\in N}, \{I_i\}_{i\in N})$. A sequence[1] $p_i = (r_i^{x_0}, r_i^{x_1}, \ldots, r_i^{x_m}, \ldots)$, where $i \in N$, will be called compatible with the play $p = (s^{y_0}, s^{y_1}, \ldots, s^{y_m}, \ldots) \in P(\Gamma\_)$ if:

(3.1) $\quad x_\ell = y_\ell$ for all $\ell \geq 0$;

(3.2) $\quad$ for all $\ell \geq 0$, $r_i^{x_\ell} \in S_i^{x_\ell}$.

Define $P_i = \{(p, p_i) : p \in P(\Gamma\_)$ and p_i is compatible with $p\}$.

In order to define replication we need to assume that there are functions $\bar{h}_i : P_i \to \mathbb{R}$ which are extensions of h_i in the sense:

(3.3) $$\bar{h}_i((s^{x_0}, s^{x_1}, \ldots, s^{x_m}, \ldots), (s_i^{x_0}, s_i^{x_1}, \ldots, s_i^{x_m}, \ldots))$$
$$= h_i((s^{x_0}, s^{x_1}, \ldots, s^{x_m}, \ldots)) .$$

We further assume

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[1]Let us adopt the convention: if $i \notin \pi(x)$ then (a) $S_i^x = \{\phi\}$ and, accordingly, (b) $s_i^x = \phi$ for $s^x \in S^x$. Since we have eliminated end-points, this should cause no confusion with the earlier notation of [1], "$\pi(x) = \phi \Longleftrightarrow S^x = \phi$" (which was used to indicate end-points).

(3.4) each $\hat{S}_i^x$ $(i \in \pi(x))$ is a convex set of some fixed normed vector space V (over the reals).

$\hat{\Gamma}$ will serve as the ambient game within which replication is defined. First suppose

(3.5) a subset $S_i^x \subset \hat{S}_i^x$ is exogenously specified for each $x \in \hat{X}$ and $i \in \pi(x)$. (S_i^x could well be finite.)

The game Γ (to be replicated) is <u>embedded</u> in $\hat{\Gamma}$, i.e., it is obtained from $\hat{\Gamma}$ by reducing the moves to S_i^x :

$$\Gamma = (N, X, \pi, \{S^x\}_{x \in X}, \Phi, \{h_i\}_{i \in N}, \{I_i\}_{i \in N}) .$$

Here $X \subset \hat{X}$; and π , Φ , h_i and I_i are to be viewed as defined by the appropriate restriction.

The r^{th} <u>replica game</u> $r\Gamma$ (with $1\Gamma \equiv \Gamma$) will be denoted:

$$r\Gamma = (rN, rX, r\pi, \{rS^x\}_{x \in rX}, r\Phi, \{rh_i\}_{i \in rN}, \{rI_i\}_{i \in rN}) .$$

The extensive form $r\Gamma_- = (rN, \ldots, r\Phi)$ is given by the condition that there exist a map $r\beta$ from rX to $\hat{X}$ satisfying (for x , y in rX):

(3.6) $rN = \bigcup_{i \in N} [i]_r$ where $[i]_r = \{(i,1), \ldots, (i,r)\}$;

(3.7) $r\pi(x) = \bigcup_{i \in \pi \cdot r\beta(x)} [i]_r$;

(3.8) $rS^x_{(i,t)} = S_i^{r\beta(x)}$ for $i \in \pi \cdot r\beta(x)$ and $1 \leq t \leq r$;

(3.9) $x \succ_r y \Longleftrightarrow r\beta(x) \succ r\beta(y)$, and $r\beta(rx_0) = x_0$;

(3.10) if $s^x \in S^x$ then $r\beta(x, s^x) = (\hat{x}, w^{\hat{x}})$, where $r\beta(x) = \hat{x}$

and $w^{\hat{x}} = \left\{\frac{1}{r}\sum_{t=1}^{r} s^x_{(i,t)}\right\}_{i\in\pi\cdot r\beta(x)}$.

