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Abstract

We find all the mixed strategy equilibria, within a certain class, of the simplified three-firm location model of Hotelling (in which the price variable is ignored).

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Martin J. Osborne** and Carolyn Pitchik***

1. Introduction

Hotelling [1929] considered the following simple model of location by firms. Two firms produce the same good, with constant unit costs. Each sets a price for its output and selects a point on some line segment at which to locate. The potential consumers of the good are uniformly distributed on the line segment. In order to buy from a firm, a consumer has to travel to the point at which it is located, and pays a constant cost per unit of distance to do so. Each consumer buys one unit of the good from the least costly source (where the cost of travelling is included). The model is appealing because it can be given a number of interesting interpretations. For example, as well as the "pure" location story told above, one can imagine two firms, each capable of producing a good with a variety of possible characteristics, and a continuum of consumers with different preferences.

Hotelling approached the problem of defining an "equilibrium" pattern of behavior in his model in the following way. He wanted to find a Nash equilibrium in prices for each pair of locations, and then use the payoffs at these equilibria to analyze the direction in which each firm would have a tendency to move. However, his calculations contain an error, and no (pure) Nash equilibrium in prices in fact exists for pairs of locations

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close to the center (see d'Aspremont et al. [1979]). Some attempts have been made to modify the model and/or change the solution concept so that an "equilibrium" always exists (see for example d'Aspremont et al. [1981]). Another line of research involves a simplified version of Hotelling's model in which the price variable is ignored (see for example Eaton and Lipsey [1975]). If Hotelling's other assumptions are maintained, this model does possess a (pure) Nash equilibrium, in which both firms locate at the center of the line segment, in accordance with Hotelling's ideas. Furthermore, some generalizations of the simplified model possess (pure) Nash equilibria, so that something can be said about the "robustness" of Hotelling's conclusion. Ideally one would want to carry out the analysis for the full Hotelling model; given that it is not clear how to solve the difficulties mentioned above, examination of the simplified model may be insightful. Also, in some contexts there is no natural "price" variable. For example, the line segment may represent the range of possible political views, and the "firms" may be political parties, which must decide which platform to adopt.

We are interested in what can be said about equilibrium behavior when the number of firms is increased, and the distribution of consumers is made non-uniform in the simplified model. If the number of firms is not equal to three and the customer distribution is uniform, Eaton and Lipsey [1975] have exhibited pure Nash equilibria (or have argued that such exist). They have have also argued that these equilibria are somewhat nonrobust to changes in the customer distribution (except in the case of just two firms). Thus, any strictly concave customer distribution—however close it is to being uniform—leads to the nonexistence of any pure Nash equilibrium. In fact, in order for there to be an equilibrium, the customer distribution must possess at least half as many local maxima as there are firms (see p.35 of Eaton and Lipsey [1975][1]). This "structural instability" casts some doubt upon the significance of the pure equilibria when they do exist. Also, the fact that there is no pure equilibrium if there are three firms (unless the customer distribution is particularly degenerate) suggests that there is the need for a solution concept which exists in a rather larger class of cases.

Dasgupta and Maskin [1977] consider the possibility of allowing the firms to use mixed strategies in the case where there are three firms and a uniform customer distribution. They modify the model in a somewhat ad hoc fashion in order to reduce the severity of certain discontinuities in the payoffs. Then, using a rather complicated notion of continuity, they claim that their modified game does indeed have a mixed strategy equilibirum (see p.18 of their paper), though they do not exhibit one.

Here we shall exhibit all the mixed strategy equilibria, within a certain class, of the unmodified model with three firms and a uniform customer distribution. The fact that these exist, of course, makes any modification to the game unnecessary[2]. There is some presumption that mixed strategy equilibria also exist if there are three firms and the customer distribution is nonuniform, and also if there are four or more firms in this case. We have, however, been unable to establish results to this effect.

It may at first seem unnatural to consider mixed strategies in this context. However, upon closer examination, there is a number of reasons why it is interesting to do so. The fact that no pure equilibrium exists means that the situation is "unstable" in some sense. The mixed strategy equilibria define precisely what sorts of "instability" can exist. To be more concrete, suppose that three political parties have to choose their platforms at a number of points in time. Our results then say that if their choices are made according to some particular probability distributions, there is no incentive for any of them to change their method of choice. Specifically, two patterns may emerge[3/]. One party may always locate in the center, while the other two follow independent probability distributions which are symmetric about the center, and put most of the weight around one quarter and three quarters of the distance along the line. Alternatively, all parties may independently choose from distributions which are uniform between the points one quarter and three quarters. On the other hand, no situation where two parties are fixed and the other randomizes is ever an equilibrium: in a three party political system, at least two of the parties will change their platforms from one period to the next.

In many games, each player can guarantee himself a higher payoff by using a mixed, rather than a pure strategy. In some games, the difference this makes is substantial, and it seems quite natural to think of the players adopting mixed strategies. For example, in "Matching Pennies"[4/] this is so, and it also is in the three person location game. Thus, in the latter case, each pure strategy guarantees a player nothing—the other two players may locate immediately to his left and right. A mixed strategy, on the other hand, can guarantee each player a positive payoff. The (mixed) strategies which guarantee the most are not equilibrium strategies (as they are in "Matching Pennies"). However, the fact that a player risks getting a payoff of zero if he plays a pure strategy may induce him to think about randomization, even if the strategy he finally adopts is not the one which guarantees the most.

The recent work on the "purification" of equilibria (see, for example, Aumann et al. [1981]) does something to make mixed strategies less artificial. The basic result is that players do not need to manufacture random variables upon which to base their actions, but can rely on existing random variables (e.g. stock market prices[5/]) which other players may also observe. Of course, the observations of the players have to possess some degree of independence, though perhaps not as much as one might have thought. The term "purification" may be a little misleading, since the "pure" strategies depend on the observations the players make, and these observations are stochastic. Nevertheless, the results augment the appeal of mixed strategy equilibria by showing that players may not need to consciously invent random variables.