Here $\succ$, $\succ_r$ is the partial order (see Section 2 of [1]) induced by Φ , $r\Phi$ on $\hat{X}$, rX . Also

$$\left\{\frac{1}{r}\sum_{t=1}^{r} s^x_{(i,t)}\right\}_{i\in\pi\cdot r\beta(x)} \in \prod_{i\in\pi\cdot r\beta(x)} S^{r\beta(x)} .$$

That such a $r\Gamma\_$ exists and is unique can be verified by building up $r\Gamma\_$ layer by layer, starting from the 0^{th} layer, and using (3.7)-(3.10). At the start rx_0 of $r\Gamma$ the player-set is $r\pi(rx_0) = \bigcup_{i\in\pi(x_0)} \{(i,1), \ldots, (i,r)\}$ by (3.9) and (3.7). Each $S^{x_0}_{(i,t)} = S^{x_0}_i$ by (3.8). For any choice of moves by the player in $r\pi(rx_0)$, the average $\frac{1}{r}(\ldots)$ yields a point in $\prod_{i\in\pi(x_0)} S^{x_0}_i$ (since each $S^{x_0}_i$ is convex), i.e., it yields a position $\hat{x}$ in $\hat{X}_1$. Now $\pi(\hat{x})$ cloned r times furnishes (by (3.7) and (3.10)) the set of players who move at $(s^{x_0}_{(i,t)})_{(i,t)\in r\pi(rx_0)} \in r\beta^{-1}(\hat{x}) \in rX_1$ in $r\Gamma$, etc.

It remains to define the payoffs and information sets of $r\Gamma$. Note that $r\beta$ yields a map $r\beta^* : P(r\Gamma\_) \to P(\hat{\Gamma}\_)$ by the rule:

$$r\beta^*(r^{y_0}, r^{y_1}, \ldots, r^{y_m}, \ldots) = (s^{x_0}, s^{x_1}, \ldots, s^{x_m}, \ldots)$$

if $r\beta(r^{y_0}, \ldots, r^{y_\ell}) = (s^{x_0}, \ldots, s^{x_\ell})$ for $\ell = 0, 1, \ldots$.

(Here, of course, $y_0 \equiv rx_0$ by assumption.)

Then if $p = (s^{rx_0}, s^{x_1}, \ldots, s^{x_m}, \ldots) \in P(r\Gamma\_)$

$$(3.11) \qquad rh_{(i,t)}(p) = \bar{h}_i(r\beta^*(p),\ (s^{x_0}_{(i,t)},\ s^{x_1}_{(i,t)}, \ldots, s^{x_m}_{(i,t)}, \ldots)) \ .$$

Finally, for $y \in rX$, put $rI'_{(i,t)} = \{x \in rX : r\beta(x) = I_i(r\beta(y))\}$. Then $\{rI_{(i,t)}(x)\}_{x \in rX}$ is the coarsest refinement of $\{rI'_i(x)\}_{x \in rX}$ which satisfies the condition of perfect recall ((2.6) of [1]). That this exists and is unique can be seen, for example, by repeating the proof of Proposition 1 of [1]. This completes the definition of $r\Gamma$.

Define δ on $\hat{X}$ as in Section 2, using the norm $\| \ \|$ on V . This in turn gives us metrics $r\delta$ on rX $(r = 1, 2, \ldots)$ by

$$r\delta(x,y) = \delta(r\beta(x),\ r\beta(y)) \ .$$

For the convergence result we will require uniform boundedness of moves

$$(3.14) \qquad \sup\{\|\alpha\| : \alpha \in \hat{S}^x_i \text{ for } x \in \hat{X} \text{ and } i \in \pi(x)\} \leq B < \infty \ .$$

<u>Proposition 2</u>. Suppose $\hat{\Gamma}$ satisfies (3.14) and Γ is embedded in $\hat{\Gamma}$. Then if $r > B/\varepsilon$ the plays produced at (ε)-N.E.'s of $\tilde{\Gamma}$ are invariant of $\tilde{\Gamma} \in \Delta(r\Gamma)$.

<u>Proof</u>. Straightforward, using Proposition 1.

The example in Section 2 is a layered game. To apply Proposition 2 in its context, note that we can take $B = 1$. Thus we see that if $n > 1/\varepsilon$, the ε-N.E. plays of $n\Gamma$ and $n\Gamma_c$ coincide.

Neither the finite-type assumption on players, nor the fact that they can observe only averages, seem crucial to a convergence result. In general one would start with an ambient <u>non-atomic</u> game in which every finite game of the sequence is embedded. The observation could be on the distribution of strategies. Proposition 1 lends itself to this general set up also. We have not, however, worked out a detailed picture.

5. THE CONVEX CASE

5.1. Primitive Nash Plays

Although our setting will remain general, the definitions and results of this section are perhaps best illustrated for layered games, to which we will refer throughout.

Put $C_i(\Gamma) = \{q : q$ is compatible with $\tilde{p}$ for some $\tilde{p} \in P(\Gamma)\}$. Consider $q = (r_i^{y_0}, r_i^{y_1}, \ldots, r_i^{y_m}, \ldots)$ in $C_i(\Gamma)$ and $p = (s^{x_0}, s^{x_1}, \ldots, s^{x_m}, \ldots)$ in $P(\Gamma)$. We will say that the pair (q,p) is play-producing if (intuitively) the moves of i given by q , along with those of others as given by p , together suffice to produce a play in $P(\Gamma)$. Precisely, we require:

(5.1) There exists a play $(w^{z_0}, w^{z_1}, \ldots, w^{z_m}, \ldots)$ in $P(\Gamma)$ such that (for $\ell = 0, 1, \ldots$)

(a) $j \in \pi(z_\ell) \setminus \{i\} \Rightarrow w_j^{z_\ell} = s_j^{x_\ell}$

(b) $i \in \pi(z_\ell) \Rightarrow w_i^{z_\ell} = r_i^{y_\ell}$.

Such a play, if it exists, is necessarily unique and will be denoted $p \oplus q$. Define, for $p = (s^{x_0}, s^{x_1}, \ldots, s^{x_\ell}, \ldots) \in P(\Gamma)$

(5.2) $C_i(p) = \{q \in C_i(\Gamma) : (q,p)$ is play-producing$\}$.

Note

(5.3) $C_i(p)$ is not empty because it always contains

$(s_i^{x_0}, s_i^{x_1}, \ldots, s_i^{x_m}, \ldots)$;

(5.4) if Γ is a layered game, then for any $\tilde{p} \in P(\Gamma)$

$$C_i(\tilde{p}) = C_i(\Gamma) = \mathop{\times}_{\ell=0}^{\infty} S_i^{x_\ell} .$$

Recall here our convention: if $i \notin \pi(x_\ell)$ then $S_i^{x_\ell} = \{\phi\}$; and $s_i^{x_\ell} = \phi$ for $s^{x_\ell} \in S^{x_\ell}$. Identify ϕ with the origin of some vector space $W \neq V$. Then each $S_i^{x_\ell} \subset V$ or W , hence $C_i(p)$ may be viewed as a subset of the vector space $\mathop{\times}_{\ell=0}^{\infty} U^\ell$, where $U^\ell = V$ if $i \in \pi(x_\ell)$ and $U^\ell = W$ if $i \notin \pi(x_\ell)$. We assume, for $p \in P(\Gamma)$ and $i \in N$:

(5.5) $C_i(p)$ is convex

(5.6) $\hat{h}_i : C_i(p) \to \mathbb{R}$, given by $\hat{h}_i(q) = h_i(p \oplus q)$, is concave.

Note that (5.5) holds if Γ is a layered game and each S_i^x (for $x \in X$, $i \in \pi(x)$) is a convex subset of V .