Finally, there is an interpretation of the model in which the random element comes from the behavior of consumers. Suppose each firm/ store can produce and sell one of a whole continuum of goods, with different characteristics, at any one time. Each consumer has a most preferred good, and, out of those available at the time he happens to shop, buys the one which is closest to his favorite. Shopping times are distributed randomly over some interval. Each firm decides over which intervals of time to produce which goods. Then, treating the continuum of characteristics as the space of location, our results identify two different sorts of equilibrium patterns of production for the firms. In one case, one firm always produces a good of type 1/2, while most of the time the other two firms produce goods of types close to 1/4 and 3/4. In the other case, all three firms distribute their product choices uniformly over [1/4, 3/4] during the interval of time. On different days the stores sell goods of different types; given the random behavior of consumers, the choices of the stores can be quite deterministic.

In the next section we shall describe the model formally and state our results. In Section 3 we provide proofs.

2. The Model

Since it is easy to describe the model with an arbitrary number of firms, we shall do so, though our results here relate only to the case of three firms. The firms are involved in a strategic game G_n with player set $N = \{1,\ldots,n\}$, strategy sets $S_i = S = [0,1]$ for each $i \in N$, and payoff functions $K_i : \times_{i \in N} S_i \to \mathbb{R}$ given by

$$K_i(s_1,\ldots,s_n) = \begin{cases} \tfrac{1}{2}(R(i) + s_i)/Q(i) & \text{if } L(i) = \emptyset,\ R(i) \neq \emptyset \\ \tfrac{1}{2}(R(i) - L(i))/Q(i) & \text{if } L(i) \neq \emptyset,\ R(i) \neq \emptyset \\ 1/Q(i) & \text{if } L(i) = R(i) = \emptyset \\ \tfrac{1}{2}(1 - (s_i + L(i)))/Q(i) & \text{if } L(i) \neq \emptyset,\ R(i) = \emptyset \end{cases}$$

where

$$L(i)\ [\text{resp. } R(i)] = \begin{cases} \text{position of closest firm on } i\text{'s left [resp. right]} & \text{if one exists} \\ \emptyset & \text{otherwise} \end{cases}$$

and $Q(i)$ is the number of firms (including i) located at s_i. The set of mixed strategies of i is the set $X_i = X$ of probability distribution functions on S_i (= [0,1]). The degenerate probability distribution function with its entire mass at the point $t \in S$ can of course be identified with the pure strategy $t \in S$; we shall simply use t to denote this mixed strategy. (Such degenerate distribution functions will always be denoted by lowercase letters, so that no confusion will occur.) Suppose that each player $j \in N$ chooses the mixed strategy $F_j \in X$. Then player i's payoff is

$$K_i(F_1,\ldots,F_n) = \int_{S_1} \cdots \int_{S_n} K_i(s_1,\ldots,s_n)dF_1(s_1)\ldots dF_n(s_n).$$

Our main result is that the game G_3 possesses equilibria. We shall in fact show the following.

Proposition 1: G_3 has no equilibrium of the type (a, b, F) (i.e. in which at least two players use pure strategies).

Proposition 2: G_3 has a unique equilibrium (up to symmetry) of the type (c, F, H) where the supports of F and H are equal to the same closed interval. It is (1/2, F, F) where the support of F is [5/24, 19/24] and

$$F(t) = \begin{cases} 1 - 2^{-2/3}(6t - 1)^{-1/3} & 5/24 \leq t \leq 1/2 \\ 2^{-2/3}(5 - 6t)^{-1/3} & 1/2 \leq t \leq 19/24. \end{cases}$$

The corresponding equilibrium payoffs are (38/96, 29/96, 29/96).

The density of the F given in this Proposition is illustrated in Diagram 1.

Proposition 3: G_3 has a unique equilibrium of the type (F_1, F_2, F_3) where the supports of each F_i are equal to the same closed interval and each F_i is differentiable on the interior of its support. It is (F, F, F),

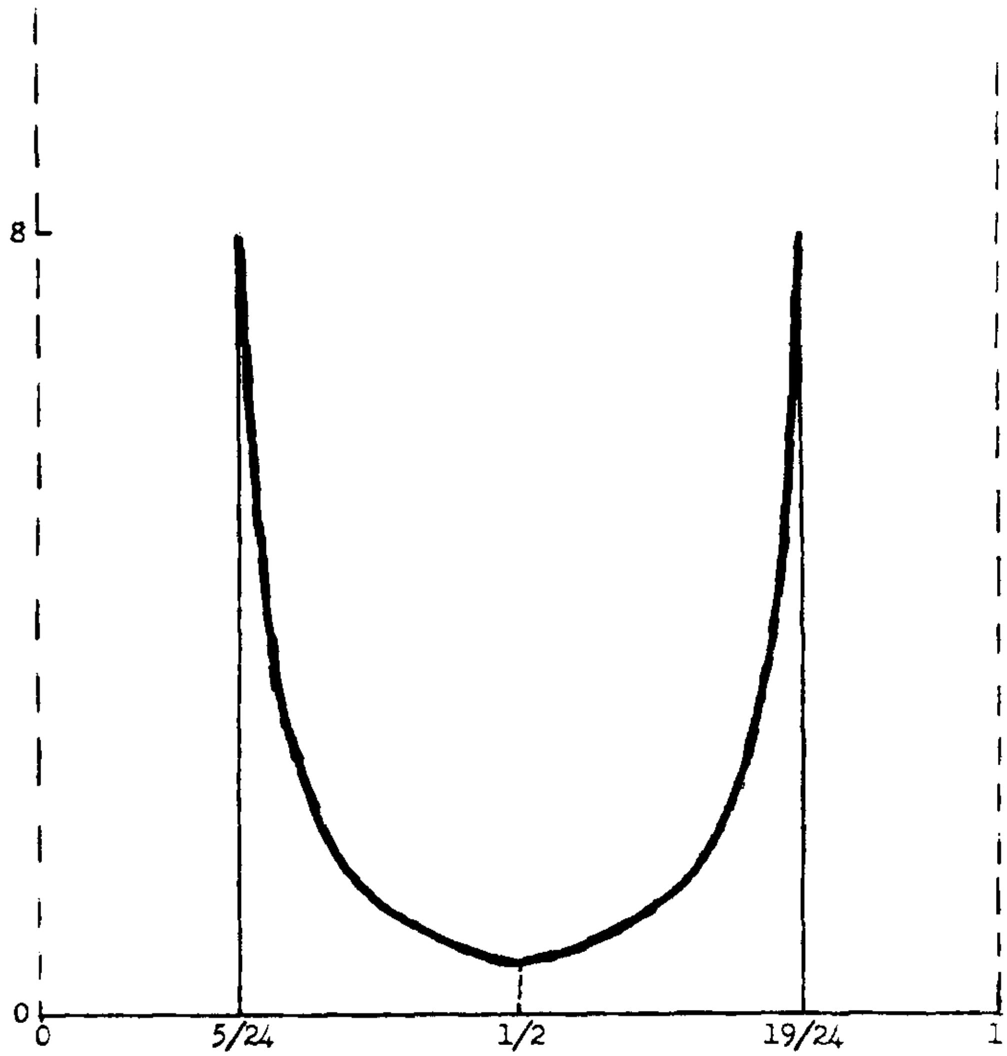

Diagram 1: The Density of the F Given in Proposition 2

where the support of F is $[1/4, 3/4]$ and $F(t) = 2t - 1/2$ for $1/4 \leq t \leq 3/4$. The corresponding equilibrium payoffs are $(1/3, 1/3, 1/3)$.

In the proof of Proposition 2 we use arguments similar to those of Karlin [1959][6] to establish that the continuous part of any equilibrium strategy in this case must be absolutely continuous, and in fact differentiable. It would be nice if we could extend these arguments to cover the case in Proposition 3, but we have been unable to do so. It would be even nicer to show that no equilibrium exists in which the support of some player's strategy is not an interval, so that there are no equilibria of G_3 except those we have found. The two-player zero-sum games which Karlin considers allow him to make a uniqueness argument of this sort[7], but once again we have not been able to extend it to encompass our three-person nonzero-sum game.

3. Proofs

Proof of Proposition 1: Without loss of generality, we may assume that $a \leq b$ and that Player 1 is using the (pure) strategy a, Player 2 the (pure) strategy b, and Player 3 the strategy F. (We already know that F cannot be a pure strategy (see, e.g., Eaton and Lipsey [1975]).) Since $R(3) = \emptyset$ if $s_3 > b$ and $L(3) = \emptyset$ if $s_3 < a$, it is easy to see that the support of F must be contained in $[a, b]$. Otherwise, Player 3 could increase his payoff by putting more weight closer to b (on the right) or by putting more weight closer to a (on the left).

We shall first show that F cannot be a discrete distribution composed simply of weights α, $1 - \alpha$ at the points a, b respectively.

This fact will allow us to conclude that the support of F must include some point $t \in (a, b)$. From there we can infer relationships between a and b which rule out any solution.

Suppose that F is discrete, with a jump of size α at a and one of size $1 - \alpha$ at b. Since a and b are jump points of F, in order for (a, b, F) to be an equilibrium we need $K_3(a, b, a) = K_3(a, b, F) = K_3(a, b, b)$. Thus, equating the first and last terms, we need $(a + b)/4 = 1/2 - (a + b)/4$, or $a + b = 1$. We also need $K_1(a, b, F) \geqq K_1(t, b, F)$ for all $t \in [0, 1]$, so that $K_1(a, b, F) \geqq \lim_{t \uparrow b} K_1(t, b, F)$ and $K_1(a, b, F) \geqq \lim_{t \uparrow a} K_1(t, b, F)$. Thus

$$\frac{\alpha}{2}\left(\frac{a + b}{2}\right) + (1 - \alpha)\left(\frac{a + b}{2}\right) \geqq \alpha\left(\frac{b - a}{2}\right) + (1 - \alpha)b,$$

so that $(2 + \alpha)a - (2 - \alpha)b \geqq 0$, and

$$\frac{\alpha}{2}\left(\frac{a + b}{2}\right) + (1 - \alpha)\left(\frac{a + b}{2}\right) \geqq \alpha a + (1 - \alpha)\left(\frac{a + b}{2}\right),$$

so that $b \geqq 3a$. Combining these last two inequalities with the fact that $a + b = 1$, we get $2 - \alpha \leqq 4a \leqq 1$, so that $\alpha \geqq 1$. But then $\alpha = 1$ and F is a pure strategy, which is impossible as noted above. Thus there exists some point in $\operatorname{supp} F \cap (a, b)$.

We shall next show that if (a, b, F) is an equilibrium then $a = 1/4$ and $b = 3/4$. Since there exists some point in $\operatorname{supp} F \cap (a, b)$ we know that $K_3(a, b, F) = (b - a)/2$, and thus $K_3(a, b, F) \geqq K_3(a, b, a)$ and $K_3(a, b, F) \geqq K_3(a, b, b)$ imply that $(b - a)/2 \geqq (a + b)/4$ and $(b - a)/2 \geqq 1/2 - (a + b)/4$, or

(1) $\qquad b \geqq 3a \quad \text{and} \quad 3b \geqq a + 2.$

We now consider the difference $K_1(s, b, F) - K_1(a, b, F)$ for some $s \in (a, b)$ which is not a jump point of F. In this case,

$$K_1(s, b, F) - K_1(a, b, F)$$

$$= \int_{[a,s)} \left(\frac{b - t}{2}\right) dF(t) + \int_{[s,b]} \left(\frac{s + t}{2}\right) dF(t) - \left(\frac{a + b}{4}\right) F(a) - \int_{(a,b]} \left(\frac{a + t}{2}\right) dF(t)$$

$$= \frac{F(a)}{4}(b - 3a) + \int_{(a,s)} \left(\frac{b - a - 2t}{2}\right) dF(t) + \int_{[s,b]} \left(\frac{s - a}{2}\right) dF(t)$$

$$= \frac{F(a)}{4}(b - 3a) + \int_{(a,s)} \left(\frac{b - a - 2t}{2}\right) dF(t) + (1 - F(s))\left(\frac{s - a}{2}\right).$$