A play $p \in P(\Gamma)$ will be called a primitive Nash play if, for any $i \in N$,

(5.7) $h_i(p) \geq h_i(p \oplus q)$ for $q \in C_i(p)$.

Note that primitive Nash plays are, by definition, invariant of the information condition. Thus Proposition 3 below shows that, if $\varepsilon > 0$, ε-N.E. plays of convex games do not proliferate with refinement of information even in the finite-player case.

Proposition 3. Assume that (3.14), (5.5) and (5.6) hold for Γ . For $\varepsilon > 0$ let p be an ε-N.E. play for Γ . Then p is a primitive Nash play.

Proof. If p is not a primitive Nash play, then

$$h_i(p \oplus q) > h_i(p)$$

for some $q \in C_i(p)$. By (5.5), (5.6) and (5.3),[1]

$$(5.8) \qquad h_i(\lambda(p \oplus q) + (1-\lambda)p) \geq h_i(p \oplus q) + (1-\lambda)h_i(p) > h_i(p) ,$$

for any $0 < \lambda < 1$. Choose $\tilde{\lambda}$ sufficiently close to 0 to ensure that:

$$(5.9) \qquad \delta^*(\tilde{\lambda}(p \oplus q) + (1-\tilde{\lambda})p, p) < \varepsilon .$$

This can be done because of (3.14). Suppose $q = (r_i^{y_0}, \ldots, r_i^{y_m}, \ldots)$ and $p = (s^{x_0}, \ldots, s^{x_m}, \ldots)$. Then, by (5.5), $\tilde{q} = (\tilde{\lambda}r_i^{y_0} + (1-\tilde{\lambda})s_i^{x_0}, \ldots, \tilde{\lambda}r_i^{y_m} + (1-\tilde{\lambda})s_i^{x_m}, \ldots)$ is in $C_i(p)$; and clearly $(p \oplus \tilde{q}) = \tilde{\lambda}(p \oplus q) + (1-\tilde{\lambda})p$ Let $(p \oplus \tilde{q}) = (w^{z_0}, \ldots, w^{z_m}, \ldots)$. Construct a strategy t_i of player i as follows:

$$t_i(x) = \begin{cases} \tilde{\lambda}r_i^{y_\ell} + (1-\tilde{\lambda})s_i^{x_\ell} \text{ if } x = z_\ell \\ \\ \text{arbitrary otherwise.} \end{cases}$$

Denoting by $z \in Z_F(\Gamma)$ the choice of strategies that produced $p = p(z)$, we see that, by (5.9),

$$p^\varepsilon(z|t_i) = \tilde{\lambda}(p \oplus q) + (1-\tilde{\lambda})p .$$

---

[1] (5.3) implies $p = p \oplus p_i$ where $p_i = (u_i^{x_0}, u_i^{x_1}, \ldots, u_i^{x_m}, \ldots)$ for $p = (u^{x_0}, u^{x_1}, \ldots, u^{x_m}, \ldots)$.

Then, by (5.8),

$$h_i(p^{\varepsilon}(z|t_i)) > h_i(p)$$

contradicting that z is an ε-N.E. of Γ .

Q.E.D.

Corollary. Let Γ be a layered game satisfying (3.14), (5.5) and (5.6). Denote the set of its primitive Nash plays of $N(\Gamma)$. Then the set of ε-N.E. plays of $\tilde{\Gamma}$ is invariant of $\tilde{\Gamma} \in \Delta(\Gamma)$ and of $\varepsilon > 0$, and coincides with $N(\Gamma)$, i.e.,

$$\eta_{\varepsilon}(\tilde{\Gamma}) = N(\Gamma)$$

for all $\tilde{\Gamma} \in \Delta(\Gamma)$, all $\varepsilon > 0$.