But then if $b > 3a$, we have $b - a - 2t > 0$ if t is close to a, so that for s sufficiently close[8] to a we have

$$\frac{F(a)}{4}(b - 3a) + \int_{(a,s)} \left(\frac{b - a - 2t}{2}\right) dF(t) + (1 - F(s))\left(\frac{s - a}{2}\right) > 0,$$

or $K_1(s, b, F) > K_1(a, b, F)$. So in order for (a, b, F) to be an equilibrium we need $b \leqq 3a$. Making a similar argument near b we obtain $3b \leqq a + 2$. Combining these with (1), we deduce that $b = 3a$ and $3b = a + 2$, or $a = 1/4$ and $b = 3/4$.

Finally, we shall show that $(a, b, F) = (1/4, 3/4, F)$ is not an equilibrium of G_3 for any F. If $s \in (a,b)$ is not a jump point of F, we have, from the calculation above (setting $a = 1/4$, $b = 3/4$),

$K_1(s, b, F) - K_1(a, b, F)$

$$= \int_{(a,s)} \left(\frac{b - a - 2t}{2}\right) dF(t) + (1 - F(s))\left(\frac{s - a}{2}\right)$$

$$= \int_{[a,s)} \left(\frac{b - a - 2t}{2}\right) dF(t) + (1 - F(s))\left(\frac{s - a}{2}\right) \quad \text{(since } b = 3a)$$

$$= \left(\frac{1 - 4s}{4}\right) F(s) + (1 - F(s))\left(\frac{4s - 1}{8}\right) \quad \text{(setting } a = 1/4,\ b = 3/4)$$

$$= (4s - 1)(1 - 3F(s))/8 > 0 \quad \text{if } F(s) < 1/3.$$

Thus $K_1(a, b, F) \geqq K_1(s, b, F)$ implies that $F(s) \geqq 1/3$ for any $s \in (a,b)$ such that s is not a jump point of F. Since F is right-continuous, this means that $F(a) \geqq 1/3$. Analogously we can show that $F(b) \leqq 2/3$.

Again, let $s \in (a,b)$ not be a jump point of F, and consider the following difference:

$K_1(s, b, F) - K_1(a, b, F)$

$$= \int_{(a,s)} \left(\frac{b - a - 2t}{2}\right) dF(t) + (1 - F(s))\left(\frac{s - a}{2}\right)$$

$$\geqq \left(\frac{1 - 4s}{4}\right)(F(s) - F(a)) + (1 - F(s))\left(\frac{4s - 1}{8}\right) \quad \text{(using } a = 1/4,\ b = 3/4)$$

$$= (4s - 1)(1 - 3F(s) + 2F(a))/8 > 0 \quad \text{if } F(s) < (1 + 2F(a))/3.$$

Thus $K_1(a, b, F) \geqq K_1(s, b, F)$ implies that $F(s) \geqq (1 + 2F(a))/3 \geqq (1 + 2/3)/3 = 5/9$ (since $F(a) \geqq 1/3$) for all $s \in (a,b)$ such that s is not a jump point of F. Thus taking limits and using the right-continuity of F we find that $F(a) \geqq 5/9$. Analogously, $F(b) \leqq 4/9$, which is impossible.

Hence there is no strategy F and pure strategies a, b such that (a, b, F) is an equilibrium of G_3, completing the proof.

In the proof of Proposition 2, we shall use the following result, which drastically limits the points at which F and H may have jumps if (c, F, H) is an equilibrium of G_3. Whenever F is a distribution function the expressions "F-measure" and "F-a.e." refer to the (unique) Borel measure defined by F.

Lemma: Suppose (c, F, H) is an equilibrium of G_3, with supp F = supp H = [a, b]. Then $c \in [a, b]$, and the only points at which F and H may have jumps are a, c/3, c, (c + 2)/3, and b; F and H cannot both have jumps at c. If F has no jump at c, then $\int_{[a,b]} x \, dF(x) = 3c - 2F(c)$.

Proof: If Player 1 locates to the right of b --i.e. $s_1 > b$ --then $R(1) = \emptyset$, so that Player 1 can increase his payoff by moving towards b from the right. Similarly if $s_1 < a$ then $L(1) = \emptyset$, so that Player 1 can increase his payoff by moving towards a from the left. Hence for (c, F, H) to be an equilibrium we must have $c \in [a, b]$.

We shall now consider the points at which F may have jumps. First note that if (c, F, H) is an equilibrium then $K_3(c, F, s) \leqq K_3(c, F, H)$ for all $s \in [a, b]$ and $K_3(c, F, s) = K_3(c, F, H)$ H-a.e.. Hence for any $t \in (a,b)$ there exists a sequence $\{s_i\} \subset (a,t)$ with $s_i \uparrow t$ such that $K_3(c, F, s_i) = K_3(c, F, H)$ for all s_i (otherwise $K_3(c, F, s) < K_3(c, F, H)$

for all s in some neighborhood of t, which must have positive H-measure since $\operatorname{supp} H = [a, b]$). Thus $\lim_{s_i \uparrow t} K_3(c, F, s_i) = K_3(c, F, H)$. Similarly there exists a sequence $\{s_i\} \subset (t, b)$ with $s_i \downarrow t$ such that $\lim_{s_i \downarrow t} K_3(c, F, s_i) = K_3(c, F, H)$.

Now suppose that F has a jump of size γ at t (i.e. $F(t) - \lim_{x \uparrow t} F(x) = \gamma$). First consider the case where $t \in (c, b)$. For any $s \in (c, t)$ we have

$$K_3(c, F, s) = \int_{[a,s)} K_3(c, x, s)dF(x) + \int_{\{s\}} K_3(c, x, s)dF(x) + \int_{(s,t)} K_3(c, x, s)dF(x)$$

$$+ \tfrac{1}{2}\gamma(t - c) + \int_{(t,b]} K_3(c, x, s)dF(x).$$

Now since F has a jump at t, $\int_{\{s\}} K_3(c, x, s)dF(x) \to 0$ as $s \uparrow t$. Hence

$$\lim_{s \uparrow t} K_3(c, F, s) = \int_{[a,t)} K_3(c, x, t)dF(x) + \tfrac{1}{2}\gamma(t - c) + \int_{(t,b]} K_3(c, x, t)dF(x).$$

Similarly, for any $s \in (t,b)$ we have

$$K_3(c, F, s) = \int_{[a,t)} K_3(c, x, s)dF(x) + \gamma(1 - \tfrac{1}{2}(s + t)) + \int_{(t,s)} K_3(c, x, s)dF(x)$$

$$+ \int_{\{s\}} K_3(c, x, s)dF(x) + \int_{(s,b]} K_3(c, x, s)dF(x).$$

Hence, as before,

$$\lim_{s \downarrow t} K_3(c, F, s) = \int_{[a,t)} K_3(c, x, t)dF(x) + \gamma(1 - t) + \int_{(t,b]} K_3(c, x, t)dF(x).$$

Thus

$$\lim_{s\uparrow t} K_3(c,\ F,\ s) - \lim_{s\downarrow t} K_3(c,\ F,\ s) = \gamma(\tfrac{1}{2}t - \tfrac{1}{2}c - 1 + t) = \tfrac{1}{2}\gamma(3t - c - 2).$$

Thus if $\gamma > 0$ we need $t = (c + 2)/3$ --i.e. the only point $t \in (c,\ b)$ at which F may have a jump is $(c + 2)/3$.

By a similar argument we can establish that the only possible position for a jump in $(a,\ c)$ is $t = c/3$. We can clearly make the same arguments for H, so it remains to show that F and H cannot have a common jump at c. If they do, then there must be sequences $\{s_i\}$ with $s_i \uparrow c$ and $\{t_i\}$ with $t_i \downarrow c$ such that

$$\lim_{s_i\uparrow c} K_3(c,\ F,\ s_i) = K_3(c,\ F,\ c) = \lim_{t_i\downarrow c} K_3(c,\ F,\ t_i) = K_3(c,\ F,\ H).$$

Letting γ be the size of the jump in F at c, we find that

$$\lim_{s_i\uparrow c} K_3(c,\ F,\ s_i) = \tfrac{1}{2}c(\gamma - F(c) + 2) - \tfrac{1}{2}\int_{[a,c)} x\ dF(x),$$

$$K_3(c,\ F,\ c) = \tfrac{1}{2}F(c) - \gamma/6 + \tfrac{1}{4}c(1 - 2F(c) + \gamma) - \tfrac{1}{4}\int_{[a,c)} x\ dF(x) + \tfrac{1}{4}\int_{(c,b]} x\ dF(x),$$

and

$$\lim_{t_i\downarrow c} K_3(c,\ F,\ t_i) = F(c) - \tfrac{1}{2}c(F(c) + 1) + \tfrac{1}{2}\int_{(c,b]} x\ dF(x).$$

Equating the first and last of these we obtain

$$\tfrac{1}{2}\int_{[a,c)\cup(c,b]} x\ dF(x) = \tfrac{1}{2}c(\gamma + 3) - F(c);$$

using this when equating the first and second we obtain $\gamma = 0$. Hence F and H cannot have a common jump at c. Finally, setting $\gamma = 0$ in the last equation we obtain

$$\int_{[a,c)\cup(c,b]} x\, dF(x) = \int_{[a,b]} x\, dF(x) = 3c - 2F(c).$$

This completes the proof of the lemma.

Proof of Proposition 2: Let (c, F, H) be an equilibrium with $\operatorname{supp} F = \operatorname{supp} H = [a,b]$. Then from the Lemma, both F and H are continuous except possibly on $J = \{a, c/3, c, (c + 2)/3, b\}$. Hence $K_3(c, F, s)$ is continuous in s on $[a, b]\setminus J$, so that we must have $K_3(c, F, s) = K_3(c, F, H) = \bar{K}_3$, say, for all $s \in [a, b]\setminus J$. But if $s \in (a, c)\setminus\{c/3\}$ then

$$K_3(c, F, s) = \int_{[a,s)} \tfrac{1}{2}(c - x)dF(x) + \int_{[s,c]} \tfrac{1}{2}(s + x)dF(x) + \int_{(c,b]} \tfrac{1}{2}(s + c)dF(x).$$

So, integrating by parts, we need

$$\bar{K}_3 = \tfrac{1}{2}(c - s)F(s) + \int_{[a,s)} \tfrac{1}{2}F(x)dx + \tfrac{1}{2}(s + c)F(c) - sF(s)$$

$$- \int_{[s,c)} \tfrac{1}{2}F(x)dx + \tfrac{1}{2}(s + c)(1 - F(c)),$$

or

$$F(s) = \frac{2}{c - 3s}\left[\bar{K}_3 - \tfrac{1}{2}(s + c) - \tfrac{1}{2}\int_{[a,s)} F(x)dx + \tfrac{1}{2}\int_{[s,c)} F(x)dx\right]$$

for all $s \in (a, c)\setminus\{c/3\}$. Hence F is absolutely continuous, and in fact differentiable on $(a, c)\setminus\{c/3\}$. A similar argument establishes that it is also differentiable on $(c, b)\setminus\{(c + 2)/3\}$.