Proof. First observe that $N(\Gamma_1) = N(\Gamma_2)$ for $\Gamma_1, \Gamma_2 \in \Delta(\Gamma)$ because the definition of primitive Nash plays does not depend on the information pattern. Hence, in particular,

$$N(\tilde{\Gamma}) = N(\Gamma_{c*}) \quad \text{for} \quad \tilde{\Gamma} \in \Delta(\Gamma)$$

where (recall) Γ_{c*} is the game with the coarsest information in $\Delta(\Gamma)$ (see Proposition 1 of [1]). By Proposition 3, $\eta_{\varepsilon}(\tilde{\Gamma}) \subset N(\tilde{\Gamma})$, thus

$$\eta_{\varepsilon}(\tilde{\Gamma}) \subset N(\Gamma_{c*}) \quad \text{for} \quad \tilde{\Gamma} \in \Delta(\Gamma) .$$

By (2.3), $\eta_{\varepsilon}(\Gamma_{c*}) \subset \eta_{\varepsilon}(\tilde{\Gamma})$. So it suffices to show: $N(\Gamma_{c*}) \subset \eta_{\varepsilon}(\Gamma_{c*})$ for any $\varepsilon > 0$. This is straightforward.

Q.E.D.

This can be applied to the example of Section 2. We get $\eta_\varepsilon(n\Gamma) = N(\Gamma) = \eta_\varepsilon(n\Gamma_{c*})$ for all n and all $\varepsilon > 0$. Thus replication was not really necessary for an asymptotic version of the non-atomic result!

Since Γ is layered, there is a coase game $\Gamma_{\bar{c}}$, which violates perfect recall and is not admitted in $\Delta(\Gamma)$, but is nevertheless useful to keep in mind. The information sets of player i in $\Gamma_{\bar{c}}$ are the sets $\{x \in X_k : i \in \pi(x)\}$, $k = 0, 1, \ldots$. Clearly Γ_{c*} is a refinement of $\Gamma_{\bar{c}}$. It can also be directly checked that:

$$\eta_\varepsilon(\Gamma_{c*}) = \eta_\varepsilon(\Gamma_{\bar{c}})$$

for all $\varepsilon \geq 0$ (note: $\varepsilon = 0$ is permitted here). Even more, we can easily establish:

$$\eta_\varepsilon(\Gamma_{c*}) = \eta_\varepsilon(\Gamma_{\bar{c}}) = \eta_0(\Gamma_{\bar{c}}) \tag{5.10}$$

for all $\varepsilon \geq 0$. Hence if Γ, besides being layered, satisfies (3.14), (5.5), (5.6) we get:

$$\eta_\varepsilon(\Gamma) = \eta_0(\Gamma_{\bar{c}}) \quad \text{for all} \quad \varepsilon > 0 . \tag{5.11}$$

5.2. <u>The Anti-Folk Theorem for Finite Games</u>

Let Γ be a normal form game, i.e., $\pi(x_0) = N$ and $\pi(s^{x_0}) = \phi$ for all $s^{x_0} \in S^{x_0}$. Assume, for any $i \in N$ and any $s^{x_0} \in S^{x_0}$:

(a) $S_i^{x_0}$ is convex

(b) The map $f : S_i^{x_0} \to \mathbb{R}$ given by $f(t_i) = h_i(s^{x_0}|t_i)$ is concave.

Denote by Γ^∞ the infinite repetition of Γ. We assume that each player can observe at least the entire past history of his own moves and payoffs.[1] Beyond this the information pattern on Γ^∞ is arbitrary; in particular it can be maximal. The payoffs in Γ^∞ are either discounted sums or lim inf of the average.