Hence $K_3(c, F, s)$ is differentiable in s on $[a, b]\setminus J$, and hence its derivative must be zero there. Differentiating $K_3(c, F, s)$ for $s \in (a, c)\setminus\{c/3\}$ and setting the result equal to zero, we obtain

(2) $$F'(s)(c - 3s) - F(s) + 1 = 0,$$

which implies that $F(s) = 1 - (1 - \alpha)(3a - c)^{1/3}(3s - c)^{-1/3}$ if $s \in (a, c)\setminus\{c/3\}$, where $\alpha = F(a)$. In order that $F'(s) \geqq 0$ we need $3a \geqq c$, or $c/3 \leqq a$, so that in fact (2) holds for all $s \in (a, c)$. We also need $K_3(c, F, s) \geqq K_3(c, F, a)$ for all $s \in (a, c)$, so that we need

$$\lim_{s\downarrow a} K_3(c, F, s) = \alpha\left(\frac{c - a}{2}\right) + \int_{(a,b]} K_3(c, t, a)dF(t)$$

$$\geqq \tfrac{1}{2}\alpha\left(\frac{c + a}{2}\right) + \int_{(a,b]} K_3(c, t, a)dF(t) = K_3(c, F, a),$$

or $\alpha(c - 3a) \geqq 0$. But from the above, $3a \geqq c$, so if $\alpha > 0$ then we have $3a = c$, in which case $F(s) = 1$ for all $s \in [a, c)$, so that F is a pure strategy, which is impossible by Proposition 1. Hence $\alpha = 0$, so that

(3) $$F(s) = 1 - (3a - c)^{1/3}(3s - c)^{-1/3} \quad \text{for } s \in [a, c).$$

We can make similar arguments for $s \in (c, b)$, to establish that $b \leqq (c + 2)/3$, and there can be no jump in F at b. We then find that

(4) $$F(s) = (2 - 3b + c)^{1/3}(2 - 3s + c)^{-1/3} \quad \text{for} \quad s \in (c, b].$$

The same arguments of course apply to H also, so that $H(s) = F(s)$ if $s \in [a, c) \cup (c, b]$, and hence for all $s \in [a, b]$. But from the Lemma, F and H cannot both have a jump at c. Since they are equal, this means that neither has a jump: $F = H$ is continuous on $[a, b]$, and differentiable except possibly at c.

We shall now find possible values for a, b, and c. Consider Player 1's payoff. If $s \neq c$ it is

(5) $$K_1(s, F, F) = 2\int_{[a,s)} [1 - \tfrac{1}{2}(t + s)]\, F(t)dF(t) + 2\int_{[a,s)}\int_{(s,b]} \tfrac{1}{2}(t - u)dF(t)dF(u) + 2\int_{[s,b]} \tfrac{1}{2}(t + s)(1 - F(t))dF(t).$$

Since F is continuously differentiable on $(a, b)\setminus\{c\}$, $K_1(s, F, F)$ is also. Hence in order that $s = c$ maximize $K_1(s, F, F)$, we need

$$\lim_{s\uparrow c} dK_1(s, F, F)/ds \geqq 0 \quad \text{and} \quad \lim_{s\downarrow c} dK_1(s, F, F)/ds \leqq 0.$$

But

(6) $$dK_1(s, F, F)/ds = F'(s)\left[2F(s) - 3s + \int_{[a,b]} x\, dF(x)\right] + \tfrac{1}{2} - F(s) = \tfrac{1}{2} - F(s),$$

using the last part of the Lemma. Since F is continuous this means that $F(c) = 1/2$. Letting $s \to c$ in (3) and (4) we then need

(7) $$(3a - c)^{1/3}(2c)^{-1/3} = 1/2$$

and

$$(8) \qquad (2 - 3b + c)^{1/3}(2(1 - c))^{-1/3} = 1/2.$$

Using (3) and (4) to calculate $\int_{[a,b]} s\, dF(s)$, the last condition in the Lemma becomes

$$(9) \qquad (2(1 - c))^{-1/3}(2 - 3b + c)^{1/3} = 2(1 - c) - (a + b)/2.$$

Hence a, b, and c must satisfy (7), (8), and (9). Solving these three equations we obtain $a = 5/24$, $c = 1/2$, and $b = 19/24$. Hence the only candidate for an equilibrium of the form (c, F, H) has $c = 1/2$ and

$$(10) \qquad F(s) = H(s) = \begin{cases} 1 - 2^{-2/3}(6s - 1)^{-1/3} & 5/24 \leqq s \leqq 1/2 \\ 2^{-2/3}(5 - 6s)^{-1/3} & 1/2 \leqq s \leqq 19/24. \end{cases}$$

To complete the proof we need to show that this (c, F, H) is in fact an equilibrium of G_3. It is clear that $K_2(1/2, s, F)$ increases to $K_2(1/2, a, F) = K_2(1/2, F, F)$ as s increases to a; similarly it increases to $K_2(1/2, b, F) = K_2(1/2, F, F)$ as s decreases to b. Hence F is a best reply of Player 2 to $(1/2, F)$; identical arguments apply to Player 3. It is also clear that $K_1(s, F, F) \leqq K_1(a, F, F)$ if $s \leqq a$ and $K_1(s, F, F) \leqq K_1(b, F, F)$ if $s \geqq b$. Thus it remains to show that $K_1(s, F, F) \leqq K_1(1/2, F, F)$ if $s \in [a, b]$. Substituting (10) into (6) we can find $dK_1(s, F, F)/ds$. Analyzing its derivative we find that dK_1/ds is negative from $s = a$ until some point, and then is positive until $s = 1/2$. Thus, the only possibilities for maxima of $K_1(s, F, F)$ on $[a, 1/2]$ are $s = a$ or $s = 1/2$. But substituting these values in (5) we find $K_1(a, F, F) = 14/48$, while $K_1(1/2, F, F) = 19/48$. Hence

$K_1(s, F, F) \leqq K_1(1/2, F, F)$ for all $s \in [a, 1/2]$. Since $F(s) = 1 - F(1 - s)$, it is immediate that $K_1(s, F, F) \leqq K_1(1/2, F, F)$ for all $s \in [1/2, b]$ also. Hence $s = 1/2$ is a best response of Player 1 to (F, F). It is easy to check that the payoffs at the equilibrium are $(19/48, 29/48, 29/48)$, completing the proof.