Thus Γ^∞ satisfies all the hypotheses of the corollary in Section 5.1. Hence, by (5.11),

$$\eta_\varepsilon(\Gamma^\infty) = \eta_0(\Gamma^\infty_{\overline{c}})$$

which is the anti-folk theorem. In the case of discounted sums as payoffs, $\eta_0(\Gamma^\infty_{\overline{c}})$ can be sharply characterized by:

$$(s^{x_0}, s^{x_1}, \ldots, s^{x_m}, \ldots) \in \eta_0(\Gamma^\infty_{\overline{c}}) \Longleftrightarrow \text{each } s^{x_\ell} \text{ is an N.E. of } \Gamma, \quad \ell = 0, 1, \ldots .$$

6. BOUNDS ON ALL OBSERVATIONS

It is impelling to consider the case when the bound on a player's capacity of observation pertains not only to (*) others' moves, but also to (**) his own payoffs and (***) his own strategies.

To take (***) into account we could make the implicit assumption that $d(\alpha,\beta) > \varepsilon$ for distinct α, β in S_i^x.

In the presence of (**), and without the constraint of (*) ((***) may, but need not, be in the background), we have: $z = \{s^x\}_{x\in X} \in Z_F(\Gamma)$ is a $[\mathfrak{d}]$-N.E. of Γ if

[1] For moves this is required by perfect recall (see (2.6) of [1]); for payoffs, this is a natural extension of (2.6) in the context of repeated games. We take (2.6) to be a fundamental defining property of extensive games as explained in [1], even though the milder assumption (2.7) of [1] would suffice (from a technical stand-point) for all the results of parts I and II except Proposition 1 of [1].

(6.1) $$h_i(p(z|t_i)) \geq h_i(p(z)) + \tilde{\varepsilon}$$

for all $t_i \in S^i$ and $i \in N$. If the ε-constraint of (*) (as in Section 3) is added, then we are left with $\underline{[\tilde{\varepsilon}]\text{-}(\varepsilon)\text{-N.E.}}$'s which is defined exactly as in (6.1) but with $p(z|t_i)$ replaced by $p^{\varepsilon}(z|t_i)$ and the added requirement that z be ε-inner. The interpretation of these modified N.E.'s is clear.

Denote the set of plays produced at (ε_1)-N.E.'s, $[\varepsilon_2]$-N.E.'s, $[\varepsilon_2]$-(ε_1)-N.E.'s of Γ by $\eta_{\varepsilon_1}(\Gamma)$, $\eta^{\varepsilon_2}(\Gamma)$, $\eta_{\varepsilon_1}^{\varepsilon_2}(\Gamma)$. ($\eta_{\varepsilon_1}(\Gamma)$ was introduced earlier.) Then

(6.2) $$\eta_{\varepsilon_1}(\Gamma) \subset \eta_{\varepsilon_1}^{\varepsilon_2}(\Gamma)$$

(6.3) If $\Gamma_1 \mathrel{-\!3} \Gamma_2$ then $\eta_{\varepsilon_1}^{\varepsilon_2}(\Gamma_1) \subset \eta_{\varepsilon_1}^{\varepsilon_2}(\Gamma_2)$

(6.4) Under the hypotheses of Proposition 1, with ε_1 in place of ε in (b), we have: if $p(z)$ is a $[\varepsilon_2]$-(ε_1)-N.E. play of $\tilde{\Gamma}$, then it is a $[\varepsilon_2]$-(ε_1)-N.E. play of $\tilde{\Gamma}$ for every $\tilde{\Gamma} \in \Delta(\Gamma)$.

(6.5) Under the hypotheses of Proposition 2, if $r > B/\varepsilon_1$, then
$$\eta_{\varepsilon_1}^{\varepsilon_2}(r\Gamma) = \eta_{\varepsilon_1}^{\varepsilon_2}(\tilde{\Gamma}) \quad \text{for any } \tilde{\Gamma} \in \Delta(r\Gamma)$$

(6.6) If Γ is layered, and satisfies (3.14), (5.5), (5.6):
$$\eta_{\varepsilon_1}^{\varepsilon_2}(\Gamma) = \eta^{\varepsilon_2}(\Gamma_{c*}) = \eta^{\varepsilon_2}(\Gamma_{\underline{c}})$$

(6.2) follows from the definition. (6.3), (6.4), (6.5), can be established in the same manner as (2.3), Proposition 1, Proposition 2. (6.6) is straightforward to check.