Proof of Proposition 3: Let the common support be $[a, b]$. We begin by showing that at most one of the distributions, say F_3, can have a jump at a. By symmetry, this will imply that at most one of the distributions can have a jump at b. Consider the payoff of Player 3 when he uses the pure strategy t:

$$(11) \quad K_3(F_1, F_2, t) = \int_a^t \left(1 - \frac{s+t}{2}\right) F_1(s)dF_2(s) + \int_a^t \left(1 - \frac{s+t}{2}\right) F_2(s)dF_1(s)$$

$$+ \int_a^t \int_t^b \left(\frac{s-u}{2}\right) dF_1(s)dF_2(u) + \int_a^t \int_t^b \left(\frac{s-u}{2}\right) dF_2(s)dF_1(u)$$

$$+ \int_t^b \left(\frac{s+t}{2}\right)(1 - F_1(s))dF_2(s) + \int_t^b \left(\frac{s+t}{2}\right)(1 - F_2(s))dF_1(s).$$

In order for F_3 to have a jump at a it is necessary that $\lim_{t \downarrow a} K_3(F_1, F_2, t)$ $(= K_3(F_1, F_2, F_3)) = K_3(F_1, F_2, a)$. Using (11) we obtain

$$\lim_{t \downarrow a} K_3(F_1, F_2, t) = 2F_1(a)F_2(a)(1 - a) + \tfrac{1}{2}F_1(a)\left[\int_{(a,b]} s \, dF_2(s) + a(1 - F_2(a))\right]$$

$$+ \tfrac{1}{2}F_2(a)\left[\int_{(a,b]} s \, dF_1(s) + a(1 - F_1(a))\right] + \tfrac{1}{2}\int_{(a,b]} (s + a)(1 - F_1(s))dF_2(s)$$

$$+ \tfrac{1}{2}\int_{(a,b]} (s + a)(1 - F_2(s))dF_1(s).$$

Also

$$K_3(F_1,\ F_2,\ a) = \tfrac{1}{3}F_1(a)F_2(a) + \tfrac{1}{2}F_1(a)\int_{(a,b]}\frac{a+s}{2}dF_2(s) + \tfrac{1}{2}F_2(a)\int_{(a,b]}\frac{a+s}{2}dF_1(s)$$

$$+ \tfrac{1}{2}\int_{(a,b]}(a+s)(1 - F_1(s))dF_2(s) + \tfrac{1}{2}\int_{(a,b]}(a+s)(1 - F_2(s))dF_1(s)$$

$$= \tfrac{1}{3}F_1(a)F_2(a) + \tfrac{1}{4}F_1(a)\left[a(1 - F_2(a)) + \int_{(a,b]} s\ dF_2(s)\right]$$

$$+ \tfrac{1}{4}F_2(a)\left[a(1 - F_1(a)) + \int_{(a,b]} s\ dF_1(s)\right] + \tfrac{1}{2}\int_{(a,b]}(a+s)(1 - F_1(s))dF_2(s)$$

$$+ \tfrac{1}{2}\int_{(a,b]}(a+s)(1 - F_2(s))dF_1(s).$$

Thus we need

$$\lim_{t\downarrow a} K_3(F_1,\ F_2,\ t) - K_3(F_1,\ F_2,\ a)$$

$$= \frac{1}{12}\left[F_1(a)F_2(a)(20 - 30a) + 3F_1(a)\left\{a + \int_{(a,b]} s\ dF_2(s)\right\} + 3F_2(a)\left\{a + \int_{(a,b]} s\ dF_1(s)\right\}\right]$$

$$= 0.$$

To analyze this expression, consider the function $g(x,\ y) = xy(20 - 30a) + 3x(a + k_1) + 3y(a + k_2)$ where $k_1 > 0$, $k_2 > 0$. We need $g(x,\ y) = 0$, i.e. $y = -3x(a + k_1)/\left[(20 - 30a)x + 3(a + k_2)\right]$. But $dy/dx = -9(a + k_1)(a + k_2)/\left[(20 - 30a)x + 3(a + k_2)\right]^2 < 0$. Thus the unique solution to $g(x,\ y) = 0$ in $[0,\ 1]\times[0,\ 1]$ is $x = y = 0$, so that the only way for F_3 to have a jump at a is for both $F_1(a)$ and $F_2(a)$ to be equal to zero (i.e. at most one of the distributions F_1, F_2, F_3 has a jump at a). Without loss of generality, we shall assume that $F_1(a) = F_2(a) = 0$ and $F_3(a) \geq 0$.

We shall now show that $F_1(t) = F_2(t) = 2t - 1/2$ is the only possible choice for F_1 and F_2. Since the support of F_i is $[a, b]$, for $i = 1, 2, 3$, it is necessary that $K_i(t, F_j, F_k)$ be constant for all $t \in (a, b)$. Thus, since we are assuming that F_i is differentiable on (a, b) $(i = 1, 2, 3)$, we need $dK_2(F_1, t, F_3)/dt = 0 = dK_1(t, F_2, F_3)/dt$ for all $t \in (a, b)$, with initial conditions $F_1(a) = F_2(a) = 0$ (F_3 given). Thus both F_1 and F_2 must solve the same differential equation $y' = f(t, y, F_3)$ with the same initial condition $y(a) = 0$. Thus $F_1(t) = F_2(t) = F(t)$ which implies that F does not have a jump at b since at most one distribution has a jump there. But now we need $dK_3(F_1, F_2, t)/dt = dK_3(F, F, t)/dt = 0$ for all $t \in (a, b)$. Thus, using (11) with $F_1 = F_2 = F$ we obtain

$$dK_3(F, F, t)/dt = F'(t)\left[2F(t) - 3t + \int_a^b s\, dF(s)\right] + 1/2 - F(t) = 0.$$