In the presence of (**) or (***) (or both), Proposition 3, and thus the anti-folk theorem of Section 5.2, breaks down. We will now show (Proposition 4 below) that it breaks in a continuous way as a function of the two additional bounds ε_2, ε_3 introduced here. Thus if their magnitude is small, an approximate version of the anti-folk theorem still obtains.

We start with an underlying game $\hat{\Gamma} = (N, \hat{X}, \pi, \{\hat{S}^x\}_{x \in \hat{X}}, \dots)$. A game Γ will be called <u>ε-embedded</u> in $\hat{\Gamma}$ if it is embedded in $\hat{\Gamma}$ (see Section 4), and satisfies (denoting $\Gamma = (N, X, \pi, \{S^x\}_{x \in X}, \dots)$):

(6.7) for any $\alpha \in \hat{S}_i^x$ there is a $\beta \in S_i^x$ with $\|\alpha-\beta\| \leq \varepsilon$.

Put $G_\varepsilon = \{\Gamma : \Gamma$ is ε-embedded in $\hat{\Gamma}\}$, $N(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \bigcup_{\Gamma \in G_{\varepsilon_3}} \eta_{\varepsilon_1}^{\varepsilon_2}(\Gamma)$.

Since $\hat{\Gamma}$ obviously belongs to G_ε, G_ε is not empty.

We shall examine the continuity properties of the set $N(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ in the variables ε_2, ε_3. Note that this set covers $\eta_{\varepsilon_1}^{\varepsilon_2}(\Gamma)$ for any $\Gamma \in G_{\varepsilon_3}$, and that $N(\varepsilon_1, 0, 0) = \eta_{\varepsilon_1}(\hat{\Gamma})$. The assumption on payoffs:

(6.8) $h_i : P(\hat{\Gamma}) \to \mathbb{R}$ is continuous for each $i \in N$

will be needed.

Proposition 4

(i) If $0 \le \varepsilon_1$, $0 \le \varepsilon_2^* \le \varepsilon_2$, $0 \le \varepsilon_3^* \le \varepsilon_3$, then:

$$N(\varepsilon_1, \varepsilon_2^*, \varepsilon_3^*) \subset N(\varepsilon_1, \varepsilon_2, \varepsilon_3) .$$

(ii) Suppose that $\hat{\Gamma}$ is a layered game satisfying (3.14), (5.5), (5.6), (6.8). Then, for fixed $\varepsilon_1 > 0$,

(a) $N(\varepsilon_1, 0, 0) = \bigcap_{\substack{\varepsilon_2>0 \\ \varepsilon_3>0}} N(\varepsilon_1, \varepsilon_2, \varepsilon_3)$;

(b) if, furthermore, each $\hat{S}_i^x$ is closed then $N(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ is upper and lower semi-continuous in ε_2 , ε_3 at $(\varepsilon_1, 0, 0)$.

Proof. (i) Consider a play p in $N(\varepsilon_1, \varepsilon_2^*, \varepsilon_3^*)$. Then there is a game Γ in $G_{\varepsilon_3^*}$ such that for some $[\varepsilon_2^*]$-(ε_1)-N.E. z of Γ , $p(z) = p$. Since $\varepsilon_3 \geqq \varepsilon_3^*$, $G_{\varepsilon_3} \subset G_{\varepsilon_3^*}$, which implies $\Gamma \in G_{\varepsilon_3}$. Furthermore, since $\varepsilon_2 \geqq \varepsilon_2^*$,

$$h_i(p(z)) + \varepsilon_2 \geqq h_i(p(z)) + \varepsilon_2^* \geqq h_i(p^{\varepsilon_1}(z|t_i))$$

for all $t_i \in S^i$, and $i \in N$. (Here S^i is the strategy set of i in Γ .)

(ii)(a) It will suffice to show that every p in $\bigcap_{\substack{\varepsilon_2>0 \\ \varepsilon_3>0}} N(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ is a primitive Nash play of $\hat{\Gamma}$, for then $N(\varepsilon_1, 0, 0) = n_{\varepsilon_1}(\hat{\Gamma})$ $= N(\hat{\Gamma}) = \bigcap_{\substack{\varepsilon_2>0 \\ \varepsilon_3>0}} N(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, using (i) and the corollary of Section 5.

We will show this by contradiction. Suppose some p in $\bigcap_{\substack{\varepsilon_2>0 \\ \varepsilon_3>0}} N(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ is not a primitive Nash play. Then there is a $q \in C_i(p)$ such that $h_i(p \oplus q) > h_i(p)$. By (5.5), (5.6),

$$h_i(\lambda(p \oplus q) + (1-\lambda)p) \geqq \lambda h_i(p \oplus q) + (1-\lambda)h_i(p) > h_i(p)$$

$$\text{for any } 0 < \lambda < 1 .$$

Choose $\tilde{\lambda}$ sufficiently close to 0 to ensure that

$$(6.9) \qquad \delta^*(\tilde{\lambda}(p \oplus q) + (1-\tilde{\lambda})p,\ p) < \varepsilon_1/2 .$$

This can be done because of (3.14). Put $(p \oplus \tilde{q}) = \lambda(p \oplus q) + (1-\lambda)p$. By (6.7) and (6.8), there exists an $\varepsilon_3 > 0$ such that for any Γ in G_{ε_3}, we can find a $\tilde{\tilde{q}} \in C_i(p)$ satisfying

$$(6.10) \qquad \delta^*(p \oplus \tilde{\tilde{q}},\ p \oplus \tilde{q}) < \varepsilon_1/2 \text{ and } h_i(p \oplus \tilde{\tilde{q}}) > h_i(p) .$$

It follows from (6.9) and (6.10) that

$$(6.11) \qquad \delta^*(p \oplus \tilde{\tilde{q}},\ p) \leqq \delta^*(p \oplus \tilde{\tilde{q}},\ p \oplus \tilde{q}) + \delta^*(p \oplus \tilde{q},\ p) < \varepsilon_1 .$$

Again choose ε_2 sufficiently close to 0 to ensure (using (6.8)) that

$$(6.12) \qquad h_i(p \oplus \tilde{\tilde{q}}) > h_i(p) + \varepsilon_2 .$$

Take any game $\Gamma \in G_{\varepsilon_3}$ such that $p \in P(\Gamma)$. Let $(p \oplus \tilde{\tilde{q}}) = (s^{x_0}, s^{x_1}, \ldots)$. Then choose $t_i \in S^i$ in such a way as to satisfy:

$$(6.13) \qquad t_i(x) = \begin{cases} s_i^x & \text{if } i \in \pi(x) \text{ and } x \in \{x_0, x_1, \ldots\} \\ \text{arbitrary otherwise.} \end{cases}$$

Clearly this can be done. Consider any z in $Z_F(\Gamma)$ for which $p(z) = p$. Then we have by (6.11) and (6.13), $p(z|t_i) = (p \oplus \tilde{\tilde{q}})$, i.e.,

$$\delta^*(p(z|t_i), p) < \varepsilon_1 .$$

By (6.12)

$$h_i(p(z|t_i)) = h_i(p \oplus \tilde{\tilde{q}}) > h_i(p) + \varepsilon_2 .$$

Therefore z is not a $[\varepsilon_2]$-(ε_1)-N.E. of Γ, i.e., $p \notin N(\varepsilon_1, \varepsilon_2, \varepsilon_3)$. This is a contradiction.

(ii)(b) Lower semi-continuity is straightforward. Upper semi-continuity can be proved by the method used in (ii)(a).

Q.E.D.

REFERENCES

[1] P. Dubey and M. Kaneko, "Information about Moves in Extensive Games: I," Cowles Foundation Discussion Paper No. 625, March 23, 1982.