Thus, F solves the differential equation

$$\frac{dy}{dx}(2y - 3x + k) + 1/2 - y = 0 \quad \text{where} \quad k = \int_a^b s\, dy(s),$$

or

$$dy(2y - 3x + k) + dx(1/2 - y) = 0.$$

Solving implicitly with the integrating factor $C(y - \frac{1}{2})^2$ we obtain

$$C(y - 1/2)^3\left[(y - 1/2)/2 - x + (1 + k)/3\right] + D = 0.$$

But $y(a) = 0$, $y(b) = 1$ implies that $y = 1/2$ for some x which implies that $D = 0$ and

$$y(x) = 2x - 1/6 - 2k/3 \quad \text{where} \quad k = \int_a^b s\,dy(s) = \int_a^b 2s\,ds = b^2 - a^2,$$

so $y(x) = 2x - 2b^2/3 + 2a^2/3 - 1/6$. But we need $y(a) = 0$, $y(b) = 1$, so that $a = 1/4$, $b = 3/4$, and $y(x) = 2x - 1/2$, i.e. $F(s) = 2s - 1/2$, supp $F = [1/4, 3/4]$. But, given F_1, F_2 we can now solve for F_3. Since $K_1(t, F_2, F_3)$ must be constant for all $t \in (a, b)$, we need $dK_1(t, F_2, F_3)/dt = 0$ for all $t \in (a, b)$. Using (11) we obtain

$$dK_1(t, F, F_3)/dt = F_3'(t)(t - 1/2)/2 + 3F(t)/2 - 4t + 3/2 - \int_a^b F_3(s)ds.$$

So F_3 must solve the differential equation

$$\frac{dy}{dx}(x - 1/2) + 3y - 8x + 3 - 2k = 0 \quad \text{where} \quad k = \int_a^b y(s)ds, \quad a = 1/4,\ b = 3/4,$$

or

$$dy(x - 1/2) + dx(3y - 8x + 3 - 2k) = 0.$$

Solving implicitly with integrating factor $C(x - 1/2)^2$, we obtain

$$C(x - 1/2)^3\left[y - 2x + 2/3 - 2k/3\right] + D = 0.$$

Since $1/4 < x < 3/4$, we must have $x = 1/2$ at some point, which implies that $D = 0$. Hence

$$y(x) = 2x - 2/3 + 2k/3 \quad \text{where} \quad k = \int_{1/4}^{3/4} (2s - 2/3 + 2k/3)ds = 1/6 + k/3,$$

which implies that $k = 1/4$, and so $y(x) = 2x - 1/2$ for $1/4 < x < 3/4$. Thus $F_3(t) = 2t - 1/2$ on $(1/4, 3/4)$. Since $0 \leqq F_3(t) \leqq 1$ in fact $F_3(t) = 2t - 1/2$ on $[1/4, 3/4]$.

So far we have shown that if (F_1, F_2, F_3) is an equilibrium point of G_3 as described in the Proposition then $F_i(t) = F(t) = 2t - 1/2$, $a = 1/4$, and $b = 3/4$, for $i = 1, 2, 3$. We now show that (F, F, F) is indeed an equilibrium point of G_3. We already know that $K_i(t, F, F) = K_i(F, F, F)$ for all $t \in [a, b]$. It remains to show that $K_i(F, F, F) \geqq K_i(t, F, F)$ for all $t \in [0, a) \cup (b, 1]$. Since F satisfies the condition $F(t) = 1 - F(1 - t)$ it is sufficient to show that $K_i(F, F, F) \geqq K_i(t, F, F)$ for all $t \in [0, a)$. But, as before, it is clear that $K_i(t, F, F)$ is increasing on $[0, a)$. Thus $K_i(F, F, F) = K_i(a, F, F) \geqq K_i(t, F, F)$ for all $t \in [0, a)$, where the middle inequality comes from the fact that F is continuous at a so that $K_i(F, F, F) = \lim_{t \downarrow a} K_i(t, F, F) = K_i(a, F, F)$. This completes the proof.

Footnotes

1/ Note that "$n \geq 2M$" about two thirds of the way down p.35 is a typographic error for "$n \leq 2M$".

2/ Presumably the idea behind the modifications which Dasgupta and Maskin make is that they produce games close (in some sense) to the original one, so that the equilibria (if any) of the new games indicate something about the way the players might behave in the original one, even if the latter do not actually possess equilibria. As far as pure strategy equilibria are concerned, this seems unlikely to be a useful procedure, since there is no presumption that close to any given game there is one which possesses a pure strategy equilibrium. As for mixed strategy equilibria, even though continuity assumptions appear in some results on existence of such equilibria, wide classes of game with discontinuities in their payoffs do in fact possess such equilibria (see, for example, Chapters 5 and 6 of Karlin [1959], and Pitchik [1982]).

3/ We are ignoring the possibility that players can choose strategies in which their action at some point depends on the actions of the other players at previous points. The equilibria we describe are equilibria of this repeated game, but there may be others.

4/ I.e. the zerosum game in which there are two players ($i = 1, 2$), each with two pure strategies ($s = 1, 2$), and the payoff to 1 when he uses the strategy j and the other player uses the strategy k is 1 if $j = k$ and 0 if $j \neq k$.

5/ It is perhaps careless to say that "stock market prices are random". But it may be true that their behavior is approximated very well by that of a random variable. An individual whose payoff depends crucially on the level of stock prices might be concerned to penetrate the apparent randomness, and develop a theory which really explains the observed behavior. However, if we consider players in a game for which the payoffs have little to do with stock prices, we can presumably conclude that none will have any reason to doubt that they behave stochastically. (One can of course generalize this argument: no phenomenon is <u>really</u> random. Even the outcome of a coin toss could be predicted, given sufficient information on the way it is carried out, and on the environment in which it occurs.)

6/ See Problem 17 on p.142 and its solution on p.294.

7/ See Problem 16 on p.142 and its solution on pp.293-4 of Karlin [1959]

8/ Such exists since F can have at most a countable number of jumps.

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